On the variational principle for the topological entropy of certain non-compact sets

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Abstract

For a continuous transformation f of a compact metric space (X,d) and any continuous function φ on X we consider sets of the form

$$K_\alpha = \Big\{ x \in X : \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \varphi(f^i(x)) = \alpha \Big\}, \quad \alpha \in \mathbb{R}.$$

For transformations satisfying the specification property we prove the following Variational Principle

$$h_{top}(f, K_\alpha) = \sup\Big(h_\mu(f) : \mu \text{ is invariant and } \int \varphi d\mu = \alpha \Big),$$

where $h_{top}(f,\cdot)$ is the topological entropy of non-compact sets. Using this result we are able to obtain a complete decription of the multifractal spectrum for Lyapunov exponents of the so-called Manneville–Pomeau map, which is an interval map with an indifferent fixed point.

1 Introduction

Often the problems of multifractal analysis of local (or pointwise) dimensions and entropies are reduced to consideration of the sets of the following form

$$K_\alpha = \Big\{ x \in X : \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \varphi(f^i(x)) = \alpha \Big\}, \quad \alpha \in \mathbb{R},$$

where $f : X \to X$ is some transformation, and $\varphi : X \to \mathbb{R}$ is a function, sometimes called observable. Typically, f is a continuous transformation of some compact metric space (X,d) and φ is sufficiently smooth.

In particular, one is interested in the "size" of these sets K_α. The following characteristics of the sets K_α have been studied in the literature:

$$\mathcal{D}_\varphi(\alpha) = \dim_H(K_\alpha), \quad \mathcal{E}_\varphi(\alpha) = h_{top}(f, K_\alpha),$$

where $\dim_H(K_\alpha)$ and $h_{top}(f, K_\alpha)$ are the Hausdorff dimension and the topological entropy of K_α, respectively. The precise definition of the topological entropy of non-compact sets will be given in section 3, but for now the topological entropy should be viewed as a dimension-like characteristic, similar to the Hausdorff dimension. The functions $\mathcal{D}_\varphi(\alpha)$, $\mathcal{E}_\varphi(\alpha)$ will be called the dimension and entropy multifractal spectra of φ.

Recently similar problems were considered in the relation with a definition of a rotational entropy [8, 10].

Multifractal analysis studies various properties of the multifractal spectra $\mathcal{D}_\varphi(\alpha)$, $\mathcal{E}_\varphi(\alpha)$ as functions of α, e.g., smoothness and convexity, and relates these spectra to other characteristics of a dynamical system. In order to obtain non-trivial results one typically has to make 2 types of assumptions: firstly, on the dynamical system (X, f), and secondly, on the properties of the observable function φ. For example,

- ([15], see also [16]) if f is a sufficiently smooth expanding conformal map, and φ is a Hölder continuous function, then $\mathcal{E}_\varphi(\alpha)$ is real-analytic and convex.
- ([21]) if f is an expansive homeomorphism with specification, and φ has bounded variation, then $\mathcal{E}_\varphi(\alpha)$ is C^1 and convex.

In both cases, $\mathcal{E}_\varphi(\alpha)$ is a Legendre transform of a pressure function $P_\varphi(q) = P(q\varphi)$, where $P(\cdot)$ is the topological pressure.

Conditions on φ in the examples above are ment to ensure the absense of phase transition, i.e., existence and uniqueness of equilibrium state for potential $q\varphi$ for every $q \in \mathbb{R}$. The main goal of this paper is to relax such conditions and to obtain results for systems exhibiting phase transitions.

A natural class of observable functions φ would be the set of all continuous function. Moreoveor, the set of all continuous functions is quite rich in the sense of possible phase tranitions. For example [20, p.52], for any set $\{\mu_1, \dots, \mu_k\}$ of ergodic shift-invariant measures on $A^{\mathbb{Z}}$, where A is a finite set, one can find a continuous function φ such that all these measures μ_i, $i = 1, \dots, k$ are equilibrium states for φ. Nevertheless, A.-H. Fan, D.-J. Feng in [7], and E. Olivier in [14], in the case of symbolic dynamics, obtained results on the spectrum $\mathcal{E}_\varphi(\alpha)$ for arbitrary continuous functions φ, similar to those mentioned above. In fact, they were studying the dimension spectrum $\mathcal{D}_\varphi(\alpha)$, but in symbolic case for every α one has $\mathcal{E}_\varphi(\alpha) = \#(A)\mathcal{D}_\varphi(\alpha)$, where $\#(A)$ is the number of elements in A.

In this paper we study the entropy spectum $\mathcal{E}_\varphi(\alpha)$ for a continuous transformation f on a compact metric space (X, d) and arbitrary continuous function φ. The main result of this paper (Theorem 5.1) states that if f is a continuous transformations with specification property, then for any α with $K_\alpha \neq \varnothing$ one has

$$\mathcal{E}_\varphi(\alpha) = H_\varphi(\alpha) = \Lambda_\varphi(\alpha),$$

where

$$H_\varphi(\alpha) := \sup\Big\{h_\mu(f) : \mu \text{ is invariant and } \int \varphi d\mu = \alpha\Big\},$$

and $\Lambda_\varphi(\alpha)$ is a special "ball"-counting dimension of K_α, similar to one introduced in [7].

Readers, familiar with Large Deviations, will recognize in $H_\varphi(\alpha)$ the so-called *rate function*. And indeed, we use the Large Deviation results for dynamical systems with specification obtained by L.-S.Young in [24].

The most intricate part of our proof is the equality $\mathcal{E}_\varphi(\alpha) = \Lambda_\varphi(\alpha)$. To show it we use a *Moran fractal structure*, inspired by one constructed in [7] for the symbolic case.

The Manneville-Pomeau map is a piecewise continuous map of a unit interval given by

$$f_s(x) = x + x^{1+s} \mod 1, \quad 0 < s < 1.$$

This map has a unique indifferent fixed point $x = 0$, and is probably the simplest example of a non-uniformly hyperbolic dynamical system. Thermodynamic properties of this transformation are quite well understood, see [19, 22, 12, 13].

In [18], M. Pollicott and H. Weiss studied the multifractal spectrum for $\varphi = \log f_s'$, i.e., the spectrum of Lyapunov exponents. They were able to obtain a partial description of this spectrum. Using our results we able to complete the picture, see section 6 for details.

A straightforward modification of our proofs shows that the results are valid in more general settings as well. Suppose $f : X \to X$ is a continuous transformation with specification property and $\varphi = (\varphi_1, \dots, \varphi_d) : X \to \mathbb{R}^d$ is a continuous function. For $\alpha \in \mathbb{R}^d$ consider the set

$$K_\alpha = \Big\{x \in X : \lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} \varphi_j(f^i(x)) = \alpha_j, \quad j = 1, \dots, d\Big\}.$$

Then

$$\mathcal{E}_\varphi(\alpha) = h_{top}(f, K_\alpha) = \sup\Big\{h_\mu(f) : \mu \text{ is invariant and } \int \varphi d\mu = \alpha\Big\}. \qquad (1)$$

In fact, even more is true. Suppose again that $\varphi : X \to \mathbb{R}^d$ is a continuous function and $\Psi : \mathrm{Im}(\varphi) \to \mathbb{R}^m$ is a continuous map defined on $\mathrm{Im}(\varphi) = \{\varphi(x) : x \in X\} \subseteq \mathbb{R}^d$. Define

$$K_\beta^{\Psi\circ\varphi} = \Big\{x \in X : \lim_{n\to\infty} \Psi\Big(\frac{1}{n}\sum_{i=0}^{n-1} \varphi(f^i(x))\Big) = \beta\Big\}.$$

Then for any β such that $K_\beta^{\Psi\circ\varphi} \neq \varnothing$ one has

$$\mathcal{E}_{\Psi\circ\varphi}(\beta) = h_{top}(f, K_\beta^{\Psi\circ\varphi}) = \sup\Big\{h_\mu(f) : \mu \text{ is invariant and } \Psi\Big(\int \varphi d\mu\Big) = \beta\Big\}. \quad (2)$$

As an imediate consequence of (1) and (2) we obtain the following result, which we call the *Contraction Principle for Multifractal Spectra*, due to the clear analogy with the well-knwon Contraction Principle in Large Deviations:

$$\mathcal{E}_{\Psi\circ\varphi}(\beta) = \sup_{\alpha:\,\Psi(\alpha)=\beta} \mathcal{E}_\varphi(\alpha).$$

For more detailed discussion and some examples see section 7.

Everywhere in the present paper $\#(C)$ denotes a cardinality of a set C. Proofs of all lemmas are collected in section 8.

2 Multifractal spectrum of continuous functions

Let $f : X \to X$ be a continuous transformation of a compact metric space (X, d). Throughout this paper we will assume that f has finite topological entropy. Suppose $\varphi : X \to \mathbb{R}$ is a continuous function. For $\alpha \in \mathbb{R}$ define:

$$K_\alpha = \Big\{ x \in X : \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \varphi(f^i(x)) = \alpha \Big\}. \tag{3}$$

We introduce the following notation

$$\mathcal{L}_\varphi = \{\alpha \in \mathbb{R} : K_\alpha \neq \varnothing\}.$$

Lemma 2.1. *The set $\mathcal{L}_\varphi$ is a non-empty bounded subset of $\mathbb{R}$.*

Definition 2.1. *A continuous transformation $f : X \to X$ satisfies specification if for any $\varepsilon > 0$ there exists an integer $m = m(\varepsilon)$ such that for arbitrary finite intervals $I_j = [a_j, b_j] \subseteq \mathbb{N}$, $j = 1, \dots, k$, such that*

$$\mathit{dist}(I_i, I_j) \geq m(\varepsilon), \quad i \neq j,$$

and any $x_1, \dots, x_k$ in X there exists a point $x \in X$ such that

$$d(f^{p+a_j}x, f^p x_j) < \varepsilon \quad \text{for all} \quad p = 0, \dots, b_j - a_j, \text{ and every } j = 1, \dots, k.$$

Following the present day tradition we do not require that x is periodic. Specification implies topological mixing. Moreover, by the Blokh theorem [2], for continuous transformations of the interval these two conditions are equivalent. Using this equivalence and the results of Jakobson [9], we conclude that for the logistic family $f_r(x) = rx(1 - x)$ the specification property holds for a set of parameters of positive Lebesgue measure.

The specification property allows us to connect together arbitrary pieces of orbits. Suppose now that for two values α_1, α_2 the corresponding sets $K_{\alpha_1}, K_{\alpha_2}$ are not empty. Using the specification property we are able to construct points with ergodic averages, converging to any number $\alpha \in (\alpha_1, \alpha_2)$. Hence, $\mathcal{L}_\varphi$ is a convex set. This implies the following:

Lemma 2.2. *If $f : X \to X$ satisfies specification, then $\mathcal{L}_\varphi$ is an interval.*

We recall that the entropy spectrum $\mathcal{E}_\varphi(\cdot)$ of φ is the map assigning to each $\alpha \in \mathcal{L}_\varphi$ the value

$$\mathcal{E}_\varphi(\alpha) = h_{top}(f, K_\alpha). \tag{4}$$

The definition and some fundamental facts about the topological entropy of non-compact sets are collected in the following section.

3 Topological entropy of non-compact sets

The generalization of the topological entropy to non-compact or non-invariant sets goes back to Bowen [3]. Later Pesin and Pitskel [17] generalized the notion of the topological pressure to the case of non-compact sets. In this paper we use an equivalent definition of the topological entropy, which can be found in [16].

