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Abstract

We investigate asymptotic properties of two–dimensional empirical point processes of exceedances (EPPE). We give a new description of a class of possible limit laws. Necessary and sufficient conditions for the weak convergence of one–dimensional EPPEs and complete convergence of two–dimensional EPPEs are established.

Keywords: *extreme values, point process, complete convergence.*

1 Introduction

The two–dimensional empirical point process of exceedances (EPPE) is a key tool for approximating probabilities of exceedances of distinct levels during different periods of time (cf. [15, 25, 8]). For instance, a stationary sequence $\{X_i\}$ of (dependent) random variables can represents claims to an insurance company. Let $N(t_i, I_i)$ denote the number of claims exceeding a level t_i in the time interval $I_i = [T_i^-; T_i^+]$. It can be of interest to approximate the probability $\mathbb{P}(N(t_1, I_1) = n_1, ..., N(t_k, I_k) = n_k)$. This question can be easily addressed if the distribution of a two–dimensional EPPE has been approximated.

In the i.i.d. case, the limiting distribution of an EPPE is necessarily Poisson (see [24, 25, 15]). In the presence of dependence, extremes may appear in clusters. As a result, the limiting distribution of a one–dimensional EPPE counting locations of extremes is necessarily compound Poisson (see [12] and Theorem A below).

One–dimensional EPPEs describing heights of extremes have not been well studied in literature before (some limit properties of such processes can be obtained if a complete convergence theorem for a two–dimensional EPPE is proved). We fill that gap in Section 4.1. We describe the class $\mathcal{P}'$ of possible limit laws and establish necessary and sufficient conditions for the weak convergence of an EPPE to a given element of $\mathcal{P}'$. We show also that corresponding jump processes have stochastically continuous trajectories.

Two–dimensional EPPEs count both locations and heights of extremes. The class $\mathcal{P}$ of possible weak limits of two–dimensional EPPEs has been described by Mori [18] as a class of infinitely divisible point processes that are invariant under certain transformations. Hsing [10] represented a point process $P \in \mathcal{P}$ in terms of the points of a two–dimensional Poisson point process and a one–dimensional point process (see Theorem B below).

In this paper we suggest a different way of defining a two–dimensional EPPE. A feature of this approach is that one has to specify a minimal level u_n such that exceedances of u_n

are considered extreme. An advantage is that a weak limit of a two–dimensional EPPE appears a natural generalisation of a compound Poisson point process. Necessary and sufficient conditions for the complete convergence of a two–dimensional EPPE to a given limit are established.

2 Background

2.1 Number of exceedances and one–dimensional EPPEs

Let $X, X_1, X_2, \ldots$ be a strictly stationary sequence of (dependent) random variables (r.v.s). Denote

$$M_n = \max_{1 \leq i \leq n} X_i \, , \; N_n(u) = \sum_{1 \leq i \leq n} \mathbb{I}\{X_i > u\} \, .$$

The random variable $N_n(u)$ is the number of exceedances over the level u by the random variables $X_1, \ldots, X_n$. Let

$$X_{n,n} \leq \ldots \leq X_{1,n} = M_n$$

be the sample order statistics. Evidently, $\{X_{k,n} \leq u\} = \{N_n(u) < k\}$.

We study asymptotic properties of the distribution of $N_n(u)$. It is clear that the sequence of threshold levels $\{u = u_n\}$ must satisfy an appropriate condition which guarantees that $\{N_n(u_n)\}$ has a non–degenerate limiting distribution. A common approach is to assume that there exists the limit

$$\lim_{n \to \infty} \mathbb{P}(M_n \leq u_n) := e^{-\lambda} \qquad (0 < \lambda < \infty) \, . \tag{2.1}$$

Let $\mathbf{\Pi}(\lambda, \zeta) \equiv \mathbf{\Pi}(\lambda, \mathcal{L}(\zeta))$ denote the compound Poisson distribution with intensity λ and multiplicity distribution $\mathcal{L}(\zeta)$, i.e.,

$$\mathbf{\Pi}(\lambda, \zeta) = \mathcal{L}\left(\sum_{i=0}^{\pi(\lambda)} \zeta_i \right) ,$$

where $\zeta_0 = 0$, $\zeta_i \stackrel{d}{=} \zeta$ $(i \geq 1)$, the random variables $\pi(\lambda), \zeta, \zeta_1, \zeta_2, \ldots$ are independent and $\pi(\lambda)$ has the Poisson $\mathbf{\Pi}(\lambda)$ distribution.

Denote by $\mathcal{R}$ the class of sequences $\{r = r_n\}$ of natural numbers such that

$$r_n \to \infty, \, r_n = o(n) \, .$$

The random variable ζ is called the *limiting cluster size* if (2.1) holds and

$$\mathcal{L}(N_r(u_n) \mid N_r(u_n) > 0) \Rightarrow \mathcal{L}(\zeta) \tag{2.2}$$

for some sequence $\{r\} \in \mathcal{R}$. Under mild conditions, the only possible limit law for $N_n(u_n)$ is compound Poisson:

$$N_n(u_n) \Rightarrow \sum_{i=0}^{\pi(\lambda)} \zeta_i \, . \tag{2.3}$$

In Section 3 we present a multilevel generalisation of this limit theorem. The only possible limit law for a vector of numbers of exceedances of distinct levels is compound Poisson.

The important particular case is a pure Poisson limit. The problem of evaluating accuracy of Poisson approximation for a sum of dependent 0's and 1's attracted significant attention (see [2, 6, 5] and references therein). The accuracy of compound Poisson approximation for $\mathcal{L}(N_n(u_n))$ has been evaluated in [3, 19, 26, 28] (see also references in [2, 19]).

The next level of generality is to consider the one–level point process of exceedances $N_n[\cdot]$, where

$$N_n[B] \equiv N_n[B, u_n] = \sum_{1 \le i \le n} \mathbb{I}\{ i/n \in B,\, X_i > u_n \} \tag{2.4}$$

for any Borel set $B \subset (0; 1]$.

The process (2.4) naturally appears when one wants to approximate the joint distribution of the numbers of exceedances *of the same level* during different periods of time.

The following result has been proved by Hsing et al. [12] under a mild mixing condition.

Theorem A [12]. *If (2.1) and (2.2) are in force then*

$$N_n[\cdot] \Rightarrow N[\cdot], \tag{2.5}$$

where $N[\cdot]$ is a compound Poisson point process with intensity rate λ and multiplicity distribution $\mathcal{L}(\zeta)$.

If $N_n[\cdot]$ converges weakly to some point process $N[\cdot]$ then $N[\cdot]$ is a compound Poisson point process on $(0; 1]$ with some multiplicity distribution $\mathcal{L}(\zeta)$ and intensity rate λ that obeys (2.1) and, whenever $\lambda > 0$, (2.2) holds.

