Possible loss and recovery of Gibbsianness during the stochastic evolution of Gibbs measures

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23rd April 2001

Abstract: We consider Ising-spin systems starting from an initial Gibbs measure ν and evolving under a spin-flip dynamics towards a reversible Gibbs measure $\mu \neq \nu$. Both ν and μ are assumed to have a finite-range interaction. We study the Gibbsian character of the measure $\nu S(t)$ at time t and show the following:
(1) For all ν and μ, $\nu S(t)$ is Gibbs for small t.
(2) If both ν and μ have a high or infinite temperature, then $\nu S(t)$ is Gibbs for all $t > 0$.
(3) If ν has a low non-zero temperature and a zero magnetic field and μ has a high or infinite temperature, then $\nu S(t)$ is Gibbs for small t and non-Gibbs for large t.
(4) If ν has a low non-zero temperature and a non-zero magnetic field and μ has a high or infinite temperature, then $\nu S(t)$ is Gibbs for small t, non-Gibbs for intermediate t, and Gibbs for large t.
The regime where μ has a low or zero temperature and t is not small remains open. This regime presumably allows for many different scenarios.

1 Introduction

Changing interaction parameters, like the temperature or the magnetic field, in a thermodynamical system is the preeminent way of studying such a system. In the theory of interacting particle systems, which are used as microscopic models for thermodynamic systems, one associates with each such interaction parameter a class of stochastic evolutions, like Glauber dynamics or Kawasaki dynamics.

In recent years there has been extensive interest in the *quenching regime*, in which one starts from a high- or infinite-temperature Gibbs state and considers the behavior of the system under a low- or zero-temperature dynamics. This is interpreted as a fast cooling procedure (which is different from the slow cooling procedure of simulated annealing). One is interested in the asymptotic behavior of the system, in particular, the occurrence of trapping in metastable frozen or semi-frozen states (see [11], [34], [35], [12], [33], [36], [5]).

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Another regime that has been intensively studied is the one where, starting from a low-non-zero-temperature Gibbs state of Ising spins in a positive magnetic field, one considers a low-non-zero-temperature negative-magnetic-field Glauber dynamics (see [38] and references therein). Under an appropriate rescaling of the time and the magnetic-field strength, one finds a metastable transition from the initial plus-state to the final minus-state.

In this paper we concentrate on the opposite case of the *unquenching regime*, in which one starts from a low-non-zero-temperature Gibbs state of Ising spins and considers the behavior of the system under a high- or infinite-temperature Glauber dynamics. This is interpreted as a fast heating procedure. As far as we know, this regime has not been studied much (see e.g. [1]), as no singular behavior was expected to occur. Although we indeed know that there is exponentially fast convergence (cf. [23], Chapter 1, Theorem 4.1, and [31], [32]) to the high- or infinite-temperature Gibbs state (i.e., the asymptotic behavior is unproblematic), we will show that *at sharp finite times there can be transitions between regimes where the evolved state is Gibbsian and regimes where the evolved state is non-Gibbsian.*

In the light of the results in [9], Chapter 4, on renormalization-group transformations, it should perhaps not come as a surprise that such transitions can happen. Indeed, we can view the time-evolved measure as a kind of (single-site) renormalized Gibbs measure. Even though the image spin $\sigma_t(x)$ at time t at site x is not a (random) function of the original spins $\sigma_0(y)$ at time 0 for y in only a finite block around x, by the Feller character of the Glauber dynamics it depends only weakly on the spins $\sigma_0(y)$ with y large. In that sense the time evolution is close to a standard renormalization-group transformation, without rescaling, and so we can expect Griffiths-Pearce pathologies.

We will prove the following:

(1) For an arbitrary initial Gibbs measure and an arbitrary Glauber dynamics, both having finite range, the measure stays Gibbs in a small time interval, whose length depends on both the initial measure and the dynamics (Theorem 4.1). This result, though somewhat surprising, essentially comes from the fact that for small times the set of sites where a spin flip has occurred consists of "small islands" that are far apart in a "sea" of sites where no spin flip has occurred.

(2) For a high- or infinite-temperature initial Gibbs measure and a high- or infinite-temperature Glauber dynamics, the measure is Gibbs for all $t > 0$ (Theorems 5.11 and 6.15).

(3) For a low-non-zero-temperature initial Gibbs measure and a high- or infinite-temperature Glauber dynamics, there is a transition from Gibbs to non-Gibbs (Theorems 5.16 and 6.18). This result is somewhat counter-intuitive: after some time of heating the system it reaches a high temperature, where a priori we would expect the measure to be well-behaved because it should be exponentially close to a Completely Analytic (see [7]) high-temperature Gibbs measure. As we will see, this intuition is wrong. However, from the results of [29] it follows that this transition does not occur when the initial measure is a rigid ground state (zero-temperature) measure (i.e., a Dirac measure).

(4) For a low-non-zero-temperature initial Gibbs measure and a high- or infinite-temperature Glauber dynamics, there possibly is a transition back from non-Gibbs to Gibbs when the Hamiltonian of the initial Gibbs measure has a non-zero magnetic field (Theorems 5.16 and 6.18).

The complementary regimes, with a low- or zero-temperature Glauber dynamics acting over large times, are left open.

In Section 2 we start by giving some basic notations and definitions, and formulating some general facts.

In Section 3 we give representations of the conditional probabilities of the time-evolved measure and clarify the link between the Gibbsian character of the time-evolved measure and the Feller property of the backwards process. These results are useful for proving the "positive side", i.e., for showing that the time-evolved measure is Gibbsian. We use a criterion of [9], Chapter 4, Step 1, or [10] to identify *bad configurations* (points of essential discontinuity of every version of the conditional probabilities) as those configurations for which the *constrained* system (i.e., the measure at time 0 conditioned on the *future* bad configuration at time $t > 0$) exhibits a phase transition. This criterion will serve for the "negative side", i.e., for showing that the time-evolved measure is non-Gibbsian.

In Section 4 we prove that for an arbitrary initial measure and an arbitrary dynamics, both having finite-range interactions, the measure at time t is Gibbs for all $t \in [0, t_0]$, where t_0 depends on the interactions.

In Section 5 we treat the case of infinite-temperature dynamics, i.e., a product of independent Markov chains. This example already exhibits all the transitions between Gibbs and non-Gibbs we are after. Moreover, it has the advantage of fitting exactly in the framework of the renormalization-group transformations: the time-evolved measure is nothing but a single-site Kadanoff transform of the original measure, where the parameter $p(t)$ of this transform varies continuously from $p(0) = \infty$ to $p(\infty) = 0$. For the case of a low-temperature initial measure we restrict ourselves to the d-dimensional Ising model.

In Section 6 we show that the results of Section 5 also apply in the case of a high-temperature dynamics. The basic ingredient is a cluster expansion in space and time, as developed in [28] and worked out in detail in [25]. This is formulated in Theorem 6.3 and is the technical tool needed to develop the "perturbation theory" around the infinite-temperature case.

In Section 7 we give a dynamical interpretation of the transition from Gibbs to non-Gibbs in terms of a change in the *most probable history of an improbable configuration*. We show that the transition is not linked with a wrong behavior in the large deviations at fixed time, and we close by formulating a number of open problems.