3.1 Definition of the topological entropy.

Once again, let (X, d) be a compact metric space, and $f : X \to X$ be a continuous transformation. For any $n \in \mathbb{N}$ we define a new metric d_n on X as follows:

$$d_n(x, y) = \max\{d(f^k(x), f^k(y)) : \ k = 0, \dots, n-1\},$$

and for every $\varepsilon > 0$ we denote by $\mathcal{B}_n(x, \varepsilon)$ an open ball of radius ε in the metric d_n around x, i.e.,

$$\mathcal{B}_n(x, \varepsilon) = \{y \in X : \ d_n(x, y) < \varepsilon\}.$$

Suppose we are given some set $Z \subseteq X$. Fix $\varepsilon > 0$. We say that an at most countable collection of balls $\Gamma = \{\mathcal{B}_{n_i}(x_i, \varepsilon)\}_i$ covers Z if $Z \subseteq \cup_i \mathcal{B}_{n_i}(x_i, \varepsilon)$. For $\Gamma = \{\mathcal{B}_{n_i}(x_i, \varepsilon)\}_i$, put $n(\Gamma) = \min_i n_i$. Let $s \geq 0$ and define

$$m(Z, s, N, \varepsilon) = \inf_{\Gamma} \sum_i \exp(-s n_i),$$

where the infinum is taken over all collections $\Gamma = \{\mathcal{B}_{n_i}(x_i, \varepsilon)\}$ covering Z and such that $n(\Gamma) \geq N$. The quantity $m(Z, s, N, \varepsilon)$ does not decrease with N, hence the following limit exists

$$m(Z, s, \varepsilon) = \lim_{N \to \infty} m(Z, s, N, \varepsilon) = \sup_{N > 0} m(Z, s, N, \varepsilon).$$

It is easy to show that there exists a critical value of the parameter s, which we will denote by $h_{top}(f, Z, \varepsilon)$, where $m(Z, s, \varepsilon)$ jumps from $+\infty$ to 0, i.e.,

$$m(Z, s, \varepsilon) = \begin{cases} +\infty, & s < h_{top}(f, Z, \varepsilon), \\ 0, & s > h_{top}(f, Z, \varepsilon). \end{cases}$$

There are no restriction on the value $m(Z,s,\varepsilon)$ for $s = h_{top}(f,Z,\varepsilon)$. It can be infinite, zero, or positive and finite. One can show [16] that the following limit exists

$$h_{top}(f,Z) = \lim_{\varepsilon\to 0} h_{top}(f,Z,\varepsilon).$$

We will call $h_{top}(f,Z)$ the topological entropy of f restricted to Z, or, simply, the topological entropy of Z, when there is no confusion about f.

3.2 Properties of the topological entropy

Here we recall some of the basic properties and important results on the topological entropy of non-compact or non-invariant sets.

Theorem 3.1 ([16]). *The topological entropy as defined above satisfies the following:*

1. $h_{top}(f,Z_1) \le h_{top}(f,Z_2)$ *for any* $Z_1 \subseteq Z_2 \subseteq X$*;*
2. $h_{top}(f,Z) = \sup_i h_{top}(f,Z_i)$*, where* $Z = \cup_{i=1}^{\infty} Z_i \subseteq X$*;*

The next theorem establishes a relation between topological entropy of a set and the measure-theoretic entropies of measures, concentrated on this set, generalizing the classical result for compact sets.

Theorem 3.2 (R. Bowen [3]). *Let* $f : X \to X$ *be a continuous transformation of a compact metric space. Suppose* μ *is an invariant measure, and* $Z \subseteq X$ *is such that* $\mu(Z) = 1$*, then*

$$h_{top}(f,Z) \ge h_\mu(f),$$

where $h_\mu(f)$ *is the measure-theoretic entropy.*

Suppose we are given an invariant measure μ. A point x is called *generic* for μ if the sequence of probability measures

$$\delta_{x,n} = \frac{1}{n}\sum_{k=0}^{n-1} \delta_{f^k(x)},$$

where δ_y is the Dirac measure at y, converges to μ in the weak topology. Denote by G_μ the set of all generic points for μ. If μ is an ergodic invariant measure, then by the Ergodic Theorem $\mu(G_\mu) = 1$. Applying the previous theorem we immediately conclude that $h_{top}(f,G_\mu) \ge h_\mu(f)$. In fact, the opposite inequality is valid as well:

Theorem 3.3 (R. Bowen [3]). *Let* μ *be an ergodic invariant measure, then*

$$h_{top}(f,G_\mu) = h_\mu(f).$$

Ya. Pesin and B. Pitskel in [17] have proved the following variational principle for non-compact sets.

Theorem 3.4. *Suppose $f : X \to X$ is a continuous transformation of a compact metric space (X, d), and $Z \subseteq X$ is an invariant set. Denote by $\mathcal{M}_f(Z)$ the set of all invariant measures μ such that $\mu(Z) = 1$. For any $x \in X$ denote by $V(x)$ the set of all limit points of the sequence $\{\delta_{x,n}\}$. Assume that for every $x \in Z$ one has*

$$V(x) \cap \mathcal{M}_f(Z) \neq \varnothing.$$

Then $h_{top}(f, Z) = \sup_{\mu \in \mathcal{M}_f(Z)} h_\mu(f)$.

The conditions of this theorem are very difficult to check in any specific situation. However, there is no hope for improving the above result for general sets Z. There are examples [16, 17] of sets, for which the condition $V(x) \cap \mathcal{M}_f(Z) \neq \varnothing$ does not hold for all $x \in Z$, and one has a strict inequality

$$h_{top}(f, Z) > \sup\{h_\mu(f) : \ \mu \in \mathcal{M}_f(X) \text{ and } \mu(Z) = 1\}.$$

In this paper we restrict our attention to the sets of a special form: namely, the sets K_α given by (3). For these particular sets we prove a variational priciple for the topological entropy, provided the transformation f satisfies specification:

Theorem 3.5. *Suppose $f : X \to X$ is a continuous transformation with the specification porperty. Let $\varphi \in C(X, \mathbb{R})$ and assume that for some $\alpha \in \mathbb{R}$*

$$K_\alpha = \Big\{ x \in X : \ \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \varphi(f^i(x)) = \alpha \Big\} \neq \varnothing,$$

then

$$h_{top}(f, K_\alpha) = \sup\Big\{ h_\mu(f) : \mu \text{ is invariant and } \int \varphi \, d\mu = \alpha \Big\}.$$

Remark 3.1. *Under the conditions of the above theorem, it is possible that for a certain parameter value α, there exists a unique invariant probability measure μ_α with $\int \varphi d\mu = \alpha$, such that*

$$h_{top}(f, K_\alpha) = h_{\mu_\alpha}(f).$$

Hence, μ_α is a measure of maximal entropy among all invariant measures μ with $\int \varphi d\mu = \alpha$. However, it is also possible, that $\mu_\alpha(K_\alpha) = 0$. This situation, for example, occurs in the family of Manneville-Pomeau maps, see Remark **??** *for more details.*

3.3 Entropy distribution principle.

The following statement will allow us to estimate the topological entropies of the sets from bellow, without constructing probability measures, which are invariant and concentrated on a given set. It is sufficient to consider only probability measures, which need not be invariant, but which satisfy some specific 'uniformity condition'. We call this result the *Entropy Distribution Principle*, by the clear analogy with a well-known *Mass Distribution Principle* [6].

Theorem 3.6 (Entropy distribution principle). *Let $f : X \to X$ be a continuous transformation. Suppose a set $Z \subseteq X$ and a constant $s \geq 0$ are such that for any $\varepsilon > 0$ one can find a Borel probability measure $\mu = \mu_\varepsilon$ satisfying*

1) $\mu_\varepsilon(Z) > 0$,

2) $\mu_\varepsilon(\mathcal{B}_n(x,\varepsilon)) \leq C(\varepsilon)e^{-ns}$ for some constant $C(\varepsilon) > 0$ and every ball $\mathcal{B}_n(x,\varepsilon)$ such that $\mathcal{B}_n(x,\varepsilon) \cap Z \neq \varnothing$.

Then $h_{top}(f,Z) \geq s$.

Proof. We are going to show that $h_{top}(f,Z,\varepsilon) \geq s$ for every sufficiently small $\varepsilon > 0$. Indeed, choose such $\varepsilon > 0$ and consider the corresponding probability measure μ_ε. Let $\Gamma = \{\mathcal{B}_{n_i}(x_i,\varepsilon)\}_i$ be some cover of Z. Without loss of generality we may assume that $\mathcal{B}_{n_i}(x_i,\varepsilon) \cap Z \neq \varnothing$ for every i. Then

$$\begin{aligned}\sum_i \exp(-sn_i) &\geq C(\varepsilon)^{-1} \sum_i \mu_\varepsilon(\mathcal{B}_{n_i}(x_i,\varepsilon)) \\ &\geq C(\varepsilon)^{-1}\mu_\varepsilon\Big(\cup_i \mathcal{B}_{n_i}(x_i,\varepsilon)\Big) \geq C(\varepsilon)^{-1}\mu_\varepsilon(Z) > 0.\end{aligned}$$

Therefore $m(Z,s,\varepsilon) > 0$, and hence $h_{top}(f,Z,\varepsilon) \geq s$. □

4 Upper estimates of $\mathcal{E}_\varphi(\alpha)$.

In this section we are going to define two auxiliary quantities $H_\varphi(\alpha)$ and $\Lambda_\varphi(\alpha)$. These quantities will be used to give an upper estimate on the multifractal spectrum $\mathcal{E}_\varphi(\alpha)$.

4.1 Definition of $H_\varphi(\alpha)$

Let us introduce some notation

$\mathcal{M}(X)$: the set of all Borel probability measures on X,

$\mathcal{M}_f(X)$: the set of all f-invariant Borel probability measures on X,

$\mathcal{M}_f^e(X)$: the set of all ergodic f-invariant Borel probability measures on X,

$\mathcal{M}_f(X,\varphi,\alpha)$: the set of all f-invariant Borel probability measures, such that
$$\int \varphi \, d\mu = \alpha.$$

We consider the weak topology on $\mathcal{M}(X)$ and also on its subsets $\mathcal{M}_f(X)$, $\mathcal{M}_f^e(X)$, etc.; as it is well known, $\mathcal{M}(X)$ is compact metrizable space in the weak topology.

Lemma 4.1. *For any $\alpha \in \mathcal{L}_\varphi$ the set $\mathcal{M}_f(X,\varphi,\alpha)$ is a non-empty, convex and closed (in the weak topology) subset of $\mathcal{M}_f(X)$.*

This result allows us to define the following quantity: for any $\alpha \in \mathcal{L}_\varphi$ put

$$H_\varphi(\alpha) = \sup\Big\{h_\mu(f) : \ \mu \in \mathcal{M}_f(X,\varphi,\alpha)\Big\}. \tag{5}$$

Lemma 4.2. *For any $\varphi \in C(X, \mathbb{R})$, $H_\varphi(\alpha)$ is a concave function on the convex hull of $\mathcal{L}_\varphi$.*

4.2 Definition of $\Lambda_\varphi(\alpha)$

Here, following the approach of [7], we introduce another dimension-like characteristic $\Lambda_\varphi(\alpha)$ of the set K_α. We use a word "dimension" in association with $\Lambda_\varphi(\alpha)$, because $\Lambda_\varphi(\alpha)$ is defined in terms similar to the definition of Hausdorff or box counting dimensions.

For $\alpha \in \mathcal{L}_\varphi$ and any $\delta > 0$ and $n \in \mathbb{N}$ put

$$P(\alpha, \delta, n) = \Big\{ x \in X : \Big| \frac{1}{n} \sum_{i=0}^{n-1} \varphi(f^i(x)) - \alpha \Big| < \delta \Big\}.$$

Clearly, for $\alpha \in \mathcal{L}_\varphi$ and any $\delta > 0$ the set $P(\alpha, \delta, n)$ is not empty for sufficiently large n.

Fix some $\varepsilon > 0$ and let $N(\alpha, \delta, n, \varepsilon)$ be the minimal number of balls $\mathcal{B}_n(x, \varepsilon)$, which is necessary for covering the set $P(\alpha, \delta, n)$. (If $P(\alpha, \delta, n)$ is empty we let $N(\alpha, \delta, n, \varepsilon) = 1$).

Obviously, $N(\alpha, \delta, n, \varepsilon)$ does not increase if δ decreases, and $N(\alpha, \delta, n, \varepsilon)$ does not decrease if ε decreases. This observation guarantees that the following limit exists

$$\Lambda_\varphi(\alpha) = \lim_{\varepsilon \to 0} \lim_{\delta \to 0} \varliminf_{n \to \infty} \frac{1}{n} \log N(\alpha, \delta, n, \varepsilon). \tag{6}$$

One can give another equivalent definition of $\Lambda_\varphi(\alpha)$. The equivalence of these definitions will be useful for subsequent arguments. Let us recall a notion of (n, ε)-separated sets: a set E is called (n, ε)-separated if for any $x, y \in E$, $x \neq y$, $d_n(x, y) > \varepsilon$.