Let $\{u_n(\cdot), n \ge 1\}$ be a sequence of functions such that $u_n(\cdot)$ is strictly decreasing for all large enough n, $u_n(0) = \infty$ and

$$\lim_{n\to\infty} \mathbb{P}\,(M_n \le u_n(t)) = e^{-t} \qquad (t > 0). \tag{2.6}$$

If $\{M_n\}$ has a limiting distribution with normalising sequences $\{a_n\}$ and $\{b_n\}$, i.e., if

$$\lim_{n\to\infty} \mathbb{P}\,((M_n - b_n)/a_n \le x) = G(x) \qquad (x \in \mathbb{R}), \tag{2.7}$$

where G is necessarily a distribution function (d.f.) from one of the three extreme value types of d.f.s (see [9, 15]) then one can put $u_n(t) = a_n G^{-1}(e^{-t}) + b_n$.

The process (2.4) counts *location* points (along the horizontal axis) where exceedances of the level u_n occur. Equivalently, it can be viewed as a jump process $\{N_n[0; s], s \in (0; 1]\}$.

A natural question is to approximate the distribution of the jump process

$$\{N_n(u_n(t)), t \in [0; T]\} \tag{2.8}$$

which describes *heights* of extremes. Necessary and sufficient conditions for the weak convergence of the process (2.8) to a compound Poisson process are given in [19]. In Section 4.1 below we describe the class $\mathcal{P}'$ of limiting distributions of the process (2.8) and present necessary and sufficient conditions for the weak convergence of the process (2.8) to a given $P' \in \mathcal{P}'$.

2.2 Two–dimensional EPPE

The two–dimensional point process of exceedances N_n^* can be defined by

$$N_n^*(A) := \sum_{i\geq 1} \mathrm{I\!I}\{\,(i/n, u_n^{-1}(X_i)) \in A\,\} \tag{2.9}$$

for any Borel set $A \subset (0;\infty) \times [0;\infty)$. If $A_t = (0;1] \times [0;t)$ then

$$N_n^*(A_t) = N_n(u_n(t)).$$

The weak convergence of N_n^* to a limiting point process is often called the *complete convergence*. Many results of Extreme Value Theory can be drawn as consequences if a complete convergence theorem is established (cf. [25, 19]).

If the r.v.s $\{X_i\}$ are independent (or dependent but without asymptotic clustering of extremes) then N^* converges weakly to a Poisson point process on $(0;\infty) \times [0;\infty)$ (see [1, 15, 24, 25]).

In the case of a stationary α–mixing sequence of random variables, the class $\mathcal{P}$ of possible limit laws for N_n^* was described by Mori [18] as a class of "infinitely divisible point processes invariant under certain transformations". Hsing [10] represented elements of $\mathcal{P}$ in terms of the points of a two–dimensional Poisson point process and a one–dimensional point process.

Theorem B [18, 10]. *Suppose that (2.6) holds and*

$$\lim_{s\to 0,\, t\to\infty} \mathrm{I\!P}(u_n(t) < X < u_n(s)) = 1$$

for all large enough n. If N_n^* converges weakly to a point process N^* then N^* is an infinitely divisible point process with the following properties:
(1) $N^* \circ g_t \stackrel{d}{=} N^*$ $(t > 0)$, where $g_t(x,y) = (x+t, y)$,
(2) $N^* \circ h_s \stackrel{d}{=} N^*$ $(s > 0)$, where $h_s(x,y) = (x/s, ys)$,
(3) $\mathrm{I\!P}(N^*((0;\varepsilon) \times (0;1)) \to 0$ as $\varepsilon \to 0$,
(4) N^* has independent increments along the horizontal axis.
The process N^* admits the representation

$$N^*(\cdot) = \sum_{i\geq 1}\sum_{j=1}^{K_i} \mathrm{I\!I}\{(Y_i, Z_i\kappa_{ij}) \in \cdot\}, \tag{2.10}$$

where $\{(Y_i, Z_i), i \geq 1\}$ *are the points of a two-dimensional homogeneous Poisson point process* η *with the Lebesgue intensity measure,* $\{\kappa_{ij}, 1 \leq j \leq K_i\}$ *are the points of a point process* κ_i *on* $[1;\infty)$, $\kappa_i \stackrel{d}{=} \kappa$ $(i \geq 1)$, *the process* κ *has an atom at 1, the processes* $\eta, \kappa_1, \kappa_2, ...$ *are mutually independent.*

Necessary and sufficient conditions for the convergence $N_n^* \Rightarrow P \in \mathcal{P}$ are given in [23] in the assumption that the sequence $\{X_i, i \geq 1\}$ possesses an *extremal index*. The idea of those conditions is that the distribution of the vector $\{N_r(u_n(t_1)), \ldots, N_r(u_n(t_k))\}$ is assumed to be "close" to that of the limiting process for any choice of $t_1 < ... < t_k$, $k \in \mathbb{N}$.

An important particular case among those possible limit laws is the compound Poisson. Necessary and sufficient conditions for the weak convergence of N_n^* to a compound Poisson point process (in terms of exceedances of two levels only) have been given in [19].

The Mori–Hsing characterisation (2.10) of the process N^* can be regarded as implicit. In Section 4 we suggest a different way of defining a two–dimensional EPPE and describe the class $\mathcal{P}_*$ of limiting point processes. We represent an element of $\mathcal{P}_*$ in terms of one–dimensional processes only (in particular, our representation immediately implies that the only possible limit law for the one–level process (2.4) is compound Poisson).

A weak limit of an EPPE is given in a form that seems to be a natural generalisation of a compound Poisson point process. Corresponding necessary and sufficient conditions for the complete convergence of a two–dimensional EPPE to a given $P_* \in \mathcal{P}_*$ are established.

Unless otherwise specified, limits are assumed as $n \to \infty$; a sum over $\emptyset$ equals zero. Below, $\{\pi_\lambda(s), s \geq 0\}$ is a Poisson process with intensity rate λ and $\pi(\cdot) \equiv \pi_1(\cdot)$.

3 Exceedances of multiple levels

3.1 Conditions

Remind that the functions $u_n(\cdot)$ are strictly decreasing for all large enough n, $u_n(0) = \infty$ and (2.6) holds. In the rest of the paper we assume that

$$\limsup n\mathbb{P}(X_n > u_n(t)) < \infty \qquad (0 < t < \infty). \tag{3.1}$$

Note that (2.6) does not imply (3.1) — for instance, consider the case $X_i \equiv X$. Denzel and O'Brien [7] give an example of an α–mixing sequence such that (2.6) holds though $n\mathbb{P}(X_n > u_n(t)) \to \infty$. On the other hand, (3.1) follows from (2.6) under a stronger mixing condition (cf. O'Brien [21], Lemma 3).

