Continuous time vertex-reinforced jump processes

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Abstract

We study the continuous time integer valued process X_t, $t \geq 0$, which jumps to each of its two nearest neighbors at the rate of one plus the total time the process has previously spent at that neighbor. We show that the proportion of the time before t which this process spends at integers j converges to positive random variables V_j, which sum to one, and whose joint distribution is explicitly described. We also show $\lim_{t\to\infty} \max_{0\leq s\leq t} X_s/\log t = 2.768\ldots$

1 Introduction

This paper introduces and studies a continuous time right-continuous integer valued stochastic process which jumps only to nearest neighbors. We call this process, which was conceived by W. Werner, a vertex-reinforced jump

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process (VRJP) and for now designate it by X_t, $t \geq 0$. Given $\{X_s, s \leq t, X_t = j\}$ and putting $A = 1 + \int_0^t I(X_s = j-1)\, ds$ and $B = 1 + \int_0^s I(X_s = j+1)\, ds$, the probability of a jump to $j-1$ $(j+1)$ at a time in $(t, t+h]$ equals $Ah + o(h)$ (respectively $Bh + o(h)$), where both $o(h)$ depend only on A and B. Thus the time elapsed after t until the first jump from j has an exponential distribution with rate $A+B$, and the probability the jump is to $j-1$ is $A/(A+B)$. This determines VRJP in the sense that the generator determines a Markov process, even though a VRJP is not a Markov process, and as with Markov processes an initial distribution needs to be specified to complete its description. It is easy to construct VRJP, started, say, at 0 from a sequence of i.i.d. exponential random variables of parameter 1. Other graphs may be considered, but in this paper we will stick to the integers. We note that the first use of exponential variables in connection with (discrete time) reinforced processes was made by Herman Rubin to couple a generalized Pólya urn with a pure birth process (see Davis [3] and Sellke [8]).

Of the discrete time reinforced random walks studied in the literature, the two that seem most fundamental are the bond-reinforced random walk first studied by Coppersmith and Diaconis in [2], the paper which originated the subject, and the vertex-reinforced random walk first studied by Pemantle in [6], and later by Pemantle and Volkov [7] and Volkov [10].

The Coppersmith-Diaconis walk on the integers starts with weight one on all the "bonds" $(i, i+1)$, and between times n and $n+1$ jumps to one of the two nearest neighbors with probabilities summing to 1 and proportional to the weights of the bonds connecting the current state with these neighbors. Coppersmith and Diaconis observed that these walks could be realized as coupled Pólya urns. This approach proves almost sure recurrence on $\mathbf{Z}^1$ (see Davis [3]), which here and elsewhere in this paper will mean that every integer is almost surely visited at arbitrarily large times. Later, in a series of intricate papers, a remarkably complete description of the limiting behavior of this and many related bond-reinforced walks was provided by Tóth (see [9]). Scaled properly (not $\sqrt{n}$, in the Coppersmith-Diaconis case) they converge to various previously unknown processes, some of them quite wild. Pemantle's vertex-reinforced random walk on the integers is the vertex-reinforced analog of the walk just described. Each integer initially has weight one, and this weight is augmented by one each time it is visited. This process jumps to one of its two nearest neighbors between times n and $n+1$, the relative weights of the neighbors giving the probabilities of the jumps. Not only is this walk not recurrent, but It was proved in Pemantle and Volkov [7] that it eventually gets stuck on a finite set of points, and with a positive probability in exactly five states! This paragraph only scratches the surface of the subject of discrete time reinforced walks. See Davis [4], Pemantle and Volkov [7], and Tóth [9] for more, including references to papers in biology and learning theory which use discrete time reinforced walks as models, and a discussion of some processes which are limits of reinforced walks which arose in other areas of probability. Both the walks described above, and VRJPs, are close in spirit to Pólya urns, although only for the Coppersmith-Diaconis walk is the connection explicit. Reinforced Brownian motions have also been studied. See [1] for references.

This paper began as an attempt to decide whether VRJP on the integers is recurrent. It is fairly easy to show that it does not get stuck in a finite number of states, but to show recurrence is a different matter. In the following two theorems X_t, $t \geq 0$, will stand for VRJP on the integers started at 0. We omit the qualification a.s. when it clearly must hold.

Theorem 1.1 . *The limits* $V_i := \lim_{t\to\infty} \frac{1}{t}\int_0^t I(X_s = i)\,ds$ *exist for each integer* i*, and are positive and sum to* 1*. There are i.i.d. random variables* U_i*,* $0 < i < \infty$ *or* $-\infty < i < 0$*, each having the density* $f_\gamma(x)$ *given by*

$$\frac{\exp\left(-\frac{1}{2}\left(\sqrt{x}-\frac{1}{\sqrt{x}}\right)^2\right)}{\sqrt{2\pi x^3}}$$

such that if we put W_i *equal to* $\prod_{k=1}^{i} U_k$ *if* $i > 0$*, equal to* $\prod_{k=i}^{-1} U_k$ *if* $i < 0$*, and equal to* 1 *if* $i = 0$*, then* $V_i = W_i / \sum_{i=-\infty}^{\infty} W_i$.

Theorem 1.2 *Let* $\alpha = 0.36\ldots.$ *be the number explicitly given in equation (5.46). Then* $\lim_{t\to\infty} \max_{0\le s\le t} X_s / \log t = \alpha^{-1} \approx 2.77$.

Of course, symmetry gives the analog of Theorem 1.2 for minimum. Thus VRRW on the integers started at 0 has range approximately a centered interval, for all large t. Each of the two theorems just stated immediately implies that VRJPs are recurrent.

2 Vertex-reinforced jump processes on $\{0, 1\}$

In exact analogy to the definition of X_t, $t \ge 0$, in the previous section, we can and do define vertex-reinforced jump processes Y on any any connected locally-finite graph, with the initial weight of each vertex v a positive number a_v, perhaps different from one, so that the weight of v at time t is here $L(t, v) := a_v + \int_0^t I(Y_s = v)\, ds$. We still call such a process a vertex-reinforced jump process (VRJP).

In this section, we study only VRJP on $\{0, 1\}$ started at 0, with initial weight a at zero and b at one, and we use $Z_t, t \ge 0$ to designate these processes. Where it might be ambiguous which initial weights we are dealing with on $\{0, 1\}$, we will use a, b as a superscript. The initial position is always 0 unless explicitly mentioned. Especially $\mathsf{P}^{a,b}$ and $\mathsf{E}^{a,b}$ refer only to VRJP started at 0. At times we will need to consider random initial weights, and we will use a similar convention.

We now recall some classical results about discrete parameter martingales. Let $f_1, f_2, \ldots$ be a martingale with difference sequence $d_1 = f_1$, $d_i = f_i - f_{i-1}$, $i > 1$. Doob's maximal inequalities ([5] (p. 308)) say

$$\mathsf{E}\left(\sup_{n\ge1} |f_n|\right)^p \le \left(\frac{p}{p-1}\right)^p \sup_{n\ge1} \mathsf{E}\,|f_n|^p, p > 1. \tag{2.1}$$

If $\mathsf{E}\, f_n^2 < \infty$ for each n, then d_i, $i \geq 1$, is an orthogonal series and thus $\mathsf{E}\,(f_{n+k} - f_n)^2 = \sum_{i=n+1}^{n+k} \mathsf{E}\, d_i^2$. In addition, the almost sure convergence of L^2-bounded and thus L^1-bounded martingales, together with the fact that f_{n+i}, $i \geq 0$, is a martingale for each i, give with (2.1)

$$\begin{aligned}\mathsf{E} \sup_{k\geq 0}(f_{n+k} - f_\infty)^2 &= \mathsf{E} \sup_{k\geq 0}[(f_{n+k} - f_n) - \lim_{k\to\infty}(f_{n+k} - f_n)]^2 \\ &\leq \mathsf{E}\, 4 \sup_{k\geq 0} |f_{n+k} - f_n|^2 \leq 16 \sum_{k>0} \mathsf{E}\, d_{n+k}^2 \\ &= 16 \lim_{k\to\infty} \mathsf{E}\,(f_{n+k} - f_n)^2, \end{aligned} \tag{2.2}$$

if $\sup_n \mathsf{E}\, f_n^2 < \infty$, where $f_\infty := \lim_{i\to\infty} f_i$.