By definition, we let $M(\alpha, \delta, n, \varepsilon)$ be the cardinality of a maximal (n, ε)-separated set in $P(\alpha, \delta, n)$. Again, we put $M(\alpha, \delta, n, \varepsilon) = 1$ if $P(\alpha, \delta, n)$ is empty. A standard argument shows that

$$N(\alpha, \delta, n, \varepsilon) \leq M(\alpha, \delta, n, \varepsilon) \leq N(\alpha, \delta, n, \varepsilon/2) \tag{7}$$

for every $n \in \mathbb{N}$ and all ε, $\delta > 0$.

Moreover, if f satisfies specification, then taking an upper limit instead of the lower limit with respect to n in the definition of $\Lambda_\varphi(\alpha)$ will give the same number.

Lemma 4.3. *If f satisfies specification, then*

$$\Lambda_\varphi(\alpha) = \lim_{\varepsilon \to 0} \lim_{\delta \to 0} \varlimsup_{n \to \infty} \frac{1}{n} \log N(\alpha, \delta, n, \varepsilon). = \lim_{\varepsilon \to 0} \lim_{\delta \to 0} \varlimsup_{n \to \infty} \frac{1}{n} \log M(\alpha, \delta, n, \varepsilon).$$

We will not use this result, and therefore, will not give a proof, which is based on establishing some sort of *subadditivity* of $N(\alpha, \delta, n, \varepsilon)$:

$$(N(\alpha, \delta, n, 4\varepsilon))^k \leq N(\alpha, 4\delta, nk + km(\varepsilon), \varepsilon)$$

for all integers $k \geq 1$ and all sufficiently large n, where m is taken from the definition of the specification property.

4.3 Upper estimate for $\mathcal{E}_\varphi(\alpha)$ in terms of $H_\varphi(\alpha)$ via $\Lambda_\varphi(\alpha)$.

Theorem 4.1. *For any $\alpha \in \mathcal{L}_\varphi$ one has*

$$\mathcal{E}_\varphi(\alpha) \le \Lambda_\varphi(\alpha) \le H_\varphi(\alpha).$$

Proof. The first inequality $\mathcal{E}_\varphi(\alpha) \le \Lambda_\varphi(\alpha)$ is quite easy. Its proof is based on a standard "box-counting" argument. Following [7], for $\alpha \in \mathcal{L}_\varphi$, $\delta > 0$ and $k \in \mathbb{N}$ consider sets

$$G(\alpha,\delta,k) = \bigcap_{n=k}^{\infty} P(\alpha,\delta,n) = \bigcap_{n=k}^{\infty} \Big\{ x \in X : \ \Big| \frac{1}{n} \sum_{i=0}^{n-1} \varphi(f^i(x)) - \alpha \Big| < \delta \Big\}.$$

It is clear, that for any $\delta > 0$

$$K_\alpha = \Big\{ x \in X : \ \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \varphi(f^i(x)) = \alpha \Big\} \subseteq \bigcup_{k=1}^{\infty} G(\alpha,\delta,k). \qquad (8)$$

We are going to show that $h_{top}(f, G(\alpha,\delta,k),\varepsilon) \le \Lambda_\varphi(\alpha)$ holds for any $k \ge 1$, implying $h_{top}(f, K_\alpha, \varepsilon) \le \Lambda_\varphi(\alpha)$ as well.

Fix arbitrary $k \ge 1$, then $G(\alpha,\delta,k)$ (as a subset of $P(\alpha,\delta,n)$ for $n \ge k$) can be covered by $N(\alpha,\delta,n,\varepsilon)$ balls $\mathcal{B}_n(x,\varepsilon)$ for all $n \ge k$. Therefore for every $s \ge 0$ and all $n \ge k$ we have

$$m(G(\alpha,\delta,k), s, \varepsilon) \le N(\alpha,\delta,n,\varepsilon) \exp(-ns). \qquad (9)$$

Suppose now that $s > \Lambda_\varphi(\alpha)$, and put $\gamma = (s - \Lambda_\varphi(\alpha))/2 > 0$. Since

$$\Lambda_\varphi(\alpha) = \lim_{\varepsilon\to 0} \lim_{\delta\to 0} \varliminf_{n\to\infty} \frac{1}{n} \log N(\alpha,\delta,n,\varepsilon),$$

for all sufficiently small $\varepsilon > 0$ and $\delta > 0$, there exists a monotonic sequence of integers $n_l \to \infty$ such that

$$N(\alpha,\delta,n_l,\varepsilon) \le \exp\big(n_l(\Lambda_\varphi(\alpha) + \gamma)\big)$$

for all $l \ge 1$.Without loss of generality we may assume that $n_1 \ge k$. Then, from (9) we obtain

$$m(G(\alpha,\delta,k), s, \varepsilon) \le \exp(-n_l \gamma),$$

and hence $m(G(\alpha,\delta,k), s, \varepsilon) = 0$. Therefore $h_{top}(f, G(\alpha,\delta,k), \varepsilon) \le s$, and

$$h_{top}(f, K_\alpha, \varepsilon) \le \sup_k h_{top}(f, G(\alpha,\delta,k), \varepsilon) \le s$$

due to (8). Thererefore, $h_{top}(f, K_\alpha) = \lim_{\varepsilon\to 0} h_{top}(f, K_\alpha, \varepsilon) \le s$ as well. Finally, since $s > \Lambda_\varphi(\alpha)$ was chosen arbitrary, we conclude that $\mathcal{E}_\varphi(\alpha) := h_{top}(f, K_\alpha) \le \Lambda_\varphi(\alpha)$.

The second inequality $\Lambda_\varphi(\alpha) \le H_\varphi(\alpha)$ is closely related to the second statement of Theorem 1 by L.-S.Young in [24], and is in fact a large deviation result. In the last stage of our proof, similar to [24], we will rely on one fact, which is established in a standard proof of the variational principle for the classical topological entropy [23].

In order to show the inequaity $\Lambda_\varphi(\alpha) \leq H_\varphi(\alpha)$, it is sufficient, for any $\gamma > 0$, to present a measure $\mu \in \mathcal{M}_f(X, \varphi, \alpha)$ (i.e., an invariant measure with $\int \varphi d\mu = \alpha$) such that

$$h_\mu(f) \geq \Lambda_\varphi(\alpha) - \gamma.$$

Fix arbitrary $\gamma > 0$. By the definition of $\Lambda_\varphi(\alpha)$, there exists a sufficiently small $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ one has

$$\Lambda_\varphi(\alpha, \varepsilon) = \lim_{\delta \to 0} \varliminf_{n \to \infty} \frac{1}{n} \log N(\alpha, \delta, n, \varepsilon) > \Lambda_\varphi(\alpha) - \frac{1}{3}\gamma.$$

Put $\varepsilon_k = \dfrac{\varepsilon_0}{2^k}$, $k \geq 1$. For any $k \geq 1$ one can find a sufficiently small δ_k, $\delta_k \to 0$, such that

$$\varliminf_{n \to \infty} \frac{1}{n} \log N(\alpha, \delta_k, n, \varepsilon_k) > \Lambda_\varphi(\alpha) - \frac{2}{3}\gamma.$$

Also, for any $k \geq 1$ we choose some $n_k \in \mathbb{N}$, $n_k \to \infty$, such that

$$N_k := N(\alpha, \delta_k, n_k, \varepsilon_k) > \exp\big(n_k(\Lambda_\varphi(\alpha) - \gamma)\big).$$

Let C_k be the centers of some minimal covering of $P(\alpha, \delta_k, n_k)$ by balls $\mathcal{B}_{n_k}(x, \varepsilon_k)$. Note, that $\#(C_k) = N_k$, and $\mathcal{B}_{n_k}(x, \varepsilon_k) \cap P(\alpha, \delta_k, n_k) \neq \varnothing$ for every $x \in C_k$. Otherwise, the covering, would not be minimal. For any $k \geq 1$ define a probability measure

$$\sigma_k = \frac{1}{N_k} \sum_{x \in C_k} \delta_x,$$

and let

$$\mu_k = \frac{1}{n_k} \sum_{i=0}^{n_k - 1} (f^{-i})^* \sigma_k = \frac{1}{N_k} \sum_{x \in C_k} \frac{1}{n_k} \sum_{i=0}^{n_k - 1} \delta_{f^i(x)}.$$

Let μ be some limit point for the sequence μ_k. By Theorem 6.9 in [23], μ is an invariant measure, and we claim that

$$\int \varphi d\mu = \alpha, \tag{10}$$

i.e., $\mu \in \mathcal{M}_f(X, \varphi, \alpha)$. Indeed, for every $k \geq 1$, one has

$$\Big| \int \varphi \, d\mu_k - \alpha \Big| \leq \frac{1}{N_k} \sum_{x \in C_k} \Big| \frac{1}{n_k} \sum_{i=0}^{n_k - 1} \varphi(f^i(x)) - \alpha \Big|.$$

However, for every $x \in C_k$ there exists $y = y(x) \in P(\alpha, \delta_k, n_k)$ such that $d_{n_k}(x, y) < \varepsilon_k$. Therefore

$$\Big| \frac{1}{n_k} \sum_{i=0}^{n_k - 1} \varphi(f^i(x)) - \alpha \Big| \leq \frac{1}{n_k} \sum_{i=0}^{n_k - 1} \big| \varphi(f^i(x)) - \varphi(f^i(y)) \big| + \delta_k \leq \mathrm{Var}(\varphi, \varepsilon_k) + \delta_k,$$

where $\mathrm{Var}(\varphi,\varepsilon_k) = \sup\big(|\varphi(x)-\varphi(y)| : d(x,y) < \varepsilon_k\big) \to 0$ as $k\to\infty$, since φ is continuous. Hence, we conclude that

$$\int \varphi d\mu_k \to \alpha, \quad k\to\infty.$$

The above invariant measure μ is a limit point for the sequence μ_k. Hence, there exists a sequence $k_j\to\infty$ such that $\mu_{k_j}\to\mu$ weakly. This in particular means that

$$\int \varphi d\mu_{k_j} \to \int \varphi d\mu.$$

Therefore we obtain (10). Finally, repeating the second half of the proof of the classical variational principle [23, Theorem 8.6, p. 189-190] we conclude that

$$h_\mu(f) \geq \overline{\lim_{k\to\infty}} \frac{1}{n_k}\log N_k \geq \underline{\lim_{k\to\infty}} \frac{1}{n_k}\log N_k \geq \Lambda_\varphi(\alpha)-\gamma.$$

This finishes the proof. □

5 Lower estimate on $\mathcal{E}_\varphi(\alpha)$.

The main result of this section is the following theorem.

Theorem 5.1. *Let $f : X \to X$ be a continuous transformation with the specification property and $\varphi\in C(X,\mathbb{R})$. Then for any $\alpha\in\mathcal{L}_\varphi$ one has*

$$\mathcal{E}_\varphi(\alpha) = \Lambda_\varphi(\alpha) = H_\varphi(\alpha). \tag{11}$$

Proof. In Theorem 4.1 we proved that for any continuous transformation f one has $\mathcal{E}_\varphi(\alpha)\leq\Lambda_\varphi(\alpha)\leq H_\varphi(\alpha)$ for all $\alpha\in\mathcal{L}_\varphi$. Hence, it is sufficient for the proof of (11) to show the opposite inequalities $\mathcal{E}_\varphi(\alpha)\geq\Lambda_\varphi(\alpha)\geq H_\varphi(\alpha)$. We start with the inequality $\Lambda_\varphi(\alpha)\geq H_\varphi(\alpha)$. Our proof relies on the proof of statement 3 of Theorem 1 in [24], but let us first recall one result of A. Katok [11].