Denote $u_n(\bar{t}) = (u_n(t_1), ..., u_n(t_k))$, and let $\mathcal{F}_{l,m} \equiv \mathcal{F}_{l,m}(u_n(\bar{t}))$ be the σ–field generated by the events $\{X_i > u_n(t_j)\}$, $l \leq i \leq m, 1 \leq j \leq k$. Put

$$\alpha(l, \{u_n(\bar{t})\}) = \sup |\mathbb{P}(AB) - \mathbb{P}(A)\mathbb{P}(B)|,$$

where the supremum is taken over $A \in \mathcal{F}_{1,m}$, $B \in \mathcal{F}_{m+l+1,n}$, $m \geq 1$ such that $\mathbb{P}(A) > 0$.

Condition $\Delta\{u_n(\bar{t})\}$ *is said to hold if*

$$\alpha(l_n, \{u_n(\bar{t})\}) \to 0$$

for some sequence $\{l_n\} \in \mathcal{R}$ (perhaps, dependent on $\{u_n(t_j)\}_{1\le j\le k}$).

We say that *condition* Δ *holds if* $\Delta\{u_n(\bar{t})\}$ *is in force for every choice of* $0 < t_1 < \ldots < t_k < \infty$, $k \in \mathbb{N}$.

Condition Δ^* *is said to hold if* $\alpha([cn], \{u_n(t_j)\}_{1\le j\le k}) \to 0$ *for every* $c \in (0;1)$ *and every choice of* $0 < t_1 < \ldots < t_k < \infty$, $k \in \mathbb{N}$ (thus, Δ^* implies Δ).

Denote by $\mathcal{R}(\bar{t}) \equiv \mathcal{R}(\{u_n(\bar{t})\})$ the class of sequences $\{r = r_n\} \in \mathcal{R}$ such that

$$n \gg r_n \gg l_n\,,\; n r_n^{-1} \alpha_n \to 0\,, \tag{3.2}$$

where $\alpha_n = \alpha(l_n, \{u_n(\bar{t})\})$. Evidently, $\mathcal{R}(\bar{t})$ is not empty: one can put

$$r_n = \max\left\{[n\sqrt{\alpha_n}]; [\sqrt{n l_n}]\right\}. \tag{3.3}$$

If the sequence $\{X_i, i \ge 1\}$ is α-mixing then (3.2) holds with one and the same sequence $\{r_n\}$ for all $\bar{t} \in \mathbb{R}_k$ (r_n can be defined by (3.3)).

3.2 Exceedances of distinct levels

In this subsection we are interested in the joint limiting distribution of the vector

$$\{N_n(u_n(t_1)), \ldots, N_n(u_n(t_k))\}$$

of the numbers of exceedances of several levels. Results on the limiting distribution of the vector $\{N_n(u_n(t_1)), \ldots, N_n(u_n(t_k))\}$ imply, in particular, the corresponding ones on the joint limiting distribution of a finite number of upper order statistics.

The class of possible limit laws for the joint limiting distribution of the first and the second maxima has been described by Welsch [30] and Mori [17] (see also [20]). Welsch's result was generalised by Hsing [11]. Under the assumption that the sequence $\{X_i, i \ge 1\}$ is α-mixing and (2.6) holds, Hsing [11] shows that the probability

$$\mathbb{P}(X_{1,n} \le u_n(s)\,,\, X_{k,n} \le u_n(t)) \equiv \mathbb{P}(N_n(u_n(s)) = 0, N_n(u_n(t)) < k) \tag{3.4}$$

converges for every $t > s > 0$ if and only if there exist functions $\rho_i(\cdot)$ and a sequence $\{r\} \in \mathcal{R}$ such that

$$\mathbb{P}(N_r(u_n(s)) = 0\,,\, N_r(u_n(t)) = i \,|\, N_r(u_n(t)) > 0) \to \rho_i \equiv \rho_i(s/t) \qquad (i \ge 1)$$

for each $t > s > 0$ and $i \in \{1, \ldots, k-1\}$. Notice that the expression suggested in [11] for the limit of probability (3.4) can be simplified to the form

$$\lim \mathbb{P}\left(N_n(u_n(s)) = 0,\, N_n(u_n(t)) < k\right) = \mathbb{P}\left(\sum_{i=1}^{\pi(t)} \zeta_i^* < k\right), \tag{3.5}$$

where the distribution of i.i.d.r.v.s $\{\zeta_i^*, i \ge 1\}$ depends on s/t: $\mathbb{P}(\zeta_1^* = i) = \rho_i(s/t)$.

Sufficient conditions for the weak convergence of the vector $\{N_n(u_n(s)), N_n(u_n(t))\}$ have been suggested in [19]. Notice that the weak convergence of M_n does not, in general, imply the weak convergence of the vector $\{N_n(u_n(s)), N_n(u_n(t))\}$. Mori [17] gives an example of a stationary sequence of one–dependent r.v.s such that (2.6) holds while $\{X_{1,n}, X_{2,n}\}$ does not converge.

It is convenient to write

$$N_n(a,t) = \sum_{1\le i\le a} \mathbb{I}\{X_i > u_n(t)\} \qquad (t \ge 0, a \ge 1). \tag{3.6}$$

Let $\bar{i} = (i_1, ..., i_k)$, $\bar{t} = (t_1, ..., t_k)$,

$$\mathbb{N}_k = \{\bar{i} \in \mathbf{Z}_+^k : i_1 + ... + i_k > 0\}, \ \mathbb{R}_k = \{\bar{t} \in \mathbb{R}^k : 0 < t_1 < ... < t_k < \infty\}.$$

For any $m \in \mathbb{N}, \bar{t} \in \mathbb{R}_k$, put

$$N_n(m,\bar{t}) = \{N_n(m,t_1), ..., N_n(m,t_k)\}.$$

Let $\zeta(\bar{t},n) \equiv \{\zeta^1(\bar{t},n), ..., \zeta^k(\bar{t},n)\}$ be a random vector with the distribution

$$\mathcal{L}(\zeta(\bar{t},n)) = \mathcal{L}(N_n(r,\bar{t}) \,|\, N_n(r,t_k) > 0). \tag{3.7}$$

Proposition 1 *Assume condition* $\Delta\{u_n(\bar{t})\}$. *If*

$$\zeta(\bar{t},n) \Rightarrow \exists \zeta(\bar{t}) \tag{3.8}$$

for some sequence $\{r\} \in \mathcal{R}(\bar{t})$ *then*

$$N_n(sn,\bar{t}) \Rightarrow \sum_{j=1}^{\pi(st_k)} \zeta_j(\bar{t}) \qquad (\forall s > 0), \tag{3.9}$$

where $\{\zeta_j(\bar{t}), j \ge 1\}$ *are independent copies of* $\zeta(\bar{t})$.