2 Notations and definitions

2.1 Configuration space

The configuration space of our system is $\Omega = \{-1, +1\}^{\mathbb{Z}^d}$, endowed with the product topology. Elements of Ω are denoted by σ, η. A configuration σ assigns to each lattice point $x \in \mathbb{Z}^d$ a spin value $\sigma(x) \in \{-1, +1\}$. The set of all finite subsets of $\mathbb{Z}^d$ is denoted by $\mathcal{S}$. For $\Lambda \in \mathcal{S}$ and $\sigma \in \Omega$, we denote by σ_Λ the restriction of σ to Λ, while Ω_Λ denotes the set of all such restrictions. A function $f : \Omega \to \mathbb{R}$ is called local if there exists a finite set $\Delta \subset \mathbb{Z}^d$ such that $f(\eta) = f(\sigma)$ for σ and η coinciding on Δ. The minimal such Δ is called the dependence set of f and is denoted by D_f. The vector space of all local functions is denoted by $\mathcal{L}$. This is a uniformly dense subalgebra of the set of all continuous functions $\mathcal{C}(\Omega)$. A local function $f : \Omega \to \mathbb{R}$ with dependence set $D_f \subset \Lambda$ can be viewed as a function on Ω_Λ. With a slight abuse of notation we use f for both objects. For $\sigma, \eta \in \Omega$ and $\Lambda \subset \mathbb{Z}^d$, we denote by $\sigma_\Lambda \eta_{\Lambda^c}$ the configuration whose restriction to Λ (resp. Λ^c) coincides with σ_Λ (resp. η_{Λ^c}). For $x \in \mathbb{Z}^d$ and $\sigma \in \Omega$, we denote by $\tau_x \sigma$ the shifted configuration defined by $\tau_x \sigma(y) = \sigma(y+x)$. A sequence of probability measures μ_Λ on Ω_Λ is said to converge to a probability measure μ on Ω (notation $\mu_\Lambda \to \mu$) if

$$\lim_{\Lambda \uparrow \mathbb{Z}^d} \int f \, d\mu_\Lambda = \int f \, d\mu \qquad \forall f \in \mathcal{L}. \tag{2.1}$$

2.2 Dynamics

The dynamics we consider in this paper is governed by a collection of spin-flip rates $c(x, \sigma)$, $x \in \mathbb{Z}^d, \sigma \in \Omega$, satisfying the following conditions:

1. *Finite range*: $c_x : \sigma \mapsto c(x, \sigma)$ is a local function of σ for all x, with $\mathrm{diam}(D_{c_x}) \leq R < \infty$.
2. *Translation invariance*: $\tau_x c_0 = c_x$ for all x.
3. *Strict positivity* : $c(x, \sigma) > 0$ for all x and σ.

Note that these conditions imply that there exist $\epsilon, M \in (0, \infty)$ such that

$$0 < \epsilon \leq c(x, \sigma) \leq M < \infty \qquad \forall x \in \mathbb{Z}^d, \sigma \in \Omega. \tag{2.2}$$

Given the rates (c_x), we consider the generator defined by

$$Lf = \sum_{x \in \mathbb{Z}^d} c_x \nabla_x f \qquad \forall f \in \mathcal{L}, \tag{2.3}$$

where

$$\nabla_x f(\sigma) = f(\sigma^x) - f(\sigma). \tag{2.4}$$

Here, σ^x denotes the configuration defined by $\sigma^x(x) = -\sigma(x)$ and $\sigma^x(y) = \sigma(y)$ for $y \neq x$. In [23], Theorem 3.9, it is proved that the closure of L on $\mathcal{C}(\Omega)$ is the generator of a unique Feller process $\{\sigma_t : t \geq 0\}$. We denote by $S(t) = \exp(tL)$ the corresponding semigroup, by $\mathbb{P}_\sigma$ the path-space measure given $\sigma_0 = \sigma$, and by $\mathbb{E}_\sigma$ expectation over $\mathbb{P}_\sigma$.

A probability measure μ on the Borel σ-field of Ω is called *invariant* if

$$\int Lf d\mu = 0 \qquad \forall f \in \mathcal{L}. \tag{2.5}$$

It is called *reversible* if

$$\int (Lf) g d\mu = \int f (Lg) d\mu \qquad \forall f, g \in \mathcal{L}. \tag{2.6}$$

Reversibility implies invariance. For spin-flip dynamics with generator L defined by (2.3), reversibility of μ is equivalent to

$$c(x, \sigma^x) \frac{d\mu^x}{d\mu} = c(x, \sigma) \qquad \forall x \in \mathbb{Z}^d, \sigma \in \Omega, \tag{2.7}$$

where μ^x denotes the distribution of σ^x when σ is distributed according to μ. Note that (2.7) implies the existence of a continuous version of the Radon-Nikodým (RN)-derivative $d\mu^x / d\mu$. This will be important in the sequel.

2.3 Interactions and Gibbs measures

A *good* interaction is a function

$$U : \mathcal{S} \times \Omega \to \mathbb{R}, \tag{2.8}$$

such that the following two conditions are satisfied:

1. *Local potentials in the interaction:* $U(A, \sigma)$ depends on $\sigma(x), x \in A$, only.
2. *Uniform summability:*

$$\sum_{A \ni x} \sup_{\sigma \in \Omega} |U(A, \sigma)| < \infty \qquad x \in \mathbb{Z}^d. \tag{2.9}$$

The set of all good interactions will be denoted by $\mathcal{B}$. A good interaction is called *translation invariant* if

$$U(A + x, \tau_{-x}\sigma) = U(A, \sigma) \qquad \forall A \in \mathcal{S}, x \in \mathbb{Z}^d, \sigma \in \Omega. \tag{2.10}$$

The set of all translation-invariant good interactions is denoted by $\mathcal{B}_{ti}$. An interaction U is called *finite-range* if there exists an $R > 0$ such that $U(A, \sigma) = 0$ for all $A \in \mathcal{S}$ with $\mathrm{diam}(A) > R$. The set of all finite-range interactions is denoted by $\mathcal{B}^{fr}$ and the set of all translation-invariant finite-range interactions by $\mathcal{B}^{fr}_{ti}$. For $U \in \mathcal{B}$, $\zeta \in \Omega$, $\Lambda \in \mathcal{S}$, we define the finite-volume Hamiltonian with boundary condition ζ as

$$H^{\zeta}_{\Lambda}(\sigma) = \sum_{A \cap \Lambda \neq \emptyset} U(A, \sigma_{\Lambda}\zeta_{\Lambda^c}) \tag{2.11}$$

and the Hamiltonian with free boundary condition as

$$H_{\Lambda}(\sigma) = \sum_{A \subset \Lambda} U(A, \sigma), \tag{2.12}$$

which depends only on the spins inside Λ. Corresponding to the Hamiltonian in (2.11) we have the finite-volume Gibbs measures $\mu^{U,\zeta}_{\Lambda}$, $\Lambda \in \mathcal{S}$, defined on Ω by

$$\int f(\xi) \mu^{U,\zeta}_{\Lambda}(d\xi) = \sum_{\sigma_{\Lambda} \in \Omega_{\Lambda}} f(\sigma_{\Lambda}\zeta_{\Lambda^c}) \frac{\exp\left[-H^{\zeta}_{\Lambda}(\sigma)\right]}{Z^{\zeta}_{\Lambda}}, \tag{2.13}$$

where Z^{ζ}_{Λ} denotes the partition function normalizing $\mu^{U,\zeta}_{\Lambda}$ to a probability measure.