In the proof of the following lemma, and throughout the paper, we adopt the usual convention that C, K etc. often stand for positive constants which may change from line to line.

Lemma 2.1 *Let f_1, f_2, ...,f_n be a martingale with differences $d_1, d_2, \ldots, d_n$ satisfying*

$$\max_{1\leq j\leq n} \mathsf{E}\left(d_j^4 \,|\, d_i,\ i < j\right) = \gamma < \infty, \tag{2.3}$$

and let $\varepsilon > 0$. Then there is a constant $K = K(\gamma, \varepsilon)$ such that

$$\mathsf{P}(\max_{1\leq j\leq n} |f_i| > \varepsilon n) < \frac{K}{n^2}.$$

Proof: This proof, which is probably known, is a close cousin to the standard proof of complete convergence of averages of i.i.d. variables with finite fourth moments. We divide the n^4 terms of the expansion for $(\sum_{i=1}^n d_i)^4$ into four groups, according to the power to which the d_i of the greatest i in that term is raised, and then rearrange the sums of the terms in the groups.

So $f_n^4 = (\sum d_i)^4 = I + II + III + IV$, where

$$(I) \quad = \quad 4\sum_{i=1}^n d_i \left(\sum_{j=1}^{i-1} d_j\right)^3,$$

$$
\begin{aligned}
(II) &= 6\sum_{i=1}^{n} d_i^2 \left(\sum_{j=1}^{i-1} d_j\right)^2, \\
(III) &= 4\sum_{i=1}^{n} d_i^3 \left(\sum_{j=1}^{i-1} d_j\right), \\
(IV) &= \sum_{i=1}^{n} d_i^4.
\end{aligned}
$$

Now $\mathsf{E}\, d_i(\sum_{j=1}^{i-1} d_j)^3 = \mathsf{E}\,[\,\mathsf{E}\,(d_i \,|\, d_j,\, j < i)(\sum d_j)^3] = \mathsf{E}\,0 = 0$, so $\mathsf{E}\,(I) = 0$. And by (2.3), $\mathsf{E}\,(d_k^2 \,|\, d_i,\, i < k)) < C(\gamma) = C$, so $\mathsf{E}\, d_i^2(\sum_{j=1}^{i-1} d_j)^2 = \mathsf{E}\;\mathsf{E}\,(d_i^2 \,|\, d_j,\, j < i)(\sum d_j)^2) < C\,\mathsf{E}\,(\sum d_j)^2 < C(i-1) < Cn$, and so we get $\mathsf{E}\,(II) < Cn^2$.

Since $\mathsf{E}\,(|d_i|^3 \,|\, d_j\, j < i) < C(\gamma) = C$ by (2.3) we similarly get $\mathsf{E}\, d_i^3 \sum_{j=1}^{i-1} d_j \leq C\,\mathsf{E}\,|\sum_{j=1}^{i-1} d_j| \leq C[\,\mathsf{E}\,(\sum d_j)^2]^{1/2} = Cn^{1/2}$, and so $\mathsf{E}\,(III) < Cn^{3/2}$.

Finally, (2.3) implies $\mathsf{E}\, d_i^4 < C$, and so $\mathsf{E}\,(IV) < Cn$.

Thus $\mathsf{E}\, f_n^4 < Cn^2$, which together with the $p = 4$ case of (2.1) and Markov's inequality, gives Lemma 2.1. ■

We recall that, to keep our notation brief, all VRJPs on $\{0,1\}$ considered in this section are started at 0. We put

$$\xi(t) = \inf\{s:\ L(s,0) = t\},$$

so that under $\mathsf{P}^{a,b}$, we have $\xi(a) = 0$.

Lemma 2.2 *For all* $t \geq a$, $\mathsf{E}^{\,a,b} L(\xi(t),1) = \frac{b}{a}t$.

A proof of this lemma is given via Laplace transforms in the appendix. We now sketch a different proof.

Proof of Lemma 2.2: We will show that $y(t) := \mathsf{E}^{\,a,b} L(\xi(t),1)$ satisfies the differential equation $y' = y/t$. Since y(a)=b, this implies Lemma 2.2. If Δ is a positive number, $L(\xi(t+\Delta),1) - L(\xi(t),1)$ is the time spent at 1 while the local time at 0 increases from t to $t+\Delta$. The probability of a jump from 0 to 1 in this time interval, given $L(\xi(t),1)$, is $L(\xi(t),1)\Delta + o(\Delta)$ as $\Delta \to 0$, and the duration of the excursion to 1 resulting from this jump has an exponential distribution with rate t between t and $t+\Delta$, so that

$$\lim_{\Delta\to 0} \frac{\mathsf{E}\, L(\xi(t+\Delta),1) - \mathsf{E}\, L(\xi(t),1)}{\Delta} = \frac{\mathsf{E}\, L(\xi(t),1)}{t},$$

and our differential equation is satisfied. It takes a little more work to show that the expectation of the sum of the durations of all the excursions beyond the first is $o(\Delta)$. This argument is omitted. ■

We put $m_t = m_t^{a,b} = \frac{L^{a,b}(\xi(t),1)}{t}$, if $t \geq a$, where the superscript means that we are studying $L(\xi(t),1)/t$ under $\mathsf{P}^{a,b}$.

Corollary 2.3 *The process m_t, $t \geq a$, is a martingale.*

Proof: This is immediate from Lemma 2.2 and the fact that given Z_s, $0 \leq s \leq \xi(t)$, the process Z_{y+t}, $y \geq 0$, has the same distribution as VRJP on $\{0,1\}$ under $\mathsf{P}^{t,L(\xi(t),1)}$. ■

Corollary 2.4 *Both the limits* $\lim_{t\to\infty} m_t^{a,b}$, *and* $\lim_{t\to\infty} \frac{L^{a,b}(t,1)}{L^{a,b}(t,0)}$ *almost surely exist and are equal and positive.*

Proof: Note that the right continuity of the paths of Z_t, $t \geq 0$, gives that if $L(t,0) = s$, then

$$\frac{L(\xi(s),1)}{s} \leq \frac{L(t,1)}{L(t,0)} \leq \lim_{r\downarrow s} \frac{L(\xi(r),1)}{s} = \lim_{r\downarrow s} \frac{L(\xi(r),1)}{r}. \tag{2.4}$$

Thus, since $\lim_{t\to\infty} m_t$ exists a.s.,

$$\lim_{t\to\infty} \frac{L^{a,b}(t,1)}{L^{a,b}(t,0)} \quad \text{exists a.s.} \tag{2.5}$$

To complete the proof we will show that the latter limit is strictly positive by showing that

$$\lim_{t\to\infty} \frac{L^{a,b}(t,0)}{L^{a,b}(t,1)} \quad \text{exists a.s.} \tag{2.6}$$

This is done by noting that if τ is the time of the first jump to 1, then, conditioned on $\{Z_t,\ 0 \le t \le \tau\}$, the distribution of $1 - \mathbf{Z}_{t+\tau},\ t \ge 0$, (i.e. we just relabel 0 as 1 and 1 as 0), has the distribution of $Z_t,\ t \ge 0$, under $\mathsf{P}^{b,a+\tau}$, so that (2.6) follows from (2.5). ■

Our treatment of the following lemma parallels that of Lemma 2.2; it is proved in the appendix, and a different proof is sketched in this section.