If the assumptions of Proposition 1 hold then (3.9) implies (3.5).

We say that the *random vector* Y *has a compound Poisson distribution if*

$$Y \stackrel{d}{=} X_1 + ... + X_{\pi(t)},$$

where vectors $\{X_i, i \ge 1\}$ *are independent,* $X_i \stackrel{d}{=} X$ $(i \ge 1)$ *and* $\pi(t)$ *is a Poisson* $\mathbf{\Pi}(t)$ *random variable.*

The following result is a multilevel generalisation of a compound Poisson limit theorem for $N_n(u_n)$.

Theorem 2 *Assume condition* $\Delta\{u_n(\bar{t})\}$. *If* $N_n(n,\bar{t})$ *converges weakly then (3.8) holds for any sequence* $\{r\} \in \mathcal{R}(\bar{t})$. *The distribution of the random vector* $\zeta(\bar{t}) = \{\zeta^1(\bar{t}),...,\zeta^k(\bar{t})\}$ *is scale–invariant:*

$$\zeta(\bar{t}) \stackrel{d}{=} \zeta(a\bar{t}) \qquad (\forall a > 0) \tag{3.10}$$

and does not depend on the choice of the sequence $\{r\} \in \mathcal{R}(\bar{t})$. *The marginal distributions of* $\zeta(\bar{t})$ *obey formula (3.12) below. The weak limit of* $N_n(n,\bar{t})$ *is necessarily a compound Poisson random vector:*

$$N_n(sn,\sigma\bar{t}) \Rightarrow \sum_{j=1}^{\pi(s\sigma t_k)} \zeta_j(\bar{t}) \qquad\qquad (s > 0,\, \sigma > 0). \tag{3.11}$$

Notice that vectors $N_n(sn,\bar{t})$ and $N_n(n,s\bar{t})$ have the same limiting distribution.

Note also that Hsing's [11] description of the joint limiting distribution of the first and the k-th sample maxima is given in the assumption that probabilities (3.4) converge for all $t > s > 0$. The feature of our result is that we assume only the convergence of $N_n(n,\bar{t})$ for a fixed $\bar{t}$, and then show that this implies the weak convergence of $N_n(sn,\sigma\bar{t})$ for all $s > 0,\, \sigma > 0$.

Let ζ be a random variable with the limiting cluster size distribution (2.2), and let $\{\zeta_i, i \geq 1\}$ be independent copies of ζ. For any $a \in (0;1]$, denote by $Z(a)$ the random variable with the distribution $\mathbb{P}(Z(a) = 0) = 1 - a$, $\mathbb{P}(Z(a) = i) = a\mathbb{P}(\zeta = i) \quad (i \geq 1)$. Note that

$$Z(a) \stackrel{d}{=} \zeta\xi(a),$$

where $\xi(a)$ is independent of ζ and has Bernoulli $\mathbf{B}(a)$ distribution.

The property $\zeta(\bar{t}) \stackrel{d}{=} \zeta(a\bar{t})$ means that the marginal distributions of the vector $\zeta(\bar{t})$ are functionals of $\mathcal{L}(\zeta)$ and ratios t_l/t_k. We show in section 5 that

$$\zeta^l(\bar{t}) \stackrel{d}{=} Z(t_l/t_k). \tag{3.12}$$

In particular, this implies that the distribution of the limiting cluster size ζ in the limit theorem (2.3) does not depend on λ.

While the limiting cluster size ζ takes values in $\mathbb{N}$, the random variable $Z(a)$ does it in $\mathbf{Z}_+ = \mathbb{N} \cup \{0\}$. In other words, clusters at a level strictly above the "basic" one can be empty. This is a feature of the multilevel situation.

Proposition 1 and Theorem 2 hint that (2.3) is not the only way of formulating the limit theorem for $N_n(u_n)$. In fact, we have a variety of ways to define the limiting cluster size and formulate a limit theorem.

Indeed, let $\{Z_i(a), i \geq 1\}$ be independent copies of $Z(a)$, $a \in (0;1]$. If Δ and (3.1) hold and $N_n(u_n(at))$ converges weakly then, according to (2.3),

$$N_n(u_n(at)) \Rightarrow \sum_{i=1}^{\pi(at)} \zeta_i. \tag{3.13}$$

If $\{Y_j\}$ are i.i.d. random vectors, $a \in [0;1]$ and $\pi(at)$ is independent of $\{Y_j\}$ then it is easy to check that

$$\sum_{i=1}^{\pi(at)} Y_i \stackrel{d}{=} \sum_{i=1}^{\pi(t)} Y_i(a)\,, \tag{3.14}$$

where $\{Y_i(a)\}$ are i.i.d. random vectors, $Y_i(a) \stackrel{d}{=} Y_i\xi(a)$ and $\pi(t)$ is independent of $\{Y_j(a)\}$. Therefore,

$$N_n(u_n(at)) \Rightarrow \sum_{i=1}^{\pi(t)} Z_i(a)\,. \tag{3.13*}$$

As one of possible applications of Theorem 2, consider the problem of approximating the distribution of the random vector

$$\nu_n(t_1,t_2) = \sum_{i=1}^{n} \mathrm{I\!I}\{u_n(t_1) \geq X_i > u_n(t_2)\}\,.$$

This problem arises, for instance, when an insurance company is interested in approximating the distribution of a number of claims varying in a specified interval.

Assume conditions of Theorem 2. If $N_n(n,\bar{t}\,)$ converges weakly then (3.11) entails

$$\nu_n(t_1,t_2) \Rightarrow \sum_{i=1}^{\pi(t_2)} Y_i\,, \tag{3.15}$$

where $\{Y_i\}$ are independent copies of $\zeta^2(\bar{t}\,) - \zeta^1(\bar{t}\,)$ and $\bar{t} = (t_1,t_2)$.

4 Main results

In Theorems 3 – 5 and Corollary 6 below we assume condition Δ.

4.1 The process $\{N_n(u_n(t)), t \in [0;T]\}$

Recall that there is a one–to–one correspondence between one–dimensional point processes and jump processes (random step functions). In this subsection we find convenient to treat one–dimensional point processes of exceedances as jump processes.

For instance, the process (2.4) may alternatively be viewed as the jump process $\{N_{[sn]}(u_n),\ s \in (0;1]\}$. It describes *locations* of exceedances of the level u_n.

