MODERATE DEVIATIONS FOR LONGEST INCREASING SUBSEQUENCES: THE LOWER TAIL

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Abstract

We derive a moderate deviation principle for the lower tail probabilities of the length of a longest increasing subsequence in a random permutation. It refers to the regime between the lower tail large deviation regime and the central limit regime. The present article together with the upper tail moderate deviation principle in [12] yields a complete picture for the whole moderate deviation regime. Other than in [12], we can directly apply estimates by Baik, Deift, and Johansson [3], who obtained a (non-standard) Central Limit Theorem for the same quantity.

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1 Introduction

Recently a problem which was invented by Ulam 40 years ago [15] has returned to the probabilists' attention: Consider the permutation group S_n on $\{1, \dots, n\}$. We say that $1 \leq i_1 < \dots < i_k \leq n$ is an increasing subsequence of length k of $\pi \in S_n$ if $\pi(i_1) < \dots < \pi(i_k)$. We denote the length of a longest increasing subsequence of a permutation π by $L_n = L_n(\pi)$; note that, in general, such a subsequence is not unique. *Ulam's problem* is: What is the typical asymptotic behavior of L_n as $n \to \infty$, if π is chosen with uniform probability $1/n!$ from S_n?

A Poissonized version of this problem is equally interesting: one replaces the deterministic number n above by a Poisson(λ)-distributed random variable N. Thus one obtains the Poissonized random variable L_N. Conditioned on the event $\{N = n\}$, L_N has the same distribution as L_n. For a geometric interpretation of L_N we refer e.g. to [2] and [12].

The probability $\mathbb{P}[L_n \leq l]$ can also be interpreted as $\int_{U_l} |\operatorname{Tr} M|^{2n} dM/n!$, where U_l is the unitary group of rank l and dM denotes the Haar measure on it. This fact and other connections of Ulam's problem to other mathematical topics can be found in two survey articles by Aldous and Diaconis [2] and Deift [5].

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Ulam conjectured that

$$c := \lim_{n\to\infty} \frac{1}{\sqrt{n}} \mathbb{E}[L_n] \tag{1.1}$$

exists. This was proved by Hammersley in 1972 [9] by an application of the subadditive ergodic theorem. The correct numerical value $c = 2$ was given by Logan and Shepp [11] and independently by Kerov and Vershik [16] in 1977. The same result was proven by different methods by Aldous and Diaconis [1], Seppäläinen [13], Johansson [10], and Groeneboom [8].

The large deviation (LD) principle to this law of large numbers was derived in two papers by Seppäläinen [13] and Deuschel and Zeitouni [6]. They proved that for all $x > 2$

$$\lim_{n\to\infty} \frac{1}{\sqrt{n}} \log \mathbb{P}\left[L_n > x\sqrt{n}\right] = -2x \operatorname{arcosh}\frac{x}{2} + 2\sqrt{x^2-4}, \tag{1.2}$$

and that for $0 < x < 2$

$$\lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}[L_n \le x\sqrt{n}] = 1 - \frac{x^2}{4} - 2\log\frac{x}{2} + 2\left(1+\frac{x^2}{4}\right)\log\left(\frac{2x^2}{4+x^2}\right). \tag{1.3}$$

In 1999 Baik, Deift, and Johansson [3] came up with a method based on the theory of matrix-valued Riemann-Hilbert problems and integrable systems to prove a non-standard Central Limit Theorem (CLT) for the quantity L_n. Their result (Theorem 1.1 in [3]) reads as follows: Rescale L_n as

$$\chi_n(\pi) := \frac{L_n(\pi) - 2\sqrt{n}}{n^{1/6}}. \tag{1.4}$$

Then χ_n converges in distribution as $n \to \infty$ to the Tracy-Widom distribution, introduced by Tracy and Widom in [14]. All moments of χ_n converge to the corresponding moments of the Tracy-Widom distribution, as well (Theorem 1.2 in [3]). This distribution is defined in the following way: Let $u(x)$ be the solution to the Painlevé II equation

$$u_{xx} = 2u^3 + xu \quad \text{with} \quad u(x) \sim -\mathrm{Ai}(x) \sim -\frac{e^{-(2/3)x^{3/2}}}{2\sqrt{\pi}x^{1/4}} \quad \text{as } x \to \infty; \tag{1.5}$$

the notation $a \sim b$ means that the quotient of both sides converges to 1, and Ai denotes the Airy function. Then the Tracy-Widom distribution has the distribution function

$$F(t) := \exp\left(-\int_t^\infty (x-t)u^2(x)dx\right). \tag{1.6}$$

Interestingly, the Tracy-Widom distribution first appeared in the context of eigenvalue statistics of the Gaussian Unitary ensemble.

The following statement is an immediate consequence of the lower tail asymptotics of the Tracy-Widom distribution (see Appendix A):

$$\lim_{t\to\infty}\lim_{n\to\infty} \frac{\log \mathbb{P}\left[L_n \le (2 - tn^{-1/3})\sqrt{n}\right]}{t^3} = -\frac{1}{12}. \tag{1.7}$$

The asymptotics (1.7) perfectly agrees with the large deviation asymptotics for the "upper end" of the lower tail, which one readily derives from (1.3):

$$\lim_{t \searrow 0} \lim_{n \to \infty} \frac{\log \mathbb{P}\left[L_n \le (2-t)\sqrt{n}\right]}{t^3 n} = -\frac{1}{12}. \tag{1.8}$$

1.1 Results

In this note we fill the gap between the estimates (1.7) and (1.8) by showing that in the lower tail moderate deviation regime the probabilities scale in very much the same way. Thus together with the results obtained in [12] we obtain a full moderate deviation principle. Our result reads as follows:

Theorem 1.1 *For all* $0 < \eta < 1/3$ *and* $t > 0$*,*

$$\lim_{n \to \infty} \frac{\log \mathbb{P}\left[L_n \le (2 - tn^{-\eta})\sqrt{n}\right]}{n^{1-3\eta} t^3} = -\frac{1}{12}. \tag{1.9}$$

Remark: Recall that in [12] the following moderate deviation principle for the upper tail was proved:

For all $0 < \eta < 1/3$ and $t > 0$:

$$\lim_{n \to \infty} \frac{\log \mathbb{P}\left[L_n > (2 + tn^{-\eta})\sqrt{n}\right]}{n^{(1-3\eta)/2} t^{3/2}} = -\frac{4}{3} \tag{1.10}$$

Observe that the moderate deviations in (1.9) have twice the speed of the moderate deviations in (1.10). This difference is in agreement with the large deviation results cited above and can be explained on an intuitive level as well: building unusually short longest increasing subsequences is much more expensive than creating extraordinarily long ones, since a very short longest increasing subsequence also restricts our choice in assembling all the other elements in a random permutation.

A more refined version of Theorem 1.1, which also covers the cases $\eta \to 0$ and $\eta \to 1/3$, will be given in Theorem 3.3 below. Both, Theorem 1.1 and Theorem 3.3, rely on the moderate deviation principle for the corresponding quantity in the Poissonized version of the problem. In order to state this moderate deviation principle, it is convenient to reparametrize the pair consisting of n (the size of the permutation group) and l (the length of a longest increasing subsequence) in the following way:

$$\gamma_{l,n} := \frac{2\sqrt{n}}{l}, \quad M_{l,n} := \frac{2\sqrt{n} - l}{l^{1/3}} = (\gamma_{l,n} - 1) l^{2/3}. \tag{1.11}$$

Note that $\gamma_{l,n}$ measures how much the length of a longest increasing subsequence deviates from its expected behavior: for large n and a typical permutation π, the quantity $l = L_n(\pi)$ will be of order $2\sqrt{n}$, so that $\gamma_{l,n}$ is close to one. On the other hand, note that the CLT is proved for the normalized quantity $n^{-1/6}(l-2\sqrt{n}) = -2^{1/3}\gamma_{l,n}^{-1/3}M_{l,n}$. Since $\gamma_{l,n}$ is typically of order 1, the variable $M_{l,n}$ measures the distance from the central limit (CL) regime. Indeed, the different lower tail asymptotic regimes can be conveniently described in terms of $\gamma_{l,n}$ and $M_{l,n}$:

CL:	$\gamma_{l,n} \to 1$ and $M_{l,n} \to M \in \mathbb{R}$.
lower end asymptotics of the CL:	first $\gamma_{l,n} \searrow 1$, second $M_{l,n} \to \infty$.
lower tail moderate deviations:	$\gamma_{l,n} \searrow 1$ and $M_{l,n} \to \infty$ simultaneously.
upper end asymptotics of the lower tail LD:	first $M_{l,n} \to \infty$, second $\gamma_{l,n} \searrow 1$.
lower tail LD:	$M_{l,n} \to \infty$ and $\gamma_{l,n} \to \gamma > 1$.

We introduce the distribution function for the Poissonized quantity L_N with $N \sim$ Poisson(λ):

$$\phi_l(\lambda) := \mathbb{P}[L_N \le l] = \sum_{n=0}^{\infty} \frac{e^{-\lambda}\lambda^n}{n!}\mathbb{P}[L_n \le l]. \tag{1.12}$$

Then we obtain the following result on $\phi_l(\lambda)$:

Theorem 1.2 *There are positive constants $c_1 \ge 2$, $c_2 \le 1/4$, and c_3, so that for all $\lambda > 0$ and $l \in \mathbb{N}$ with $M_{l,\lambda} \ge c_1$ and $1 < \gamma_{l,\lambda} \le 1 + c_2$ the following holds:*

$$\frac{\log \phi_l(\lambda)}{M_{l,\lambda}^3} = \frac{4\gamma_{l,\lambda} - \gamma_{l,\lambda}^2 - 3 - 2\log\gamma_{l,\lambda}}{4(\gamma_{l,\lambda}-1)^3} + \epsilon_{l,\lambda} \tag{1.13}$$

with an error term $\epsilon_{l,\lambda}$ bounded by

$$|\epsilon_{l,\lambda}| \le c_3 M_{l,\lambda}^{-3} \log M_{l,\lambda}. \tag{1.14}$$

Remark: Note that

$$\frac{4\gamma - \gamma^2 - 3 - 2\log\gamma}{4(\gamma-1)^3} = -\frac{1}{6} + O(\gamma - 1) \qquad \text{as } \gamma \to 1, \tag{1.15}$$

such that (1.13) states that under the conditions of Theorem 1.2, $N \sim$ Poisson(λ), and for $\gamma_{l,n} \to 1$

$$\frac{\log \mathbb{P}[L_N \le l]}{M_{l,\lambda}^3} = -\frac{1}{6} + \epsilon_{l,\lambda} + O(\gamma_{l,\lambda} - 1) \tag{1.16}$$

holds.

1.2 Incorporating an estimate of Baik, Deift, and Johansson

Our proof of Theorem 1.2 is based on an estimate in [3]. We first quickly indicate how to translate questions about longest increasing subsequences into problems about orthonormal polynomials to which the techniques in [3] and [12] apply. More precise explanations can be found in [12] and in [3].