Theorem 5.2. *Let $f : X\to X$ be a continuous transformation on a compact metric space, and ν be an ergodic invariant measure. For $\varepsilon>0$, $\delta>0$ denote by $N_f^\nu(\delta,\varepsilon,n)$ the minimal number of ε-balls in the d_n-metric which cover a set of measure at least $1-\delta$. Then, for each $\delta\in(0,1)$, we have*

$$h_\nu(f) = \lim_{\varepsilon\to0}\overline{\lim_{n\to\infty}}\frac{1}{n}\log N_f^\nu(\delta,\varepsilon,n) = \lim_{\varepsilon\to0}\underline{\lim_{n\to\infty}}\frac{1}{n}\log N_f^\nu(\delta,\varepsilon,n).$$

Remark 5.1. *Suppose ν is ergodic and $Y\subseteq X$ is such, that $\nu(Y)\geq1-\delta$. Denote by $S(Y,\varepsilon,n)$ the maximal cardinality of an (n,ε)-separated set in Y. Similar to (7) we conclude that $S(Y,\varepsilon,n)\geq N_f^\nu(\delta,\varepsilon,n)$.*

To prove the inequality $\Lambda_\varphi(\alpha)\geq H_\varphi(\alpha)$, it is sufficient to show that for any $\gamma>0$ and every $\mu\in\mathcal{M}_f(X,\varphi,\alpha)$ one has

$$\Lambda_\varphi(\alpha)\geq h_\mu(f)-4\gamma.$$

Choose arbitrary $\gamma>0$, and let $\varepsilon>0$, $\delta>0$ be so small, that the following holds

1) $\gamma > \delta$;

2) $d(x, y) < \varepsilon \Rightarrow |\varphi(x) - \varphi(y)| < \delta$;

3) $\varliminf_{n\to\infty} \frac{1}{n} \log N(\alpha, 3\delta, n, \varepsilon) < \Lambda_\varphi(\alpha) + \gamma$.

We can approximate μ by an invariant measure ν with the following properties (see [24, p.535]):

a) $\nu = \sum_{i=1}^{k} \lambda_i \nu_i$, where $\lambda_i > 0$, $\sum_i \lambda_i = 1$, and ν_i is an ergodic invariant measure for every $i = 1, \ldots, k$;

b) $h_\nu(f) \geq h_\mu(f) - \gamma$;

c) $\left| \int \varphi d\nu - \int \varphi d\mu \right| < \delta$.

Since ν_i is ergodic for every i, there exists a sufficiently large N such that the set of points

$$Y_i(N) = \left\{ x \in X : \left| \frac{1}{n} \sum_{j=0}^{n-1} \varphi(f^j(x)) - \int \varphi d\nu_i \right| < \gamma \quad \text{for all } n > N \right\}$$

has a ν_i-measure at least $1 - \gamma$ for every $i = 1, \ldots, k$.

Therefore, according to Theorem 5.2, there exist integers N_i such that for all $n_i > N_i$ the minimal number of 4ε-balls in d_{n_i}-metric, which is necessary to cover $Y_i(N)$ is greater than or equal to $\exp(n_i(h_{\nu_i}(f) - \gamma))$. This implies, according to the remark 5.1, that the cardinality of a maximal $(n_i, 4\varepsilon)$-separated set in $Y_i(N)$ is greater than or equal to $\exp(n_i(h_{\nu_i}(f) - \gamma))$. Finally, choose a sufficeintly large integer N_0 such that for every $n > N_0$ one has

$$n_i := [\lambda_i n] > \max(N_i, N)$$

for all $i = 1, \ldots, k$, also denote by $C(n_i, 4\varepsilon)$ some maximal $(n_i, 4\varepsilon)$-separated set in $Y_i(N)$. For every k-tuple $(x_1, \ldots, x_k)$, where $x_i \in C(n_i, 4\varepsilon)$, find a point $y = y(x_1, \ldots, x_k) \in X$ such that it shadows pieces of orbits $\{x_i, \ldots, f^{n_i-1}x_i | i = 1, \ldots, k\}$ within the distance ε and the gap $m = m(\varepsilon)$. Put $\hat{n} = m(k-1) + \sum_i n_i$. Firstly, we observe that to different $(x_1, \ldots, x_k) \in C_{n_1} \times \ldots \times C_{n_k}$ correspond different points $y = y(x_1, \ldots, x_k)$. This is indeed the case, because for $y = y(x_1, \ldots, x_k)$ and $y' = y(x'_1, \ldots, x'_k)$ one has

$$d_{\hat{n}}(y, y') > 2\varepsilon. \tag{12}$$

Secondly, for every $y = y(x_1, \ldots, x_k)$ one has

$$\left| \frac{1}{\hat{n}} \sum_{p=0}^{\hat{n}-1} \varphi(f^p(y)) - \alpha \right| < 2\delta + \frac{km}{\hat{n}} ||\varphi||_{C^0}.$$

Hence, for sufficiently large $\hat{n}$ (i.e., large n) every point $y = y(x_1, \ldots, x_k)$ is in $P(\alpha, 3\delta, \hat{n})$.

On the other hand, due to (12), one would need at least

$$\#(C_{n_1}) \times \ldots \times \#(C_{n_k}) \geq \exp\Big([\lambda_1 n](h_{\nu_1}(f) - \gamma) + \ldots + [\lambda_k n](h_{\nu_k}(f) - \gamma)\Big)$$
$$\geq \exp\big(n(h_\nu(f) - 2\gamma)\big) \geq \exp\big(n(h_\mu(f) - 3\gamma)\big)$$

ε-balls in the $d_{\hat{n}}$-metric to cover $P(\alpha, 3\delta, \hat{n})$. Therefore

$$\varliminf_{n\to\infty} \frac{1}{\hat{n}} \log N(\alpha, 3\delta, \hat{n}, \varepsilon) \geq h_\mu(f) - 3\gamma.$$

Hence, due to the choice of $\varepsilon, \delta > 0$, we have $\Lambda_\varphi(\alpha) + \gamma > h_\mu(f) - 3\gamma$. This finishes the proof of our first inequality $\Lambda_\varphi(\alpha) \geq H_\varphi(\alpha)$.

A much more difficult inequality to prove is the the remaining one: $\mathcal{E}_\varphi(\alpha) \geq \Lambda_\varphi(\alpha)$. In order to show it we will construct a *Moran fractal*, suitable for the purposes of computation of topological entropy. Roughly speaking Moran fractal is a limit set of a following geometric construction: consider a monotonic sequence of compact sets $\{F_k\}$, $F_{k+1} \subseteq F_k$, such that F_k is a union of N_k closed sets $\Delta_i^{(k)}$, $i = 1, \ldots, N_k$, of approximately the same size. Moreover, the sets $\Delta_i^{(k+1)}$ forming the $(k+1)$-level of the construction are somewhat similar to the sets $\Delta_i^{(k)}$ of the k-th level. The Moran fractal associated to this consturction is the set F

$$F = \bigcap_k F_k.$$

One could think of a Moran fractal as a generalization of a standard middle-third Cantor set. A particular choice of F_k will ensure that the limit set F will be a closed subset of K_α, but also will allow us to construct a probability measure μ on F, satisfying the conditions of the Entropy Distribution Principle with $s = \Lambda_\varphi(\alpha) - \gamma$ for any $\gamma > 0$. Thus the topological entropy of F will be larger or equal than s. Since $F \subseteq K_\alpha$, the same will be true for the topological entropy of K_α.

Fix some $\gamma > 0$, and choose a sufficiently small $\varepsilon > 0$ such that

$$\lim_{\delta\to 0} \varliminf_{n\to\infty} \frac{1}{n} \log M(\alpha, \delta, n, 8\varepsilon) \geq \Lambda_\varphi(\alpha) - \gamma/2.$$

We assumed that f satisfies specification, let $m = m(\varepsilon)$ be as in the definition of the specification property, and let

$$m_k = m(\varepsilon/2^k), \quad k \geq 1.$$

Choose also some sequence $\delta_k \downarrow 0$ and a sequence $n_k \uparrow +\infty$ such that

$$M_k := M(\alpha, \delta_k, n_k, 8\varepsilon) > \exp\big(n_k(\Lambda_\varphi(\alpha) - \gamma)\big), \quad \text{and} \quad n_k \geq 2^{m_k}.$$

To shorten the notation we put $s = \Lambda_\varphi(\alpha) - \gamma$.

By definition M_k is the cardinality of a maximal $(n_k, 8\varepsilon)$-separated set in $P(\alpha, \delta_k, n_k)$. Denote by $C_k = \{x_i^k |\ i = 1, \dots, M_k\}$ one of these maximal $(n_k, 8\varepsilon)$-separated sets.

Step 1. Construction of intermidiate sets D_k. We start by choosing some sequence of integers $\{N_k\}$ such that $N_1 = 1$ and two following conditions are satisfied:

1) $N_k \geq 2^{n_{k+1}+m_{k+1}}$ for $k \geq 2$;

2) $N_{k+1} \geq 2^{N_1 n_1 + \ldots + N_k(n_k+m_k)}$ for $k \geq 1$.

Then this sequence N_k is growing very fast, and in particular

$$\lim_{k\to\infty} \frac{n_{k+1} + m_{k+1}}{N_k} = 0 \quad \text{and} \quad \lim_{k\to\infty} \frac{N_1 n_1 + \ldots + N_k(n_k + m_k)}{N_{k+1}} = 0. \tag{13}$$

For any N_k-tuple $(i_1, \dots, i_{N_k}) \in \{1, \dots, M_k\}^{N_k}$ let $y(i_1, \dots, i_{N_k})$ be some point which shadows pieces of orbits $\{x_{i_j}^k, fx_{i_j}^k, \dots, f^{n_k-1}x_{i_j}^k\}$, $j = 1, \dots, N_k$, with a gap m_k, i.e.,

$$d_{n_k}(x_{i_j}, f^{a_j} y(i_1, \dots, i_{N_k})) < \frac{\varepsilon}{2^k},$$

where $a_j = (n_k + m_k)(j-1)$, $j = 1, \dots, N_k$. Such point $y(i_1, \dots, i_{N_k})$ exists, because f satisfies specification. Collect all such points into the set

$$D_k = \{y(i_1, \dots, i_{N_k})|\ i_1, \dots, i_{N_k} \in \{1, \dots, M_k\}\}. \tag{14}$$

We claim that different tuples $(i_1, \dots, i_{N_k})$ produce different points $y(i_1, \dots, i_{N_k})$, and that these points are sufficienly separated in the metric d_{t_k}, where

$$t_k = N_k n_k + (N_k - 1) m_k.$$

This is the content of the following lemma.

Lemma 5.1. *If $(i_1, \dots, i_{N_k}) \neq (j_1, \dots, j_{N_k})$, then*

$$d_{t_k}(y(i_1, \dots, i_{N_k}), y(j_1, \dots, j_{N_k})) > 6\varepsilon. \tag{15}$$

Hence, $\#(D_k) = M_k^{N_k}$.

Since $N_1 = 1$, without loss of generality we may assume that $D_1 = C_1$.

Step 2. Construction of L_k. Here we construct inductively a sequence of finite sets L_k. Points of L_k will be the centers of a balls forming the k-th level of our Moran construction.

Let $L_1 = D_1$ and put $l_1 = n_1$. Suppose we have already defined a set L_k, now we present a construction of L_{k+1}. We let

$$l_{k+1} = l_k + m_{k+1} + t_{k+1} = N_1 n_1 + N_2(n_2 + m_2) + \ldots + N_{k+1}(n_{k+1} + m_{k+1}). \tag{16}$$

For every $x \in L_k$ and $y \in D_{k+1}$ let $z = z(x,y)$ be some point such that

$$d_{l_k}(x,z) < \frac{\varepsilon}{2^{k+1}}, \quad \text{and} \quad d_{t_{k+1}}(y, f^{l_k+m_{k+1}}z) < \frac{\varepsilon}{2^{k+1}}. \tag{17}$$

Such a point exists due to the specification property of f. Collect all these points into the set

$$L_{k+1} = \Big\{ z = z(x,y) | \; x \in L_k, \; y \in D_{k+1} \Big\}. \tag{18}$$

Similar to the proof of Lemma 5.1 we can show that different pairs (x,y), $x \in L_k$, $y \in D_{k+1}$, produce different points $z = z(x,y)$. Hence, $\#(L_{k+1}) = \#(L_k)\#(D_{k+1})$. Therefore, by induction

$$\#(L_k) = \#(D_1)\ldots\#(D_k) = M_1^{N_1}\ldots M_k^{N_k}.$$

It immediately follows from (15) and (17), that for every $x \in L_k$ and any $y, y' \in D_{k+1}$, $y \neq y'$, one has

$$d_{l_k}(z(x,y), z(x,y')) < \frac{\varepsilon}{2^k}, \quad \text{and} \quad d_{l_{k+1}}(z(x,y), z(x,y')) > 5\varepsilon. \tag{19}$$

There is an obvious *tree* structure in the construction of the sets L_k. We will say that a point $z \in L_{k+1}$ *descends* from $x \in L_k$ if there exists $y \in D_{k+1}$ such that $z = z(x,y)$. We also say that a point $z \in L_{k+p}$ *descends* from $x \in L_k$ if there exists a sequence of points $(z_k, \ldots, z_{k+p})$, $z_k = x$, $z_{k+p} = z$, and $z_t \in L_t$, such that z_{l+1} descends from z_l in the above sense for every $l = k, \ldots, k+p-1$.