Lemma 2.5 *For all* $r \ge a$

$$\mathsf{E}^{\,a,b} L(\xi(r),1)^2 = -\frac{b}{a} + \frac{ab^2+b}{a^3} r^2.$$

Proof: We have $L(\xi(r+dr),1) = L(\xi(r)) + \nu\eta$, where ν is Bernoulli $(L(\xi(r),1)dr)$ and η is exponential (r), and ν and η are independent given $L(\xi(r),1)$. Thus, noting $\nu^2 = \nu$, we have

$$\begin{aligned}
\mathsf{E}\, L(\xi(r+dr),1)^2 &= \mathsf{E}\, L(\xi(r),1)^2 + 2\,\mathsf{E}\,\{L(\xi(r+dr),1)\,\mathsf{E}\,(\nu \,|\, L(\xi(r),1))\}\,\mathsf{E}\,\eta \\
&+ \mathsf{E}\,\eta^2\,\mathsf{E}\,\mathsf{E}\,(\nu^2 \,|\, L(\xi(r),1)) \\
&= \mathsf{E}\, L(\xi(r),1)^2 + \frac{2}{r}\,\mathsf{E}\, L(\xi(r),1)^2 dr + \frac{2}{r^2}\,\mathsf{E}\, L(\xi(r),1) dr \\
&= \mathsf{E}\, L(\xi(r),1)^2 + \frac{2}{r}\,\mathsf{E}\, L(\xi(r),1)^2 dr + \frac{2b}{ar} dr,
\end{aligned}$$

using Lemma 2.2 in the last line.

Thus $\mathsf{E}\, L(\xi(r),1)^2$ satisfies $y' = \frac{2}{r}y + \frac{2b}{ar}$, and $y(a) = b^2$, and Lemma 2.5 follows. ■

Lemma 2.5 immediately gives the L^2 norm of the martingale $m_t^{a,b},\ t \ge a$, is finite, since

$$\mathsf{E}^{\,a,b}(m_r)^2 = \frac{ab^2+b}{a^3} - \frac{b}{ar^2},\ \ r \ge a. \tag{2.7}$$

The continuous version of (2.2) with m_r playing the role of f_n, along with (2.7) give, putting $m_\infty := \lim_{t\to\infty} m_t$,

$$\mathsf{E}^{\,a,b} \sup_{s\ge a}(m_s - m_\infty)^2 \le 16 \sup_{s\ge a}\,\mathsf{E}\,(m_s - m_a)^2 = 16 \sup_{s\ge a}(\,\mathsf{E}\, m_s^2 - m_a^2) \le 16\frac{b}{a^3},$$

which, together with (2.4), gives

$$\mathsf{E}^{a,b} \sup_{t\geq 0} \left(\frac{L(t,1)}{L(t,0)} - m_\infty\right)^2 \leq 16\frac{b}{a^3}. \tag{2.8}$$

Also, we have,

Lemma 2.6 $\mathsf{E}^{a,b} \sup_{y\geq 0} \left|\log \frac{L(y,1)}{L(y,0)}\right|^4 < C(a,b)$.

Proof: Since $(\log x)^4 < x^2$, $x > 1$, (2.4) together with (2.7) and the continuous version of (2.1) in the case $p = 2$ give, if $r^+ = \max(r, 0)$,

$$\mathsf{E}^{a,b} \sup_{y\geq 0} \left[\left(\log \frac{L(y,1)}{L(y,0)}\right)^+\right]^4 \leq 4\frac{ab^2+b}{a^3}. \tag{2.9}$$

Let τ be the time of the first jump of Z_t to 1. Then, given τ, the process $\frac{L(y+\tau,0)}{L(y+\tau,1)}$, $y \geq 0$, has the same distribution as $\frac{L(y,1)}{L(y,0)}$, $y \geq 0$, under $\mathsf{P}^{b+\tau,a}$, and from this and (2.9) it is easy to conclude that

$$\mathsf{E}^{a,b} \sup_{y\geq 0} \left[\left(\log \frac{L(y,0)}{L(y,1)}\right)^+\right]^4 < C(a,b), \tag{2.10}$$

since the interval $0 \leq y \leq \tau$ is easily handled. Together (2.9) and (2.10) give Lemma 2.6. ∎

We use the superscript $^{1,1+\exp 1}$ to indicate we are starting VRJP on $\{0,1\}$ with a random initial weight of 1 plus an exponential (1) random variable at one, and 1 at zero, so that the VRJP behaves like VRJP on $\{0,1\}$ with initial weights 1 at both zero and one, started at one, after the first jump to zero. It is easy to conclude from Lemma 2.6 and the concavity of $\log x$ that $\log m_t$, $t \geq 0$, is a supermartingale under $\mathsf{E}^{1,1+\exp 1}$, satisfying

$$\sup_{t\geq 0} \mathsf{E}^{1,1+\exp 1} |\log m_t|^4 < \infty. \tag{2.11}$$

Furthermore, we will calculate in the appendix the number $\alpha \approx 0.36$, defined by

$$\alpha = \mathsf{E}^{1,1+\exp 1} \log m_\infty. \tag{2.12}$$

Now $\mathsf{E}^{1,1+\exp 1} \log m_t$ is non-increasing as t increases, and (2.11) implies this convergence is dominated, so that

$$\mathsf{E}^{1,1+\exp 1} \log m_t \downarrow \alpha \text{ as } t \to \infty. \tag{2.13}$$

3 VRJP on the nonnegative integers

In this section, Y_t, $t \geq 0$, exclusively stands for VRJP on $\{0, 1, 2, \ldots\}$ started at 0, with initial weights all 1, and we use $L^Y(t, k)$ to denote $1 + \int_0^t I(Y_s = k)\, ds$, often omitting the superscript. We let $T_n = \inf\{t : Y_t = n\}$. We will need the following, which is immediate from the construction of VRJP.

Restriction principle. VRJP observed only at the times when it stays on some subset of consecutive integers A, behaves the same way as VRJP restricted to the set A. Moreover, it is independent of either the path VRJP to the right of A or the path of VRJP to the left of A.

More precisely, let W_t, $t \geq 0$, be a VRJP (initial weights are 1) on any set of consecutive integers, and let A be a subset of consecutive integers of those integers. Let $T = \inf\{t \geq 0 : W_t \in A\}$ be a stopping time, and $k = W_T \in A$ be the "port of entry". Put $\delta^A(a) = \sup\{t : \int_0^t I(W_s \in A)\, ds = a\}$. Then $H_a := W_{\delta(a)}$ is a VRJP on A started at k. If B represents the states to the left or to the right of A, then $W_{\delta^B(a)}$, $a \geq 0$, is independent of $W_{\delta^A(a)}$, $a \geq 0$.