It is convenient to study the Poissonized quantity L_N first, $N \sim \text{Poisson}(\lambda)$. Once we have obtained the moderate deviation behavior for L_N, we derive that of L_n by a de-Poissonization procedure. Roughly speaking, we compare L_N with varying Poisson parameters λ with L_n. λ is chosen such that n lies typically in the central regime or the moderate deviation regime *of the Poisson variable* N; this allows us to separate moderate deviation effects caused by atypically small values of N from those caused by permutations with an unusually short longest increasing subsequence. The details are described in Section 3.

The reason why the Poissonization helps is an identity by Gessel [7]:

$$\phi_l(\lambda) = e^{-\lambda} D_{l-1}(\lambda), \tag{1.17}$$

where $D_{l-1}(\lambda)$ is an $l \times l$ Toeplitz determinant:

$$D_{l-1}(\lambda) = \det \left(\int_{-\pi}^{\pi} e^{-i(k-j)\theta} e^{2\sqrt{\lambda}\cos\theta} \frac{d\theta}{2\pi} \right)_{0 \le k,j \le l-1}. \tag{1.18}$$

Baik, Deift, and Johansson [3] analyze $\phi_l(\lambda)$ by examining the asymptotics of $D_{l-1}(\lambda)$ when $\lambda \to \infty$ and $l \sim 2\sqrt{\lambda}$. The above Toeplitz determinants are related to certain orthogonal polynomials: let

$$p_{l,\lambda}(z) = \sum_{j=0}^{l} \kappa_{l,j}(\lambda) z^j, \qquad \kappa_l(\lambda) := \kappa_{l,l}(\lambda) > 0 \tag{1.19}$$

be the l^{th} orthonormal polynomial with respect to the weight function $e^{2\sqrt{\lambda}\cos\theta} \frac{d\theta}{2\pi}$ on the unit circle, i.e.,

$$\int_{-\pi}^{\pi} \overline{p_{l,\lambda}(e^{i\theta})} p_{k,\lambda}(e^{i\theta}) e^{2\sqrt{\lambda}\cos\theta} \frac{d\theta}{2\pi} = \delta_{l,k}, \quad l, k \ge 0, \tag{1.20}$$

where $\delta_{l,k}$ denotes Kronecker's delta. Then one can show (see (1.24) in [3])

$$\kappa_l^2(\lambda) = \frac{D_{l-1}(\lambda)}{D_l(\lambda)}, \tag{1.21}$$

which leads to (see (1.25) in [3])

$$\log \phi_l(\lambda) = \sum_{k=l}^{\infty} \log \kappa_k^2(\lambda). \tag{1.22}$$

Baik, Deift, and Johansson [3] connect $\kappa_k^2(\lambda)$ to the solution of certain 2×2-matrix Riemann-Hilbert problems. As we will just use one of Baik, Deift, and Johansson's estimates, but (other than in [12]) not the Riemann-Hilbert techniques themselves, we will refrain from explaining them here in detail and just refer the interested reader to the explanations in [3] and [12].

With the help of these Riemann-Hilbert techniques, Baik, Deift, and Johansson [3] derive (among others) the following estimate:

Lemma 1.3 (See [3], Lemma 6.3., part (ii)) *For some positive constants c_4, c_5 (sufficiently large) and c_2 (sufficiently small), the following holds: if $\lambda > 0$ and $q \in \mathbb{N}$ fulfill*

$$1 + c_5 q^{-2/3} \leq \gamma_{q,\lambda} \leq 1 + c_2 \tag{1.23}$$

or equivalently

$$M_{q,\lambda} \geq c_5 \text{ and } \gamma_{q,\lambda} \leq 1 + c_2, \tag{1.24}$$

then

$$\kappa_{q-1}^2 = \exp\{q(-\gamma_{q,\lambda} + \log \gamma_{q,\lambda} + 1)\}\gamma_{q,\lambda}^{-1/2} e^{o_{q,\lambda}} \tag{1.25}$$

with an error term

$$|o_{q,\lambda}| \leq \frac{c_4}{q(\gamma_{q,\lambda} - 1)} \leq 1. \tag{1.26}$$

Note that $q(\gamma_{q,\lambda} - 1)$ is large if (1.23) holds and c_2 and c_3^{-1} are large enough.

The rest of this paper is organized as follows: In Section 2 we prove Theorem 1.2. Section 3 contains the de-Poissonization procedure which allows us to derive the moderate deviations of L_n from those of L_N. In an appendix we show that our moderate deviation result is compatible with the CLT derived in [3].

2 Summation

This section is devoted to the proof of Theorem 1.2. As a main ingredient we use Lemma 1.3. Positive constants c_j keep their meaning globally during the whole article. If c_j depends on an additional parameter, this is denoted explicitly.

Proof of Theorem 1.2. In Lemma 1.3 above, we may assume without loss of generality that $c_5 \geq 1$, and $c_2 \leq 1/4$. We set $c_1 := 2c_5 \geq 2$. Let l and λ fulfill the hypothesis of Theorem 1.2. Then

$$\lambda = \gamma_{l,\lambda}^2 M_{l,\lambda}^3 (\gamma_{l,\lambda} - 1)^{-3}/4 \geq c_1^3 c_2^{-3}/4. \tag{2.1}$$

As a consequence of (1.22),

$$\log \frac{\phi_l(\lambda)}{\phi_b(\lambda)} = \sum_{q=l+1}^{b} \log \kappa_{q-1}^2(\lambda) \tag{2.2}$$

holds for all $b > l$, $b \in \mathbb{N}$.