The process $\{N_n\,(u_n(t))\,, t \in [0;T]\}$ describes *heights* of extremes. In this subsection we investigate asymptotic properties of the distribution of the process $\{N_n\,(u_n(\cdot))\}$.

Necessary and sufficient conditions for the weak convergence of $\{N_n(u_n(\cdot))\}$ to a compound Poisson process are given in [19]. According to Mori's [18] result, the family $\mathcal{P}'$ of weak limits of $\{N_n(u_n(\cdot))\}$ is wider than the class of compound Poisson processes.

In this subsection we present necessary and sufficient conditions for the weak convergence of $\{N_n(u_n(\cdot))\}$ to a given process $P' \in \mathcal{P}'$. We show that every element $P' \in \mathcal{P}'$ is a sum of a Poisson number of jump processes.

More generally, we will study the limiting behaviour of the jump process

$$\{N_n(sn,t), t \in [0;T]\},$$

where $s > 0$ is a fixed number.

Note that $N_n(sn,t) = N_n^*((0;s] \times [0;t))$. According to Theorem B, if N_n^* converges to a point process N^* then

$$N_n(sn,\cdot) \Rightarrow N_s'(\cdot) \equiv N^*((0;s] \times [0;\cdot)), \tag{4.1}$$

where the process $\{N_s'(t), t > 0\}$ has the following properties:

(a) $N_{as}'(t) \stackrel{d}{=} N_s'(at)$

(b) $\mathbb{P}(N_\varepsilon'(1) > 0) = \mathbb{P}(N_1'(\varepsilon) > 0) \to 0$ as $\varepsilon \to 0$

Notice that (b) is a consequence of (2.6): $\mathbb{P}(N_1'(\varepsilon) = 0) = \lim \mathbb{P}(N_n(u_n(\varepsilon)) = 0) = e^{-\varepsilon} \to 1$ as $\varepsilon \to 0$.

In Theorems 3 and 4 below, s and τ are fixed positive numbers. Denote

$$\mathbb{R}_k^1 = \{\bar{t} \in \mathbb{R}_k : t_k = 1\}.$$

Theorem 3 *Suppose that there exists a jump process $\{\gamma(t), t \in [0;1]\}$ with stochastically continuous trajectories such that for arbitrary $k \geq 1$ and $\bar{t} \in \mathbb{R}_k^1$,*

$$\zeta(\bar{t}, n) \Rightarrow (\gamma(t_1), ..., \gamma(t_k)) \tag{4.2}$$

for some $\{r\} \in \mathcal{R}$. Then

$$\{N_n(sn, \tau t), t \in [0;1]\} \Rightarrow \{N_\tau(s,t), t \in [0;1]\}, \tag{4.3}$$

where

$$N_\tau(s,t) = \sum_{j=1}^{\pi_\tau(s)} \gamma_j(t), \tag{4.4}$$

$\{\gamma_j(\cdot)\}$ are independent copies of $\gamma(\cdot)$. The process (4.4) has the following property:

$$N_\tau(as,\cdot) \stackrel{d}{=} N_\tau(s, a\cdot) \qquad (\forall a \in [0;1]). \tag{4.5}$$

Evidently, (4.3) can be rewritten as follows:

$$\{N_n(sn,t), t \leq \tau\} \Rightarrow \left\{ \sum_{j=1}^{\pi_\tau(s)} \gamma_j(t/\tau), t \leq \tau \right\}. \tag{4.3*}$$

The process $\left\{\sum_{j=1}^{\pi(s)} \gamma_j(\cdot)\right\}$ can be called *Poisson cluster process* or *compound Poisson process of the second order* (regarding the standard compound Poisson process as a "compound Poisson process of the first order"). If $\{X_i\}$ are i.i.d.r.v.s then one can take $l = 0$ and $r = 1$ to show that $N_n(n,\cdot)$ converges to a pure Poisson process with intensity rate 1 (it admits the representation $\sum_{j=1}^{\pi(1)} \gamma_j(\cdot)$, where $\gamma(t) = \mathbb{I}\{\xi < t\}$ and ξ has a uniform $U[0;1]$ distribution — cf. [27], ch. 1.).

While the random variable ζ represents the limiting cluster size, the process γ describes the variability of heights of cluster members.

Theorem 4 *If* $\{N_n(n,t), t \in [0;1]\}$ *converges weakly to some jump process* P *then there exists a jump process* $\{\gamma(t), t \in [0;1]\}$ *with stochastically continuous trajectories such that (4.2) holds and*

$$N_n(sn, \tau\cdot) \Rightarrow N_\tau(s,\cdot) \tag{4.6}$$

for every $s > 0$, $\tau > 0$. *The marginal distributions of the process* γ *obey*

$$\gamma(t) \stackrel{d}{=} Z(t) \qquad (0 \le t \le 1)\,. \tag{4.7}$$

Theorems 3 and 4 show that the class $\mathcal{P}'$ of possible weak limits of the process (2.8) consists of processes $\sum_{j=1}^{\pi(T)} \gamma_j(\cdot/T)$, where $\gamma(\cdot)$ is a jump process on $[0;1]$ with stochastically continuous trajectories such that $\mathbb{P}(\gamma(1) \ge 1) = 1$.

4.2 Complete convergence

Let T be a fixed positive number. We define the two-dimensional process $N_T^* \equiv N_T^*(\gamma)$ on $(0;1] \times [0;1)$ as a point process with the following properties (evidently, it suffices defining N_T^* on unions of rectangles):

(P1) N_T^* has independent increments along the horizontal axis,

(P2) $N_T^*((a;b] \times B) \stackrel{d}{=} N_T^*((0;b-a] \times B)$ for any Borel set $B \subset [0;1)$,

(P3) $\{N_T^*((0;a] \times [0;t)),\, t \in [0;1)\} \stackrel{d}{=} \{N_T(a,t),\, t \in [0;1)\}$.

Otherwise, N_T^* can be viewed as a random measure

$$N_T^*(A) = \int_A N_T^*(dx \times dy)\,,$$

where A is a Borel set in $(0;1] \times [0;1)$ and

$$N_T^*(dx \times dy) = \sum_{\pi_T(x) < j \le \pi_T(x+dx)} (\gamma_j(y+dy) - \gamma_j(y))\,.$$

Note that the two-dimensional process N_T^* is constructed via one-dimensional processes.

Evidently, N_T^* has properties (1), (3) and (4) of Theorem B; property (2) follows from (4.5). Besides, it is easy to see that $N_T^*\big((0;as] \times [0;b)\big) \stackrel{d}{=} N_{sT}^*\big((0;a] \times [0;b)\big)$.

