Stochastic Taylor Expansions for Poisson Processes and Applications towards Risk Management

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February 16, 2001

Abstract

The Taylor expansion is a powerful tool in the analysis of deterministic functions. In the case of stochastic processes there exists already a stochastic Taylor expansion for diffusion processes and some general existence results for other classes of processes. We explicitly calculate a stochastic Taylor expansion for multivariate Poisson processes. An extension to diffusion processes with Poisson jumps is straightforward. The expansion is used for two financial applications in the context of Risk Management.

1 Introduction

If we study local properties of a deterministic function the Taylor expansion plays a crucial role. But stochastic processes do not behave in the same way as deterministic function, therefore we can not do a deterministic Taylor expansion for stochastic processes. There exist already some results about stochastic Taylor expansions for diffusion processes (see for example [2] or [9]). Strongly related to the question of Taylor expansions is the *Chaotic Representation Property* (CRP) derived for normal martingales, i.e. for martingales X such that $\langle X, X\rangle_t = ct$, for some constant $c > 0$ in [4], or for Lévy processes with exponential moments in [11]. The CRP gives us nice existence results of decompositions of L^2 random variables on some probability space in terms of orthogonal sequences of martingales. Although it is sometimes in principle possible to calculate the coefficients of the expansion (see [10]), the calculations may become quite involved.
In this paper we give a simple and explicit calculation of a stochastic Taylor expansion of a function of Poisson processes at a fixed time t. In the next section we explain the idea in the one-dimensional case. In Section 3 we extend this result to a

multivariate Poisson process. Then we put the expansion of [9] and the expansion for multivariate Poisson processes together. Finally we apply our results to various examples related to Risk Management in Section 4.

The main tool we use is Itô's formula. One possible general formulation can be given as follows (see [15, p. 70ff]).

Theorem 1.1 (Itô Formula). *Let $X = (X^1, \dots, X^n)$ be an n-tuple of semimartingales and let $f : \mathbb{R}^n \to \mathbb{R}$ have continuous second order partial derivatives. Then $f(X)$ is again a semimartingale and the following formula holds:*

$$
\begin{aligned}
f(X_t) = f(X_0) &+ \sum_{i=1}^{n} \int_{0+}^{t} \frac{\partial f}{\partial x_i}(X_{s-}) dX_s^i \\
&+ \frac{1}{2} \sum_{1 \le i,j \le n} \int_{0+}^{t} \frac{\partial^2 f}{\partial x_i \partial x_j}(X_{s-}) d[X^i, X^j]_s^c \\
&+ \sum_{0<s\le t} \left(f(X_s) - f(X_{s-}) - \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(X_{s-}) \Delta X_s^i \right). \qquad (1)
\end{aligned}
$$

Remark. The Poisson process is a finite variation process. Furthermore it can only jump finitely many times in each finite time interval. For such processes formula (1) reduces to a much simpler form. As one easily checks we have:

$$
f(X_t) = f(X_0) + \sum_{s \le t} f(X_s) - f(X_{s-}). \qquad (2)
$$

Now the function f need not be C^2. In fact we do not have to put any restriction on the function f. Similar to the treatment of diffusion processes in [9] we recursively use formula (2) for Poisson processes. Only in the end when we generalize to jump-diffusions we have to use formula (1).

2 The univariate case

One basic object in this paper are Poisson processes:

Definition 2.1. *A Poisson process N_t with parameter λ is a counting process starting at 0 with stationary, independent increments and $N_t \sim$Poisson(λt).*

We introduce the following notation:

Notation:
Given a function $f : \mathbb{R} \to \mathbb{R}$ we define the difference operator Δ by: $\Delta f(x) = f(x+1) - f(x)$. We then extend this notation. Set $\Delta^0 f(x) = f(x)$, $\Delta^1 f(x) = \Delta f(x)$ and define $\Delta^i f(x) = \Delta(\Delta^{i-1} f(x))$ recursively.

Remark. The difference operator Δf should not be mixed up with (ΔN_t) which denotes the jump process of N_t. The meaning should always be clear from the context.

We now explain the main idea of the expansion. In the case of a Poisson process one can easily rewrite Itô's formula as has been done in Remark 1. For an arbitrary function f we obtain the following:

$$\begin{aligned} f(N_t) &= f(0) + \sum_{0<s\leq t} (f(N_s) - f(N_{s-})) \\ &= f(0) + \sum_{0<s\leq t} (f(N_s) - f(N_{s-}))\Delta N_s \\ &= f(0) + \sum_{0<s\leq t} (f(N_{s-}+1) - f(N_{s-}))\Delta N_s \\ &= f(0) + \int_{0+}^{t} (f(N_{s-}+1) - f(N_{s-}))dN_s \\ &= f(0) + \int_{0+}^{t} \Delta f(N_{s-})dN_s. \end{aligned} \tag{3}$$

The second and third equality follow from the fact that for a Poisson process $\Delta N_s = 1$ in the case of a jump at time s or 0 otherwise. Then we use the definition of the integral in the Poisson case and finally the definition of the difference operator Δ.
In the same way as above one can apply Itô's formula to the function $\Delta f(N_s)$.

$$\Delta f(N_s) = \Delta f(0) + \int_{0+}^{s} \Delta(\Delta f(N_{u-}))dN_u. \tag{4}$$

Plugging (4) into (3) leads to:

$$\begin{aligned} f(N_t) &= f(0) + \int_0^t \left(\Delta f(0) + \int_{0+}^{s-} \Delta(\Delta f(N_{u-}))dN_u \right) dN_s \\ &= f(0) + \Delta f(0) \int_{0+}^{t} dN_s + \int_{0+}^{t}\int_{0+}^{s-} \Delta^2 f(N_{u-})dN_u dN_s. \end{aligned} \tag{5}$$

We then go on inductively. To simplify notation and to get rid of multiple integrals which show up in the expansion we use some well-known results related to multiple integrals with respect to a Poisson process.

2.1 Multiple stochastic integrals

Multiple stochastic integrals with respect to the Poisson process show up in the expansion. We use the following notation.

$$\begin{aligned} P_t^{(n)}(f) &= \int_{0+}^{t}\int_{0+}^{t_1-}\cdots\int_{0+}^{t_{n-1}-} f(N_{t_n-})dN_{t_n}\dots dN_{t_1}, \\ P_t^{(0)}(f) &= f(t). \end{aligned} \tag{6}$$

We simply write $P_t^{(n)}$ in (6) for $f \equiv 1$.

Example. We give some easy examples for this notation.

$$\begin{aligned} P_t^{(0)} =& 1, \\ P_t^{(1)} =& \int_{0+}^{t} dN_{t_1} = N_t, \\ P_t^{(2)}(f) =& \int_{0+}^{t} \int_{0+}^{t_1-} f(N_{t_2-}) dN_{t_2} dN_{t_1}. \end{aligned}$$

We cite a result of [5, p. 23] which can easily be proved by induction.

Proposition 2.2. *Given a Poisson process $(N_t)_{0\leq t}$. Then we have for all $n \geq 0$ that:*

$$P_t^{(n)} = \binom{N_t}{n}. \tag{7}$$

Proof. See [5] $\square$

Example. This proposition gives an easy way of calculating multiple stochastic integrals with respect to a Poisson process.

$$\begin{aligned} P_t^{(2)} &= \int_{0+}^{t} \int_{0+}^{s-} dN_u dN_s = \frac{1}{2} N_t(N_t - 1), \\ P_t^{(3)} &= \int_{0+}^{t} \int_{0+}^{s-} \int_{0+}^{u-} dN_v dN_u dN_s = \frac{1}{6} N_t(N_t - 1)(N_t - 2). \end{aligned}$$

2.2 The stochastic Taylor expansion

We can now prove the following theorem.