Step 3. The Moran fractal F. For every k put

$$F_k = \bigcup_{x \in L_k} \overline{\mathcal{B}}_{l_k}\Big(x, \frac{\varepsilon}{2^{k-1}}\Big),$$

where $\overline{\mathcal{B}}_l(x,\delta)$ is the closed ball around x of radius δ in the metric d_l, i.e.,

$$\overline{\mathcal{B}}_l(x,\delta) = \{y \in X : \; d_l(x,y) \leq \delta\}.$$

Lemma 5.2. *For every k the following is satisfied:*

1) for any $x, x' \in L_k$, $x \neq x'$, the sets $\overline{\mathcal{B}}_{l_k}\Big(x, \frac{\varepsilon}{2^{k-1}}\Big)$, $\overline{\mathcal{B}}_{l_k}\Big(x', \frac{\varepsilon}{2^{k-1}}\Big)$ are disjoint;

2) if $z \in L_{k+1}$ descends from $x \in L_k$, then

$$\overline{\mathcal{B}}_{l_{k+1}}\Big(z, \frac{\varepsilon}{2^k}\Big) \subseteq \overline{\mathcal{B}}_{l_k}\Big(x, \frac{\varepsilon}{2^{k-1}}\Big).$$

Hence, $F_{k+1} \subseteq F_k$.

Finally, we put

$$F = \bigcap_{k \geq 1} F_k.$$

It is clear that F is a non-empty closed subset of X.

Lemma 5.3. *For every $x \in F$ one has*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \varphi(f^i(x)) = \alpha.$$

Therefore $F \subseteq K_\alpha$.

Step 4. A special probability measure μ. For every $k \geq 1$ define an atomic probability measure μ_k as follows

$$\mu_k(\{z\}) = \frac{1}{\#(L_k)} \quad \text{for every } z \in L_k.$$

Obviously, $\mu_k(F_k) = 1$.

Lemma 5.4. *A sequence of probability measures $\{\mu_k\}$ converges in a weak topology. Denote the limiting measure by μ, then $\mu(F) = 1$.*

An important property of the limiting measure μ is formulated in the next lemma.

Lemma 5.5. *For every sufficiently large n and every point $x \in X$ such that*

$$\mathcal{B}_n(x, \varepsilon/2) \cap F \neq \varnothing$$

one has

$$\mu(\mathcal{B}_n(x, \varepsilon/2)) \leq e^{-n(s-\gamma)}. \tag{20}$$

Summarizing all from above we see that for every positive γ and every sufficiently small $\varepsilon > 0$, we have constructed a compact set F, $F \subseteq K_\alpha$, and a measure μ such that (20) holds. ¿From the Entropy Distribution Principle and the fact that $\subseteq K_\alpha$, we conclude

$$\Lambda_\varphi(\alpha) - 2\gamma = s - \gamma \leq h_{top}(f, F, \varepsilon/2) \leq h_{top}(f, K_\alpha, \varepsilon/2),$$

and hence

$$\mathcal{E}_\varphi(\alpha) = h_{top}(f, K_\alpha) = \lim_{\varepsilon\to 0} h_{top}(f, K_\alpha, \varepsilon) \geq \Lambda_\varphi(\alpha) - 2\gamma.$$

Since $\gamma > 0$ is arbitrary, we finally conclude that $\mathcal{E}_\varphi(\alpha) \geq \Lambda_\varphi(\alpha)$, which finishes the proof of Theorem 5.1. □

6 Manneville-Pomeau map

Before we start we the detailed discussion of the multifractal spectrum for Lyapunov exponents of the Manneville-Pomeau maps, let us establish a general relation between the multifractal spectra in general and the Legendre transform of the pressure function.

For a continuous function $\varphi : X \to \mathbb{R}$, and $q \in \mathbb{R}$ let $P_\varphi(q) = P(q\varphi)$, where $P(\cdot)$ is the topological pressure. By the classical Variational Principle one has

$$P(\psi) = \sup\Big\{ h_\mu(f) + \int \psi d\mu : \mu \in \mathcal{M}_f(X) \Big\}.$$

Since we have assumed that the topological entropy of f is finite, $P(\psi)$ is finite for every continuous ψ. Moreover, $P(\cdot)$ is convex, Lipschitz continuous, increasing and $P(c + \psi + \xi - \xi \circ f) = c + P(\psi)$, whenever $c \in \mathbb{R}$, and $\psi, \xi \in C(X, \mathbb{R})$.

For any $\alpha \in \mathbb{R}$ define the Legendre transform $P_\varphi^*(\alpha)$ by

$$P_\varphi^*(\alpha) = \inf_{q \in \mathbb{R}} \Big(P_\varphi(q) - q\alpha \Big).$$

Note, that $P_\varphi^*(\alpha) < +\infty$ for all $\alpha \in \mathbb{R}$, however, it is possible that $P_\varphi^*(\alpha) = -\infty$.

Theorem 6.1. *Let $f : X \to X$ be a continuous transformation with specification, and $\varphi : X \to \mathbb{R}$ be a continuous function. Then*

(i) for any $\alpha \in \mathcal{L}_\varphi$, one has

$$H_\varphi(\alpha) \le P_\varphi^*(\alpha);$$

(ii) if, moreover, f is such that the entropy map $\mu \to h_\mu(f)$ is upper semi-continuous, then for any α from the interior of $\mathcal{L}_\varphi$ one has

$$H_\varphi(\alpha) = P_\varphi^*(\alpha).$$

Remark 6.1. *Transformations $f : X \to X$ with an upper semi-continuous entropy map*

$$H(\mu) : \mathcal{M}_f(X) \to [0, +\infty) : \ \mu \to h_\mu(f)$$

play a special role in the theory of equilibrium states. This class of transformations includes, for example, all expansive maps [23]. A useful property of such transfomations is that every continuous function ψ has a least one equilibrium state.

Proof of Theorem 6.1. (i) For any $\alpha \in \mathcal{L}_\varphi$ and any $q \in \mathbb{R}$ one has

$$\begin{aligned} H_\varphi(\alpha) &= \sup\Big\{ h_\mu(f) : \ \mu \in \mathcal{M}_f(X), \ \int \varphi \, d\mu = \alpha \Big\} \\ &= \sup\Big\{ h_\mu(f) + q \int \varphi \, d\mu : \ \mu \in \mathcal{M}_f(X), \ \int \varphi \, d\mu = \alpha \Big\} - q\alpha \\ &\le \sup\Big\{ h_\mu(f) + q \int \varphi \, d\mu : \ \mu \in \mathcal{M}_f(X) \Big\} - q\alpha = P(q\varphi) - q\alpha, \end{aligned}$$

where the last equality follows to the Variational Principle for topological pressure. Hence, $H_\varphi(\alpha) \le \inf_q \big(P(q\varphi) - q\alpha \big) = P_\varphi^*(\alpha)$.

(ii) It was shown by O. Jenkinson [10], that if the entropy map is upper semi-continuous, then for any α from the interior of $\mathcal{L}_\varphi$, there exists $q^* \in \mathbb{R}$ and an invariant measure ν, which is an equilibrium state for $q^*\varphi$ such that

$$\int \varphi d\nu = \alpha.$$

Hence

$$H_\varphi(\alpha) = \sup\Big\{h_\mu(f): \ \mu \in \mathcal{M}_f(X), \ \int \varphi \, d\mu = \alpha\Big\} \geq h_\nu(f) = P(q^*\varphi) - q^*\alpha.$$

Therefore $H_\varphi(\alpha) \geq P^*_\varphi(\alpha)$ and the result follows. □

The following theorem is an immediate corollary of Theorems 5.1 and 6.1.

Theorem 6.2. *Suppose $f : X \to X$ is a continuous transformation with specification property such that the entropy map is upper semi-continuous. Then for any $\alpha \in (\inf \mathcal{L}_\varphi, \sup \mathcal{L}_\varphi)$ one has*

$$\mathcal{E}_\varphi(\alpha) = P^*_\varphi(\alpha).$$

Remark 6.2. *Note that for transformations with the specification property, $\mathcal{L}_\varphi$ is an interval.*

Let us consider in greater detail an application of the above theorem to the multifractal analysis of the Manneville-Pomeau maps.

For a given number s, $0 < s < 1$, a corresponding Manneville-Pomeau map is given by

$$f : [0,1] \to [0,1] : x \to x + x^{1+s} \mod 1.$$

The map f is topologically conjugated to a one-sided shift on two symbols, and thus satisfies the specification property. Morevoer, f is expansive, and hence the entropy map is upper semi-continuous. Let $\varphi(x) = \log f'(x)$. With such choice the level sets K_α are preciesly the level sets of pointwise Lyapunov exponents, which are defined (provided the limit exists, of course) as

$$\lambda(x) = \lim_{n\to\infty} \frac{1}{n} \log |(f^n)'(x)|, \quad \text{and} \quad K_\alpha = \big\{x : \ \lambda(x) = \alpha\big\}.$$

Due to the fact that $x = 0$ is an indifferent fixed point for the Manneville-Pomeau map, there exist points x with $\lambda(x)$ arbitrary close to 0, and hence $\inf \mathcal{L}_\varphi = 0$.

Let us discuss some thermodynamic properties of the Manneville-Pomeau maps. First of all, there exists a unique absolutely continuous f-invariant measure μ. Moreover, μ is an equilibrium state for the potential $-\varphi$ and μ is ergodic. However, there exists another equilibrium state for $-\varphi$, namely, the Dirac masure at 0, δ_0. The coexistence of two equilibrium states results in a non-analytic behaviour of the pressure function $P_\varphi(q) := P(q\varphi)$. Namely, it was shown in [19, 22] that $P_\varphi(q)$ is positive and strictly convex for $q > -1$, and $P_\varphi(q) \equiv 0$ for $q \leq -1$, see Figure 1.

Since f satisfies specification and is expansive, Theorem 6.2 is applicable and hence $\mathcal{E}_\varphi(\alpha) = P^*_\varphi(\alpha)$. The graph of $P^*_\varphi(\alpha)$ is shown in Figure 1.

The entropy spectrum $\mathcal{E}_\varphi(\alpha)$ is concave, but not strictly concave. The graph of $\mathcal{E}_\varphi(\alpha)$ contains a piece of a straight line.

We represent the interval $[\inf L_\varphi, \sup L_\varphi] = [0, \bar\alpha]$ as the union of two intervals $[0, \alpha_0]$ and $(\alpha_0, \bar\alpha]$, where α_0 is the largest α such that $P^*(\alpha) = \alpha$, i.e., $P^*(\cdot)$ is linear on $[0, \alpha_0]$. In fact,

$$\alpha_0 = h_\mu(f) = \int \log f' d\mu,$$

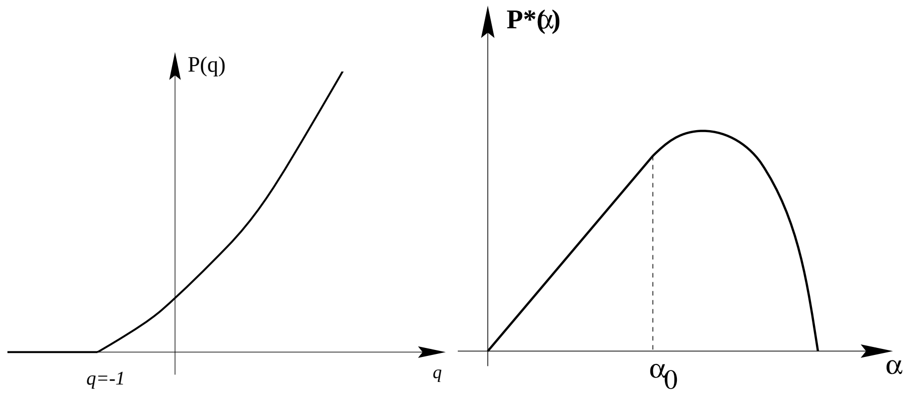

Figure 1: The pressure function $P_\varphi(q)$ and its Legendre transform $P^*(\alpha) = \mathcal{E}_\varphi(\alpha)$.

where μ is an absolutely continuous invariant measure.