For a probability measure μ on Ω, we denote by μ^{ζ}_{Λ} the conditional probability distribution of $\sigma(x), x \in \Lambda$, given $\sigma_{\Lambda^c} = \zeta_{\Lambda^c}$. Of course, this object is only defined on a set of μ-measure one. For $\Lambda \in \mathcal{S}, \Gamma \in \mathcal{S}$ and $\Lambda \subset \Gamma$, we denote by $\mu_{\Gamma}(\sigma_{\Lambda}|\zeta)$ the conditional probability to find σ_{Λ} inside Λ, given that ζ occurs on $\Gamma \setminus \Lambda$. For $U \in \mathcal{B}$, we call μ a Gibbs measure with interaction U if its conditional probabilities coincide with the ones prescribed in (2.13), i.e., if

$$\mu^{\zeta}_{\Lambda} = \mu^{U,\zeta}_{\Lambda} \qquad \mu - a.s. \qquad \Lambda \in \mathcal{S}, \zeta \in \Omega. \tag{2.14}$$

We denote by $\mathcal{G}(U)$ the set of all Gibbs measures with interaction U. For any $U \in \mathcal{B}$, $\mathcal{G}(U)$ is a non-empty compact convex set. The set of all Gibbs measures is

$$\mathcal{G} = \bigcup_{U \in \mathcal{B}} \mathcal{G}(U). \tag{2.15}$$

Note that $\mathcal{G}$ is not a convex set, since for U and V in $\mathcal{B}_{ti}$, convex combinations of $\mathcal{G}(U)$ and $\mathcal{G}(V)$ are not in $\mathcal{G}$ unless $\mathcal{G}(U) = \mathcal{G}(V)$ (see [9] section 4.5.1).

Remark: We will often use the notation $H = \sum_A U(A, \cdot)$ for the "Hamiltonian" corresponding to the interaction U. This formal sum has to be interpreted as the collection of "energy differences", i.e., if σ and η agree outside a finite volume Λ, then:

$$H(\eta) - H(\sigma) = \sum_{A \cap \Lambda \neq \emptyset} [U(A, \eta) - U(A, \sigma)]. \tag{2.16}$$

Definition 2.17 *A measure μ is called* **Gibbsian** *if $\mu \in \mathcal{G}$, otherwise it is called* **non-Gibbsian**.

2.4 Gibbsian and non-Gibbsian measures

In this paper we study the time-dependence of the Gibbsian property of a measure under the stochastic evolution $S(t)$. In other words, we want to investigate whether or not $\nu S(t) \in \mathcal{G}$ at a given time $t > 0$.

Proposition 2.18 *The following three statements are equivalent:*

1. *$\mu \in \mathcal{G}$.*
2. *μ admits a continuous and strictly positive version of its conditional probabilities μ_Λ^ζ, $\Lambda \in \mathcal{S}, \zeta \in \Omega$.*
3. *μ admits a continuous version of the RN-derivatives $d\mu^x/d\mu$, $x \in \mathbb{Z}^d$.*

Proof. See [21] and [39]. ∎

We will mainly use item 3 and look for a continuous version of the RN-derivatives $d\mu^x/d\mu$ by approximating them uniformly with local functions.

A necessary and sufficient condition for μ not to be Gibbsian ($\mu \notin \mathcal{G}$) is the existence of a bad configuration, i.e., a point of essential discontinuity. This is defined as follows:

Definition 2.19 *A configuration $\eta \in \Omega$ is called* **bad** *for a probability measure μ if there exists $\epsilon > 0$ and $x \in \mathbb{Z}^d$ such that for all $\Lambda \in \mathcal{S}$ there exist $\Gamma \supset \Lambda$ and $\xi, \zeta \in \Omega$ such that:*

$$\left| \mu_\Gamma(\sigma(x) | \eta_{\Lambda \setminus \{x\}} \zeta_{\Gamma \setminus \Lambda}) - \mu_\Gamma(\sigma(x) | \eta_{\Lambda \setminus \{x\}} \xi_{\Gamma \setminus \Lambda}) \right| > \epsilon. \tag{2.20}$$

Note that in this definition only the finite-dimensional distributions of μ enter. It is clear that a bad configuration is a point of discontinuity of *every* version of the conditional probabilities of μ. Conversely, a measure that has no bad configurations is Gibbsian (see e.g. [27]).

2.5 Main question

Our starting points in this paper are the following ingredients:

1. **A translation invariant initial measure** $\nu \in \mathcal{G}(U_\nu)$, corresponding to a finite-range translation-invariant interaction $U_\nu \in \mathcal{B}_{ti}^{fr}$ as introduced in Section 2.3.

2. **A spin-flip dynamics**, with flip rates as introduced in Section 2.2. This dynamics has a *reversible* measure μ, which satisfies

$$\frac{d\mu^x}{d\mu} = \frac{c(x,\sigma)}{c(x,\sigma^x)}. \tag{2.21}$$

Hence, by Proposition 2.18 there exists an interaction $U_\mu \in \mathcal{B}$ such that $\mu \in \mathcal{G}(U_\mu)$. Since the rates are translation invariant and have finite range, this interaction can actually be chosen in $\mathcal{B}_{ti}^{fr}$ and satisfies (recall (2.11) and (2.14))

$$\frac{d\mu^x}{d\mu} = \exp\left(\sum_{A \ni x}[U_\mu(A,\sigma) - U_\mu(A,\sigma^x)]\right). \tag{2.22}$$

Without loss of generality we can take the rates $c(x,\sigma)$ of the form

$$c(x,\sigma) = \exp\left(\frac{1}{2}\sum_{A \ni x}[U_\mu(A,\sigma) - U_\mu(A,\sigma^x)]\right). \tag{2.23}$$

A finite-volume approximation of the rates in (2.23) that we will often use is given by

$$c_\Lambda(x,\sigma) = \exp\left[H_\Lambda^\mu(\sigma) - H_\Lambda^\mu(\sigma^x)\right], \tag{2.24}$$

where H_Λ^μ is the Hamiltonian with free boundary condition associated with the interaction U_μ (recall (2.12)). These rates generate a pure-jump process on $\Omega_\Lambda = \{-1,+1\}^\Lambda$ with generator

$$(L_\Lambda f)(\cdot) = \sum_{x\in\Lambda} c_\Lambda(x,\cdot)\nabla_x f(\cdot) \qquad \forall f \in \mathcal{L}. \tag{2.25}$$

Since $L_\Lambda f$ converges to Lf as $\Lambda \uparrow \mathbb{Z}^d$ for any local function $f \in \mathcal{L}$, the corresponding semigroup $S_\Lambda(t)$ converges strongly in the uniform topology on $\mathcal{C}(\Omega)$ to the semigroup $S(t)$, i.e., $S_\Lambda(t)f \to S(t)f$ as $\Lambda \uparrow \mathbb{Z}^d$ in the uniform topology for any $f \in C(\Omega)$. Therefore we have the following useful approximation result. Let ν be a probability measure on Ω and ν_Λ its restriction to Ω_Λ (viewed as a subset of Ω). Then

$$\lim_{\Lambda\uparrow\mathbb{Z}^d} \nu_\Lambda S_\Lambda(t) = \nu S(t), \tag{2.26}$$

where the limit is in the sense of (2.1). If $\nu \in \mathcal{G}(U_\nu)$ is a Gibbs measure, then we can replace the finite-volume restriction ν_Λ by the free-boundary-condition finite-volume Gibbs measure (in the case of no phase transition), or by the appropriate finite-volume Gibbs measure with generalized boundary condition that approximates ν (in the case of a phase transition).