Theorem 2.3. *(***Stochastic Taylor expansion for the Poisson process***). Given $(N_t)_{0\leq t}$ a Poisson process and $f : \mathbb{R} \to \mathbb{R}$. Then for all $m \in \mathbb{N}_0$ the following expansion holds:*

$$f(N_t) = \sum_{i=0}^{m} \Delta^i f(0) \binom{N_t}{i} + P_t^{(m+1)}(\Delta^{m+1} f). \tag{8}$$

Proof. By induction.
From (3) we have that $f(N_t) = f(0) + \int_{0+}^{t} \Delta f(N_{s-}) dN_s = f(0) + P_t^{(1)}(\Delta^1 f)$.
This is the statement for $m = 0$.
Assume the statement is true for $m - 1$, i.e.

$$f(N_t) = \sum_{i=0}^{m-1} \Delta^i f(0) \binom{N_t}{i} + P_t^{(m)}(\Delta^m f).$$

From (4) we expand the function $\Delta^m f(N_{t_m})$ and get:

$$\begin{aligned}\Delta^m f(N_{t_m}) =& \Delta^m f(0) + \int_{0+}^{t_m} \Delta^{m+1} f(t_{m+1}-) dN_{t_{m+1}} \\ \Rightarrow P_t^{(m)}(\Delta^m f) =& \Delta^m f(0) P_t^{(m)} + P_t^{(m+1)}(\Delta^{m+1} f).\end{aligned}$$

□

Example. Theorem 2.3 shows, how to calculate stochastic Taylor expansions for arbitrary functions. Set for example $m = 2$, then:

$$f(N_t) = f(0) + (f(1) - f(0))N_t + (f(2) - 2f(1) + f(0))\frac{1}{2}N_t(N_t - 1) + R.$$

When we drop the remainder term R, the approximation is exact on the set $\{N_t \leq 2\}$ as one can easily check.

The property of exactness of the truncated stochastic Taylor expansion for the Poisson process holds more generally. Define the truncated Taylor expansion $f_m(N_t)$ as:

$$f_m(N_t) = \sum_{i=0}^{m} \Delta^k f(0) \binom{N_t}{k}. \tag{9}$$

Proposition 2.4. *The truncated stochastic Taylor expansion $f_m(N_t)$ for the Poisson process is exact on the set $\{N_t \leq m\}$.*

Proof. By induction.
$f_0(N_t) = f(0)$ is exact on $\{N_t = 0\}$.
$m \geq 1$. Assume the statement is true for $m - 1$. On the set $\{N_t \leq m - 1\}$ we have that $\binom{N_t}{m} = 0$, hence $f_{m-1} = f_m$ on $\{N_t \leq m - 1\}$. By assumption f_{m-1} is exact on this set. It remains to show that $f_m(m) = f(m)$.
A simple calculation shows that:

$$\Delta^k f(0) = \sum_{i=0}^{k} (-1)^{k+i} \binom{k}{i} f(i).$$

Therefore we can write

$$\begin{aligned} f_m(m) &= \sum_{k=0}^{m} \Delta^k f(0) \binom{m}{k} = \sum_{k=0}^{m} \sum_{i=0}^{k} (-1)^{k+i} \binom{k}{i} f(i) \binom{m}{k} \\ &= \sum_{i=0}^{m-1} \left(\sum_{k=i}^{m} (-1)^{k+i} \binom{k}{i} \binom{m}{k} \right) f(i) + f(m) \\ &= \sum_{i=0}^{m-1} \left(\binom{m}{i} \underbrace{\sum_{k=0}^{m-i} (-1)^{k+2i} \binom{m-i}{k}}_{(1-1)^{m-i}=0} \right) f(i) + f(m), \end{aligned}$$

and hence the Proposition is proved. □

Corollary 2.5. *i)* $f_m(N_t) \longrightarrow f(N_t)$ *almost surely for* $m \to \infty$.
ii) *If* $\|f\|_{L^\infty} \le M$, *then we also have, that* $f_m(N_t) \longrightarrow f(N_t)$ *in* L^1 *for* $m \to \infty$.

Proof. To prove i) we use the following:

$$\{\sup_{n\ge m} |f_n - f| > \epsilon\} \subset \bigcup_{n\ge m}\{|f_n - f| > \epsilon\} \subset \bigcup_{n\ge m}\{f_n \ne f\} \overset{\text{Prop.2.4}}{\subset} \bigcup_{n\ge m}\{N_t > n\}$$

But $\bigcup_{n\ge m}\{N_t > n\} = \{N_t > m\}$.

$$\begin{aligned}\Rightarrow P[\sup_{n\ge m}|f_n - f| > \epsilon] \le & P[N_t > m] = \sum_{n>m} e^{-\lambda t}\frac{(\lambda t)^n}{n!} \\ \le & \frac{(\lambda t)^{m+1}}{(m+1)!} \to 0 \text{ as } m \to \infty.\end{aligned}$$

We use Proposition 2.4 again, to show ii).

$$\begin{aligned}
E[|f_m - f|] =& E[|f_m - f|, N_t > m] \\
\le& E[|f_m|, N_t > m] + E[|f|, N_t > m] \\
E[|f|, N_t > m] \le& MP[N_t > m] \to 0 \text{ as } m \to \infty \\
|\Delta^l f(0)| \le& M\sum_{i=0}^{l}\binom{l}{i} = M2^l \\
\Rightarrow |f_m(N_t)| \le& M\sum_{l\le m} 2^l\binom{N_t}{l} \\
\Rightarrow E[|f_m|, N_t > m] \le& M\sum_{l\le m}2^l\sum_{k>m}\binom{k}{l}e^{-\lambda t}\frac{(\lambda t)^k}{k!} \\
=& M\sum_{k>m}e^{-\lambda t}\frac{(\lambda t)^k}{k!}\sum_{l\le m}\binom{k}{l}2^l \\
\le& M\sum_{k>m}e^{-\lambda t}\frac{(\lambda t)^k}{k!}\sum_{l\le k}\binom{k}{l}2^l \\
=& M\sum_{k>m}e^{-\lambda t}\frac{(\lambda t)^k}{k!}3^k = Me^{-\lambda t}\sum_{k>m}\frac{(3\lambda t)^k}{k!} \overset{m\to\infty}{\longrightarrow} 0\,.
\end{aligned}$$

$\square$

Remark. We can even prove the stronger assertion that the whole trajectory converges uniformly, because $|f(N_s) - f_m(N_s)| = 0$ on $\{N_t \le m\}$ for $s \le t$. Therefore we know that $\sup_{s\le t}|f(N_s) - f_m(N_s)| = 0$ on $\{N_t \le m\}$ and the proof is analogous to the proof for i) above.

3 The multivariate case

In this section we extend the result from Theorem 2.3 to the multivariate case. We start with a d-dimensional Poisson process $N_t = (N_t^{(1)}, \ldots, N_t^{(d)})$ where $N^{(i)},\ N^{(j)}$ are independent Poisson processes for $i \neq j$ with intensities λ_i. We use now Itô's lemma again. As the main ideas become already clear in the two-dimensional case we let $d = 2$. In the same way as in (3) we can write for any function $f : \mathbb{R}^2 \to \mathbb{R}$:

$$
\begin{aligned}
f(N_t) &= f(0) + \sum_{0<s\leq t} f(N_s) - f(N_{s-}) \\
&= f(0) + \sum_{0<s\leq t} \Big(f(N_{s-}^{(1)} + 1, N_{s-}^{(2)}) - f(N_{s-}^{(1)}, N_{s-}^{(2)}) \Big) \Delta N_s^{(1)} \\
&\quad + \sum_{0<s\leq t} \Big(f(N_{s-}^{(1)}, N_{s-}^{(2)} + 1) - f(N_{s-}^{(1)}, N_{s-}^{(2)}) \Big) \Delta N_s^{(2)} \\
&= f(0) + \int_{0+}^{t} \Delta_1 f(N_{s-}) dN_s^{(1)} + \int_{0+}^{t} \Delta_2 f(N_{s-}) dN_s^{(2)}, \qquad (10)
\end{aligned}
$$

where we have introduced the notation $\Delta_1 f(x, y) = f(x+1, y) - f(x, y)$ and $\Delta_2 f(x, y) = f(x, y+1) - f(x, y)$ respectively. The reasoning for the first equality is exactly the same as in the univariate case. For the second equality we have to note that $\Delta N_t^{(1)} \Delta N_t^{(2)} = 0$ a.s., as two independent Poisson processes do not jump at the same times almost surely. Therefore we can divide the sum in the first line and we have a $f(N_s) = f(N_{s-}^{(1)} + 1, N_{s-}^{(2)})$ in the sum in the second line and the equivalent in the the sum in the third line. The last equation follows from the definition of the integral as above.