Additional considerations show that:

- For each $\alpha \in (0, \alpha_0)$ there exists a unique invariant measure $\mu_\alpha \in \mathcal{M}_f([0,1], \varphi, \alpha)$ such that

$$h_{\mu_\alpha}(f) = \sup\Big\{ h_\nu(f) : \nu \text{ is invariant and } \int \varphi d\nu = \alpha \Big\},$$

  i.e., μ_α is a measure of maximal entropy in $\mathcal{M}_f([0,1], \varphi, \alpha)$, and hence

$$h_{top}(f, K_\alpha) = h_{\mu_\alpha}(f);$$

- Moreover, for any $\alpha \in (0, \alpha_0)$ one has

$$\mu_\alpha = \alpha\mu + (1-\alpha)\delta_0,$$

  where μ is the absolutely continuous invariant measure mentioned above.

Since μ, δ_0 are ergodic, and K_α are invariant sets, we conclude that

$$\mu_\alpha(K_\alpha) = 0$$

for all $\alpha \in (0, \alpha_0)$. This is a new phenomenon, because until a typical situation in multifractal analysis would be $\mu_\alpha(K_\alpha) = 1$ for the "maximal" measure μ_α. And indeed, for all $\alpha \in (\alpha_0, \alpha_1]$, the measures μ_α of maximal entropy in $\mathcal{M}_f([0,1], \varphi, \alpha)$ exist as well, but

$$\mu_\alpha(K_\alpha) = 1.$$

The explanation of this phenomenon lies in fact that the pressure function has a phase transition of the first order at $q = -1$.

7 Multidimensional spectra and Contraction Principle

Suppose $f : X \to X$ is a continuous transformation of a compact metric space (X, d) satisfying specification property, and $\varphi : X \to \mathbb{R}^d$ is a continuous function. Suppose also that we are given a continuous map

$$\Psi : U \to \mathbb{R}^m.$$

where $U \subseteq \mathbb{R}^d$ is such that $\mathrm{Im}(\varphi) = \big\{\varphi(x) : \ x \in X\big\} \subseteq U$. For any $\beta \in \mathbb{R}^m$ define a set

$$K^{\Psi\circ\varphi}(\beta) = \Big\{x \in X : \ \lim_{n\to\infty} \Psi\Big(\frac{1}{n}(S_n\varphi)\Big) = \beta\Big\}.$$

We are interested in the entropy spectrum of $\Psi \circ \varphi$, i.e., the function

$$\mathcal{E}_{\Psi\circ\varphi}(\beta) = h_{top}\big(f, K^{\Psi\circ\varphi}(\beta)\big),$$

defined on a set $\mathcal{L}_{\Psi\circ\varphi} = \{\beta : \ K^{\Psi\circ\varphi}(\beta) \neq \varnothing\}$. Our claim is

Theorem 7.1. *Let f be a continuous transformation satisfying the specification property, and $\varphi : X \to \mathbb{R}^d, \Psi : \mathbb{R}^d \to \mathbb{R}^m$ be continuous map such that $\Phi \circ \varphi$ is well defined. Then that for every $\beta \in \mathcal{L}_{\Psi\circ\varphi}$ one has*

$$\mathcal{E}_{\Psi\circ\varphi}(\beta) = \sup\Big\{h_\mu(f) : \ \mu \ \text{ is invariant and } \ \Psi\Big(\int \varphi d\mu\Big) = \beta\Big\}. \tag{21}$$

The proof of this fact is a generalization of the 1-dimensional proof presented in the previous sections.

We would like to discuss now some corollaries of Theorem 7.1. First of all, by taking Ψ to be identity we immediately conclude that

$$\mathcal{E}_\varphi(\alpha) = h_{top}(f, K_\alpha^\varphi) = \sup\Big\{h_\mu(f) : \ \mu \ \text{ is invariant and } \ \int \varphi d\mu = \alpha\Big\}. \tag{22}$$

A second corollary is the following theorem, which we call the **Contraction Principle for entropy spectra** due to a clear analogy to a well-known Contraction Principle from the theory of Large Deviations, see e.q. [5].

Theorem 7.2. *Under conditions of Theorem 7.1, for any $\beta \in \mathcal{L}_{\Psi\circ\varphi}$ one has*

$$\mathcal{E}_{\Psi\circ\varphi}(\beta) = \sup_{\alpha:\,\Psi(\alpha)=\beta} \mathcal{E}_\varphi(\alpha). \tag{23}$$

Proof. The statement follows from the variational descriptions (21), (22) of the entropy spectra $\mathcal{E}_{\Psi\circ\varphi}(\beta)$ and $\mathcal{E}_\varphi(\alpha)$. Indeed, to prove the claim we have to show that

$$\begin{aligned}\sup\Big\{h_\mu(f) : \ \mu \in \mathcal{M}_f(X) \text{ and } \Psi\Big(\int \varphi d\mu\Big) = \beta\Big\}\\ = \sup_{\alpha:\Psi(\alpha)=\beta} \sup\Big\{h_\mu(f) : \ \mu \in \mathcal{M}_f(X) \text{ and } \int \varphi d\mu = \alpha\Big\}.\end{aligned} \tag{24}$$

A proof of (24) is straightforward. □

In our opinion, it is an interesting question whether the contraction principle (23) is valid for systems without specification.

For transformations f with the specification property the domain $\mathcal{L}_\varphi$ is a convex set, and $\mathcal{E}_\varphi(\alpha)$ is a concave function. Theorems 7.1, 7.2 can be used to produce multifractal spectra $\mathcal{E}_{\Psi\circ\varphi}$ which are not concave, or defined on a non-convex domains $\mathcal{L}_{\Psi\circ\varphi}$. For another setup which also leads to a non-concave multifractal spectra see [1, Proposition 10].

8 Proofs

Proof of Lemma 2.1. Any continuous transformation of a compact metric space admits an invariant probability measure. Moreover, there exist ergodic invariant measures. Suppose μ is ergodic, then by Ergodic Theorem

$$\frac{1}{n}\sum_{i=0}^{n-1}\varphi(f^i(x)) \to \int \varphi d\mu, \quad \text{as} \quad n\to\infty$$

for μ-a.e. $x \in X$. Hence, $\mathcal{L}_\varphi \neq \varnothing$. Clearly, $\mathcal{L}_\varphi \subseteq \big[-||\varphi||_{C^0}, ||\varphi||_{C^0}\big]$, where $||\varphi||_{C^0} = \max_x |\varphi(x)| < \infty$. □

Proof of Lemma 2.2. Suppose $K_{\alpha_i} \neq \varnothing$, $i = 1,2$. let $t \in (0,1)$ and put $\alpha = t\alpha_1 + (1-t)\alpha_2$. Choose some $x_i \in K_{\alpha_i}$ and take any $\mu_i \in V(x_i)$, $i = 1,2$, where $V(x)$ is the set of limit points for the sequence of probability measure

$$\delta_{x,n} = \frac{1}{n}\sum_{k=0}^{n-1}\delta_{f^k(x)}.$$

Then μ_i is an invariant measure with $\int\varphi d\mu_i = \alpha_i$, $i = 1,2$ (see the proof of Lemma 4.1 below). Put $\mu = t\mu_1 + (1-\alpha)\mu_2$. Obviously, $\int \varphi d\mu = \alpha$. Now, we apply [4, Proposition 21.14], which says that for a transformation with the specification property every invariant measure (not, necessarily ergodic!) has a generic point, i.e., there exists a point $x \in X$ such that $\delta_{x,n} \to \mu$ as $n \to \infty$. Hence, for the same point x one has

$$\int \varphi\, d\delta_{x,n} = \frac{1}{n}\sum_{i=0}^{n-1}\varphi(f^i(x)) \to \int \varphi d\mu = \alpha,$$

and therefore, $K_\alpha \neq \varnothing$. □

Proof of Lemma 4.1. We start by showing that $\mathcal{M}_f(X,\varphi,\alpha)$ is not empty for any $\alpha \in \mathcal{L}_\varphi$. Take any $x \in K_\alpha$, and denote by $V(x)$ the set of all limit points of the sequence $\{\delta_{x,n}\}_{n\geq 1}$. Due to compactness of $\mathcal{M}(X)$ the set $V(x)$ is not empty. Moreover, $V(x) \subseteq \mathcal{M}_f(X)$ [23, Theorem 6.9]. Consider an arbitrary measure

$\mu \in V(x)$. By the construction of $V(x)$, there exists a sequence $n_k \to \infty$ such that $\delta_{x,n_k} \to \mu$ weakly. Hence

$$\frac{1}{n_k} \sum_{i=0}^{n_k-1} \varphi(f^i(x)) \to \int \varphi d\mu, \quad k \to \infty.$$

Since $x \in K_\alpha$, we obtain that $\int \varphi d\mu = \alpha$, and hence, $\mu \in \mathcal{M}_f(X, \varphi, \alpha)$. Convexity and closedness of $\mathcal{M}_f(X, \varphi, \alpha)$ are trivial. $\square$

Proof of Lemma 4.2. Convexity of $H_\varphi(\alpha)$ is an obvious consequence of the affinity of the entropy map $h_\mu(f) : \mathcal{M}_f(X) \to [0, +\infty]$, [4]. $\square$

Proof of Lemma 5.1. If $(i_1, \ldots, i_{N_k}) \neq (j_1, \ldots, j_{N_k})$, there exist l such that $i_l \neq j_l$. By the construction of $y(i_1, \ldots, i_{N_k})$ and $y(j_1, \ldots, j_{N_k})$ we have

$$d_{n_k}(x^k_{i_l}, f^{a_l} y(i_1, \ldots, i_{N_k})) < \varepsilon, \quad \text{and} \quad d_{n_k}(x^k_{j_l}, f^{a_l} y(j_1, \ldots, j_{N_k})) < \varepsilon.$$

Since $x^k_{i_l}, x^k_{j_l}$ are different points in the $(n_k, 8\varepsilon)$-separated set, one has

$$\begin{aligned} &d_{n_k}(f^{a_l} y(i_1, \ldots, i_{N_k}), f^{a_l} y(j_1, \ldots, j_{N_k})) \\ &\quad \geq d_{n_k}(x^k_{i_l}, x^k_{j_l}) - d_{n_k}(x^k_{i_l}, f^{a_l} y(i_1, \ldots, i_{N_k})) - d_{n_k}(x^k_{j_l}, f^{a_l} y(j_1, \ldots, j_{N_k})) \\ &\quad > 8\varepsilon - \varepsilon - \varepsilon = 6\varepsilon. \end{aligned}$$

Since

$$d_{t_k}(y(i_1, \ldots, i_{N_k}), y(j_1, \ldots, j_{N_k})) \geq d_{n_k}(f^{a_l} y(i_1, \ldots, i_{N_k}), f^{a_l} y(j_1, \ldots, j_{N_k})),$$

the proof is finished. $\square$

Proof of Lemma 5.2. 1) By (19) for $x, x' \in L_k$, $x \neq x'$, one has $d_{l_k}(x, x') > 5\varepsilon$. Hence

$$\overline{\mathcal{B}}_{l_k}\left(x, \frac{\varepsilon}{2^{k-1}}\right) \bigcap \overline{\mathcal{B}}_{l_k}\left(x', \frac{\varepsilon}{2^{k-1}}\right) = \varnothing.$$

2) For $x \in L_k$ and $z \in L_{k+1}$ such that z descends from x, by (19) one has $d_{l_k}(x, z) < \varepsilon/2^k$. Hence, $\overline{\mathcal{B}}_{l_k}(z, \varepsilon/2^k) \subseteq \overline{\mathcal{B}}_{l_k}(x, \varepsilon/2^{k-1})$. Finally, since $l_{k+1} > l_k$, one has

$$\overline{\mathcal{B}}_{l_{k+1}}(z, \varepsilon/2^k) \subseteq \overline{\mathcal{B}}_{l_k}(z, \varepsilon/2^k).$$

$\square$

Proof of Lemma 5.3.