We define the EPPE $N_{n,T}^*$ on $(0;1] \times [0;1)$ by the equation

$$N_{n,T}^*(A) = \sum_{i=1}^{n} \mathbb{I}\{(i/n, T^{-1}u_n^{-1}(X_i)) \in A\} \tag{4.8}$$

for any Borel set $A \subset (0;1] \times [0;1)$. In other words, we restrict our attention to the interval $X_i \in (u_n(T);\infty)$. The level $u_n(T)$ can be seen as a minimal threshold u such that X_i is considered "extreme" if it exceeds u.

Theorem 5 and Corollary 6 below show that processes N_T^* are the only possible weak limits for $N_{n,T}^*$.

Theorem 5 *Suppose that there exists a jump process* $\{\gamma(t), t \in [0;1]\}$ *with stochastically continuous trajectories such that (4.2) holds. Then*

$$N^*_{n,T} \Rightarrow N^*_T(\gamma). \tag{4.9}$$

From Theorems 4 and 5 we deduce

Corollary 6 *If* $N^*_{n,T}$ *converges weakly to some point process then there exists a jump process* $\{\gamma(t), t \in [0;1]\}$ *with stochastically continuous trajectories such that* $N^*_{n,T} \Rightarrow N^*_T(\gamma)$.

5 Proofs

The following well known fact (cf. [14, 22]) will be used in the proofs of Proposition 1 and Theorem 2.

Proposition 7 *Suppose that condition* $\Delta\{u_n(t)\}$ *holds for some* $t > 0$. *If (2.6) is in force then*

$$\mathbb{P}(N_r(u_n(t)) > 0) \sim tr/n \tag{5.1}$$

for any sequence $\{r\} \in \mathcal{R}(t)$. *If (5.1) is valid for a sequence* $\{r\} \in \mathcal{R}(t)$ *then*

$$\lim \mathbb{P}(M_{sn} \leq u_n(t)) = e^{-st} \qquad (\forall s > 0). \tag{5.2}$$

Proof of Proposition 1. Let $\{r\} \in \mathcal{R}(t)$. Denote $m = [sn/r]$. Using Bernstein's blocks method (cf. Lemma 2.2 in [12]) and taking into account conditions $\Delta\{u_n(\bar{t})\}$ and (3.1), it is easy to show that

$$|\mathbb{E}e^{ivN_n(sn,\bar{t})} - \mathbb{E}^m e^{ivN_n(r,\bar{t})}| \to 0 \tag{5.3}$$

for any $v \in \mathbb{R}^k$. Note that

$$\mathbb{E}e^{ivN_n(r,\bar{t})} = \mathbb{P}(N_n(r,t_k) = 0) + \mathbb{E}\left\{e^{ivN_n(r,\bar{t})}\middle|N_n(r,t_k) > 0\right\}\mathbb{P}(N_n(r,t_k) > 0).$$

Hence

$$\mathbb{E}e^{ivN_n(sn,\bar{t})} = \exp\left(m\mathbb{P}(N_n(r,t_k) > 0)\mathbb{E}\left\{e^{ivN_n(r,\bar{t})} - 1|N_n(r,t_k) > 0\right\}\right) + o(1).$$

According to Proposition 7, $\mathbb{P}(N_r(u_n(t_k)) > 0) \sim t_k r/n$. Therefore,

$$\mathbb{E}e^{ivN_n(sn,\bar{t})} = \exp\left(st_k\left[\mathbb{E}e^{iv\zeta(\bar{t},n)} - 1\right]\right) + o(1). \tag{5.4}$$

Relation (3.9) follows from (3.8) and (5.4). □

Proof of Theorem 2. Assume that $N_n(sn,\bar{t}) \Rightarrow \exists N$ for some $s > 0$. This evidently implies

$$\lim \mathbb{E}e^{ivN_n(sn,\bar{t})} = \mathbb{E}e^{ivN} \qquad (\forall v \in \mathbb{R}^k).$$

According to (5.4), there exists the limit $\lim \mathbb{E}e^{iv\zeta(\bar{t},n)} := \varphi_o(v)$. As a limit of a sequence of characteristic functions, it is a characteristic function itself. Hence (3.8) holds and

$$\mathbb{E}e^{ivN} = \exp(st_k[\varphi_o(v) - 1]),$$

i.e., N is a compound Poisson random vector with intensity st_k and multiplicity distribution $\mathcal{L}(\zeta)$ such that $\mathbb{E}e^{iv\zeta} = \varphi_o(v)$.

The distribution of the vector $\zeta(\bar{t})$ does not depend on the choice of a sequence $\{r\}$. Indeed, let $u_n = u_n(t)$, $t > 0$. If $\mathcal{L}(N_{r'}(u_n) \mid N_{r'}(u_n) > 0) \Rightarrow \mathcal{L}(\exists\zeta')$ for another sequence $\{r' = r'_n\} \in \mathcal{R}(\bar{t})$ then (3.9) entails $N_n(u_n) \Rightarrow \sum_{j=0}^{\pi(t)} \zeta_j$ and $N_n(u_n) \Rightarrow \sum_{j=0}^{\pi(t)} \zeta'_j$. Hence $\zeta \stackrel{d}{=} \zeta'$.

In order to show that the distribution of the vector $\zeta(\bar{t})$ is scale–invariant, denote $\bar{t}_* = (t_1/t_k, t_2/t_k, ..., 1)$. By (3.9), $N_n(sn, \bar{t})$ converges for all $s > 0$. According to Lemma 8 below, $N_n(n, s\bar{t})$ converges for all $s > 0$ as well, and the limiting distributions of $N_n(sn, \bar{t})$ and $N_n(n, s\bar{t})$ coincide. Since

$$N_n(n, \bar{t}) \Rightarrow \sum_{j=1}^{\pi(t_k)} \zeta_j(\bar{t}), \quad N_n(t_k n, \bar{t}_*) \Rightarrow \sum_{j=1}^{\pi(t_k)} \zeta_j(\bar{t}_*)$$

according to (3.9), we have $\sum_{j=1}^{\pi(t_k)} \zeta_j(\bar{t}) \stackrel{d}{=} \sum_{j=1}^{\pi(t_k)} \zeta_j(\bar{t}_*)$. Hence $\zeta(\bar{t}) \stackrel{d}{=} \zeta(\bar{t}_*)$.

Formula (3.11) follows from (3.8) – (3.10).