It is now again clear how to go on. We define $g_i(N_s) = \Delta_i f(N_s)$ for $i = 1, 2$. Then we apply (10) to the functions g_i. Putting these expressions in the integrals above leads to:

$$
\begin{aligned}
f(N_t) &= f(0) + \Delta_1 f(0) \int_{0+}^{t} dN_s^{(1)} + \Delta_2 f(0) \int_{0+} dN_s^{(2)} \\
&\quad + \int_{0+}^{t} \int_{0+}^{s-} \Delta_1(\Delta_1 f(N_{u-})) dN_u^{(1)} dN_s^{(1)} \\
&\quad + \int_{0+}^{t} \int_{0+}^{s-} \Delta_1(\Delta_2 f(N_{u-})) dN_u^{(2)} dN_s^{(1)} \\
&\quad + \int_{0+}^{t} \int_{0+}^{s-} \Delta_2(\Delta_1 f(N_{u-})) dN_u^{(1)} dN_s^{(2)} \\
&\quad + \int_{0+}^{t} \int_{0+}^{s-} \Delta_2(\Delta_2 f(N_{u-})) dN_u^{(2)} dN_s^{(2)}. \qquad (11)
\end{aligned}
$$

We can now certainly expand each of the remainder integrals further but we do this in a "symmetric" way, because then we are able to get rid of the integrals in the expansion. We explain this treatment in the general d-dimensional case. To do this we need some further notation.

Notation: A *multi-index* a of length $l := l(a) \in \{0, 1, \dots\}$ is a row vector $a = (a_1, \dots, a_l)$, where $a_k \in \{1, \dots, d\}$. The set of multi-indices is denoted by $\mathcal{M} = \{(a_1, \dots, a_l) : a_i \in \{1, \dots, d\},\ i \in \{1, \dots, l\},\ l = 1, 2, 3, \dots\} \cup \{\nu\}$, where ν is the multi-index of length zero.
We further define some operations on multi-indices which prove to be useful: Given $a = (a_1, \dots, a_l) \in \mathcal{M} \setminus \{\nu\}$ and $b = (b_1, \dots, b_m),\ c = (c_1, \dots, c_n) \in \mathcal{M}$ we set:

$$\begin{aligned} -a &= (a_2, \dots, a_l), \\ a- &= (a_1, \dots, a_{l-1}) \text{ and} \\ b * c &= (b_1, \dots, b_m, c_1, \dots, c_n). \end{aligned}$$

We define the *permutation invariant set of multi-indices of order n with parameters* $i_1, \dots, i_d \in \mathbb{N}_0$, where $n \geq 1$ and $i_1 + \dots + i_d = n$ as follows:

$$\mathcal{S}_{i_1 \dots i_d} = \{a = (a_1, \dots, a_n) : \#\{a_j = k\} = i_k \text{ for } j = 1, \dots, n,\ k = 1, \dots, d\}.$$

We also define the difference operator Δ_a for $a = (a_1, \dots, a_l) \in \mathcal{M}$ as:

$$\Delta_a f(x) = \prod_{j=1}^{l} \Delta_{a_j} f(x).$$

One can easily check, that the difference operators commute. Therefore we have for $a \in \mathcal{S}_{i_1 \dots i_d}$:

$$\Delta_a f(x) = \prod_{j=1}^{d} \Delta_j^{i_j} f(x) = \sum_{k_1, \dots, k_d}^{i_1, \dots, i_d} (-1)^{l(\alpha) + \sum_{j=1}^{d} k_j} \prod_{r=1}^{d} \binom{i_r}{k_r} f(k_1, \dots, k_d). \quad (12)$$

Sometimes only the number of each coordinate matters. Therefore we denote by $s_{i_1 \dots i_d}$ a representative of each set $\mathcal{S}_{i_1 \dots i_d}$ (one can for example choose the element of $\mathcal{S}_{i_1 \dots i_d}$ such that $a_1 \leq a_2 \leq \dots \leq a_n$). And finally we define the stochastic integral of an element $a = (a_1, \dots, a_l) \in \mathcal{M}$ and a function f recursively as:

$$P_a(f)_t = \begin{cases} f(N_t) & n = 0, \\ \int_{0+}^{t} P_{a-}(f)_{s-} dN_s^{(a_l)} & n \geq 1. \end{cases}$$

If $f \equiv 1$, $P_a(f)_t$ is denoted by $P_{a,t}$ as before. The following generalization of Proposition 2.2 holds:

Proposition 3.1. *Given a d-dimensional Poisson process $(N_t^{(1)}, \dots, N_t^{(d)})$ where $N^{(i)}$, $N^{(j)}$ are independent Poisson processes for $i \neq j$. Then the following holds:*

$$\sum_{a \in \mathcal{S}_{i_1 \dots i_d}} P_{a,t} = \prod_{j=1}^{d} \binom{N_t^{(j)}}{i_j}. \quad (13)$$

Remark. If $i_j = n$ for some k then we have that $\mathcal{S}_{i_1 \dots i_d} = \{(j, \dots, j)\}$ and we are back in the univariate case. The statement of the proposition reduces to Proposition 2.2.

Proof. (By induction on the order of $\mathcal{S}_{i_1 \dots i_d}$.)
$n = 1$: $\mathcal{S}^1_{i_1 \dots i_d} = \{(a_1)\}$, where $a_1 = j$.

$$\sum_{a \in \mathcal{S}^1_{i_1 \dots i_d}} P_{a,t} = P_{(j),t} = N_t^{(j)},$$

which is the statement for $n = 1$.
Assume now that the statement is true for $n - 1$. By Proposition 2.2 we have that:

$$d\left(\binom{N_t^{(j)}}{i_j}\right) = \begin{cases} \binom{N_t^{(j)}}{i_j - 1} dN_t^{(j)} & \text{if } i_j \neq 0 \\ 0 & \text{if } i_j = 0 \end{cases}$$

We now use the integration by parts formula [15, p. 76] to derive the differential of the right hand side of (13). The bracket terms drop out because of the independence of the Poisson processes.

$$\begin{aligned} d\left(\prod_{j=1}^{d} \binom{N_t^{(j)}}{i_j}\right) &= \sum_{\substack{k=1 \\ i_k \neq 0}}^{d} \left(\prod_{\substack{j=1 \\ j \neq k}}^{d} \binom{N_{t-}^{(j)}}{i_j}\right) \binom{N_{t-}^{(k)}}{i_k - 1} dN_t^{(k)} \\ &= \sum_{\substack{k=1 \\ i_k \neq 0}}^{d} \sum_{a \in \mathcal{S}_{i_1 \dots (i_k - 1) \dots i_d}} P_{a,t-} dN_t^{(k)} \\ &= \sum_{a \in \mathcal{S}_{i_1 \dots i_d}} dP_{a,t}. \end{aligned}$$

In the second equality we have used the induction assumption and the last equality follows from the definition of $P_{a,t}$. □

We proceed as follows. We expand the remainder terms appearing in (11) in such a way that Proposition 3.1 helps us to take many integrals together. A necessary condition for this procedure is the commutativity property of the difference operators Δ.

3.1 The stochastic Taylor expansion

We have now all tools to state a stochastic Taylor expansion for d-dimensional Poisson processes.