Estimate on D_k. Let us introduce some notation: for any $c > 0$ put

$$\mathrm{Var}(\varphi, c) = \sup\{|\varphi(x) - \varphi(y)| : \ d(x, y) < c\}.$$

Note, that due to compactness of X, $\mathrm{Var}(\varphi, c) \to 0$ as $c \to 0$ for any continuous function φ. Also, if $d_n(x, y) < c$, then

$$\Big| \sum_{i=0}^{n-1} \varphi(f^i(x)) - \sum_{i=0}^{n-1} \varphi(f^i(y)) \Big| \leq \sum_{i=0}^{n-1} \Big| \varphi(f^i(x)) - \varphi(f^i(y)) \Big| \leq n \mathrm{Var}(\varphi, c).$$

Suppose now that $y \in D_k$, let us estimate $\left|\sum_{p=0}^{t_k-1} \varphi(f^p(y)) - t_k\alpha\right|$. By the definition of D_k, there exist a N_k-tuple $(i_1, \ldots, i_{N_k})$, and points $x_{i_j}^k \in C_k$ for $j = 1, \ldots N_k$, such that

$$d_{n_k}(x_{i_j}^k, f^{a_j} y) < \frac{\varepsilon}{2^k}$$

where $a_j = (n_k + m_k)(j-1)$. Hence,

$$\left|\sum_{p=0}^{n_k-1} \varphi(f^p x_{i_j}^k) - \sum_{p=0}^{n_k-1} \varphi(f^{a_j+p} y)\right| \leq n_k \mathrm{Var}\big(\varphi, \frac{\varepsilon}{2^k}\big).$$

Since $x_{i_j}^k \in C_k \subseteq P(\alpha, \delta_k, n_k)$ we have

$$\left|\sum_{p=0}^{n_k-1} \varphi(f^{a_j+p} y) - n_k\alpha\right| \leq n_k\Big(\mathrm{Var}\big(\varphi, \frac{\varepsilon}{2^k}\big) + \delta_k\Big). \tag{25}$$

To estimate $\left|\sum_{p=0}^{t_k-1} \varphi(f^p(y)) - t_k\alpha\right|$ we represent the interval $[0, t_k - 1]$ as the union

$$\bigcup_{j=0}^{N_k-1} [a_j, a_j + n_k - 1] \bigcup \bigcup_{j=0}^{N_k-2} [a_j + n_k, a_j + n_k + m_k - 1].$$

On the intervals $[a_j, a_j + n_k - 1]$ we will use the estimate (25), and on the intervals $[a_j + n_k, a_j + n_k + m_k - 1]$ we use that

$$\left|\sum_{p=0}^{m_k-1} \varphi(f^{a_j+n_k+p} y) - m_k\alpha\right| \leq m_k(||\varphi||_{C^0} + |\alpha|) \leq 2m_k||\varphi||_{C^0},$$

since $\alpha \in \mathcal{L}_\varphi \subseteq \big[-||\varphi||_{C^0}, ||\varphi||_{C^0}\big]$. Therefore

$$\left|\sum_{p=0}^{t_k-1} \varphi(f^p(y)) - t_k\alpha\right| \leq N_k n_k\Big(\mathrm{Var}(\varphi, \frac{\varepsilon}{2^k}) + \delta_k\Big) + 2(N_k - 1)m_k||\varphi||_{C^0}. \tag{26}$$

Estimate on L_k. Introduce

$$R_k = \max_{z \in L_k} \left|\sum_{p=0}^{l_k-1} \varphi(f^p(z)) - l_k\alpha\right|.$$

Let us obtain by induction an upper estimate on R_k.

If $k = 1$, then $L_1 = D_1 = C_1 \subseteq P(\alpha, \delta_1, n_1)$ (note, that $l_1 = n_1$), therefore we have

$$R_1 \leq l_1\delta_1.$$

By the defintion of L_{k+1} every $z \in L_{k+1}$ is obtained by shadowing of some points $x \in L_k$ and $y \in D_{k+1}$:

$$d_{l_k}(x, z) < \frac{\varepsilon}{2^{k+1}}, \quad d_{t_{k+1}}(y, f^{l_k+m_{k+1}} z) < \frac{\varepsilon}{2^{k+1}}.$$

Hence,

$$\Big|\sum_{p=0}^{l_{k+1}-1}\varphi(f^p(z))-l_{k+1}\alpha\Big|\le\Big|\sum_{p=0}^{l_k-1}\varphi(f^p(z))-\sum_{p=0}^{l_k-1}\varphi(f^p(x))\Big|+\Big|\sum_{p=0}^{l_k-1}\varphi(f^p(x))-l_k\alpha\Big|$$
$$+\Big|\sum_{p=l_k}^{l_k+m_{k+1}}\varphi(f^p(z))-m_{k+1}\alpha\Big|$$
$$+\Big|\sum_{p=0}^{t_{k+1}-1}\varphi(f^{l_k+m_{k+1}+p}(z))-\sum_{p=0}^{t_{k+1}-1}\varphi(f^p(y))\Big|+\Big|\sum_{p=0}^{t_{k+1}-1}\varphi(f^p(y))-t_{k+1}\alpha\Big|$$
$$\le l_k\mathrm{Var}\big(\varphi,\frac{\varepsilon}{2^{k+1}}\big)+R_k+2m_{k+1}||\varphi||_{C^0}+t_{k+1}\mathrm{Var}\big(\varphi,\frac{\varepsilon}{2^{k+1}}\big)$$
$$+N_{k+1}n_{k+1}\Big(\mathrm{Var}(\varphi,\frac{\varepsilon}{2^{k+1}})+\delta_{k+1}\Big)+2(N_{k+1}-1)m_{k+1}||\varphi||_{C^0},$$

where we have used the estimate (26) for $\big|\sum_{p=0}^{t_{k+1}-1}\varphi(f^p(y))-t_{k+1}\alpha\big|$. Hence

$$R_{k+1}\le R_k+2l_{k+1}\mathrm{Var}(\varphi,\frac{\varepsilon}{2^{k+1}})+l_{k+1}\delta_{k+1}+2N_{k+1}m_{k+1}||\varphi||_{C^0},$$

and by induction

$$R_k\le 2\sum_{p=1}^{k}l_p\Big(\mathrm{Var}\big(\varphi,\frac{\varepsilon}{2^p}\big)+\delta_p+\frac{N_pm_p}{l_p}||\varphi||_{C^0}\Big). \tag{27}$$

Let us analyse the obtained expression for R_k. We claim that $R_k/l_k\to 0$ as $k\to\infty$. We start by observing that, $\mathrm{Var}\big(\varphi,\frac{\varepsilon}{2^k}\big)\to 0$ since φ is continuous. By the choice of the sequence $\{\delta_k\}$ one has $\delta_k\to 0$ as well. Moreover, since $l_k\ge N_k(n_k+m_k)$ and the sequence $\{n_k\}$ is such that $n_k\to\infty$ as $k\to\infty$, and $n_k\ge 2^{m_k}$, we conclude that $m_k/n_k\to 0$ as well. Therefore, we can rewrite (27) as

$$R_k\le\sum_{p=1}^{k}l_pc_p,$$

where $c_k\to 0$ as $k\to\infty$. By the choice of N_k (13), we have $l_k\ge 2^{l_{k-1}}$, hence for suffiently large k one has

$$\frac{R_k}{l_k}\le c_k+\frac{1}{k}\sum_{p=1}^{k-1}c_p,$$

and hence $R_k/l_k\to 0$ as $k\to\infty$.

Estimate on F. Now, suppose $x\in F$, $n\in\mathbb{N}$ and $n>l_1$. Then there exists a unique $k\ge 1$ such that

$$l_k<n\le l_{k+1}.$$

Also, there exist a unique j, $0\le j\le N_{k+1}-1$ such that

$$l_k+j(n_{k+1}+m_{k+1})<n\le l_k+(j+1)(n_{k+1}+m_{k+1})$$

Since $x \in F$ there exists $z \in L_{k+1}$ such that

$$d_{l_{k+1}}(x,z) < \frac{\varepsilon}{2^k}.$$

On the other hand since $z \in L_{k+1}$ there exist $\bar{x} \in L_k$ and $y \in D_{k+1}$ such that

$$d_{l_k}(\bar{x},z) < \frac{\varepsilon}{2^{k+1}}, \quad d_{t_{k+1}}(y, f^{l_k+m_{k+1}}z) < \frac{\varepsilon}{2^{k+1}}.$$

Therefore

$$d_{l_k}(x,\bar{x}) < \frac{\varepsilon}{2^{k-1}}, \quad d_{t_{k+1}}(f^{l_k+m_{k+1}}x, y) < \frac{\varepsilon}{2^{k-1}}.$$

Moreover, if $j > 0$, then by the definition of D_{k+1} there exist points $x_{i_1}^{k+1}, \ldots, x_{i_j}^{k+1} \in C_{k+1}$ such that

$$d_{n_{k+1}}(x_{i_t}^{k+1}, f^{a_t}y) < \frac{\varepsilon}{2^{k+1}},$$

where $a_t = (n_{k+1} + m_{k+1})(t-1)$, $t = 1, \ldots, j$, and hence

$$d_{n_{k+1}}(x_{i_t}^{k+1}, f^{l_k+m_{k+1}+a_t}x) < \frac{\varepsilon}{2^{k-2}}. \tag{28}$$

We represent $[0, n-1]$ as the union

$$[0, l_k - 1] \bigcup \bigcup_{t=1}^{j} [l_k + (t-1)(m_{k+1} + n_{k+1}), l_k + t(m_{k+1} + n_{k+1}) - 1]$$
$$\bigcup [l_k + j(m_{k+1} + n_{k+1}), n-1].$$

One has

$$\Big|\sum_{p=0}^{l_k-1} \varphi(f^p x) - l_k\alpha\Big| \leq \Big|\sum_{p=0}^{l_k-1} \varphi(f^p x) - \sum_{p=0}^{l_k-1} \varphi(f^p \bar{x})\Big| + \Big|\sum_{p=0}^{l_k-1} \varphi(f^p \bar{x}) - l_k\alpha\Big|$$
$$\leq l_k \mathrm{Var}\big(\varphi, \frac{\varepsilon}{2^{k-1}}\big) + R_k$$

On each of the intervals $[a_t, a_t + (m_{k+1} + n_{k+1}) - 1]$, where $a_t = l_k + (t-1)(m_{k+1} + n_{k+1})$, we estimate

$$\Big|\sum_{p=a_t}^{a_t+m_{k+1}+n_{k+1}-1} \varphi(f^p x) - (m_{k+1} + n_{k+1})\alpha\Big|$$
$$\leq 2m_{k+1}||\varphi||_{C^0} + n_{k+1}\delta_{k+1} + n_{k+1}\mathrm{Var}(\varphi, \varepsilon/2^{k-2}),$$

because of (28) and the fact that $x_{i_j}^{k+1} \in C_{k+1} \subseteq P(\alpha, \delta_{k+1}, n_{k+1})$.

Finally, on $[l_k + j(m_{k+1} + n_{k+1}), n-1]$ we have

$$\Big|\sum_{p=l_k+j(m_{k+1}+n_{k+1})}^{n-1} \varphi(f^p x) - \big(n - l_k - j(m_{k+1} + n_{k+1})\big)\alpha\Big|$$
$$\leq 2(n - l_k - j(m_{k+1} + n_{k+1}))||\varphi||_{C^0} \leq 2(n_{k+1} + m_{k+1})||\varphi||_{C^0}.$$

Collecting all estimates together one has

$$\Big|\sum_{p=0}^{n-1}\varphi(f^px)-n\alpha\Big|\le R_k+(l_k+jn_{k+1})\mathrm{Var}\big(\varphi,\frac{\varepsilon}{2^{k-2}}\big)$$
$$+2\Big(n_{k+1}+(j+1)m_{k+1}\Big)||\varphi||_{C^0}+jn_{k+1}\delta_{k+1}.$$

Now, since $n>l_k+j(n_{k+1}+m_{k+1})$, and $l_k>N_k$, we obtain

$$\Big|\frac{1}{n}\sum_{p=0}^{n-1}\varphi(f^px)-\alpha\Big|<\frac{R_k}{l_k}+\mathrm{Var}\big(\varphi,\frac{\varepsilon}{2^{k-2}}\big)+2\Big(\frac{n_{k+1}+m_{k+1}}{N_k}+\frac{m_{k+1}}{n_{k+1}}\Big)||\varphi||_{C^0}+\delta_{k+1}.$$

Since the right hand side tends to 0 as $k\to\infty$, and $k\to\infty$ for $n\to\infty$, we finally conclude that

$$\lim_{n\to\infty}\frac{1}{n}\sum_{p=0}^{n-1}\varphi(f^px)=\alpha$$

for all $x\in F$, and hence, $F\subseteq K_\alpha$. □

Proof of Lemma 5.4. We are going to show that for every continuous function ψ there exist a limit

$$I(\psi)=\lim_{k\to\infty}\int\psi\,d\mu_k. \tag{29}$$

Obviously, if $I(\psi)$ is well defined, then I is a positive linear functional on $C(X,\mathbb{R})$. Hence by the Riesz theorem there exist a unique probability measure μ on X such that

$$I(\psi)=\int\psi\,d\mu\quad\text{for every }\psi\in C(X,\mathbb{R}),$$

and thus, $\mu_k\to\mu$ weakly.