We choose a "reference point" $b \in \mathbb{N}$, such that $M_{b,\lambda} \in [c_5, c_1[$; such a number b exists: to see this, one observes $M_{\sqrt{\lambda},\lambda} = \lambda^{1/3} > c_1$, $M_{2\sqrt{\lambda},\lambda} = 0 < c_5$, and for $q \in [\sqrt{\lambda}, 2\sqrt{\lambda}]$: $|\partial M_{q,\lambda}/\partial q| \leq (2/3)|q^{-1/3} + \lambda^{1/2} q^{-4/3}| \leq 4\lambda^{-1/6}/3 < 1 \leq c_5 = |[c_5, c_1[|$. Furthermore, $M_{b,\lambda} < c_1 \leq M_{l,\lambda}$ implies $l < b$. As a consequence of $M_{l,\lambda} \geq c_5$, $M_{b,\lambda} \geq c_5$, and $\gamma_{b,\lambda} \leq \gamma_{l,\lambda} \leq 1 + c_2$, the hypothesis (1.24) is fulfilled for $q = l$ and $q = b$. Hence, using the monotonicity of $q \mapsto M_{q,\lambda}$ and $q \mapsto \gamma_{q,\lambda}$, this hypothesis is fulfilled for all $q \in [l, b]$, too. From the formulas (2.2) and (1.25) we obtain:

$$\log \frac{\phi_l(\lambda)}{\phi_b(\lambda)} \quad = \quad \sum_{q=l+1}^{b} q(-\gamma_{q,\lambda} + \log \gamma_{q,\lambda} + 1) - \frac{1}{2} \sum_{q=l+1}^{b} \log \gamma_{q,\lambda} + \sum_{q=l+1}^{b} o_{q,\lambda}. \tag{2.3}$$

We examine the first sum on the right-hand side of (2.3), using the trapezoidal rule with error estimates:

$$\sum_{q=l+1}^{b} f(q) = \int_l^b f(x)\, dx + \frac{f(b) - f(l)}{2} + \int_l^b K(x) f''(x)\, dx \tag{2.4}$$

with $f \in C^2[l, b]$ and

$$K(x) := \frac{1}{2}\{x\}(1 - \{x\}); \tag{2.5}$$

here $\{x\} = x - \max\{n \in \mathbb{Z} \,:\, n \leq x\}$ denotes the fractional part of x. We get

$$\begin{aligned}
&\sum_{q=l+1}^{b} q(-\gamma_{q,\lambda} + \log \gamma_{q,\lambda} + 1) \qquad (2.6)\\
&= 2\sqrt{\lambda} \int_l^b \left(\frac{x}{2\sqrt{\lambda}} - 1 - \frac{x}{2\sqrt{\lambda}} \log \frac{x}{2\sqrt{\lambda}} \right) dx \\
&\quad + \frac{b}{2}(-\gamma_{b,\lambda} + \log \gamma_{b,\lambda} + 1) - \frac{l}{2}(-\gamma_{l,\lambda} + \log \gamma_{l,\lambda} + 1) - \int_l^b \frac{K(x)}{x}\, dx \\
&= \lambda \left(4\gamma_{l,\lambda}^{-1} - 3\gamma_{l,\lambda}^{-2} - 2\gamma_{l,\lambda}^{-2} \log \gamma_{l,\lambda} - 4\gamma_{b,\lambda}^{-1} + 3\gamma_{b,\lambda}^{-2} + 2\gamma_{b,\lambda}^{-2} \log \gamma_{b,\lambda} \right) \\
&\quad + \frac{b}{2}(\log \gamma_{b,\lambda} + 1) - \frac{l}{2}(\log \gamma_{l,\lambda} + 1) - \int_l^b \frac{K(x)}{x}\, dx.
\end{aligned}$$

Using $0 \leq K(x) \leq \frac{1}{8}$, we see

$$0 \leq \int_l^b \frac{K(x)}{x}\, dx \leq \frac{1}{8} \log \frac{b}{l}. \tag{2.7}$$

To estimate the second term on the right-hand side of (2.3) we note that

$$\prod_{q=l+1}^{b} \gamma_{q,\lambda} = \prod_{q=l+1}^{b} \frac{2\sqrt{\lambda}}{q} = (2\sqrt{\lambda})^{b-l}\frac{l!}{b!}. \tag{2.8}$$

Using Stirling's formula $n! = \sqrt{2\pi}n^{n+1/2}e^{-n+\theta(n)}$ with $\lim_{n\to\infty}\theta(n) = 0$, we obtain

$$\prod_{q=l+1}^{b} \gamma_{q,\lambda} = (2\sqrt{\lambda})^{b-l}e^{b-l}l^{l+1/2}b^{-b-1/2}e^{\theta(l)-\theta(b)}, \tag{2.9}$$

and we conclude

$$\begin{aligned} -\frac{1}{2}\sum_{q=l+1}^{b} \log\gamma_{q,\lambda} &= \frac{l-b}{2}\log\left(2\sqrt{\lambda}\right) - \left(\frac{l}{2}+\frac{1}{4}\right)\log l + \left(\frac{b}{2}+\frac{1}{4}\right)\log b \\ &\quad + \frac{l-b}{2} + \frac{\theta(b)-\theta(l)}{2} \\ &= \frac{l}{2}(\log\gamma_{l,\lambda}+1) - \frac{b}{2}(\log\gamma_{b,\lambda}+1) + \frac{1}{4}\log\frac{b}{l} + \frac{\theta(b)-\theta(l)}{2}. \end{aligned} \tag{2.10}$$