Theorem 3.2. *(***The stochastic Taylor expansion a for d-dimensional Poisson process***) Given a d-dimensional Poisson process* (N_t) *as above and a function* $f : \mathbb{R}^d \to \mathbb{R}$*. The following expansion holds:*

$$\begin{aligned} f(N_t) = & \sum_{k=0}^{m} \sum_{\substack{i_1,\dots,i_d \in \mathbb{N}_0 \\ i_1+\dots+i_d=k}} \Delta_{\mathcal{S}^k_{i_1\dots i_d}} f(0) \prod_{j=1}^{d} \binom{N_t^{(j)}}{i_j} \\ & + \sum_{i_1+\dots+i_d=m+1} \sum_{a \in \mathcal{S}_{i_1\dots i_d}} P_a(\Delta_a f)_t. \end{aligned} \tag{14}$$

Proof. By induction.
$m = 0$: When we make a similar expansion as in (10) for d Poisson processes we get:

$$\begin{aligned} f(N_t) &= f(0) + \sum_{j=1}^{d} \int_{0+}^{t} \Delta_j f(N_{s-}) dN_s^{(j)} \\ &= f(0) + \sum_{i_1+\dots+i_d=1} \sum_{a \in \mathcal{S}_{i_1\dots i_d}} P_a(\Delta_a f)_t, \end{aligned}$$

which is the statement for $m = 0$.
$m \geq 1$. Assume that the statement is true for $m - 1$, i.e.

$$\begin{aligned} f(N_t) = & \sum_{k=0}^{m-1} \sum_{\substack{i_1,\dots,i_d \in \mathbb{N}_0 \\ i_1+\dots+i_d=k}} \Delta_{\mathcal{S}_{i_1\dots i_d}} f(0) \prod_{j=1}^{d} \binom{N_t^{(j)}}{i_j} \\ & + \sum_{i_1+\dots+i_d=m} \sum_{a \in \mathcal{S}^m_{i_1\dots i_d}} P_a(\Delta_a f)_t. \end{aligned}$$

By (10) we expand the functions $\Delta_a f(N_t)$. For each $a \in \mathcal{S}_{i_1\dots i_d}$ we have that:

$$\begin{aligned} \Delta_a f(N_t) &= \Delta_a f(0) + \sum_{j=1}^{d} \int_{0+}^{t} \Delta_j \Delta_a f(N_s) dN_s^{(j)} \\ \Rightarrow P_a(\Delta_a f)_t &= \Delta_a f(0) P_{a,t} + \sum_{j=1}^{d} P_{(j)*a}(\Delta_{(j)*a} f)_t. \end{aligned}$$

Each possible combinations of $i_1, \dots, i_d \in \mathbb{N}_0,\ i_1 + \dots + i_d = m$ contributes by Proposition 3.1 $\Delta_{\mathcal{S}^m_{i_1\dots i_d}} f(0) \prod_{j=1}^{d} \binom{N_t^{(j)}}{i_j}$ and for the remainder we have that:

$$\sum_{i_1+\dots+i_d=m} \sum_{a \in \mathcal{S}_{i_1\dots i_d}} \sum_{j=1}^{d} P_{(j)*a}(\Delta_{(j)*a} f)_t = \sum_{i_1+\dots+i_d=m+1} \sum_{a \in \mathcal{S}_{i_1\dots i_d}} P_a(\Delta_a f)_t.$$

Hence Theorem 3.2 is proved. ∎

In the same way as for the one-dimensional case we can show the exactness of the truncated multivariate stochastic Taylor expansion $f_m(N_t)$ on certain sets, where obviously $f_m(N_t)$ now is defined as

$$f_m(N_t) = \sum_{k=0}^{m} \sum_{\substack{i_1,\dots,i_d \in \mathbb{N}_0 \\ i_1+\cdots+i_d=k}} \Delta_{\mathfrak{s}_{i_1\dots i_d}} f(0) \prod_{j=1}^{d} \binom{N_t^{(j)}}{i_j} \tag{15}$$

$$\Big(= \sum_{\substack{i_1,\dots,i_d \in \mathbb{N}_0 \\ i_1+\cdots+i_d\le m}} \Delta_{\mathfrak{s}_{i_1\dots i_d}} f(0) \prod_{j=1}^{d} \binom{N_t^{(j)}}{i_j} \Big).$$

The following generalization of Proposition 2.4 holds:

Proposition 3.3. *The truncated stochastic Taylor expansion $f_m(N_t)$ is exact on the set $\mathcal{E}_m = \bigcup \{N_t^{(1)} \le i_1, \dots, N_t^{(d)} \le i_d\} \supset \{N_t^{(1)} \le \lfloor m/d \rfloor, \dots, N_t^{(d)} \le \lfloor m/d \rfloor\}$, where the union has to be taken over all combinations $i_1+\cdots+i_d \le m$.*

Proof. It suffices to show the following: $f_m(N_t)$ is exact on the set $\{N_t^{(1)} = n_1, \dots, N_t^{(d)} = n_d\}$ for any $n_1, \dots, n_d$ such that $n_1 + \cdots + n_d \le m$.
We define $m' = n_1+\cdots+n_d$, $\bar{n} = (n_1, \dots, n_d)$ and $\bar{k} = (k_1, \dots, k_d)$. We examine (15) and remark that if there exists a j such that $n_j < i_j$ then $\prod \binom{n_j}{i_j} = 0$, i.e. all combinations of $i_1, \dots, i_j$ with this property do not contribute to f_m.

$$\begin{aligned}
\Rightarrow\ f_m(\bar{n}) &= \sum_{k=0}^{m'} \sum_{\substack{i_1+\cdots+i_d=k \\ i_j\le n_j}} \Delta_{\mathfrak{s}^k_{i_1\dots i_d}} f(0) \prod_{j=1}^{d} \binom{n_j}{i_j} \\
&\overset{(12)}{=} \sum_{k=0}^{m'} \sum_{\substack{i_1+\cdots+i_d=k \\ i_j\le n_j}} \sum_{k_1,\dots,k_d}^{i_1,\dots,i_d} (-1)^{\sum i_j+k_j} \prod_{r=1}^{d} \binom{i_r}{k_r} f(\bar{k}) \prod_{j=1}^{d} \binom{n_j}{i_j} \\
&= \sum_{k=0}^{m'} \sum_{\substack{k_1,\dots,k_d \\ \sum k_j\le k}}^{n_1,\dots,n_d} f(\bar{k}) \sum_{\substack{i_1+\cdots+i_d=k \\ k_j\le i_j\le n_j}} (-1)^{\sum i_j+k_j} \prod_{r=1}^{d} \binom{i_r}{k_r} \prod_{j=1}^{d} \binom{n_j}{i_j} \\
&= \sum_{k=0}^{m'} \sum_{\substack{k_1,\dots,k_d \\ \sum k_j\le k}}^{n_1,\dots,n_d} f(\bar{k}) \binom{n_j}{n_j-k_j} \sum_{\substack{i_1+\cdots+i_d=k \\ k_j\le i_j\le n_j}} (-1)^{\sum i_j+k_j} \prod_{j=1}^{d} \binom{n_j-k_j}{i_j-k_j} \\
&= \sum_{k=0}^{m'} \sum_{\substack{k_1,\dots,k_d \\ \sum k_j\le k}}^{n_1,\dots,n_d} f(\bar{k}) \prod_{j=1}^{d} \binom{n_j}{n_j-k_j} \sum_{\substack{i_1+\cdots+i_d=k-k' \\ 0\le i_j\le n_j-k_j}} (-1)^{\sum i_j+2k_j} \prod_{j=1}^{d} \binom{n_j-k_j}{i_j},
\end{aligned}$$

where $k' = \sum k_j$. But now we fix $k_1 \le n_1, \dots, k_d \le n_d$ such that $k_1 + \cdots + k_d =$

$k' \le k$ and examine the coefficient of $f(k_1, \ldots, k_d)$.

$$\begin{aligned}
&\prod_{j=1}^{d} \binom{n_j}{n_j - k_j} \sum_{k=0}^{m'} \sum_{\substack{i_1+\cdots+i_d=k-k' \\ 0\le i_j \le n_j - k_j}} (-1)^{\sum i_j} \prod_{j=1}^{d} \binom{n_j - k_j}{i_j} \\
&= \prod_{j=1}^{d} \binom{n_j}{n_j - k_j} \sum_{k=k'}^{m'} \sum_{\substack{i_1+\cdots+i_d=k-k' \\ 0\le i_j \le n_j - k_j}} (-1)^{\sum i_j} \prod_{j=1}^{d} \binom{n_j - k_j}{i_j} \\
&= \prod_{j=1}^{d} \binom{n_j}{n_j - k_j} \sum_{k=0}^{m'-k'} \sum_{\substack{i_1+\cdots+i_d=k \\ 0\le i_j \le n_j - k_j}} \prod_{j=1}^{d} (-1)^{i_j} \binom{n_j - k_j}{i_j} \\
&= \prod_{j=1}^{d} \binom{n_j}{n_j - k_j} \sum_{0\le i_j \le n_j - k_j} \prod_{j=1}^{d} (-1)^{i_j} \binom{n_j - k_j}{i_j} \\
&= \prod_{j=1}^{d} \binom{n_j}{n_j - k_j} \prod_{j=1}^{d} (1-1)^{n_j - k_j}.
\end{aligned}$$

If $k_j = n_j$ for all j then the coefficient is 1, in all other cases the coefficient vanishes, which proves the Proposition. $\square$

Corollary 3.4. $i)$ $f_m(N_t) \longrightarrow f(N_t)$ *almost surely as* $m \to \infty$.
$ii)$ *If* $\|f\|_{L^\infty} \le M$, *then we also have, that* $f_m(N_t) \longrightarrow f(N_t)$ *in* L^1 *as* $m \to \infty$.