The main question that we will address in this paper is the following:

Question:

Is $\nu S(t) = \nu_t$ a Gibbs measure?

In order to study this rather general question we have to distinguish between different regimes, as defined next.

Definition 2.27 *$U \in \mathcal{B}$ is a high-temperature interaction if*

$$\sup_{x\in\mathbb{Z}^d}\sum_{A\ni x}(|A|-1)\sup_{\sigma,\sigma'\in\Omega}|U(A,\sigma)-U(A,\sigma')|<2. \tag{2.28}$$

Equation (2.28) implies the Dobrushin uniqueness condition for the associated conditional probabilities $\mu_\Lambda^{U,\zeta}$, $\Lambda \in \mathcal{S}, \zeta \in \Omega$. In particular, it implies that $|\mathcal{G}(U)| = 1$ (i.e., no phase transition). Note that it is independent of the "single-site part" of the interaction, i.e., of the interactions $U(\{x\},\sigma)$.

Remark: We interpret the above norm as an inverse temperature, so small norm means high temperature.

Definition 2.29 *We call:*

1. *an initial measure ν "high-temperature" if it has an interaction satisfying (2.28), and write $T_\nu >> 1$.*
2. *an initial measure ν "infinite-temperature" if it is a product measure, (i.e., if the corresponding interaction U_ν satisfies $U_\nu(A,\sigma) = 0$ for all A with $|A| > 1$), and write $T_\nu = \infty$.*
3. *a dynamics "high-temperature" if the associated reversible Gibbs measure μ has an interaction U_μ satisfying (2.28), and write $T_\mu >> 1$.*
4. *a dynamics "infinite-temperature" if the associated reversible measure μ is a product measure (i.e., if the corresponding interaction U_μ satisfies $U_\mu(A,\sigma) = 0$ for all A with $|A| > 1$), and write $T_\mu = \infty$.*

As we will see in Section 5, the study of infinite-temperature dynamics is particularly instructive, since it can be treated essentially completely and already contains all the interesting phenomena we are after.

3 General facts

3.1 Representation of the RN-derivative

As summarized in Proposition 2.18, an object of particular use in the investigation of the Gibbsian character of a measure is its RN-derivative $d\mu^x/d\mu$ w.r.t. a spin flip at site x. In this section we show how to exploit the reversibility of the dynamics in order to obtain a sequence of continuous functions converging to the RN-derivative of the time-evolved measure $\nu_t = \nu S(t)$ w.r.t. spin flip. Let us first consider the finite-volume case. We start from the finite-volume generator

$$L_\Lambda f(\sigma) = \sum_{x\in\Lambda} c_\Lambda(x,\sigma)(f(\sigma^x) - f(\sigma)), \tag{3.1}$$

where the finite-volume rates $c_\Lambda(x,\cdot)$ are given by (2.24). Suppose that our starting measure $\nu \in \mathcal{G}(U_\nu)$ is such that $|\mathcal{G}(U_\nu)| = 1$, which implies that the free-boundary-condition finite-volume approximations ν_Λ converge to ν. The free-boundary-condition finite-volume Gibbs measure μ_Λ, corresponding to the interaction U_μ, is the reversible measure of the generator L_Λ. We can then compute, using reversibility,

$$\begin{aligned}
\frac{d\nu_\Lambda S_\Lambda(t)^x}{d\nu_\Lambda S_\Lambda(t)}(\sigma) &= \left(\frac{d\nu_\Lambda S_\Lambda(t)^x}{d\mu_\Lambda S_\Lambda(t)^x}(\sigma)\right)\left(\frac{d\mu_\Lambda S_\Lambda(t)^x}{d\mu_\Lambda S_\Lambda(t)}(\sigma)\right)\left(\frac{d\mu_\Lambda S_\Lambda(t)}{d\nu_\Lambda S_\Lambda(t)}(\sigma)\right)\\
&= \left(\frac{d\nu_\Lambda S_\Lambda(t)}{d\mu_\Lambda S_\Lambda(t)}(\sigma^x)\right)\left(\frac{d\mu_\Lambda^x}{d\mu_\Lambda}(\sigma)\right)\left(\frac{d\mu_\Lambda S_\Lambda(t)}{d\nu_\Lambda S_\Lambda(t)}(\sigma)\right)\\
&= \left[S_\Lambda(t)\left(\frac{d\nu_\Lambda}{d\mu_\Lambda}\right)(\sigma^x)\right]\left(\frac{d\mu_\Lambda^x}{d\mu_\Lambda}(\sigma)\right)\left[S_\Lambda(t)\left(\frac{d\nu_\Lambda}{d\mu_\Lambda}\right)(\sigma)\right]^{-1}. \qquad (3.2)
\end{aligned}$$

Definition 3.3 $H_\Lambda^{\mu,\nu}(\sigma) = \sum_{A\subset\Lambda}[U_\mu(A,\sigma) - U_\nu(A,\sigma)]$. *Note that this "difference Hamiltonian" depends on both the initial measure and the dynamics.*

Using this definition, we may rewrite (3.2) as

$$\frac{d\nu_\Lambda S_\Lambda(t)^x}{d\nu_\Lambda S_\Lambda(t)}(\sigma) = \frac{d\mu_\Lambda^x}{d\mu_\Lambda}(\sigma)\frac{\mathbb{E}^\Lambda_{\sigma^x}\left(\exp[H_\Lambda^{\mu,\nu}(\sigma_t)]\right)}{\mathbb{E}^\Lambda_{\sigma}\left(\exp[H_\Lambda^{\mu,\nu}(\sigma_t)]\right)}, \qquad (3.4)$$

where $\mathbb{E}^\Lambda_\sigma$ denotes the expectation for the process with semigroup $S_\Lambda(t)$ starting from σ. Since this semigroup converges to the semigroup $S(t)$ of the infinite-volume process as $\Lambda \to \mathbb{Z}^d$, we obtain the following:

Proposition 3.5 *For any $\sigma \in \Omega$ and $t \geq 0$,*

$$\frac{d\nu S(t)^x}{d\nu S(t)}(\sigma) = \frac{d\mu^x}{d\mu}(\sigma)\lim_{\Lambda\uparrow\mathbb{Z}^d}\frac{\mathbb{E}_{\sigma^x}\left(\exp[H_\Lambda^{\mu,\nu}(\sigma_t)]\right)}{\mathbb{E}_{\sigma}\left(\exp[H_\Lambda^{\mu,\nu}(\sigma_t)]\right)}, \qquad (3.6)$$

where this equality is to be interpreted as follows: if the limit in the RHS of (3.6) is a limit in the uniform topology, then it defines a continuous version of the LHS.