Finally we estimate the sum of the error terms in (2.3), using (1.26) and $2\sqrt{\lambda} - b = M_{b,\lambda}^{3/2}(\gamma_{b,\lambda}-1)^{-1/2} \geq 1$:

$$\begin{aligned} \left|\sum_{q=l+1}^{b} o_{q,\lambda}\right| &\leq \sum_{q=l+1}^{b} \frac{c_4}{q(\gamma_{q,\lambda}-1)} = \sum_{q=l+1}^{b} \frac{c_4}{2\sqrt{\lambda}-q} \leq 2c_4\int_l^b \frac{dq}{2\sqrt{\lambda}-q} \\ &= 2c_4\log\frac{2\sqrt{\lambda}-l}{2\sqrt{\lambda}-b} = \frac{2}{3}c_4\log\frac{l}{b} + 2c_4\log\frac{M_{l,\lambda}}{M_{b,\lambda}}. \end{aligned} \tag{2.11}$$

Combining (2.6), (2.7), (2.10), and (2.11) with (2.3), we get:

$$\begin{aligned} &\log\frac{\phi_l(\lambda)}{\phi_b(\lambda)} \\ &= \lambda\left(4\gamma_{l,\lambda}^{-1} - 3\gamma_{l,\lambda}^{-2} - 2\gamma_{l,\lambda}^{-2}\log\gamma_{l,\lambda} - 4\gamma_{b,\lambda}^{-1} + 3\gamma_{b,\lambda}^{-2} + 2\gamma_{b,\lambda}^{-2}\log\gamma_{b,\lambda}\right) + \epsilon(l,b,\lambda), \end{aligned} \tag{2.12}$$

with an error term bounded by

$$|\epsilon(l,b,\lambda)| \leq (1+c_4)\left|\log\frac{b}{l}\right| + 2c_4\left|\log\frac{M_{l,\lambda}}{M_{b,\lambda}}\right| + \left|\frac{\theta(l)-\theta(b)}{2}\right| \leq c_6 + 2c_4\log M_{l,\lambda} \tag{2.13}$$

for some positive constant c_6; note that $b/l = \gamma_{l,\lambda}/\gamma_{b,\lambda} \in [1/2, 2]$. We estimate the b-dependent part in (2.12) using $\lambda = \gamma_{b,\lambda}^2 M_{b,\lambda}^3(\gamma_{b,\lambda}-1)^{-3}/4$:

$$\begin{aligned} \lambda\left|1 - 4\gamma_{b,\lambda}^{-1} + 3\gamma_{b,\lambda}^{-2} + 2\gamma_{b,\lambda}^{-2}\log\gamma_{b,\lambda}\right| &= M_{b,\lambda}^3\frac{\left|\gamma_{b,\lambda}^2 - 4\gamma_{b,\lambda} + 3 + 2\log\gamma_{b,\lambda}\right|}{4(\gamma_{b,\lambda}-1)^3} \\ &\leq c_7 M_{b,\lambda}^3 \leq c_7 c_1^3 \end{aligned} \tag{2.14}$$

for some constant $c_7 > 0$; here we have used the convergence in (1.15) as well as $\gamma_{b,\lambda} \in [1, 5/4]$. Using our notation, part (iii) of Lemma 7.1. in [3] states the following: There is a constant $c_8 > 0$, such that for all sufficiently large $M > 0$ there is $C(M) > 0$, so that for all $\lambda > 0$ and $q \in \mathbb{N}$ with $-M \leq M_{q,\lambda} \leq M$ we have

$$\left|\log \phi_{q-1}(\lambda) - \log F(2^{1/3} M_{q,\lambda})\right| \leq C(M) q^{-1/3} + c_8 e^{-(M/2)^{3/2}}; \tag{2.15}$$

see also the last line of the proof of Theorem 1.1 of [3] (page 1170). Here F denotes the distribution function of the Tracy-Widom distribution. As a consequence of (2.15), there is a constant $c_9 > 0$ such that

$$|\log \phi_b(\lambda)| \leq c_9; \tag{2.16}$$

to see this, one may choose $q = b+1$ in (2.15), then use that $0 < M_{b+1,\lambda} < M_{b,\lambda} \leq c_1$, and finally use that $|\log F|$ is bounded on bounded intervals. The estimates (2.12), (2.13), (2.14), (2.16) and $M_{l,\lambda} \geq c_1 > 1$ together imply

$$\log \phi_l(\lambda) = \lambda \left(-1 + 4\gamma_{l,\lambda}^{-1} - 3\gamma_{l,\lambda}^{-2} - 2\gamma_{l,\lambda}^{-2} \log \gamma_{l,\lambda}\right) + M_{l,\lambda}^3 \epsilon_{l,\lambda}, \tag{2.17}$$

with some error term $\epsilon_{l,\lambda}$ bounded by (1.14) for some constant $c_3 > 0$. Hence the claim (1.13) follows, using the fact $\lambda = \gamma_{l,\lambda}^2 M_{l,\lambda}^3 (\gamma_{l,\lambda} - 1)^{-3}/4$, and we have proved Theorem 1.2.