Lemma 3.1 *If $n > 0$, $\mathsf{P}(T_n < \infty) = 1$.*

Proof: Since $\sum_{i=0}^{\infty} [L(t, i) - 1] = t$, if $\mathsf{P}(T_n < \infty) < 1$ there must be a j, $0 \leq j < n$, such that $\mathsf{P}(L(\infty, j) = \infty, L(\infty, j+1) < \infty) > 0$. Now we use the restriction principle on $\{j, j+1\}$, together with Corollary 2.4, with j relabeled as zero and $j+1$ relabeled as one, to get a contradiction. ■

Next we observe the following

Lemma 3.2 *Let $1 \le j < n$. Then given $L(T_n, i)$, $i \ge j+1$, the distribution of $L(T_n, j)/L(T_n, j+1)$ is the distribution of $m^{1,1+\exp 1}_{L(T_n,j+1)}$.*

Proof: Note that $L(T_{j+1}, j)$ has the distribution $1 + \exp 1$, while of course $L(T_{j+1}, j+1) \equiv 1$. The rest of the argument follows from the restriction principle applied to $\{j, j+1\}$, together with the observation that what happens on excursions of Y_t to the right of $j+1$ is not influenced by what happens to Y_t while it is on $\{0, 1, 2, \ldots, j+1\}$. ■

The following proposition establishes the recurrence of VRJP on non-negative integers.

Proposition 3.3 *For all $j \ge 0$, $L(\infty, j) = \infty$ a.s.*

Proof: Suppose that $\mathsf{P}(L(\infty, j) < \infty) > 0$ for some j. Then $\mathsf{E}\, 1/L(\infty, j) > 0$. By restriction principle applied to $\{j, j+1, \ldots\}$, and Lemma 3.1, this expectation must be the same for all $j \ge 0$.

We claim that $\mathsf{E}\,(m^{1,1+\exp 1}_t)^{-1} < 1$ for all $t > 0$. We know that (see the appendix)

$$\phi_{a,b}(\lambda) = \mathsf{E}\, \exp(-\lambda m^{a,b}_t) = e^{b(a - \sqrt{\lambda^2/t^2 + 2\lambda + a^2})}.$$

Since $\int_0^\infty \exp(-\lambda m^{a,b}_t) d\lambda = \frac{1}{m^{a,b}_t}$, we have

$$\begin{aligned}
\mathsf{E}\, \frac{1}{m^{1,1+\exp 1}_t} &= \int_0^\infty \phi_{1,1+\exp 1}(\lambda) d\lambda \\
&= \int_0^\infty d\lambda \int_0^\infty \exp(-u) du \exp((u+1)(1 - \sqrt{\lambda^2/t^2 + 2\lambda + 1})) \\
&< \int_0^\infty d\lambda \int_0^\infty du \exp(1 - (u+1)(1 - \sqrt{2\lambda + 1})) \\
&= \int_0^\infty \frac{\exp(1 - \sqrt{2\lambda+1})}{\sqrt{2\lambda+1}} d\lambda = 1.
\end{aligned}$$

On the other hand, if $\mathsf{P}(L(\infty, j) < \infty) > 0$,

$$\mathsf{E} \quad \left(\frac{1}{L(\infty, j-1)} - \frac{1}{L(\infty, j)} \right)$$

$$= \mathsf{E} \quad \left[\mathsf{E} \left(\frac{1}{L(\infty, j-1)} - \frac{1}{L(\infty, j)} \,|\, L(\infty, j) = t \right) \right] < 0,$$

since by restriction principle, after relabeling $j-1$ as one and j as zero,

$$\begin{aligned} \mathsf{E} \left(\frac{1}{L(\infty, j-1)} - \frac{1}{L(\infty, j)} \,|\, L(\infty, j) = t \right) &= \frac{1}{t} \mathsf{E}^{\,1,1+\exp 1} \left(\frac{t}{L(\xi(t), 1)} - 1 \right) \\ &= \frac{1}{t} \left(\mathsf{E} \, \frac{1}{m_t^{1,1+\exp 1}} - 1 \right) < 0, \end{aligned}$$

yielding a contradiction. ∎

Now let $R_i^t = \frac{L(t,i)}{L(t,i+1)}$ and $R_i^n := R_i^{T_n}$. The notational ambiguity will not cause trouble. For $i \geq 0$ put $Z_i = \lim_{t\to\infty} R_i^t$. Recall that α is defined in (2.12) and calculated in (5.46). Put both R_{-1}^t and Z_{-1} equal to 1.

Lemma 3.4 *The following hold.*
i) Z_i, $i \geq 0$, are i.i.d., $\mathsf{E} \log Z_i = \alpha$, and the density of Z_i^{-1} is the function $f_\gamma(x)$ given in the statement of Theorem 1.1.
ii) $\sum_{k=0}^{\infty} \prod_{i=-1}^{k-1} Z_i^{-1} < \infty$ a.s.
iii) $\mathsf{E} \left(\log R_j^n \,|\, L(T_n, i),\ j+1 \leq i \leq n \right) > \alpha$, $0 \leq j < n$.

Proof: The first two statements of i), and iii) follow from Lemma 3.2, the definition of α, and, in the case of iii), (2.13). For the last statement of i) see formula (5.45) in the appendix. And ii) follows almost immediately from i) and the SLLN, which enables the bounding of the terms of the sum by a geometric series. ∎

Next we state the main result of this section, the almost sure convergence, in l_1, of the empirical occupational time distribution, Put

$$p_k = \frac{\prod_{i=-1}^{k-1} Z_i^{-1}}{\sum_{k=0}^{\infty} \prod_{i=-1}^{k-1} Z_i^{-1}}.$$

Theorem 3.5 *The following holds.*

$$\lim_{t\to\infty} \sum_{k=0}^{\infty} \left| \frac{L(t,k) - 1}{t} - p_k \right| = 0 \ \ a.s.$$

Before proving Theorem 3.5 we sketch for motivation a short proof of a weaker result. We have

$$L(t,k) = L(t,0) \prod_{j=-1}^{k-1} (R_j^t)^{-1} =: \Gamma_k^t, \;\; k \geq 0.$$

Putting $\Theta_k = \prod_{i=-1}^{k-1} Z_i^{-1}$, the definition of Z_i gives

$$\lim_{t\to\infty} \frac{\Gamma_k^t}{\Gamma_{k+1}^t} = \frac{\Theta_k}{\Theta_{k+1}} \text{ a.s.}, \;\; k \geq 0.$$

Now if n is fixed and a_k^t, $0 \leq k \leq n$, $t \geq 0$, and b_k, $0 \leq k \leq n$ are positive numbers such that

$$\lim_{t\to\infty} \frac{a_k^t}{a_{k+1}^t} = \frac{b_k}{b_{k+1}},$$

then

$$\lim_{t\to\infty} \frac{a_k^t}{\sum_{i=0}^n a_i^t} = \frac{b_k}{\sum_{i=0}^n b_i}, \;\; 0 \leq k \leq n,$$

which shows

$$\lim_{t\to\infty} \frac{L(k,t)}{\sum_{i=0}^n L(i,t)} = \frac{\Theta_k}{\sum_{i=0}^n \Theta_i}.$$

The last equality together with Proposition 3.3 implies that if $0 \leq k \leq n$

$$\lim_{t\to\infty} \frac{L(k,t)-1}{\sum_{i=0}^n (L(i,t)-1)} = \frac{\Theta_k}{\sum_{i=0}^n \Theta_i},$$

a junior version of Theorem 3.5, since $\sum_{i=0}^\infty (L(i,t)-1) = t$.

The following lemma is a more precise version of the simple fact about sequences just used.