Proof. i) By Proposition 3.3 we know that the truncated Taylor expansion is exact on the set $\mathcal{E}_n = \{\sum N_t^{(i)} \le m\}$. It is a well-known property of Poisson processes that $\sum N_t^{(i)} \sim \text{Poisson}(\sum \lambda_i t)$. Therefore we use an analogous argument as in the proof of Corollary 3.4 to prove i).
ii) Define $\mathcal{G}_n = \{N_t^{(1)} \le \lfloor n/d \rfloor, \ldots, N_t^{(d)} \le \lfloor n/d \rfloor\} = \{\max_{j=1,\ldots,d} N_d^{(j)} \le \lfloor n/d \rfloor\}$ and $\mathcal{M}_m = \sum N_t^j \le \lfloor m/d \rfloor$. Obviously we have the following inclusions: $\mathcal{M}_m \subset \mathcal{G}_m \subset \mathcal{E}_m$. Therefore

$$\begin{aligned}
E[|f_m - f|] =& E[|f_m - f|, \mathcal{E}_m^C] \le E[|f_m - f|, \mathcal{G}_m^C] \\
&\le E[|f_m|, \mathcal{G}_m^C] + E[|f|, \mathcal{G}_m^C] \le E[|f_m|, \mathcal{M}_m^C] + M \underbrace{P[\mathcal{G}_m^C]}_{\to 0}.
\end{aligned}$$

By (12) we have that $|\Delta_\alpha f(0)| \leq 2^{l(\alpha)} M$.

$$
\begin{aligned}
E[|f_m|, \mathcal{M}_m^C] \leq & M E[\sum_{k=0}^{m} 2^k \sum_{i_1+\cdots+i_d=k} \binom{N_t^{(j)}}{i_j}, \mathcal{M}_m^C] \\
\leq & M E[\sum_{k=0}^{m} 2^k \left(\sum_{j=1}^{d} N_t^{(j)} \right)^k, \mathcal{M}_m^C] \\
\leq & M E[\sum_{k=0}^{m} 2^k S_{d,t}^k, S_{d,t} > \lfloor m/d \rfloor] = M \sum_{k=0}^{m} 2^k \sum_{l > \lfloor m/d \rfloor} \frac{l^k}{k!} e^{-\tilde{\lambda}} \frac{\tilde{\lambda}^l}{l!},
\end{aligned}
$$

where we used the abbreviation $S_{d,t} = \sum_{j=1}^{d} N_t^{(j)}$, but $S_{d,t}$ is again Poisson distributed with parameter $\tilde{\lambda} = t \sum \lambda_j$. The sums in the last expression are interchangeable and therefore we have:

$$
\begin{aligned}
E[|f_m|, \mathcal{M}_m^C] \leq & M \sum_{l > \lfloor m/d \rfloor} e^{-\tilde{\lambda}} \frac{\tilde{\lambda}^l}{l!} \sum_{k=0}^{\infty} \frac{(2l)^k}{k!} \\
= & M \sum_{l > \lfloor m/d \rfloor} e^{-\tilde{\lambda}} \frac{\tilde{\lambda}^l}{l!} e^{2l} = M e^{-\tilde{\lambda}} \sum_{l > \lfloor m/d \rfloor} \frac{(\tilde{\lambda} e^2)^l}{l!}.
\end{aligned}
$$

This last expression again tends to 0 as $m \to \infty$ and the proof is finished $\square$

Remark. - The infinite sum exists and gives us a chaos expansion in terms of the non-compensated Poisson process. If we then turn to the compensated Poisson process and adjust the coefficients we have the explicit form of the chaos expansion for the compensated Poisson process. This expansion has the advantage that it consists of a decomposition in terms of strongly orthogonal processes which may simplify many calculations.

- Theorem 3.2 shows in particular that a standard Taylor expansion is completely useless. As the Poisson process can only jump up one unit the important parts of the function f are of course the values of the function at the positive integers and zero.

- It is of course possible to expand some Poisson processes further than others and it is possible to write an expansion which is no longer symmetric. For the ease of exposition this special treatment has been chosen. The general case is a straightforward extension of our results.

3.2 The Diffusion-Poisson case

The obvious next step is to extend the method to a multivariate process X_t, $X_t = (X_t^{(1)}, X_t^{(2)})$, where $X_t^{(1)}$ is an r-dimensional diffusion process, i.e.

$$
X_t^{(1)} = X_0^{(1)} + \int_0^t a(s, X_s^{(1)}) ds + \int_0^t b(s, X_s^{(1)}) dB_s, \tag{16}
$$

with $a_i(t,x)$, $b_{ij}(t,x)$ Borel measurable functions; $1 \le i \le r$, $1 \le j \le m$, $(B_t)_{t\ge 0}$ m-dimensional Brownian motion on a fixed probability space $(\Omega, \mathcal{F}, P)$ with filtration $(\mathcal{F}_t)$. The coefficients $a(t,x)$ and $b(t,x)$ satisfy the global Lifschitz and linear growth conditions (see [8, p. 281ff]) to ensure existence and uniqueness of the solution of (16).
The process $X_t^{(2)} = (N_t^{(1)}, \ldots, N_t^{(d)})$ is a d-dimensional Poisson process as in the previous section. In the following we change the notations a little bit and follow the notation of [9]. The components of a multi-index $\alpha = (a_1, \ldots, a_l)$ can now vary between 0 and m+d. We define the following operators:

$$L^0 = \frac{\partial}{\partial t} + \sum_{i=1}^{r} a_i(t,x)\frac{\partial}{\partial x_i} + \frac{1}{2}\sum_{i,k=1}^{r}\sum_{j=1}^{m} b_{ij}(t,x)b_{kj}(t,x)\frac{\partial^2}{\partial x_i \partial x_k}, \tag{17}$$

$$L^j = \sum_{i=1}^{r} b_{ij}(t,x)\frac{\partial}{\partial x_i} \quad \text{for } 1 \le j \le m, \tag{18}$$

$$L^k = \Delta_{k-m+r} \quad \text{for } m+1 \le k \le m+d, \tag{19}$$

Of course we define the operators L^a for $a \in \mathcal{M}$ recursively as $L^a = L^{a_1}L^{-a}$. We define also the following multiple Itô Integrals with respect to "suitable" functions f and a multi-index $a = (a_1, \ldots, a_l)$.

$$I_\alpha[f(\cdot, X_\cdot)]_t := \begin{cases} f(t, X_t) & : \; l = 0 \\ \int_0^t I_{\alpha-}[f(\cdot, X_\cdot,)]_{s-} ds & : \; l \ge 1, a_l = 0, \\ \int_0^t I_{\alpha-}[f(\cdot, X_\cdot)]_{s-} dB_s^{(a_l)} & : \; l \ge 1, \; 1 \le a_l \le m, \\ \int_0^t I_{\alpha-}[f(\cdot, X_\cdot)]_{s-} dN_s^{(a_l - r)} & : \; l \ge 1, \; m+1 \le a_l \le m+d. \end{cases}$$

Remark. The only restrictions on "suitable" functions stem from the integrals with respect to Brownian motion. We define similar classes of functions as in [9]. Namely:

$$\mathcal{H}_v = \{f : \forall t \ge 0, \; |f(t,\omega)| < \infty \text{ P a.s.}\} \tag{20}$$

$$\mathcal{H}_{(0)} = \{f : \forall t \ge 0, \; \int_0^t |f(t,\omega)| ds < \infty \text{ P a.s.}\} \tag{21}$$

$$\mathcal{H}_{(1)} = \{f : \forall t \ge 0, \; \int_0^t |f(t,\omega)|^2 ds < \infty \text{ P a.s.}\} \tag{22}$$

$\mathcal{H}_{(j)} = \mathcal{H}_{(1)}$ for all $j = 1, \ldots, m$. For $\alpha \in \mathcal{M}$ with at least one component less then m we define $\alpha_c = (a_{j_1}, \ldots, a_{j_k})$ the collapsed multi-index, where we deleted all components bigger than m.
$\mathcal{H}_\alpha$ is recursively defined to be the set of càdlàg processes such that the process $(I_{\alpha_c-}[f(\cdot)]_{0,\cdot}, t \ge 0)$ satisfies $I_{\alpha_c-}[f(\cdot)]_{0,\cdot} \in \mathcal{H}_{(a_{j_l})}$.