□

3 De-Poissonization

We split the de-Poissonization considerations into two parts: an upper and a lower estimate. For the upper bound, we use a result of Baik, Deift, and Johansson, while for the lower bound, we compare L_n with L_N, $N \sim \text{Poisson}(\lambda)$, with varying values of λ. Let

$$q_{l,n} := \mathbb{P}[L_n \leq l] \tag{3.1}$$

denote the cumulative distribution function of L_n. We start with the upper bound:

Lemma 3.1 *There exist positive constants c_{10}, c_{11}, c_{12} such that for all $n \in \mathbb{N}$ and $l \in \mathbb{N}$, $l \leq n$, satisfying $M_{l,n} \geq c_{12}$ and $1 < \gamma_{l,n} \leq 1 + c_2$*

$$\frac{\log q_{l,n}}{M_{l,n}^3} \leq \frac{\log \phi_l(n)}{M_{l,n}^3} + \frac{c_{11}}{M_{l,n}^{3/2}} + c_{10}(\gamma_{l,n} - 1). \tag{3.2}$$

Proof. By Lemma 8.3 of [3], there exist $c_{13} > 0$ and $c_{12} \geq 2c_1$ (sufficiently large) such that for all $n > c_{12}$ and all $l \in \mathbb{N}$, $l \leq n$:

$$q_{l,n} \leq c_{13} \phi_l(n - \sqrt{n}). \tag{3.3}$$

Let l and n fulfill the hypothesis of the lemma. Using $\gamma_{l,n} = 2\sqrt{n}/l > 1$ we conclude $n > l^2/4 = M_{l,n}^3(\gamma_{l,n} - 1)^{-3}/4 \geq c_{12}^3 c_2^{-3}/4 \geq c_{12}$. Hence we can apply (3.3): Taking logarithms on both sides of the inequality and dividing by $M_{l,n}^3$ we obtain

$$\frac{\log q_{l,n}}{M_{l,n}^3} \leq \frac{\log c_{13}}{M_{l,n}^3} + \frac{\log \phi_l(n - \sqrt{n}) - \log \phi_l(n)}{M_{l,n}^3} + \frac{\log \phi_l(n)}{M_{l,n}^3}. \tag{3.4}$$

Note that

$$M_{l,n-\sqrt{n}}^3 = \left(2\sqrt{n - \sqrt{n}} - l\right)^3 l^{-1} = (2\sqrt{n} - l - \delta_n)^3 l^{-1} \tag{3.5}$$

with some $\delta_n \in [1, 2]$. Together with

$$2\sqrt{n} - l = l^{1/3} M_{l,n} \geq 2c_1 \geq 4 \geq 2\delta_n \tag{3.6}$$

this implies

$$\frac{1}{2} \leq \frac{M_{l,n-\sqrt{n}}}{M_{l,n}} = \frac{\gamma_{l,n-\sqrt{n}} - 1}{\gamma_{l,n} - 1} \leq 1. \tag{3.7}$$

In particular, it follows that $M_{l,n-\sqrt{n}} \geq M_{l,n}/2 \geq c_{12}/2 \geq c_5$ and $1 \leq \gamma_{l,n-\sqrt{n}} \leq \gamma_{l,n} \leq 1 + c_2$. Let

$$g(\gamma) := \frac{1}{6} + \frac{4\gamma - \gamma^2 - 3 - 2\log\gamma}{4(\gamma - 1)^3}. \tag{3.8}$$

By (1.15), there exists a constant $c_{14} > 0$ such that

$$|g(\gamma)| \leq c_{14}(\gamma - 1) \tag{3.9}$$

holds for $1 \leq \gamma \leq 1 + c_2$. With the help of Theorem 1.2 we estimate the second term on the right-hand side of (3.4):

$$\frac{\log \phi_l(n - \sqrt{n}) - \log \phi_l(n)}{M_{l,n}^3} = -\frac{M_{l,n-\sqrt{n}}^3 - M_{l,n}^3}{6M_{l,n}^3} + \frac{M_{l,n-\sqrt{n}}^3}{M_{l,n}^3}\rho_{l,n-\sqrt{n}} - \rho_{l,n} \tag{3.10}$$

with an error term $\rho_{l,n} := \epsilon_{l,m} + g(\gamma_{l,m})$ satisfying

$$|\rho_{l,m}| \leq c_3 M_{l,m}^{-3} \log M_{l,m} + c_{14}(\gamma_{l,m} - 1), \qquad m \in \{n - \sqrt{n}, n\}. \tag{3.11}$$

By (3.5), (3.6), and $2\sqrt{n} - l = l^{1/3} M_{l,n} = M_{l,n}^{3/2}(\gamma_{l,n} - 1)^{-1/2} \geq M_{l,n}^{3/2}$ we obtain

$$\left|\frac{M_{l,n-\sqrt{n}}^3 - M_{l,n}^3}{6M_{l,n}^3}\right| = \left|\frac{(2\sqrt{n} - l - \delta_n)^3 - (2\sqrt{n} - l)^3}{6(2\sqrt{n} - l)^3}\right| \leq \frac{c_{15}}{2\sqrt{n} - l} \leq \frac{c_{15}}{M_{l,n}^{3/2}} \tag{3.12}$$

for some constant $c_{15} > 0$. Using (3.7), (3.11), the bound

$$M_{l,n-\sqrt{n}}^{-3} \log M_{l,n-\sqrt{n}} \leq 8 M_{l,n}^{-3} \log M_{l,n}, \tag{3.13}$$

and the monotonicity of $m \mapsto \gamma_{l,m}$, we conclude

$$\left| \frac{M_{l,n-\sqrt{n}}^3}{M_{l,n}^3} \rho_{l,n-\sqrt{n}} \right| + |\rho_{l,n}| \leq 9 c_3 M_{l,n}^{-3} \log M_{l,n} + 2 c_{14} (\gamma_{l,n} - 1). \tag{3.14}$$

Combining (3.10), (3.12), and (3.14) with (3.4), we obtain (3.2) for some $c_{11} > 0$ and $c_{10} := 2c_{14}$.