Lemma 3.6 *Let a_i and b_i, $0 \leq i \leq n$, be positive numbers, and let $\varepsilon > 0$. Put $A = \sum_{i=0}^n a_i$ and $B = \sum_{i=0}^n b_i$. Then*

$$\sum_{i=0}^n \frac{|a_i - b_i|}{a_i} < \varepsilon \textit{ implies } \sum_{i=0}^n \left| \frac{a_i}{A} - \frac{b_i}{B} \right| < \frac{2\varepsilon}{1-\varepsilon}.$$

Proof: The hypotheses imply $|A-B| < \varepsilon A$, and so

$$\begin{aligned}\sum_{i=0}^{n}\left|\frac{a_i}{A}-\frac{b_i}{B}\right| &= \sum_{i=0}^{n}\left|\frac{a_i(B-A)+(a_i-b_i)A}{AB}\right| \le \frac{|B-A|}{B}+\frac{\sum|b_i-a_i|}{B} \\ &< \frac{\varepsilon}{1-\varepsilon}+\frac{\varepsilon}{1-\varepsilon}.\end{aligned}$$

■

Now since $L(T_n, n) = 1$, we have, recalling $R^{T_n}_{n-i}$ is shortened to R^n_{n-i}, that $\log L(T_n, n-k) = \sum_{i=0}^{k} \log R^n_{n-i}$. Put $\Delta^n_i = \mathsf{E}\,(\log R^n_{n-i} \,|\, R^n_{n-j},\ 0 \le j < i)$, $1 \le i \le n$, and $D^n_i = \log R^n_{n-i} - \Delta^n_i$. (Sometimes we drop the superscript.) Then Lemma 3.4 iii) implies

$$\Delta^n_i \ge \alpha \text{ a.s.},\ 1 \le i \le n. \tag{3.14}$$

and of course D^n_i, $1 \le i \le n$, is a martingale difference sequence. Furthermore, Lemma 3.2 and (2.11) imply $\mathsf{E}\,(|\log R^n_{n-i}|^4 \,|\, R^n_{n-j},\ 1 \le j < i) < C$, $1 \le i \le n$, which in turn implies

$$\mathsf{E}\,(D^4_i \,|\, R^n_{n-j},\ 1 \le j < i) < C,\ 1 \le i \le n,$$

where the C is absolute, especially it does not depend on i or n.

Thus for any $\varepsilon > 0$, according to Lemma 2.1,

$$\mathsf{P}(|\sum_{i=1}^{n} D_i| > \varepsilon n) < \frac{C(\varepsilon)}{n^2}, \tag{3.15}$$

which with (3.14) implies $\mathsf{P}(\sum_{i=1}^{n} D_i + \Delta_i < (\alpha-\varepsilon)n) < C(\varepsilon)/n^2$, or equivalently,

$$\mathsf{P}(\log L(T_n, 0) < (\alpha-\varepsilon)n) < \frac{C(\varepsilon)}{n^2}. \tag{3.16}$$

This inequality, Borel-Cantelli, and the fact that $L(T_n, 0) - 1 < T_n$, imply

$$\mathsf{P}(\liminf_{n\to\infty} \log T_n/n \ge \alpha) = 1, \tag{3.17}$$

or equivalently

$$\limsup_{t\to\infty} \frac{\max_{0\le s\le t} X_s}{\log t} \le \alpha^{-1} \text{ a.s.} \tag{3.18}$$

In the following, if $a < b$ are not necessarily integers, we use $\sum_{i=a}^{b} r_i$ to designate the sum of those r_i for all i satisfying $a \le i \le b$. We let θ be a fixed number between 0 and 1 satisfying $\theta \log 2 < \frac{1}{4}$, which guarantees

$$\frac{2^{(n+1)\theta}}{e^{n/4}} < 2\beta^n \text{ where } \beta := e^{\theta \log 2 - \frac{1}{4}} < 1. \tag{3.19}$$

The proof of Theorem 3.5 will be completed by establishing the following two limits. We have

$$\sup_{t\ge T_n} \sum_{i=0}^{n(1-\theta)} \left| \frac{L(t,i)-1}{\sum_{j=0}^{n(1-\theta)}(L(t,j)-1)} - p_i \right| \to 0 \text{ as } n\to\infty \tag{3.20}$$

and

$$\sum_{i=n(1-\theta)}^{n+1} \frac{L(T_{n+1},i)}{e^{n/4}} \to 0 \text{ as } n\to\infty. \tag{3.21}$$

To see that (3.20) and (3.21) imply Theorem 3.5, observe that to prove Theorem 3.5, it suffices to prove

$$\sup_{T_n\le t\le T_{n+1}} \sum_{k=0}^{n+1} \left| \frac{L(t,k)-1}{\sum_{j=0}^{n+1}(L(t,j)-1)} - p_k \right| \to 0 \text{ as } n\to\infty$$

since $L(T_{n+1},k) = 1$ if $k > n+1$, and $\sum_{k=0}^{\infty} p_k = 1$. Now (3.20) obviously implies

$$\sup_{T_n\le t\le T_{n+1}} \sum_{k=0}^{n(1-\theta)} \left| \frac{L(t,k)-1}{\sum_{j=0}^{n(1-\theta)}(L(t,j)-1)} - p_k \right| \to 0 \text{ as } n\to\infty.$$

Furthermore, if $n \ge 1$,

$$\sup_{T_n\le t\le T_{n+1}} \sum_{k=n(1-\theta)}^{n+1} \left| \frac{L(t,k)-1}{\sum_{j=0}^{n+1}(L(t,j)-1)} - p_k \right|$$

$$\leq \sup_{T_n \leq t \leq T_{n+1}} \sum_{k=n(1-\theta)}^{n+1} \left| \frac{L(t,k)-1}{\sum_{j=0}^{n+1}(L(t,j)-1)} \right| + \sum_{k=n(1-\theta)}^{n+1} p_k$$

$$\leq \frac{\sum_{k=n(1-\theta)}^{n+1} L(T_n,k)}{T_n} + \sum_{k=n(1-\theta)}^{\infty} p_k.$$

The second sum here clearly approaches 0 as $n \to \infty$, and since $T_n \geq e^{n/4}$ for large enough n, by (3.17) and (5.46), (3.21) gives that the first does also.

We first prove (3.21), then (3.20). Using Lemma 3.2 and Corollary 2.3 we have, for $0 \leq j \leq n$,

$$\begin{aligned} \mathsf{E}\, L(T_n,j) &= \mathsf{E}\,\mathsf{E}\,(L(T_n,j) \,|\, L(T_n,j+1)) \\ &= \mathsf{E}\, L_n(T_n,j+1)\,\mathsf{E}\left(\frac{L(T_n,j)}{L(T_n,j+1)} \,|\, L(T_n,j+1)\right) \\ &= \mathsf{E}\, L(T_n,j+1)\,\mathsf{E}^{\,1,1+\exp 1} m_{L(T_n,j+1)} \\ &= \mathsf{E}\, L(T_n,j+1)\,\mathsf{E}^{\,1,1+\exp 1} m_1 = 2\,\mathsf{E}\, L(T_n,j+1). \end{aligned}$$

Together with $\mathsf{E}\, L(T_n,n) = 1$ this gives $\mathsf{E}\, L(T_n,k) = 2^{n-k}$, $0 \leq k \leq n$.

Thus

$$\sum_{n=1}^{\infty} \sum_{k=n(1-\theta)}^{n+1} \frac{\mathsf{E}\, L(k,T_{n+1})}{e^{n/4}} < \infty,$$

using (3.19), and (3.21) follows.