We recursively use Itô's lemma to derive a stochastic Taylor expansion for processes X_t defined as above. Using formula (1) we have that:

$$
\begin{aligned}
f(t, X_t) = f(0, X_0) &+ \int_{0+}^{t} \frac{\partial f}{\partial t}(s, X_{s-})ds + \sum_{i=1}^{r} \int_{0+}^{t} a_i(s, X_s^{(1)}) \frac{\partial f}{\partial x_i}(s, X_{s-})ds \\
&+ \sum_{j=1}^{m} \int_{0+}^{t} \sum_{i=1}^{r} b_{ij}(s, X_s^{(1)}) \frac{\partial f}{\partial x_i}(s, X_{s-})dB_s^{(j)} \\
&+ \frac{1}{2} \sum_{i,k=1}^{r} \sum_{j=1}^{m} \int_{0+}^{t} b_{ij}(s, X_s^{(1)}) b_{kj}(s, X_s^{(1)}) \frac{\partial^2 f}{\partial x_i \partial x_k}(s, X_{s-})ds \\
&+ \sum_{i=m+1}^{m+d} \int_{0+}^{t} \Delta_i f(s, X_{s-})dN_s^{(i-m)}
\end{aligned}
$$

With the help of the notations defined above we can write this in a more condensed form, namely:

$$
f(t, X_t) = f(0, X_0) + \sum_{j=0}^{m+r} I_{(j)}[L^{(j)} f(\cdot, X_\cdot)]_t \tag{23}
$$

Similar to the strategy of the proof of the Itô-Taylor expansion in [9] one can prove the following theorem.

Theorem 3.5. *Let $\mathcal{A} \subset \mathcal{M}$ be a hierarchical set, and let $f : \mathbb{R}^+ \times \mathbb{R}^r \times \mathbb{R}^d \to \mathbb{R}$, then the Itô-Taylor expansion*

$$
f(t, X_t) = \sum_{\alpha \in \mathcal{A}} I_\alpha[f_\alpha(0, X_0)]_t + \sum_{\alpha \in \mathcal{B}(\mathcal{A})} I_\alpha[f_\alpha(\cdot, X_\cdot)]_t \tag{24}
$$

holds, provided all of the derivatives of f, a and b and all of the multiple Itô integrals appearing in (24) exist.

Proof. The proof of this theorem is analogous to the proof in [9] and the proofs of the previous theorems. □

Remark. It is straightforward to extend this result to the case of bounded stopping times, i.e. instead of considering the interval $[0, t]$ we can look at $[\rho, \tau]$, where ρ and τ be two stopping times with:

$$
t_0 \leq \rho(\omega) \leq \tau(\omega) \leq T \quad \text{w. p. 1.}
$$

The task of establishing convergence results for the Poisson-diffusion case has still to be resolved.

4 Applications to Risk Management

The first application considers a simple jump-diffusion. We assume that the log-returns of some financial risky asset are given by:

$$X_t = \mu t + \sigma B_t + c(N_t^{(1)} - N_t^{(2)}), \tag{25}$$

where B is a standard Brownian motion, independent of two independent Poisson processes with the same intensity λ. We assume that the annual volatility is 0.2, the annual drift is 0.07. Furthermore the jump-size c is 0.05 and 75% of the variance are explained by the Brownian motion part which implies that $\lambda = 2$. Our market consists of this risky asset together with a riskless asset with price process $Y_t = \exp(rt)$.

We consider the case of a Call option in this market. As the market is incomplete, one may argue about the proper way to price this option. We do not want to discuss problems of option pricing in incomplete markets and simply take the Esscher measure (see for example [6]) for pricing. So the price of the Call can be given as:

$$C(x,t) = \mathrm{E}_{\mathrm{Q}}[e^{-r(T-t)}(S_T - K)_+ | S_t = S_0 e^x], \tag{26}$$

where the Radon-Nykodym derivative of Q with respect to P is given by $\mathrm{dQ}/\mathrm{dP} = \exp(\theta X - K(\theta))$. The parameter θ is the solution of $K(\theta+1) - K(\theta) = r$, where K denotes the cumulant generating function of X under the physical measure P. For the following S_0 and K are chosen equal to 1 and the risk-free interest rate $r = 0.05$, the time to maturity $T = 1/2$. A stochastic Taylor expansion $\hat{C}(t)$ for this function can easily be done according to Theorem 3.5. As hierarchical set we choose $\mathcal{A} = \{n(\alpha) + l(\alpha) \le 2\}$, $n(\alpha)$ denotes the number of zeros of α and $l(\alpha)$ denotes the length.

$$\begin{aligned}
\hat{C}(t) =& C(0,0) + (C_{(0)} + \mu C_{(1)})t + C_{(1)} B_t + \frac{C_{(11)}}{2} B_t^2 + C_{(2)} N_t^{(1)} + C_{(3)} N_t^{(2)} \\
&+ B_t (C_{(12)} N_t^{(1)} + C_{(13)} N_t^{(2)}) + C_{(22)} \binom{N_t^{(1)}}{2} + C_{(33)} \binom{N_t^{(2)}}{2} \\
&+ C_{(23)} N_t^{(1)} N_t^{(2)}, \text{ where} \qquad (27) \\
C_a =& L^a C(0,t) \text{ and } a \in \mathcal{A}. \qquad (28)
\end{aligned}$$

This expansion is then used for short-time risk management. Assume we are interested in the possible changes of the value of the option over $t = 5$ days. The function $C(x,t)$ is strictly monotonically increasing in x and therefore we can approximate the exact density of $C(X_t, t)$ by conditioning on the number of jumps and truncating the infinite sum at an appropriate level. We compare the exact density with the results from simulation. We simulate 500'000 times the three-dimensional vector $(W_t, N_t^{(1)}, N_t^{(2)})$ and perform a non-parametric density estimation with the kernel $k(x) = 15/8(1-4x^2)^2$ for $|x| \le 1/2$, and 0 else. Figure 1 shows the exact density and the non-parametric density estimation and the relative error of the approximation.

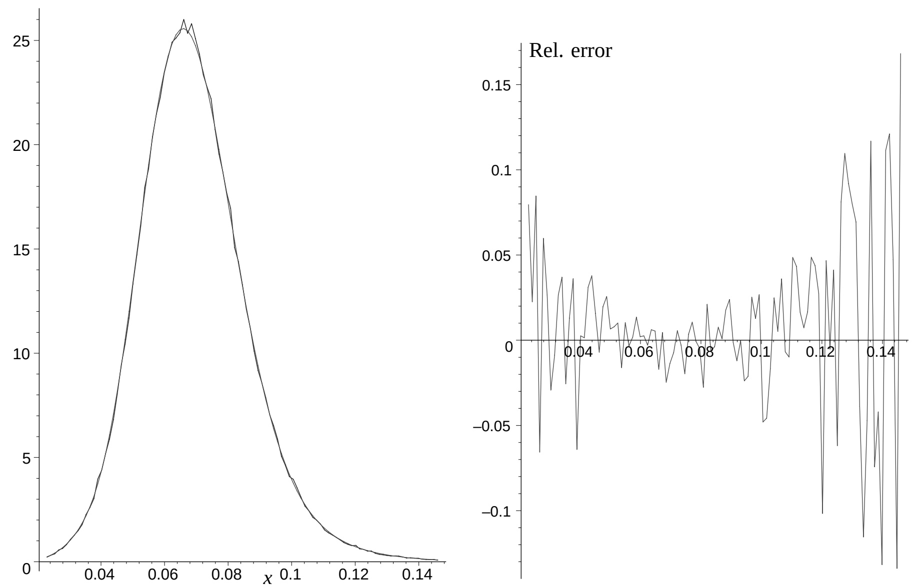

Figure 1: Exact density and non-parametric density estimation for the Call option and relative error of the approximation with respect to the exact density.