Proof. The claim follows from a combination of (2.26) and (3.4) with Lemma 3.7 below. ∎

Lemma 3.7 *If $\nu_n \to \nu$ weakly as $n \to \infty$, and $d\nu_n^x/d\nu_n \in \mathcal{C}(\Omega)$ exists for any $n \in \mathbb{N}$ and converges uniformly to a continuous function Ψ, then*

$$\Psi = \lim_{n\uparrow\infty}\frac{d\nu_n^x}{d\nu_n} = \frac{d\nu^x}{d\nu}. \qquad (3.8)$$

Proof. Let $f : \Omega \to \mathbb{R}$ be a continuous function. Define $\theta_x : \Omega \to \Omega$ by $\theta_x(\sigma) = \sigma^x$. Then also $f \circ \theta_x : \Omega \to \mathbb{R}$ is a continuous function. Therefore

$$\begin{aligned}
\int f d\nu^x &= \int (f\circ\theta_x(\sigma))\,\nu(d\sigma)\\
&= \lim_{n\uparrow\infty}\int (f\circ\theta_x(\sigma))\,\nu_n(d\sigma)\\
&= \lim_{n\uparrow\infty}\int \frac{d\nu_n^x}{d\nu_n}(\sigma)f(\sigma)\nu_n(d\sigma)\\
&= \lim_{n\uparrow\infty}\int \Psi(\sigma)f(\sigma)\nu_n(d\sigma)\\
&= \int \Psi f d\nu, \qquad (3.9)
\end{aligned}$$

where the fourth equality follows from

$$\lim_{n\uparrow\infty}\int\Big|\frac{d\nu_n^x}{d\nu_n}(\sigma)-\Psi(\sigma)\Big|f(\sigma)\nu_n(d\sigma)\leq\lim_{n\uparrow\infty}\|f\|_\infty\|\frac{d\nu_n^x}{d\nu_n}-\Psi\|_\infty=0. \tag{3.10}$$

Since (3.9) holds for any continuous function f, the statement of the lemma follows from the Riesz representation theorem. ∎

Proposition 3.5, combined with Proposition 2.18, will be used in Sections 4–6 to prove Gibbsianness.

3.2 Path-space representation of the RN-derivative

An alternative representation of the RN-derivative $d\nu_t^x/d\nu_t$ is obtained by observing that $\nu_t = \nu S(t)$ is the restriction of the path-space measure $\mathbb{P}_\nu^{[0,t]}$ to the "layer" $\{t\}\times\Omega$. In some sense, this path-space measure can be given a Gibbsian representation with the help of Girsanov's formula. The "relative energy for spin flip" of this path-space measure is a well-defined (though unbounded) random variable. Conditioning the path-space measure RN-derivative for a spin flip at site x in the layer $\{t\}\times\Omega$, we get the RN-derivative $d\nu_t^x/d\nu_t$. More formally, let us denote by π_t the projection on time t in path space, i.e., $\pi_t(\omega)=\omega_t$ with $\omega\in D([0,t],\Omega)$ the Skorokhod space. By a spin flip at site x in path space we mean a transformation

$$\Theta_x : D([0,t],\Omega)\to D([0,t],\Omega) \tag{3.11}$$

such that

$$(\pi_t(\omega))^x=\pi_t(\Theta_x(\omega)). \tag{3.12}$$

Different choices are possible, but in this section we choose

$$(\Theta_x(\omega))(s,y)=\begin{cases}-\omega(s,x) & \text{for } y=x,\ 0\leq s\leq t,\\ \omega(s,y) & \text{otherwise.}\end{cases} \tag{3.13}$$

Let $\mathcal{F}_{[t]}$ denote the σ-field generated by the projection π_t. Then we can write the following formula:

$$\frac{d\nu S(t)^x}{d\nu S(t)}=\mathbb{E}_\nu^{[0,t]}\left(\frac{d\mathbb{P}_\nu^{[0,t]}\circ\Theta_x}{d\mathbb{P}_\nu^{[0,t]}}\mid\mathcal{F}_{[t]}\right). \tag{3.14}$$

This equation is useful because of the Gibbsian form of the RHS of (3.14) given by Girsanov's formula, as shown in the proof of the following:

Proposition 3.15 *Let ν be a Gibbs measure on Ω. For any $t>0$,*

$$\nu S(t)^x << \nu S(t) \tag{3.16}$$

and the RN-derivative can be written in the form

$$\frac{d\nu S(t)^x}{d\nu S(t)}=\mathbb{E}_\nu^{[0,t]}\left[\left(\frac{d\nu^x}{d\nu}\circ\pi_0\right)\Psi_x\mid\mathcal{F}_{[t]}\right], \tag{3.17}$$

where $\Psi_x : D([0,t],\Omega)\to\mathbb{R}$ is a continuous function on path space (in the Skorokhod topology).

Proof. We first approximate our process by finite-volume pure-jump processes and use Girsanov's formula to obtain the densities of these processes w.r.t. the independent spin-flip process. Indeed, denote by $\mathbb{P}^{\Lambda}_{\sigma}$ the path-space measure of the finite-volume approximation with generator (2.25) and by $\mathbb{P}^{\Lambda,0}_{\sigma}$ the path-space measure of the independent spin-flip process in Λ, i.e., the process with generator

$$L^0_{\Lambda} f = \sum_{x\in\Lambda} \nabla_x f \qquad f \in \mathcal{L}. \tag{3.18}$$

We have for $f : \Omega \to \mathbb{R}$ such that $D_f \subset \Lambda$,

$$\begin{aligned} \int f(\sigma)\ \nu S(t)^x(d\sigma) &= \lim_{\Lambda\uparrow\mathbb{Z}^d} \int \nu(d\sigma) \int \mathbb{P}^{\Lambda}_{\sigma}(d\omega)\, f\left(\pi_t\left(\Theta_x(\omega)\right)\right) \\ &= \lim_{\Lambda\uparrow\mathbb{Z}^d} \int \nu(d\sigma) \int \mathbb{P}^{\Lambda,0}_{\sigma}(d\omega) \frac{d\mathbb{P}^{\Lambda}_{\sigma}}{d\mathbb{P}^{\Lambda,0}_{\sigma}}(\omega)\, f\left(\pi_t\left(\Theta_x(\omega)\right)\right). \end{aligned} \tag{3.19}$$

Since $\mathbb{P}^{\Lambda,0}_{\sigma}$ is the path-space measure of the independent spin-flip process, the transformed measure $\mathbb{P}^{\Lambda,0}_{\sigma} \circ \Theta_x$ equals $\mathbb{P}^{\Lambda,0}_{\sigma^x}$. Abbreviate

$$F_{\Lambda}(\omega) = \frac{d\mathbb{P}^{\Lambda}_{\omega_0}}{d\mathbb{P}^{\Lambda,0}_{\omega_0}}(\omega). \tag{3.20}$$