Next we prove (3.20). We observe

$$\begin{aligned} &\sup_{t \geq T_n} \sum_{i=0}^{n(1-\theta)} \left| \frac{L(t,i)-1}{\sum_{j=0}^{n(1-\theta)}(L(t,j)-1)} - \frac{L(t,i)}{\sum_{j=0}^{n(1-\theta)} L(t,j)} \right| \\ \leq\ &\sup_{t \geq T_n} \frac{n(1-\theta)+2}{\sum_{j=0}^{n(1-\theta)}(L(t,j)-1)}, \end{aligned} \tag{3.22}$$

which follows immediately by putting the difference of the quotients on the LHS of (3.22) over a common denominator. Since $\sum_{j=0}^{n(1-\theta)} L(t,j) - 1 \geq T_{\lceil n(1-\theta) \rceil}$, if $t \geq T_n$, where $\lceil \cdot \rceil$ is the greatest integer function, (3.17) shows that that suprema to the right of the inequality in (3.22) approaches 0 as

$n \to \infty$, and thus the suprema to the left does. This implies that the following inequality is equivalent to (3.20).

$$\sup_{t \geq T_n} \sum_{i=0}^{n(1-\theta)} \left| \frac{L(t,i)}{\sum_{j=0}^{n(1-\theta)} L(t,j)} - p_i \right| \to 0 \text{ as } n \to \infty. \tag{3.23}$$

To prove (3.23), we rewrite it as

$$\sup_{t \geq T_n} \sum_{k=0}^{n(1-\theta)} \left| \frac{\prod_{i=-1}^{k-1} (R_i^t)^{-1}}{\sum_{j=0}^{n(1-\theta)} \prod_{i=-1}^{j-1} (R_i^t)^{-1}} - \frac{\prod_{i=-1}^{k-1} Z_i^{-1}}{\sum_{j=0}^{n(1-\theta)} \prod_{i=-1}^{j-1} Z_i^{-1}} \right| \to 0$$

as $n \to \infty$, and we note that using Lemma 3.6, to prove (3.23) it suffices to prove

$$\sup_{t \geq T_n} \sum_{k=0}^{n(1-\theta)} \left| \frac{\prod_{i=-1}^{k-1} (R_i^t)^{-1} - \prod_{i=-1}^{k-1} Z_i^{-1}}{\prod_{i=-1}^{k-1} Z_i^{-1}} \right| \to 0 \text{ as } n \to \infty$$

which reduces to

$$\sup_{t \geq T_n} \sum_{k=0}^{n(1-\theta)} \left| 1 - \prod_{i=-1}^{k-1} \left(1 + \frac{Z_i - R_i^t}{R_i^t} \right) \right| \to 0 \text{ as } n \to \infty. \tag{3.24}$$

Now $|1 - \prod_{i=0}^{m} (1 + a_i)| < \exp(\sum_{i=0}^{m} |a_i|) - 1 \leq 2\Sigma|a_i|$ if $\Sigma|a_i| < 0.1$, and so (3.24) follows from

$$n \cdot \sup_{t \geq T_n} \sum_{k=0}^{n(1-\theta)} \left| \frac{Z_i - R_i^t}{R_i^t} \right| \to 0 \text{ as } n \to \infty, \tag{3.25}$$

The initial n in (3.25) is an upper bound (if n is large) for the number of summands k in (3.24), $k = 0, 1, \ldots, n(1-\theta)$. Now exactly as we proved (3.16), we have

$$\mathsf{P}\left(\sum_{i=1}^{\theta n - 1} D_i + \Delta_i < 0.2\, \theta n \alpha \right) < \frac{C}{n^2} \tag{3.26}$$

Also, Lemma 2.1 and (3.14), or even the weaker version of (3.14) with α replaced by zero, imply

$$\mathsf{P}(\inf_{\theta n \leq k \leq n} \sum_{i=\theta n}^{k} D_i + \Delta_i < -0.1 \theta (1-\theta) n \alpha) < \frac{C}{n^2}. \tag{3.27}$$

Together (3.26) and (3.27) give

$$\mathsf{P}(\inf_{\theta n-1\le k\le n}\sum_{i=0}^{k} D_i+\Delta_i<0.1\theta n\alpha)<\frac{C}{n^2},$$

or, equivalently,

$$\mathsf{P}(L(T_n,i)\ge e^{0.1\theta n},\ 0\le i\le n(1-\theta)+1)>1-\frac{C}{n^2}. \tag{3.28}$$

Let $G_i=G_i^n=\{L(T_n,i)>e^{0.1\theta n}\}$. Then (3.28) may be restated as

$$\mathsf{P}\left(\bigcup_{i=0}^{n(1-\theta)+1} G_i^c\right)<\frac{C}{n^2}, \tag{3.29}$$

where the superscript ^{c} denotes complement.

Now conditioned on $L(T_n,i)$ and $L(T_n,i+1)$, the distribution of Y_t, $t\ge T_n$, restricted to $\{i,i+1\}$ has the distribution of the two state walk of Section 2, Z_t, $t\ge 0$, under $\mathsf{P}^{L(T_n,i+1),L(T_n,i)}$ if we relabel $i+1$ as 0 and i as 1. Thus (2.8) implies

$$\begin{aligned}
&\mathsf{E}\ \sup\nolimits_{t\ge T_n}(Z_i-R_i^t)^2 I(G_{i+1})\\
=\ &\mathsf{E}\ \mathsf{E}\ \left(\sup\nolimits_{t\ge T_n}(Z_i-R_i^t)^2\,|\,L(T_n,i+1),L(T_n,i)\right) I(G_{i+1}) \qquad (3.30)\\
\le\ &\mathsf{E}\,\frac{16L(T_n,i)}{L(T_n,i+1)^3} I(G_{i+1})\le 16\,(e^{-0.1\theta n})^2\ \mathsf{E}\,R_i^n=32\,e^{-0.2\theta n},
\end{aligned}$$

using Corollary 2.3 and the fact that R_i^n has the distribution of $m_{L(T_n,i+1)}$ under $\mathsf{P}^{1,1+\exp 1}$, so $\mathsf{E}\,R_i^n=\mathsf{E}^{\,1,1+\exp 1}m_0=2$.

Thus,

$$\begin{aligned}
\mathsf{E}\ \sup\nolimits_{t\ge T_n}\left|\tfrac{Z_i-R_i^t}{R_i^t}\right| I(G_{i+1}) \ &\le\ \left[\mathsf{E}\ \sup\nolimits_{t\ge T_n}(Z_i-R_i^t)^2 I(G_{i+1})\right]^{\frac12}\\
&\times\ \left[\mathsf{E}\ \sup\nolimits_{t\ge 0}\tfrac{1}{(R_i^t)^2}\right]^{\frac12}\le Ce^{-0.1\theta n},
\end{aligned} \tag{3.31}$$

using (3.30). That $\mathsf{E}\ \sup_{t\ge T_n}(R_i^t)^{-2}$ is finite follows from the restriction principle and the fact that $\sup_{t\ge 0}(R_i^t)^{-1}$ has the same distribution as $\sup_{s\ge 1} m_s$

under $P^{1,1}$, using the continuous version of Lemma 2.1 and the sentence which includes (2.7). Finally, to complete the proof of (3.25) and thus (3.20), we have, for large enough n,

$$\{n \sup_{t\geq T_n} \sum_{i=0}^{n(1-\theta)} \left|\frac{Z_i - R_i^t}{R_i^t}\right| \geq e^{-0.1\theta n}\} \subset \left(\bigcup_{i=0}^{n(1-\theta)} G_i^c\right)$$
$$\cup \bigcup_{i=0}^{n(1-\theta)} \{\sup_{t\geq T_n} \left|\frac{Z_i - R_i^t}{R_i^t}\right| I(G_{i+1}) > n^{-1}e^{-0.05\theta n}\} \tag{3.32}$$

And

$$\begin{aligned} &\mathsf{P}\left(\bigcup_{i=0}^{n(1-\theta)} \left\{\sup_{t\geq T_n} \left|\frac{Z_i - R_i^t}{R_i^t}\right| I(G_{i+1}) > n^{-1}e^{-0.05\theta n}\right\}\right) \\ \leq & \sum_{i=0}^{n(1-\theta)} \mathsf{P}\left(\sup_{t\geq T_n} \left|\frac{Z_i - R_i^t}{R_i^t}\right| I(G_{i+1}) > n^{-1}e^{-0.05\theta n}\right) \\ \leq & \sum_{i=0}^{n(1-\theta)} ne^{0.05\theta n}\, \mathsf{E} \sup_{t\geq T_n} \left|\frac{Z_i - R_i^t}{R_i^t}\right| I(G_{i+1}) \leq Cn^2 e^{-0.05\theta n} < \frac{C}{n^2}, \end{aligned}$$

using (3.31). And this inequality, together with (3.29) and (3.32) and Borel-Cantelli, establish (3.25).