The interval shown captures 99.8% of the mass of the distribution. Figure 1 indicates that in this case we can approximate most risk measures which only depend on the distribution.

Next, we apply the approximation to a more sophisticated model. We now exploit the well-known approximation of a Lévy process X with characteristic triplet (γ, σ, ν) by discretising the Lévy measure ν. We choose some small ϵ and let all jumps smaller than ϵ contribute to the diffusion part of X. Then we make a partition of $\mathbb{R} \setminus [-\epsilon, \epsilon]$ of the following form. We choose real numbers $a_0 < a_1 < \cdots < a_k = -\epsilon, \epsilon = a_{k+1} < a_{k+2} < \cdots < a_{d+1}$. Define $X^{(d)}$ by:

$$X_t^{(d)} = \gamma t + \sigma_n B_t + \sum_{i=1}^{d} c_i (N_t^{(i)} - \lambda_i t) \text{ with} \tag{29}$$

$$\sigma_n = \sigma + \int_{-\epsilon}^{\epsilon} x^2 \nu(dx) \tag{30}$$

$$\lambda_i = \begin{cases} \nu([a_{i-1}, a_i[) & \text{for } i \leq k \\ \nu([a_i, a_{i+1}[) & \text{for } k+1 \leq i \leq d \end{cases} \tag{31}$$

$$c_i^2 \lambda_i = \begin{cases} \int_{a_{i-1}}^{a_i -} x^2 \nu(dx) & \text{for } i \leq k \\ \int_{a_i}^{a_{i+1}-} x^2 \nu(dx) & \text{for } k+1 \leq i \leq d \end{cases} \tag{32}$$

Remark. The procedure is explained for example in [16]. By [13] one can show that $X^{(d)} \to X$ weakly in the sense of the Skorohod topology for $d \to \infty$.

The further strategy is similar to the strategy for the jump-diffusion. We want to do risk management for a Call option. Again the Esscher measure is used for pricing and Formula (26) holds. We calculate the exact density of the Call and the result from a non-parametric density estimation obtained from 500'000 simulations of the approximation for two Lévy processes, namely a NIG Lévy process (see for example [16]) and a Lévy process of Meixner type (see [7] or [18]). These two processes are both pure jump processes. We recall the density, the cumulant generating function, the drift and the Lévy measure for these two processes.

$$f_{\mathrm{NIG}}(x;\alpha,\beta,\delta,\mu) = \frac{\delta\,\alpha\, e^{\delta\sqrt{\alpha^2-\beta^2}+\beta\,(x-\mu)}\mathrm{K}_1(\alpha\, g(x-\mu,\delta))}{\pi\, g(x-\mu,\delta)}, \tag{33}$$

$$g(x;\delta) = \sqrt{x^2+\delta^2}$$

$$K_{\mathrm{NIG}}(\theta;\alpha,\beta,\delta,\mu) = \mu\theta + \delta\left(\sqrt{\alpha^2-\beta^2} - \sqrt{\alpha^2-(\beta+\theta)^2}\right), \tag{34}$$

$$\gamma_{\mathrm{NIG}}(\alpha,\beta,\delta,\mu) = \mu + \frac{2\delta\alpha}{\pi}\int_0^1 \sinh(\beta x) K_1(\alpha x)\,dx, \tag{35}$$

$$\nu_{\mathrm{NIG}}(dx;\alpha,\beta,\delta,\mu) = \frac{\delta\,\alpha\, e^{\beta\, x}\mathrm{K}_1(\alpha\ |x|)}{\pi\ |x|}dx. \tag{36}$$

The function $K_1(x)$ denotes the modified Bessel function of the third kind of order 1 (see for example [12]). And now for the Meixner case:

$$f_{\mathrm{Mei}}(x;\alpha,\beta,\delta,\mu) = \frac{(2\ \cos\frac{\beta}{2})^{2\,\delta} e^{\frac{\beta\,(x-\mu)}{\alpha}} \left(\left|\Gamma(\delta+\frac{i(x-\mu)}{\alpha})\right|\right)^2}{\pi\,\alpha\,\Gamma(2\,\delta)}, \tag{37}$$

$$K_{\mathrm{Mei}}(\theta;\alpha,\beta,\delta,\mu) = \mu\,\theta + 2\delta\left(\log\cos\frac{\beta}{2} - \log\cos\frac{\alpha\,\theta+\beta}{2}\right), \tag{38}$$

$$\gamma_{\mathrm{Mei}}(\alpha,\beta,\delta,\mu) = \mu + \alpha\,\delta\ \tan\frac{\beta}{2} - 2\,\delta\int_1^\infty \frac{\sinh\frac{\beta\,x}{\alpha}}{\sinh\frac{\pi\,x}{\alpha}}\,dx \tag{39}$$

$$\nu_{\mathrm{Mei}}(dx;\alpha,\beta,\delta,\mu) = \frac{\delta\, e^{\frac{\beta\,x}{\alpha}}}{x\sinh\frac{\pi\,x}{\alpha}}dx. \tag{40}$$

From the form of the cumulant generating functions one easily deduces that the density at any time t can be calculated by multiplying the parameters δ and μ by t for both cases. The parameters for the Esscher transforms are also easily found. In the NIG case one only has to shift β to $\beta+\theta$ to get the density under the measure P_θ and similarly from β to $\beta+\alpha\theta$ for the Meixner case.

The parameters are chosen from moment estimates of the Nikkei index over the period 01.01.97 to 31.12.99, which give us in total 737 daily observations. The estimated parameters are $(\alpha,\beta,\delta,\mu) = (49.78,\ 1.57,\ 0.013,\ -0.00044)$ in the NIG case and $(\alpha,\beta,\delta,\mu) = (0.030,\ 0.13,\ 0.57,\ -0.0011)$ in the Meixner case for

daily returns. The statstical justification for this specific data set can be found for the case of the Meixner process in [17]. Similar investigation has been made for the NIG case for example in [14].
For the discretisation we choose in both cases $\epsilon = 0.005$ and the a_i are chosen symmetric with $|a_i| \in \{\epsilon, 0.008, 0.0125, 0.02, 0.04, 0.08, 0.2, \infty\}$. Figure 2 shows the comparison of a non-parametric density approximation obtained from 500'000 simulations of (29) in this case and the true density in the NIG case, the Meixner case being similar. The interval shown for the relative error captures over 99.5% of the mass of the distribution. The price of a Call option in this market can be written

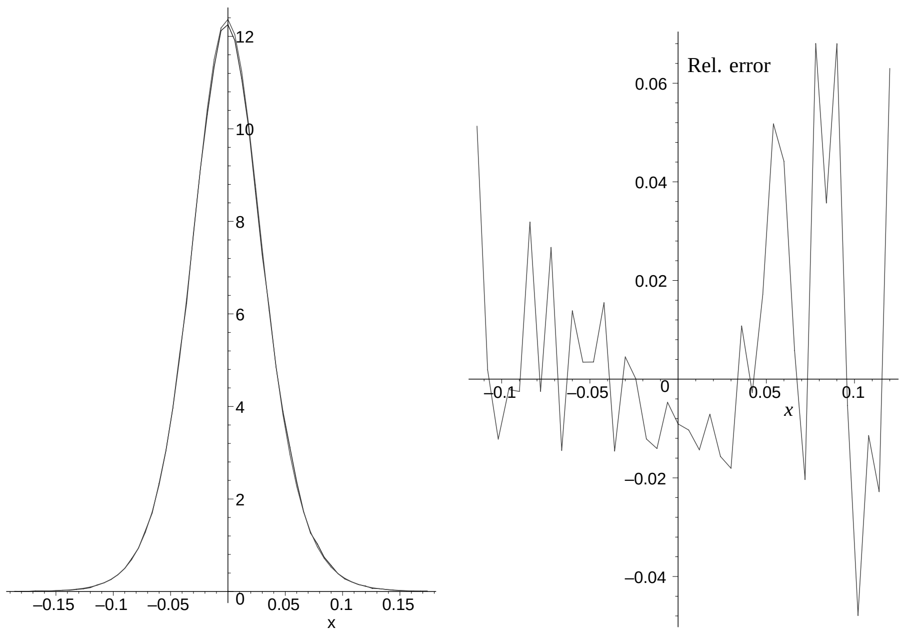

Figure 2: Density and non-parametric density estimation from simuating (29) for the NIG process and the corresponding relative error of the approximation with respect to the true density.