Let us prove (29). It is sufficient to show that for every $\delta>0$ there exists $K=K(\delta)>0$ such that for all $k_1,k_2>K$ one has

$$\Big|\int\psi\,d\mu_{k_1}-\int\psi\,d\mu_{k_2}\Big|=\Big|\frac{1}{\#(L_{k_1})}\sum_{x\in L_{k_1}}\psi(x)-\frac{1}{\#(L_{k_2})}\sum_{y\in L_{k_2}}\psi(y)\Big|<\delta.$$

Without loss of generality we may assume that $k_1>k_2$. Then

$$\Big|\frac{1}{\#(L_{k_1})}\sum_{x\in L_{k_1}}\psi(x)-\frac{1}{\#(L_{k_2})}\sum_{y\in L_{k_2}}\psi(y)\Big|\le\frac{1}{\#(L_{k_1})}\sum_{x\in L_{k_1}}\Big|\psi(x)-\psi(y(x))\Big|,$$

where $y(x)\in L_{k_2}$ is a uniqie point in L_{k_2} such that x descends from $y(x)$. Taking into account the way the sets L_k were constructed, we conclude that

$$d(x,y(x))\le\frac{\varepsilon}{2^{k_1}}.$$

Hence, for $k_1,k_2>K$ one has

$$\Big|\int\psi\,d\mu_{k_1}-\int\psi\,d\mu_{k_2}\Big|\le\sup\Big(|\psi(x)-\psi(y)|:\ d(x,y)<\frac{\varepsilon}{2^K}\Big)\to0\quad\text{as }K\to\infty.$$

Now, we have to show that $\mu(F) = 1$. Note, that $\mu_{k+p}(F_k) = 1$ for all $p \geq 0$, since $F_{k+p} \subseteq F_k$ and $\mu_{k+p}(F_{k+p}) = 1$ by construction. Since μ is the weak limit of $\{\mu_k\}$, and F_k are closed, using the properties of weak convergence of measures we obtain

$$\mu(F_k) \geq \overline{\lim_{p\to\infty}} \mu_{k+p}(F_k) = 1,$$

and hence $\mu(F_k) = 1$. Finally, since $F = \bigcap_k F_k$, one has $\mu(F) = 1$. □

Proof of Lemma 5.5. By the definition, $\mathcal{B}_n(x,\varepsilon)$ is an open set, thus, since $\mu_k \to \mu$, we have

$$\mu(\mathcal{B}_n(x,\varepsilon)) \leq \underline{\lim}_{k\to\infty} \mu_k(\mathcal{B}_n(x,\varepsilon)) = \underline{\lim}_{k\to\infty} \frac{1}{\#(L_k)} \#(\{z \in \mathcal{L}_k : \ z \in \mathcal{B}_n(x,\varepsilon)\}).$$

Suppose $n \geq l_1 = n_1$, then there exists $k \geq 1$ such that

$$l_k < n \leq l_{k+1}.$$

As in the proof of Lemma 5.3, let $j \in \{0, \dots, N_{k+1} - 1\}$ be such that

$$l_k + (n_{k+1} + m_{k+1})j < n \leq l_k + (n_{k+1} + m_{k+1})(j+1).$$

We start by showing that $\#(\mathcal{B}_n(x,\varepsilon) \cap L_k) \leq 1$, and thus $\mu_k(\mathcal{B}_n(x,\varepsilon)) \leq \#(L_k)^{-1}$. Indeed, suppose there two points $z_1, z_2 \in L_k$ such that $z_1, z_2 \in \mathcal{B}_n(x,\varepsilon)$ as well. This means that $d_n(z_1, z_2) < 2\varepsilon$. However, from (19) we know that $d_{l_k}(z_1, z_2) > 5\varepsilon$. Hence, we have arrived at contradiction, since $n > l_k$ and thus $d_n(z_1, z_2) \geq d_{l_k}(z_1, z_2)$.

We continue by showing that $\mu_{k+1}(\mathcal{B}_n(x,\varepsilon))$ does not exceed $(\#(L_k) \times M_{k+1}^{j+1})^{-1}$. Suppose, two points $z_1, z_2 \in L_{k+1}$ are in $\mathcal{B}_n(x,\varepsilon)$ as well. Therefore, there exist points $x_1, x_2 \in L_k$ and $y_1, y_2 \in D_{k+1}$ such that

$$z_1 = z(x_1, y_1), \quad z_2 = z(x_2, y_2).$$

All the points in D_{k+1} are obtained by shadowing certain combinatations of points from C_{k+1} (see (14)), i.e.,

$$y_1 = y(i_1, \dots, i_{N_{k+1}}), \quad y_2 = y(i'_1, \dots, i'_{N_{k+1}}),$$

where $(i_1, \dots, i_{N_{k+1}}), (i'_1, \dots, i'_{N_{k+1}}) \in \{1, \dots, M_{k+1}\}^{N_{k+1}}$.

We claim that necessarily $x_1 = x_2$ and $(i_1, \dots, i_j) = (i'_1, \dots, i'_j)$. Indeed, if $x_1 \neq x_2$ then

$$\begin{aligned} d_{l_k}(x_1, x_2) &\leq d_{l_k}(x_1, z_1) + d_{l_k}(z_1, x) + d_{l_k}(x, z_2) + d_{l_k}(z_2, x_2) \\ &\leq \frac{\varepsilon}{2^k} + \varepsilon + \varepsilon + \frac{\varepsilon}{2^k} \leq 5\varepsilon, \end{aligned}$$

and thus we have a contradiction with (19). Similary we proceed with our second claim. If $j = 0$ there is nothing to prove. Suppose $j > 0$ and there exists t,

$1 \leq t \leq j$, such that $i_t \neq i'_t$. Since $y_1 = y(i_1, \dots, i_{N_{k+1}})$, and $y_2 = y(i'_1, \dots, i'_{N_{k+1}})$, one has

$$d_{n_{k+1}}(x_{i_t}^{k+1}, f^{a_t} y_1) < \frac{\varepsilon}{2^{k+1}}, \quad d_{n_{k+1}}(x_{i'_t}^{k+1}, f^{a_t} y_2) < \frac{\varepsilon}{2^{k+1}}.$$

Moreover,

$$d_{t_{k+1}}(z_1, y_1) < \frac{\varepsilon}{2^{k+1}}, \quad d_{t_{k+1}}(z_2, y_2) < \frac{\varepsilon}{2^{k+1}},$$

and hence

$$\begin{aligned} d_{n_{k+1}}(x_{i_t}^{k+1}, x_{i'_t}^{k+1}) \leq & d_{n_{k+1}}(x_{i_t}^{k+1}, f^{a_t} y_1) + d_{t_{k+1}}(y_1, f^{l_k + m_{k+1}} z_1) + \\ & d_n(z_1, z_2) + d_{t_{k+1}}(f^{l_k + m_{k+1}} z_2, y_2) + d_{n_{k+1}}(f^{a_t} y_2, x_{i'_t}^k) \\ \leq & \frac{\varepsilon}{2^{k+1}} + \frac{\varepsilon}{2^{k+1}} + 2\varepsilon + \frac{\varepsilon}{2^{k+1}} + \frac{\varepsilon}{2^{k+1}} < 6\varepsilon, \end{aligned}$$

which contradicts the fact that $d_{n_{k+1}}(x_{i_t}^{k+1}, x_{i'_t}^{k+1}) > 8\varepsilon$, since $x_{i_t}^{k+1}, x_{i'_t}^{k+1}$ are different points in a $(n_{k+1}, 8\varepsilon)$-separated set C_{k+1}.

Since $(i_1, \dots, i_j)$ is the same for all points $z = z(x, y(i_1, \dots, i_j, \dots, i_{N_{k+1}}))$ which can lie in $\mathcal{B}_n(x, \varepsilon)$, we easily conclude that there are at most $M_{k+1}^{N_{k+1}-j}$ such points. Hence

$$\mu_{k+1}(\mathcal{B}_n(x, \varepsilon)) \leq \frac{1}{\#(L_k) M_{k+1}^{N_{k+1}}} M_{k+1}^{N_{k+1}-j} = \frac{1}{\#(L_k) M_{k+1}^j}$$

For any $p > 1$ one has

$$\mu_{k+p}(\mathcal{B}_n(x, \varepsilon/2)) \leq \frac{1}{\#(L_k) M_{k+1}^j}$$

as well. This is indeed the case, because the points of L_{k+p}, which lie in $\mathcal{B}_n(x, \varepsilon/2)$, can only descend from the points of L_{k+1}, which are in $\mathcal{B}_n(x, \varepsilon)$. We provethis finally by contradiction. Suppose we can find points $z_1 \in L_{k+1}$ and $z_2 \in L_{k+p}$, z_2 descends from z_1 such that

$$d_n(z_2, x) < \varepsilon/2 \quad \text{and} \quad d_n(z_1, x) > \varepsilon.$$

This implies that $d_n(z_1, z_2) \geq d_n(z_1, x) - d_n(x, z_2) > \varepsilon/2$. The latter however is not possible, since

$$d_n(z_1, z_2) \leq d_{l_{k+1}}(z_1, z_2) \leq \frac{\varepsilon}{2^{k+2}} + \frac{\varepsilon}{2^{k+3}} + \dots = \frac{\varepsilon}{2^{k+1}}.$$

Hence there are exactly $\#(D_{k+2}) \dots \#(D_{k+p})$ points in L_{k+p}, $p \geq 2$, which descend from a given point in L_{k+1}. Hence

$$\mu_{k+p}(\mathcal{B}_n(x, \varepsilon/2)) \leq \frac{M_{k+1}^{N_{k+1}-j} \#(D_{k+2}) \dots \#(D_{k+p})}{\#(L_k) M_{k+1}^{N_{k+1}} \#(D_{k+2}) \dots \#(D_{k+p})} = \frac{1}{\#(L_k) M_{k+1}^j}.$$

And therefore

$$\mu(\mathcal{B}_n(x, \varepsilon/2)) \leq \varliminf_{p \to \infty} \mu_{k+p}(\mathcal{B}_n(x, \varepsilon/2)) \leq \frac{1}{\#(L_k) M_{k+1}^j}.$$

Now, by the choice of k and j we have

$$n - l_k - j(n_{k+1} + m_{k+1}) \leq n_{k+1} + m_{k+1},$$

where $l_k = N_1 n_1 + N_2(n_2 + m_2) + \ldots + N_k(n_k + m_k)$. Therefore

$$\frac{n - l_k - j(n_{k+1} + m_{k+1})}{l_k + j(n_{k+1} + m_{k+1})} \leq \frac{n_{k+1} + m_{k+1}}{N_k} \to 0 \text{ as } k \to \infty$$

because of the choice of N_k. Since M_k has been chosen in a such way that $M_k \geq \exp(sn_k)$, and m_k are much smaller than n_k, for large k we obtain

$$\begin{aligned}
\#(L_k)M_{k+1}^j = M_1^{N_1} \ldots M_k^{N_k} M_{k+1}^j &\geq \exp\Big(s(N_1 n_1 + N_2 n_2 + \ldots + N_k n_k + j n_{k+1})\Big) \\
&\geq \exp\Big((s - \gamma/2)(N_1 n_1 + \ldots + N_k(n_k + m_k) + j(n_{k+1} + m_{k+1})\Big) \\
&\geq \exp\big((s - \gamma)n\big)
\end{aligned}$$

Therefore, since $k \to \infty$ as $n \to \infty$, for all sufficiently large n one has

$$\mu(\mathcal{B}_n(x, \varepsilon/2)) \leq \exp(-n(s - \gamma))$$

for every x such that $\mathcal{B}_n(x, \varepsilon/2) \cap F \neq \varnothing$. □

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