Then we obtain

$$\begin{aligned} &\int \nu(d\sigma) \int \mathbb{P}^{\Lambda,0}_{\sigma}(d\omega) F_{\Lambda}(\omega) f(\pi_t(\Theta_x(\omega))) \\ &= \int \nu(d\sigma) \int \mathbb{P}^{\Lambda,0}_{\sigma^x}(d\omega) F_{\Lambda}(\Theta_x(\omega)) f(\pi_t(\omega)) \\ &= \int \nu(d\sigma) \int \mathbb{P}^{\Lambda}_{\sigma^x}(d\omega) \frac{d\mathbb{P}^{\Lambda,0}_{\sigma^x}}{d\mathbb{P}^{\Lambda}_{\sigma^x}}(\omega) F_{\Lambda}(\Theta_x(\omega)) f(\pi_t(\omega)) \\ &= \int \nu(d\sigma) \frac{d\nu^x}{d\nu}(\sigma) \int \mathbb{P}^{\Lambda}_{\sigma}(d\omega) \left(\Psi_{x,\Lambda}(\omega) f(\pi_t(\omega))\right), \end{aligned} \tag{3.21}$$

where Ψ_{Λ} can be computed from Girsanov's formula (see [24] p. 314) and for Λ large enough reads

$$\Psi_{x,\Lambda}(\omega) = \exp\left[\sum_{|y-x|\le R} \int_0^t \log \frac{c(y,\omega^x_s)}{c(y,\omega_s)} dN^y_s(\omega) + \sum_{|y-x|\le R} \int_0^t [c(y,\omega_s) - c(y,\omega^x_s)] ds\right], \tag{3.22}$$

where $N^y_t(\omega)$ is the number of spin flips at site y up to time t along the trajectory ω. We thus obtain the representation of (3.17) by observing that $\Psi_{x,\Lambda}$ does not depend on Λ for Λ large enough and using the convergence of $\mathbb{P}^{\Lambda}_{\sigma}$ to $\mathbb{P}_{\sigma}$ as $\Lambda \uparrow \mathbb{Z}^d$. Indeed, the only point to check is that

$$\left(\frac{d\nu^x}{d\nu} \circ \pi_0\right) \Psi_x \in L^1(\mathbb{P}_{\nu}), \tag{3.23}$$

so that the conditional expectation in (3.17) is well-defined. However, this is a consequence of the following two observations:

1. $d\nu^x/d\nu$ is uniformly bounded because $\nu \in \mathcal{G}$.

2. For Ψ_x we have the bound

$$|\Psi_x(\omega)| \leq e^{2Ct} \left(\frac{M}{\epsilon}\right)^{N_t^{R,x}(\omega)}, \tag{3.24}$$

where, as before, M and ϵ are the maximum and minimum rates, $N_t^{R,x}(\omega)$ is the total number of spin flips in the region $\{y : |y - x| \leq R\}$ up to time t along the trajectory ω. Since the rates are bounded from above, the expectation of the RHS of (3.24) over $\mathbb{P}_\sigma$ is finite uniformly in σ.

∎

3.3 Backwards process

Proposition 3.15 provides us with a representation of the RN-derivative $d\nu_t^x/d\nu_t$ that can be interpreted as the expectation of a continuous function on path space *in the backwards process.* The backwards process is the Markov process with a time-dependent transition operator given by

$$(T_\nu(s,t)f)(\sigma) = \mathbb{E}_\nu(f \circ \pi_s | \sigma_t = \sigma) \qquad 0 \leq s \leq t, \tag{3.25}$$

where $(\cdot|\sigma_t = \sigma)$ is conditional expectation with respect to the σ-field at time t. Notice that this transition operator depends on the initial Gibbs measure ν and is a function of s and t (time-inhomogeneous process). Although the evolution has a reversible measure μ, at any finite time the distribution at time t is not μ. This causes essential differences between the forward and the backwards process.

The dependence of $T_\nu(s,t)$ on ν is crucial and shows that even for innocent dynamics, like the independent spin-flip process, the transition operators of the backwards process may fail to be Feller for certain choices of ν (see Section 5 below). In general, the independence of the Poisson clocks that govern where the spins are flipped (in the backwards process this means *were flipped*) is lost.

In order to have continuity of the RN-derivative $d\nu_t^x/d\nu_t$, it is sufficient that the operators $T_\nu(s,t)$ have the Feller property, i.e., map continuous functions to continuous functions.

Proposition 3.26 *If ν is a Gibbs measure, then:*

$$T_\nu(s,t)C(\Omega) \subset C(\Omega) \;\forall\; 0 \leq s < t \leq t_0 \qquad \Longrightarrow \qquad \nu S(t) \in \mathcal{G} \;\forall\; 0 \leq t \leq t_0. \tag{3.27}$$

Proof. This is an immediate consequence of Proposition 3.15. See also [20]. ∎

As in Section 3.1, we can thus hope to approximate the transition operators of the backwards process by "local operators" (operators mapping $\mathcal{L}$ onto $\mathcal{L}$).

Proposition 3.28 *For any $\sigma \in \Omega$ and $0 \leq s < t$,*

$$(T_\nu(s,t)f)(\sigma) = \lim_{\Lambda \uparrow \mathbb{Z}^d} \frac{\mathbb{E}_\sigma\left(\exp[H_\Lambda^{\mu,\nu}(\sigma_t)]f(\sigma_{t-s})\right)}{\mathbb{E}_\sigma\left(\exp[H_\Lambda^{\mu,\nu}(\sigma_t)]\right)}, \tag{3.29}$$

where this equality is to be interpreted as follows: if the limit in the RHS of (3.29) is a limit in the uniform topology, then it defines a continuous version of the LHS.

Proof. Let us first compute $T_\nu(s,t)$ in the case of the finite-volume reversible Markov chain with generator (2.25). For the sake of notational simplicity, we omit the indices Λ referring to the finite volume, and abbreviate $\nu_s = \nu S(s)$:

$$\begin{aligned}
(T_\nu(s,t)f)(\sigma) &= \sum_\eta p_{t-s}(\eta,\sigma)\frac{\nu_s(\eta)}{\nu_t(\sigma)}f(\eta) \\
&= \frac{\mu_t(\sigma)}{\nu_t(\sigma)}\sum_\eta p_{t-s}(\sigma,\eta)\frac{\nu_s(\eta)}{\mu_s(\eta)}f(\eta) \\
&= \left[S(t)\left(\frac{d\nu}{d\mu}\right)(\sigma)\right]^{-1}\sum_\eta p_{t-s}(\sigma,\eta)\left[S(s)\left(\frac{d\nu}{d\mu}\right)(\eta)\right]f(\eta) \\
&= \frac{S(t-s)\left(S(s)\left(\frac{d\nu}{d\mu}\right)f\right)}{S(t)\left(\frac{d\nu}{d\mu}\right)}(\sigma) \\
&= \frac{\mathbb{E}_\sigma\left(\exp[H^{\mu,\nu}_\Lambda(\sigma_t)]f(\sigma_{t-s})\right)}{\mathbb{E}_\sigma\left(\exp[H^{\mu,\nu}_\Lambda(\sigma_t)]\right)},
\end{aligned} \tag{3.30}$$

where $H^{\mu,\nu}_\Lambda$ is defined in Definition 3.3. ∎

Propositions 3.26 and 3.28 are the analogues of Propositions 2.18 and 3.5. We will not actually use them, but they provide useful insight.