again as (26). A stochastic Taylor expansion $\hat{C}(t)$ for this function can easily be done according to Theorem 3.5. As hierarchical set we choose $\mathcal{A} = \{n(\alpha)+l(\alpha) \leq 2\}$. So the form of the stochastic Taylor expansion can be given as (27) with the obvious difference that now we have 14 independent Poisson processes. The results in the NIG case are shown in Figure 3 and the results in the Meixner case are shown in Figure 4. These two processes are chosen in order to get rid of any problem concerning Fourier inversion, as we can explicitly give the density for any time horizon as outlined above. Therefore we can compare the results obtained from simulations with the exact results. The interval shown captures 99.6% of the mass of the distribution. Figure 3 and Figure 4 show that the stochastic Taylor expansion can be used in this context. Suppose one has to assess the magnitude of possible losses for a more complicated derivative in a Lévy type model. If one

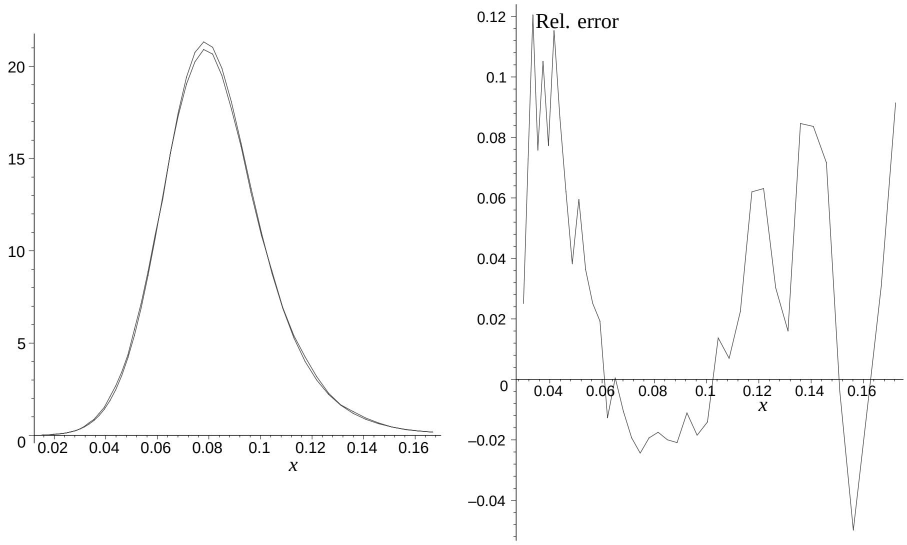

Figure 3: Density and non-parametric density estimation for the Call option and relative error of the approximation with respect to the true density for a NIG Lévy process.

has to rely on simulations, it can be very time consuming to evaluate the derivative for each simulation. Using the method outlined above one roughly has to evaluate the derivative $d(d+5)/2$ times where d denotes the number of Poisson processes used for the approximation. This can often lead to a significant speed-up of the procedure.

Finally we compare two approximated risk measures obtained from simulations with the exact calculations obtained from the NIG Lévy process. The accuracy of this approach is compared to results obtained from the Black-Scholes model, i.e. we fit the daily log-returns to a normal distribution. The two risk measures considered are the Value-at-Risk (VaR) and the Tail conditional expectation (TCE). The VaR of a random variable X (which stands for the random outcome of some financial position over a fixed time horizon) for some fixed level α is defined as:

$$\mathrm{VaR}_\alpha(X) = -q_\alpha(X), \tag{41}$$

the negative α-quantile of X, which is defined unambiguously for our example.

The TCE with level α, denoted by TCE_α, is the risk measure defined by:

$$\mathrm{TCE}_\alpha(X) = -\mathrm{E}[X|X \leq -\mathrm{VaR}_\alpha(X)]. \tag{42}$$

For further particulars of these risk measures we refer to [1] or [3]. The following tables give an overview over the obtained results, where we calculate the risk measures for X being the value of a Call option in the NIG market after 5 days. The column “relative error” calculates the relative error of the empirical quantile obtained from the simulation of the approximation with respect to the true quantile

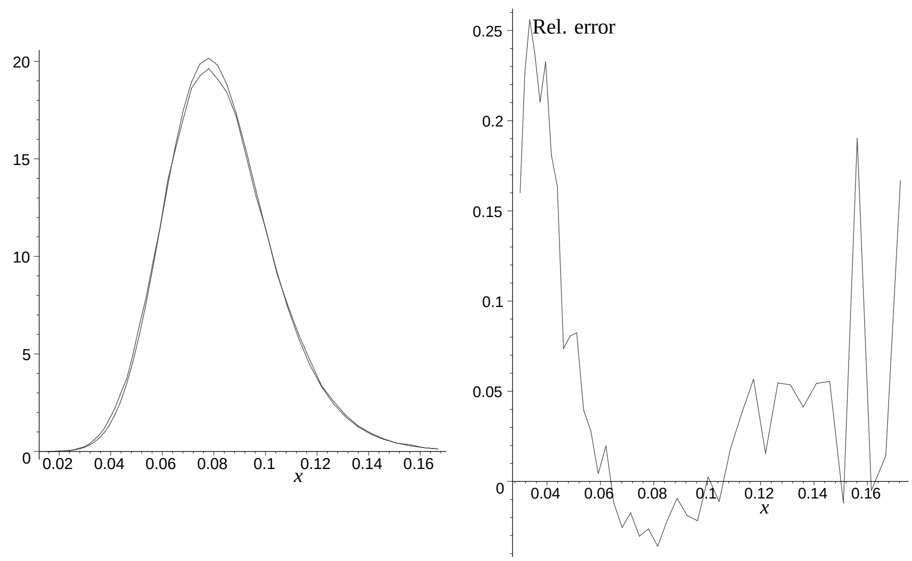

Figure 4: Density and non-parametric density estimation for the Call option and relative error of the approximation with respect to the true density for a Meixner Lévy process.

in percent. The column “quotient with BS” shows the quotient of the relative error for the Black-Scholes approximation divided by the relative error for the approximation. With the exception of the 0.005-quantile and the TCE for the level 0.01 all errors are smaller for the approximation outlined in this chapter than for a simple normal approximation, which is admittedly the least we should expect. The biggest percentage error for these risk measures is only about 2%.

level	exact quantile	relative error	quotient with BS
0.001	0.0273	1.91%	5.24
0.002	0.0307	1.66%	2.97
0.005	0.0358	0.694%	-.617
0.01	0.0401	0.363%	-9.90
0.02	0.0449	0.00309%	-1970
0.05	0.0520	0.108%	-74.6

Table 1: Comparison of lower quantiles for different levels.

level	exact quantile	relative error	quotient with BS
0.95	0.121	- 0.431%	37.2
0.98	0.135	- 0.135%	102.
0.99	0.146	0.477%	-23.1
0.995	0.158	0.646%	-13.9
0.998	0.173	0.865%	-8.26
0.999	0.186	0.894%	-5.92

Table 2: Comparison of upper quantiles for different levels.

level	exact TCE	relative error	quotient with BS
0.001	0.0235	1.40%	11.7
0.002	0.0266	0.669%	15.2
0.005	0.0310	0.198%	19.3
0.01	0.0348	- 0.407%	0.752
0.02	0.0391	- 0.998%	3.79
0.05	0.0452	- 0.939%	7.14

Table 3: Comparison of TCE for lower quantiles for different levels.

level	exact TCE	relative error	quotient with BS
0.95	0.137	- 1.92%	15.4
0.98	0.152	- 1.15%	19.2
0.99	0.163	- 0.426%	37.0
0.995	0.176	0.0459%	- 253.0
0.998	0.192	- 0.0296%	287.0
0.999	0.205	0.171%	- 32.5

Table 4: Comparison of TCE for upper quantiles for different levels.

Acknowledgements

The first author is Postdoctoral Fellow of the Fund for Scientific Research-Flanders (Belgium) (F.W.O.-Vlaanderen)

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