It remains to show that $\zeta^l(\bar{t}) \stackrel{d}{=} Z(t_l/t_k)$. Indeed, (3.9) entails $N_n(u_n(t_l)) \Rightarrow \sum_{j=0}^{\pi(t_k)} \zeta_j^l(\bar{t})$. According to (2.3), $N_n(u_n(t_l)) \Rightarrow \sum_{j=0}^{\pi(t_l)} \zeta_j$. Note that $Z(1) \stackrel{d}{=} \zeta$ and $\mathbf{\Pi}(as, \zeta) = \mathbf{\Pi}(s, Z(a))$ for any $s > 0$. Hence $\sum_{j=0}^{\pi(t_l)} \zeta_j \stackrel{d}{=} \sum_{j=0}^{\pi(t_k)} Z(t_l/t_k)$. Comparing the characteristic functions of $\sum_{j=0}^{\pi(t_k)} \zeta_j^l(\bar{t})$ and $\sum_{j=0}^{\pi(t_k)} Z(t_l/t_k)$, we get (3.12). □

Let $\bar{t} \in \mathbb{R}_k$, and let I be an open interval in $(0; \infty)$. Denote

$$P_1(n, s) = \mathbb{P}(N_{sn}(u_n(t_1)) < i_1, ..., N_{sn}(u_n(t_k)) < i_k), \tag{5.5}$$

$$P_2(n, s) = \mathbb{P}(N_n(u_n(st_1)) < i_1, ..., N_n(u_n(st_k)) < i_k). \tag{5.6}$$

The following lemma follows the corresponding lines in Hsing [10, 11] but our mixing condition is weaker than the condition Δ^* assumed in [10] or the α–mixing condition assumed in [11].

Lemma 8 *Assume condition $\Delta\{u_n(\bar{t})\}$. If one of the probabilities (5.5) or (5.6) converges for each $s \in I$ then so does the other, and the limits coincide.*

Proof of Lemma 8. Let $s' > s > s''$ be the points from I. As was noticed in [11], $u_{[n/s']}(t) < u_n(s''t)$ for all sufficiently large n (this follows from (5.2)).

Suppose that the limit $g(s) \equiv g(s, \bar{t}) = \lim P_2(n, s)$ exists $(s \in I)$. Then

$$\begin{aligned} &\limsup P_1(n, s') = \limsup P_1([n/s'], s') \\ &= \limsup \mathbb{P}(N_n(u_{[n/s']}(t_1)) < i_1, ..., N_n(u_{[n/s']}(t_k)) < i_k) \\ &\leq \limsup \mathbb{P}(N_n(u_n(st_1)) < i_1, ..., N_n(u_n(st_k)) < i_k) = \lim P_2(n, s). \end{aligned} \tag{5.7}$$

Similarly

$$\lim P_2(n,s) \le \liminf P_1(n,s''). \tag{5.8}$$

Therefore,

$$\limsup P_1(n,s') \le g(s_1) \le g(s_2) \le \liminf P_1(n,s'')$$

if $s' > s_1 > s_2 > s''$. Notice that

$$\begin{aligned}&\liminf P_1(n,s'') - \limsup P_1(n,s') \le \limsup [P_1(n,s'') - P_1(n,s')]\\ &\le \sum_{i=1}^{k} \lim \mathbb{P}\left(N_{[s''n]-[s'n]}(u_n(t_i)) > 0\right) \le \sum_{i=1}^{k} ([s''n]-[s'n])\, \mathbb{P}(X > u_n(t_i)).\end{aligned} \tag{5.9}$$

This and (3.1) imply

$$0 \le \liminf P_1(n,s'') - \limsup P_1(n,s') \to 0$$

as $s' - s'' \to 0$. Hence the function $g(s)$ is uniformly continuous in I.

If $s_1 > s > s_2$ are the points from I then (5.7) and (5.8) entail

$$g(s_1) = \lim P_2(n,s_1) \le \liminf P_1(n,s) \le \limsup P_1(n,s) \le \lim P_2(n,s_2) = g(s_2).$$

Letting $s_1 \to s$ and $s_2 \to s$, we observe that the limit $\lim P_1(n,s)$ exists and equals $g(s)$.

Suppose that for every $s \in I$ there exists the limit $h(s) \equiv h(s,\bar t) = \lim P_1(n,s)$. Similarly to (5.7) and (5.8),

$$\lim P_1(n,s') \le \liminf P_2(n,s) \le \limsup P_2(n,s) \le \lim P_1(n,s'').$$

From (5.9) we derive that $\lim P_1(n,s'') - \lim P_1(n,s') \to 0$ as $s' - s'' \to 0$. This entails the limit $\lim P_2(n,s)$ exists and equals $h(s)$. The proof is complete. □

Proof of Theorem 3. Let $\bar t \in \mathbb{R}^1_k$, and denote $N_\tau(s,\bar t) = \{N_\tau(s,t_1),\dots,N_\tau(s,t_k)\}$. Relation (3.8) holds with $\zeta(\bar t) \stackrel{d}{=} (\gamma(t_1),\dots,\gamma(1))$. Proposition 1 and Theorem 2 entail

$$\{N_n(sn,\tau t_1),\dots,N_n(sn,\tau)\} \Rightarrow \{N_\tau(s,t_1),\dots,N_\tau(s,1)\}$$

for every $\bar t \in \mathbb{R}^1_k$. Thus, finite-dimensional distributions of $\{N_n(sn,\tau t), t \in [0;1]\}$ converge to those of $\{N_\tau(s,t),\, t \in [0;1]\}$. In view of [13], ch. 4, this implies the weak convergence $N_n(sn,\tau\cdot) \Rightarrow N_\tau(s,\cdot)$.

In order to check (4.5), we must show that finite-dimensional distributions of the processes coincide:

$$\left\{\sum_{j=1}^{\pi_\tau(as)} \gamma_j(t_1),\dots,\sum_{j=1}^{\pi_\tau(as)} \gamma_j(t_k)\right\} \stackrel{d}{=} \left\{\sum_{j=1}^{\pi_\tau(s)} \gamma_j(at_1),\dots,\sum_{j=1}^{\pi_\tau(s)} \gamma_j(at_k)\right\}. \tag{5.10}$$

By (3.11), the left-hand side of (5.10) is the weak limit of $N_n(asn,\tau\bar t)$. Let $\tilde t = \{a\bar t, 1\}$. Then $N_n(sn,\tau\tilde t) \Rightarrow N_\tau(s,\tilde t)$ and hence $N_n(sn,\tau a\bar t) \Rightarrow N_\tau(s,a\bar t)$, the right-hand side of (5.10). According to Theorem 2, the weak limits of $N_n(asn, \tau\bar{t})$ and $N_n(sn, \tau a\bar{t})$ coincide. This implies (5.10) and (4.5). □