The next theorem is a one-sided version of Theorem 1.2.

Theorem 3.7 *The following holds.*

$$\lim_{t\to\infty} \frac{\max_{0\leq s\leq t} X_s}{\log t} = \alpha^{-1} \ \ a.s. \tag{3.33}$$

Proof: We will show that given $\varepsilon > 0$, there is a constant $C(\varepsilon)$ such that

$$P\left(T_n > e^{n(\alpha+\varepsilon)}\right) < \frac{C(\varepsilon)}{n^2} \ n \geq 1. \tag{3.34}$$

This inequality together with Borel-Cantelli shows $\limsup_{n\to\infty} \log T_n/n \leq \alpha$, which implies

$$\liminf_{t\to\infty} \frac{\max_{0\leq s\leq t} X_s}{\log t} \geq \alpha^{-1},$$

and which, with (3.18), gives (3.33).

We know $\mathsf{E}^{1,1+\exp 1}\log m_t$ decreases to α. Let $K = \mathsf{E}^{1,1+\exp 1}\log m_1$. Let $\gamma > 0$, and let the integer N satisfy

$$\mathsf{E}^{1,1+\exp 1}\log m_{e^{0.1\theta N}} < (1+\gamma)\alpha. \tag{3.35}$$

Here and below we use the notation of the proof of Theorem 3.5.

Now (3.35) implies $\Delta_i I(G_{i+1}) < (1+\gamma)\alpha$, $0 \le i \le n$, and so on the intersection of the G_i, $1 \le i \le n$,

$$\sum_{i=1}^{n}\Delta_i = \sum_{i=1}^{\theta n}\Delta_i + \sum_{i=\theta n+1}^{n}\Delta_i. \le K\theta n + \sum_{i=\theta n+1}^{n}\Delta_i.$$

Thus if $n \ge N$,

$$\begin{aligned}\sum_{i=1}^{n}\Delta_i I\left(\bigcap_{i=1}^{n(1-\theta)+1} G_i\right) &\le K\theta n + [(1-\theta)n+1](1+\gamma)\alpha \\ &=: nf(\theta,\gamma),\end{aligned} \tag{3.36}$$

where $f(\theta,\gamma) = K\theta + (1-\theta)(1+\gamma)\alpha + \Theta(1/n)$.

Lemma 2.1 gives

$$\mathsf{P}\left(\sup_{1\le k\le n}\sum_{i=1}^{k} D_i > \gamma n\right) < \frac{C(\gamma)}{n^2},$$

and so

$$\mathsf{P}\left(\sup_{1\le k\le n}\sum_{i=1}^{k} D_i + \Delta_i > nf(\theta,\gamma) + n\gamma\,,\ \bigcap_{i=1}^{n(1-\theta)+1} G_i\right) < \frac{C(\gamma)}{n^2}$$

when $n \ge N$. Together with (3.29), this gives that if $n \ge N$,

$$\mathsf{P}\left(\sum_{i=1}^{k} D_i + \Delta_i < nf(\theta,\gamma) + n\gamma,\ 1 \le k \le n\right) > 1 - \frac{C}{n^2},$$

so that

$$\mathsf{P}(\log L(T_n, i) < nf(\theta,\gamma) + n\gamma,\ 0 \le i \le n) > 1 - \frac{C}{n^2}.$$

Since $\sum_{i=0}^{n-1}(L(T_n,i)-1)=T_n$, we get

$$\mathsf{P}\left(T_n < C_1 n e^{n[f(\theta,\gamma)+\gamma]}\right) > 1-\frac{C}{n^2},\ n\geq N.$$

Now if we choose θ and γ so small that $f(\theta,\gamma)+\gamma<\alpha+\varepsilon$, this implies (3.34). ■

4 VRJP on the integers.

We begin this section by describing the random variables V_i of Theorem 1.1. Then we prove Theorem 1.1 and use it and Theorem 3.7 to prove Theorem 1.2. Let $X=X_t$, $t\geq 0$, be a VRJP on the integers started at 0. Let X_s^+, $s\geq 0$, be X restricted to the nonnegative integers, and let X_s^-, $s\geq 0$, be X restricted to the non-positive integers. Then both X^+ and $(-X^-)$ are VRJP's on the nonnegative integers. Let Z_i^+, $i>0$ be the variables defined for X^+ exactly as the variables Z_i were defined for Y_t, $t\geq 0$, in Section 3, and let Z_i^- be the analogous variables for X^-. Then by Theorem 3.5 $\{Z_i^+,\ 1\leq i<\infty,\ Z_i^-,\ 1\leq i<\infty\}$ are i.i.d. random variables, each having the density function given in the statement of Theorem 1.1, as shown in the appendix.

Put

$$W_k=\begin{cases}\prod_{i=1}^k (Z_i^+)^{-1}, & k>0,\\ 1, & k=0,\\ \prod_{i=1}^k (Z_i^-)^{-1}, & k<0,\end{cases}$$

and put $W=\sum_{i=-\infty}^{\infty} W_i$, and $V_k=W_i/W$. We now prove Theorem 1.1, with V_i as just constructed.

Let $\delta(t)=\int_0^t I(X_s=0)\,ds$, $\eta(t)=\int_0^t I(X_s>0)\,ds$, and $\mu(t)=\int_0^t I(X_s<0)\,ds$. Then

$$\delta(t)+\eta(t)+\mu(t)=t, \tag{4.37}$$

and using the restriction principle we get both

$$\lim_{t\to\infty} \frac{\int_0^t I(X_s = j)\, ds}{\eta(t) + \delta(t)} = \frac{W_j}{\sum_{i=0}^{\infty} W_i}, \ j \geq 0, \tag{4.38}$$

and

$$\lim_{t\to\infty} \frac{\int_0^t I(X_s = j)\, ds}{\mu(t) + \delta(t)} = \frac{W_j}{\sum_{i=0}^{\infty} W_{-i}}, \ j \leq 0. \tag{4.39}$$

Let $\bar{\delta}(t)$, $\bar{\eta}(t)$, and $\bar{\mu}(t)$ stand for $\delta(t)/t$, $\eta(t)/t$, and $\mu(t)/t$ respectively. Then (4.37) and the versions of (4.38) and (4.39) for $j = 0$ give the following three equations:

$$\begin{aligned} \bar{\delta}(t) + \bar{\eta}(t) + \bar{\mu}(t) &= 1, \\ \lim_{t\to\infty} \frac{\bar{\delta}(t)}{\bar{\delta}(t) + \bar{\eta}(t)} &= \frac{W_0}{\sum_{i=0}^{\infty} W_i}, \\ \lim_{t\to\infty} \frac{\bar{\delta}(t)}{\bar{\delta}(t) + \bar{\mu}(t)} &= \frac{W_0}{\sum_{i=0}^{\infty} W_{-i}}. \end{aligned} \tag{4.40}$$

Thus

$$\begin{aligned} \lim_{t\to\infty} \bar{\delta}(t) &= W_0/W, \\ \lim_{t\to\infty} \bar{\eta}(t) &= \sum_{i=1}^{\infty} W_i/W, \\ \lim_{t\to\infty} \bar{\mu}(t) &= \sum_{i=1}^{\infty} W_{-i}/W. \end{aligned} \tag{4.41}$$

The equations (4.38), (4.39), and (4.41) imply Theorem 1.1.