3.4 Criterion for Gibbsianness of $\nu S(t)$

A useful tool to study whether $\nu S(t) \in \mathcal{G}$ is to consider the joint distribution of (σ_0, σ_t), where σ_0 is distributed according to ν. Let us denote this joint distribution by $\hat\nu_t$, which can be viewed as a distribution on $\{-1,+1\}^S$ with $S = \mathbb{Z}^d \oplus \mathbb{Z}^d$ consisting of two "layers" of $\mathbb{Z}^d$. The correspondence between $\hat\nu_t$ and $\nu S(t)$ is made explicit by the formula

$$\int \hat\nu_t(d\sigma, d\eta) f(\sigma) g(\eta) = \int \nu(d\sigma)(f S(t) g)(\sigma) \qquad f, g \in \mathcal{L}. \tag{3.31}$$

Now, for reasons that will become clear later, $\hat\nu_t$ has more chance of being Gibbsian than $\nu S(t)$. The latter can then be viewed as the restriction of a Gibbs measure of a two-layer system to the second layer. Restrictions of Gibbs measures have been studied e.g. in [37], [29] [10], [27], [26], and it is well-known that they can fail to be Gibbsian, and most examples of non-Gibbsian measures can be viewed as restrictions of Gibbs measures. Formally, the Hamiltonian of $\hat\nu_t$ is

$$H_t(\sigma,\eta) = H_\nu(\sigma) - \log p_t(\sigma,\eta), \tag{3.32}$$

where $p_t(\sigma,\eta)$ is the transition kernel of the dynamics. Of course, the object $\log p_t(\sigma,\eta)$ has to be interpreted in the sense of the formal sums $\sum_A U(A,\sigma)$ introduced in Section 2.3. More precisely, if $\delta_\sigma S(t)$ is a Gibbs measure for any σ, then $\log p_t(\sigma,\eta)$ is the Hamiltonian of this Gibbs measure. In order to prove or disprove Gibbsianness of the measure $\nu S(t)$, one has to study the Hamiltonian (3.32) for *fixed* η. Let us denote by $\mathcal{G}(H^t_\eta)$ the set of Gibbs measures associated with the Hamiltonian $H^t_\eta = H_t(\cdot,\eta)$. From [10] we have the following:

Proposition 3.33 *For any $t \geq 0$:*

1. *If $|\mathcal{G}(H_\eta^t)| = 1$ for all $\eta \in \Omega$, then $\nu S(t)$ is a Gibbs measure.*

2. *For monotone specifications, if $|\mathcal{G}(H_\eta^t)| \geq 2$, then η is a bad configuration for $\nu S(t)$, so $\nu S(t)$ is not a Gibbs measure (by Proposition 2.18).*

Proof. See [10]. Part 2 is expected to be true without the requirement of monotonicity but this has not been proved. ∎

A monotone specification arises e.g. when the Hamiltonian of (3.32) comes from a ferromagnetic pair potential and an arbitrary single-site part (possibly an inhomogeneous magnetic field).

In the case of a high-temperature dynamics ($T_\mu >> 1$), $\delta_\sigma S(t)$ converges to μ for any σ. This implies that for large t we can view the Hamiltonian of (3.32) as follows:

$$H_t(\sigma, \eta) = H_\nu(\sigma) + H_\mu(\eta) + o_{\sigma,\eta}(t), \tag{3.34}$$

where $o_{\sigma,\eta}(t)$ means some Hamiltonian with corresponding interaction converging to zero as $t \uparrow \infty$ in $\mathcal{B}$. Therefore, if H_ν does not have a phase transition, then H_η^t should not have a phase transition either for large t. On the other hand, if H_ν does have a phase transition, then the $o_{\sigma,\eta}(t)$-term will be important to *select one of the phases*. In Sections 5–6 we will come back to this description in more detail.

The case of independent spin flips corresponds to $H_\mu = 0$.

4 Conservation of Gibbsianness for small times

Having put the technical machinery in place in Sections 2–3, we are now ready to formulate and prove our main results in Sections 4–6.

In this section we prove that for every finite-range spin-flip dynamics starting from a Gibbs measure ν corresponding to a finite-range interaction the measure $\nu S(t)$ remains Gibbsian in a small interval of time $[0, t_0]$. The intuition behind this theorem is that for small times the set of sites where a spin flip has occurred consists of "small islands" that are far apart in a "sea" of sites where no spin flip has occurred. This means that sites that are far apart have more or less disjoint histories.

Theorem 4.1 *Let both the initial measure ν and the reversible measure μ be Gibbs measures for finite-range interactions U_ν resp. U_μ. Then there exists $t_0 = t_0(\mu, \nu) > 0$ such that $\nu S(t)$ is a Gibbs measure for all $0 \leq t \leq t_0$.*

Proof. During the proof we abbreviate $H_\Lambda = H_\Lambda^{\mu,\nu}$. We prove that the limit

$$\lim_{\Lambda \uparrow \mathbb{Z}^d} \frac{\mathbb{E}_{\sigma^x}\left(\exp[H_\Lambda(\sigma_t)]\right)}{\mathbb{E}_\sigma\left(\exp[H_\Lambda(\sigma_t)]\right)} \tag{4.2}$$

converges uniformly in $t \in [0, t_0]$ for t_0 small enough when $U_\nu, U_\mu \in \mathcal{B}^{fr}$. The t_0 depends on both U_ν and U_μ.

Let us write R_ν, R_μ to denote the range of U_ν, U_μ (see Section 2.2).

I: $R_\nu < \infty$**,** $R_\mu = 0$**.**

To warm up, we first deal with unbiased independent spin-flip dynamics. For this dynamics the distribution of σ_t under $\mathbb{P}^0_{\sigma^x}$ coincides with the distribution of σ^x_t under $\mathbb{P}^0_\sigma$. Therefore we can write

$$\begin{aligned}\frac{\mathbb{E}^0_\sigma\left(\exp[H_\Lambda(\sigma^x_t)]\right)}{\mathbb{E}^0_\sigma\left(\exp[H_\Lambda(\sigma_t)]\right)} &= \frac{\sum_{A\subset\Lambda}\delta_t^{|A|}(1-\delta_t)^{|\Lambda|-|A|}\exp[(H^{A\Delta\{x\}}-H)(\sigma)]}{\sum_{A\subset\Lambda}\delta_t^{|A|}(1-\delta_t)^{|\Lambda|-|A|}\exp([H^{A}-H)(\sigma)]}\\ &= \left(\frac{\sum_{A\subset\Lambda}\left(\frac{\delta_t}{1-\delta_t}\right)^{|A|}\exp[(H^{A\Delta\{x\}}-H^{\{x\}})(\sigma)]}{\sum_{A\subset\Lambda}\left(\frac{\delta_t}{1-\delta_t}\right)^{|A|}\exp[(H^{A}-H)(\sigma)]}\right)\Psi_x(\sigma), \qquad (4.3)\end{aligned}$$

where

$$\Psi_x(\sigma)=\exp[(H^{\{x\}}-H)(\sigma)] \qquad (4.4)$$

is a continuous function of σ, the sum runs over

$$A=\{y\in\Lambda:\sigma_t(y)\neq\sigma_0(y)\}, \qquad (4.5)$$

while

$$\delta_t=\mathbb{P}^0_\sigma(\sigma_t(x)\neq\sigma_0(x))=1-e^{-2t}. \qquad (4.6)$$

The notation H^A, $A\subset\Lambda$, is defined by

$$H^A(\sigma)=H(\sigma^A) \qquad (4.7)$$

with σ^A the configuration obtained from σ by flipping all the spins in A.