Proof of Theorem 4. Suppose that the process $N_n(n, \cdot)$ converges weakly to some jump process P. Let $k \in \mathbb{N}$, $\bar{t} \in \mathbb{R}^1_k$. Then

$$(N_n(n, t_1), ..., N_n(n, 1)) \Rightarrow (P(t_1), ..., P(1)). \tag{5.11}$$

Theorem 2 and (5.11) imply (3.8). A comparison of (3.11) with (5.11) yields $P(\cdot) \stackrel{d}{=} N_1(1, \cdot)$. Moreover, (3.8) and (3.11) imply

$$(N_n(sn, \tau t_1), ..., N_n(sn, \tau)) \Rightarrow (N_\tau(s, t_1), ..., N_\tau(s, 1)) \qquad (\forall s > 0, \tau > 0). \tag{5.12}$$

Since the distributions (3.7) are consistent, so are the distributions of $\zeta(\bar{t}), \bar{t} \in \mathbb{R}^1_k, k \geq 1$. By Kolmogorov's theorem, there exists a process $\gamma = \{\gamma(t), t \in [0;1]\}$ such that $\{\mathcal{L}(\zeta(\bar{t})), \bar{t} \in \mathbb{R}^1_k\}_{k\geq 1}$ are the finite–dimensional distributions of γ. Evidently, γ is a jump process. The weak convergence (4.6) follows from (5.12) and Proposition 9 below.

In order to show that $\gamma(t) \stackrel{d}{=} Z(t)$ for any $t \in [0;1]$, remind that

$$N_n(n, t) \Rightarrow \sum_{j=1}^{\pi(t)} \zeta_j \stackrel{d}{=} \sum_{j=1}^{\pi(1)} Z_j(t)$$

and $N_n(n, \bar{t}) \Rightarrow \sum_{j=1}^{\pi(1)} \bar{\zeta}_j$, where $\bar{t} = (t, 1)$ and $\bar{\zeta}_j = (\gamma_j(t), \gamma_j(1))$. Hence $N_n(n, t) \Rightarrow \sum_{j=1}^{\pi(1)} \gamma_j(t)$. Therefore, $\sum_{j=1}^{\pi(1)} Z_j(t) \stackrel{d}{=} \sum_{j=1}^{\pi(1)} \gamma_j(t)$. This entails (4.7). □

Evidently, Theorem 4 remains valid if its assumption is replaced by the following one: "for some $T > 0$, the process $\{N_n(n, t), t \in [0; T]\}$ converges weakly to some jump process P_T".

Proposition 9 *Let $\{r\} \in \mathcal{R}$. If, for arbitrary $k \geq 1$ and $\bar{t} \in \mathbb{R}^1_k$, (4.2) holds for some $\{r\} \in \mathcal{R}$ then trajectories of the process γ are stochastically continuous on $[0;1]$.*

Define the random measure $Q\{\cdot\}$ by the equation

$$Q\{[s;t]\} = \gamma(t) - \gamma(s) \qquad (0 \leq s < t \leq 1) \tag{5.13}$$

(Q is defined on intervals in $[0;1]$ and thus on all Borel sets in $[0;1]$). Note that (5.13) stipulates a one–to–one correspondence between $\gamma(\cdot)$ and the point process Q (if we had a point process Q on $[0;1]$ then we could define a jump process $\{\gamma(t), t \in [0;1]\}$ by the equation $\gamma(t) = Q\{[0;t]\}$. Proposition 9 states that

$$\mathbb{P}(Q\{t\} > 0) = 0 \qquad (\forall t \in [0;1]). \tag{5.14}$$

Proof of Proposition 9. Evidently, $\gamma(0) = 0$. The fact that $\mathbb{P}(Q\{0\} > 0) = 0$ (equivalently, $\mathbb{P}(\gamma(s) > 0) \to 0$ as $s \to 0$) follows from (4.7).

Let $t \in (0;1]$. If $\mathbb{P}(Q\{t\} > 0) > 0$ then (3.15) implies

$$\mathbb{P}(\nu_n(\sigma t-, \sigma t+) > 0) \;\to\; \mathbb{P}\left(\sum_{j=1}^{\pi(\sigma t)} Q_j\{t\} > 0\right) \qquad (5.15)$$
$$= \; 1 - \exp(-\sigma t \mathbb{P}(Q\{t\} > 0)) > 0$$

for every $\sigma \in (0;1]$, where $\{Q_j, j \geq 1\}$ are independent copies of Q.

Denote by γ_t the weak limit of the process $\{N_n(n, \sigma t), \sigma \in (0;1]\}$, and let Q_t be the corresponding point process. Relation (5.15) means that the set $\{\sigma : \mathbb{P}(Q_t\{\sigma\} > 0) > 0\}$ is uncountable. This contradicts to [16], Proposition 1.1.5. Hence (5.14) holds. □

Proof of Theorem 5. Proposition 9 ensures that $\mathbb{P}(N_T^*((0;1] \times \{b\}) > 0) = 0$ for any $b \in [0;1)$. Because of (3.1),

$$\mathbb{P}(N_T^*(\{a\} \times [0;1)) > 0) = \lim_{\varepsilon \to 0} \lim_{n \to \infty} \mathbb{P}\left(N_{[\varepsilon n]}(u_n(T)) > 0\right) = 0$$

for any $a \in (0;1]$. Thus, $\mathbb{P}(N_T^*(\partial A) > 0) = 0$ if A is a rectangle on $(0;1] \times [0;1)$. Therefore (see [13], chapter 4), (4.9) follows if we show that

$$\{N_{n,T}^*(A_1), \ldots, N_{n,T}^*(A_k)\} \Rightarrow \{N_T^*(A_1), \ldots, N_T^*(A_k)\} \qquad (5.16)$$

for any array $\{A_1, \ldots, A_k\}$ of finite unions of rectangles.

Splitting rectangles in a proper way, we observe that it suffices to prove (5.16) in the case $A_i = (a_i; b_i] \times \bigcup_{j=1}^{m_i} [c_{ij}; d_{ij})$, where the intervals $(a_i; b_i]$ are disjoint and for each i, the intervals $[c_{ij}; d_{ij})$ are disjoint too.

Property (P1) implies the random variables $\{N_T^*(A_i)\}$ are independent. By standard arguments (cf. [10, 11, 19]), the random variables $\{N_{n,T}^*(A_i)\}$ are asymptotically independent as well. Thus, it remains to show that

$$N_{n,T}^*(A) \Rightarrow N_T^*(A) \qquad (5.17)$$

for any set $A = (a; b] \times \bigcup_{j=1}^{m} [c_j; d_j) \subset (0;1] \times [0;1)$, where the intervals $[c_j; d_j)$ are disjoint.

Theorem 3 establishes (5.17) in the case $(a; b] = (0; 1]$. The arguments are evidently valid for an arbitrary interval $(a; b] \subset (0; 1]$. □

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