□

Lemma 3.2 *For every fixed number $0 < \alpha < 1/2$ there is a constant $c_{16}(\alpha) > 0$, such that for every $n, l \in \mathbb{N}$ with $0 < \gamma_{n,l} - 1 \leq c_{16}(\alpha)$ and $M_{n,l} \geq c_1$ we have*

$$\frac{\log q_{l,n}}{M_{l,n}^3} \geq \frac{\log \phi_l(n)}{M_{l,n}^3} - c_{17} (\gamma_{l,n} - 1)^\alpha - c_{18} \frac{\log M_{l,n}}{M_{l,n}^3} \tag{3.15}$$

with positive constants c_{17} and c_{18} independent of α.

Proof of Lemma 3.2. We choose a fixed number $0 < \alpha < 1/2$. Given n and l such that $0 < \gamma_{l,n} - 1 \leq c_2/2$ and $M_{n,l} \geq c_1$ holds, we define $\xi := (\gamma_{l,n} - 1)^{1+\alpha} \in]0,1]$ and set $\lambda := (1+\xi)n$. For $\mu > 0$ we denote by $\mathbb{P}_\mu$ the Poisson measure on $\mathbb{N}_0$ with parameter μ, and we denote by N the identity map on $\mathbb{N}_0$. Furthermore we set

$$\upsilon := \frac{d\mathbb{P}_\lambda}{d\mathbb{P}_n}(n) = e^{n-\lambda} \lambda^n n^{-n} = \exp\{n(\log(1+\xi) - \xi)\} \in [e^{-n\xi^2/2}, e^{-n\xi^2/4}]. \tag{3.16}$$

For fixed l, the map $n \mapsto q_{l,n}$ is monotonically decreasing. Using this and the fact that the density $d\mathbb{P}_n / d\mathbb{P}_\lambda$ is monotonically decreasing (because $n < \lambda$), we obtain

$$\begin{aligned} q_{l,n} &\geq q_{l,n}(1-\upsilon) \\ &= \mathbb{E}_\lambda \left[q_{l,n} \left(1 - \upsilon \frac{d\mathbb{P}_n}{d\mathbb{P}_\lambda} \right) \right] \geq \mathbb{E}_\lambda \left[q_{l,N} \left(1 - \upsilon \frac{d\mathbb{P}_n}{d\mathbb{P}_\lambda} \right) \right] \\ &= \mathbb{E}_\lambda [q_{l,N}] - \upsilon \mathbb{E}_n [q_{l,N}] = \phi_l(\lambda) - \upsilon \phi_l(n) \\ &\geq \phi_l(\lambda) - e^{-n\xi^2/4} \phi_l(n). \end{aligned} \tag{3.17}$$

The heuristic idea behind the remaining part of the proof is that $\phi_l(\lambda)$ is "close" to $\phi_l(n)$ in the sense that $\phi_l(\lambda)/\phi_l(n)$ is "close" to 1 (on a rather rough scale), and $e^{-n\xi^2/4}$ is "close" to 0.

We observe that $\gamma_{l,\lambda} - 1 = \gamma_{l,n}\sqrt{1+\xi} - 1 \leq \gamma_{l,n} - 1 + \xi\gamma_{l,n}/2 \leq \gamma_{l,n} - 1 + \xi = (1 + (\gamma_{l,n} - 1)^\alpha)(\gamma_{l,n} - 1) \leq 2(\gamma_{l,n} - 1) \leq c_2$ and $M_{l,\lambda} \geq M_{l,n} \geq c_1$. Hence the assumptions of Theorem 1.2 are satisfied for l, n and l, λ. We estimate

$$1 \leq \frac{M_{l,\lambda}}{M_{l,n}} = \frac{\gamma_{l,\lambda} - 1}{\gamma_{l,n} - 1} \leq 1 + (\gamma_{l,n} - 1)^\alpha. \tag{3.18}$$

By Theorem 1.2, (3.8/3.9), and the above estimates, we obtain

$$\begin{aligned}\frac{\phi_l(\lambda)}{\phi_l(n)} &= \exp\left\{\frac{M_{l,n}^3}{6}\left(1-\frac{M_{l,\lambda}^3}{M_{l,n}^3}\right)+g(\gamma_{l,\lambda})M_{l,\lambda}^3-g(\gamma_{l,n})M_{l,n}^3+\epsilon_{l,\lambda}M_{l,\lambda}^3-\epsilon_{l,n}M_{l,n}^3\right\}\\ &\geq \exp\left\{-c_{17}M_{l,n}^3(\gamma_{l,n}-1)^\alpha-c_{19}\log M_{l,n}\right\} \qquad (3.19)\end{aligned}$$

for some positive constants c_{17} and c_{19}. Substituting this in (3.17) and using

$$n\xi^2=\frac{\gamma_{l,n}^2}{4}M_{l,n}^3(\gamma_{l,n}-1)^{2\alpha-1}, \qquad (3.20)$$

we get:

$$\begin{aligned}q_{l,n} &\geq \left(\exp\{-c_{17}M_{l,n}^3(\gamma_{l,n}-1)^\alpha-c_{19}\log M_{l,n}\}-e^{-n\xi^2/4}\right)\phi_l(n) \qquad (3.21)\\ &= \left(1-\exp\left\{M_{l,n}^3\left(-\frac{\gamma_{l,n}^2}{16}(\gamma_{l,n}-1)^{2\alpha-1}+c_{17}(\gamma_{l,n}-1)^\alpha+c_{19}\frac{\log M_{l,n}}{M_{l,n}^3}\right)\right\}\right)\\ &\quad\cdot\exp\left\{-c_{17}M_{l,n}^3(\gamma_{l,n}-1)^\alpha-c_{19}\log M_{l,n}\right\}\phi_l(n).\end{aligned}$$