Proof of Theorem 1.2: From (3.33) and the restriction principle we have

$$\lim_{t\to\infty} \frac{\max_{0\leq s\leq t} X_s}{\log[\delta(t) + \eta(t)]} = \alpha^{-1} \text{ a.s.} \tag{4.42}$$

Equations (4.41) together with (4.42) prove Theorem 1.2. ■

5 Appendix

To describe the distribution of $L(\xi(t), 1)$, defined in Section 2 immediately before Lemma 2.2, we will calculate its Laplace transform $\phi^{a,b}(\lambda, t) = \mathsf{E}^{a,b} e^{-\lambda L(\xi(t),1)}$, $\lambda \geq 0$. Further we will omit the superscript ^{a,b} unless it makes our arguments ambiguous.

Denote $w(t) := L(\xi(t), 1)$ and observe that

$$w(t + dt) = w(t) + \nu\eta$$

where ν is a Bernoulli $(w(t)dt)$ random variable and η is an exponential (t) random variable, which, given w and t, are independent of anything. Hence

$$\phi(\lambda, t + dt) = \mathsf{E}\,(e^{-\lambda w - \lambda\eta\nu}) = \mathsf{E}\,\left[e^{-\lambda w}\,\mathsf{E}\,(e^{-\lambda\eta\nu} \mid w)\right] \tag{5.43}$$

The inner conditional expectation is easy to compute:

$$\begin{aligned} \mathsf{E}\,(e^{-\lambda\eta\nu} \mid w) &= (1 - w\,dt) \times 1 + w\,dt \times \mathsf{E}\,(e^{-\lambda\eta}) \\ &= (1 - w\,dt) + w\,dt \times \frac{t}{\lambda + t} = 1 - w\,dt\frac{\lambda}{\lambda + t} \end{aligned}$$

Plugging this into (5.43) yields

$$\phi(\lambda, t + dt) - \phi(\lambda, t) = -\frac{\lambda}{\lambda + t}\,\mathsf{E}\,(we^{-\lambda w})\,dt,$$

whence, since $\mathsf{E}\,(w(t)e^{-\lambda w(t)} \mid t) = -\partial\phi(\lambda, t)/\partial\lambda$,

$$\frac{\partial\phi}{\partial t} = \frac{\lambda}{\lambda + t} \cdot \frac{\partial\phi}{\partial\lambda}.$$

The natural boundary conditions are

$$\begin{aligned} \phi(\lambda, a) &= e^{-\lambda b}, \\ \phi(0, \cdot) &= 1. \end{aligned}$$

Solving this (see Section 5.1) we obtain

$$\phi^{a,b}(\lambda, t) = e^{b\left(a - \sqrt{\lambda^2 + 2\lambda t + a^2}\right)}. \tag{5.44}$$

Though we are not able to invert Laplace transform (5.44) for every t, it still gives us Lemmas 2.2 and 2.5

$$\begin{aligned} \mathsf{E}^{a,b} w(t) &= \frac{b}{a} \cdot t, \\ \mathsf{E}^{a,b} [w(t)]^2 &= \frac{b(t^2 - a^2 + t^2 ab)}{a^3} \end{aligned}$$

by differentiating $\phi^{a,b}(\lambda, t)$ once and twice at $\lambda = 0$.

Next, we want to calculate explicitly the distribution of $\gamma := m_\infty^{1,1} = \lim_{t\to\infty} \frac{w(t)}{t}$ which exists by Corollary 2.4. By interchanging the integration and the limit we obtain

$$\mathsf{E}^{1,1} e^{-\lambda\gamma} = \lim_{t\to\infty} \phi^{1,1}(\lambda/t, t) = e^{1-\sqrt{1+2\lambda}}.$$

Using Laplace transform, we can, for example, compute moments of γ:

$$\mathsf{E}\,\gamma = 1, \quad \mathsf{E}\,\gamma^2 = 2, \quad \mathsf{E}\,\gamma^3 = 7, \quad \mathsf{E}\,\gamma^4 = 37, \quad \mathsf{E}\,\gamma^5 = 266, \quad \ldots.$$

The inversion of $\mathsf{E}^{1,1} e^{-\lambda\gamma}$ requires an integration on a complex plane. We omit these calculations, presenting only the result. The density of the distribution of $\gamma = m_\infty^{1,1}$ for $x > 0$ is

$$f_\gamma(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{1-\sqrt{-2i\lambda+1}} e^{-i\lambda x}\, d\lambda = \frac{e^{1-\frac{1}{2}(x+x^{-1})}}{\sqrt{2\pi x^3}} \tag{5.45}$$

(one can quite easily verify that its Laplace transform coincides with $e^{1-\sqrt{1+2\lambda}}$). This density is also the density of $m_\infty^{1,1+\exp 1}$, and is the density f_γ of Theorem 1.1. Moreover, we can present the formula for c.d.f. of γ:

$$F_\gamma(x) = 1 - \Phi\left(\frac{1}{\sqrt{x}} - \sqrt{x}\right) + e^2 \left[1 - \Phi\left(\frac{1}{\sqrt{x}} + \sqrt{x}\right)\right], \quad x > 0$$

where $\Phi(\cdot)$ is a c.d.f. of a normal zero-one distribution.

To calculate α in (2.12) observe that $m_\infty^{1,1+\exp 1}$ has the same distribution as $1/m_\infty^{1,1} = 1/\gamma$. Consequently,

$$\alpha = \int_0^\infty \log(x) f_{1/\gamma}\, dx = \int_0^\infty \log x \cdot \frac{\exp(1 - \frac{x}{2} - \frac{1}{2x})}{\sqrt{2\pi x}}\, dx = 0.3613\ldots \tag{5.46}$$

5.1 Solution of the equation $(-x)\phi'_x + (x+y)\phi'_y = 0$

This equation is a linear PDE to which we can apply a standard technique. We will look for a solution in the area where $x \geq 0$ and $y \geq 1$ with a boundary condition

$$\phi(x, a) = e^{-bx}$$

Let $\mathbf{v}(x, y) = (-y, x+y)$ be a column vector, then the equation is equivalent to

$$\mathbf{v} \cdot \nabla\phi = 0,$$

where $\cdot$ denotes a scalar product and $\nabla\phi$ is a gradient of ϕ. Thus, $\phi(x, y)$ must be constant along the solutions of the equation $\dot{\mathbf{z}} = \mathbf{v}$, where $\mathbf{z} = (x, y)$. Solving the system

$$\begin{cases} \dot{x} &= -x \\ \dot{y} &= x + y \end{cases}$$

we obtain $x(t) = C_1 e^{-t}$, $y(t) = C_2 e^t - \frac{1}{2} C_1 e^{-t}$. Hence, $2C_1 C_2 = x(x+2y)$, and any solution $\phi(x, y)$ to the PDE must be a function of one argument $x(x+2y)$.

If this curve $(x(t), y(t))$ intersects the horizontal line $y = a$ at point $\tilde{x} \geq 0$, then $x(x+2y) = \tilde{x}(\tilde{x}+2a)$ and $\tilde{x} = -a + \sqrt{x(x+2y)+a^2}$ (we took a positive sign at the square root since $\tilde{x}$ must be non-negative). On the other hand, $\phi(\tilde{x}, a) = e^{-\tilde{x}b}$, therefore

$$\phi(x, y) = \phi(\tilde{x}, a) = e^{b\left(a - \sqrt{x(x+2y)+a^2}\right)}.$$

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