We observe: there is a positive constant $c_{16}(\alpha)$ (sufficiently small) such that the assumptions $M_{l,n}\geq c_1$ and $0<\gamma_{l,n}-1\leq c_{16}(\alpha)$ imply

$$M_{l,n}^3\left(-\frac{\gamma_{l,n}^2}{16}(\gamma_{l,n}-1)^{2\alpha-1}+c_{17}(\gamma_{l,n}-1)^\alpha+c_{19}\frac{\log M_{l,n}}{M_{l,n}^3}\right)\leq -1. \qquad (3.22)$$

Note that $c_{17}(\gamma_{l,n}-1)^\alpha+c_{19}M_{l,n}^{-3}\log M_{l,n}$ is bounded by a constant and $(\gamma_{l,n}-1)^{2\alpha-1}$ can be made arbitrarily large by choosing $c_{16}(\alpha)$ sufficiently small because $2\alpha-1<0$. Hence, by (3.21),

$$q_{l,n}\geq\exp\{-c_{17}M_{l,n}^3(\gamma_{l,n}-1)^\alpha-c_{19}\log M_{l,n}-1\}\phi_l(n), \qquad (3.23)$$

and thus we get the claim (3.15) for some constant $c_{18}>c_{19}$.

□

We combine the upper and lower de-Poissonization estimates:

Theorem 3.3 *As $M_{l,n}\to\infty$ and $\gamma_{l,n}\searrow 1$ (independently of each other),*

$$\frac{\log q_{l,n}}{M_{l,n}^3}\longrightarrow -\frac{1}{6}. \qquad (3.24)$$

More precisely, we have the following speed of convergence: for every fixed $\alpha\in]0,1/2[$, there exist positive constants c_{20}, c_{21}, c_{22}, and $c_{23}(\alpha)$ such that for all natural numbers $l\leq n$ with $1<\gamma_{l,n}\leq 1+c_{23}(\alpha)$ and $M_{l,n}\geq c_{22}$ the following holds:

$$\left|\frac{\log q_{l,n}}{M_{l,n}^3}+\frac{1}{6}\right|\leq c_{20}(\gamma_{l,n}-1)^\alpha+c_{21}M_{l,n}^{-3/2}. \qquad (3.25)$$

Proof of Theorem 3.3. The theorem is an immediate consequence of the Lemmata 3.1, 3.2, and Theorem 1.2.

□

Proof of Theorem 1.1. Given fixed numbers $t > 0$ and $\eta \in]0, 1/3[$, we define $l(n)$ implicitly by the equation

$$tn^{1/3-\eta} = \frac{2\sqrt{n} - l(n)}{n^{1/6}}. \tag{3.26}$$

(In general $l(n) \notin \mathbb{N}$, however, this causes no serious problem.) Using

$$M^3_{l(n),n} = \frac{\gamma_{l(n),n}}{2} t^3 n^{1-3\eta} \stackrel{n\to\infty}{\longrightarrow} \infty \tag{3.27}$$

and

$$1 < \gamma_{l(n),n} \stackrel{n\to\infty}{\longrightarrow} 1, \tag{3.28}$$

the claim (1.9) follows from Theorem 3.3.

□

A Asymptotic behavior of the Tracy-Widom distribution

Even though the lower tail asymptotics of the Tracy-Widom distribution seems to be well known, we could not find a reference. Therefore we briefly describe it here.

Lemma A.1

$$\log F(t) = \exp\left(\frac{t^3}{12} + O(|t|)\right) \; \textit{for } t \to -\infty \tag{A.1}$$

Proof. Recall that u denotes the solution of the Painlevé II equation given by (1.5). It is known (see for example [4], Theorem 1.28) that there exist constants $c_{24}, c_{25} > 0$ such that

$$u^2(x) \le c_{24} e^{-x} \qquad \text{for all } x \ge -c_{25} \tag{A.2}$$

$$u^2(x) = -\frac{x}{2} + \frac{\epsilon(x)}{x^2} \qquad \text{for all } x \le -c_{25} \text{ with } \sup_{x \le -c_{25}} |\epsilon(x)| \le c_{24}. \tag{A.3}$$

By Definition (1.6) of the Tracy-Widom distribution,

$$\log F(t) = -\int_t^\infty (x-t) u^2(x) dx. \tag{A.4}$$

We write the last integral for $t < -c_{25}$ as a sum of two integrals splitting the domain of integration into the two intervals $[t, -c_{25}]$ and $]-c_{25}, \infty[$. Using (A.2) we obtain

$$\left| \int_{-c_{25}}^{\infty} (x-t)u^2(x)dx \right| \le c_{24} \int_{-c_{25}}^{\infty} (|x|+|t|)e^{-x}dx = O(|t|) \quad \text{as } t \to -\infty. \tag{A.5}$$

Using (A.3) we obtain

$$-\int_{t}^{-c_{25}} (x-t)u^2(x)dx = \int_{t}^{0} \frac{(x-t)x}{2}dx + I(t) = \frac{t^3}{12} + I(t) \tag{A.6}$$

with

$$|I(t)| \le \left| \int_{0}^{-c_{25}} \frac{(x-t)x}{2}dx \right| + \left| \int_{t}^{-c_{25}} \frac{(x-t)\epsilon(x)}{x^2}dx \right| = O(|t|) \quad \text{as } t \to -\infty. \tag{A.7}$$

□

Baik, Deift, and Johansson's nonstandard central limit theorem together with Lemma A.1 imply (1.7). This asymptotics is also compatible with Theorem 3.3.

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