Suppose first that $R_\nu=1$. Then

$$H^{A\cup B}-H^A=H^B-H \qquad \forall\ A,B:\ d(A,B)>1. \qquad (4.8)$$

For $A\subset\Lambda$ we can decompose A into disjoint nearest-neighbor connected subsets $\gamma_1,\ldots,\gamma_k$ and thus rewrite (4.3) as follows:

$$\frac{\mathbb{E}^0_\sigma\left(\exp[H_\Lambda(\sigma^x_t)]\right)}{\mathbb{E}^0_\sigma\left(\exp[H_\Lambda(\sigma_t)]\right)}=\left(\frac{\sum_{n=0}^\infty\frac{1}{n!}\sum_{\gamma_1,\ldots,\gamma_n\subset\Lambda,\gamma_i\cap\gamma_j=\emptyset}\prod_{i=1}^n w^x_\sigma(\gamma_i)}{\sum_{n=0}^\infty\frac{1}{n!}\sum_{\gamma_1,\ldots,\gamma_n\subset\Lambda,\gamma_i\cap\gamma_j=\emptyset}\prod_{i=1}^n w_\sigma(\gamma_i)}\right)\Psi_x \qquad (4.9)$$

with

$$\begin{aligned}w^x_\sigma(\gamma) &= \epsilon_t^{|\gamma|}\exp[H^{\gamma\Delta\{x\}}(\sigma)-H^{\{x\}}(\sigma)]\\ w_\sigma(\gamma) &= \epsilon_t^{|\gamma|}\exp[H^{\gamma}(\sigma)-H(\sigma)]\end{aligned} \qquad (4.10)$$

and $\epsilon_t=\delta_t/(1-\delta_t)$. Note that $w^x_\sigma(\gamma)=w_\sigma(\gamma)$ for all γ that do not contain x.

Next, since

$$|(H^\gamma-H)(\sigma)|\le|\gamma|C \qquad (4.11)$$

with

$$C=2\sup_\Lambda\sup_\sigma\frac{|H_\Lambda(\sigma)|}{|\Lambda|}<\infty, \qquad (4.12)$$

we have the estimate

$$|w_\sigma(\gamma)|\le\exp(-\alpha_t|\gamma|) \qquad \text{with } \alpha_t=-C+\log(1/\epsilon_t). \qquad (4.13)$$

A similar estimate holds for $|w^x_\sigma(\gamma)|$. Since $\alpha_t \uparrow \infty$ as $t \downarrow 0$, it follows that for t small enough we can expand the logarithm of both the numerator and the denominator in (4.9) in a uniformly convergent cluster expansion:

$$
\begin{aligned}
\log\left(\sum_{n=0}^{\infty}\frac{1}{n!}\sum_{\gamma_1,\ldots,\gamma_n\subset\Lambda,\gamma_i\cap\gamma_j=\emptyset}\prod_{i=1}^{n} w^x_\sigma(\gamma_i)\right) &= \sum_{\Gamma} a(\Gamma)w^x_\sigma(\Gamma),\\
\log\left(\sum_{n=0}^{\infty}\frac{1}{n!}\sum_{\gamma_1,\ldots,\gamma_n\subset\Lambda,\gamma_i\cap\gamma_j=\emptyset}\prod_{i=1}^{n} w_\sigma(\gamma_i)\right) &= \sum_{\Gamma} a(\Gamma)w_\sigma(\Gamma). \qquad (4.14)
\end{aligned}
$$

By the estimate (4.13) we have, for t small enough,

$$\limsup_{\Lambda\uparrow\mathbb{Z}^d}\sum_{\Gamma\ni x,\Gamma\not\subset\Lambda}\sup_{\sigma}|a(\Gamma)(w^x_\sigma(\Gamma)-w_\sigma(\Gamma))| = 0 \qquad \forall x\in\mathbb{Z}^d \tag{4.15}$$

and hence we obtain uniform convergence of the limit in (4.2).

The case $R_\nu < \infty$ is treated in the same way. We only have to redefine the γ_i's as the R_ν-connected decomposition of A. Note that t_0 depends on R_ν and converges to zero when $R_\nu \uparrow \infty$.

II: $R_\nu < \infty$, $R_\mu < \infty$.

Next we prove that the limit (4.2) converges uniformly if both interactions U_μ, U_ν are finite range. For the sake of notational simplicity we first restrict ourselves to the case $R_\nu = R_\mu = 1$.

We abbreviate $U = U_\mu - U_\nu$. The idea is that we go back to the independent spin-flip dynamics via Girsanov's formula. After that we can again set up a cluster expansion, which includes additional factors in the weights due to the dynamics.

The first step is to rewrite (4.2) in terms of the independent spin-flip dynamics:

$$
\begin{aligned}
&\frac{\mathbb{E}_{\sigma^x}\left(\exp[H_\Lambda(\sigma_t)]\right)}{\mathbb{E}_{\sigma}\left(\exp[H_\Lambda(\sigma_t)]\right)} \qquad (4.16)\\
&= \frac{\mathbb{E}^0_\sigma\left(\exp\left(\sum_{y\in\Lambda}\int_0^t \log c(y,\sigma^x_s)dN^y_s + \int_0^t(1-c(y,\sigma^x_s))ds\right)\exp[H_\Lambda(\sigma^x_t)]\right)}{\mathbb{E}^0_\sigma\left(\exp\left(\sum_{y\in\Lambda}\int_0^t \log c(y,\sigma_s)dN^y_s + \int_0^t(1-c(y,\sigma_s))ds\right)\exp[H_\Lambda(\sigma_t)]\right)}.
\end{aligned}
$$

For a given realization ω of the independent spin-flip process, we say that a site y is ω-*active* if the spin at that site has flipped at least once. The set of all ω-active sites is denoted by $J(\omega)$. Let $\bar\sigma$ denote the trajectory that stays fixed at σ over the time interval $[0,t]$. For $A\subset\Lambda$, define

$$
\begin{aligned}
\mathcal{U}_1(A,\omega) &= \int_0^t \log c(y,\omega_s)dN^y_s(\omega) + \int_0^t(1-c(y,\omega_s))ds \quad \text{if } A = D_{c_y}\\
&= 0 \quad \text{if } A \neq D_{c_y}, \qquad (4.17)\\
\mathcal{U}_2(A,\omega) &= U(A,\omega_t),
\end{aligned}
$$

and put

$$\mathcal{U}(A,\omega) = \mathcal{U}_1(A,\omega) + \mathcal{U}_2(A,\omega). \tag{4.18}$$

Also define

$$\mathcal{U}^x(A,\omega) = \mathcal{U}(A,\omega^x), \tag{4.19}$$

where the trajectory ω^x is defined as

$$(\omega^x)_s = (\omega_s)^x \qquad 0 \le s \le t. \tag{4.20}$$

With this notation we can rewrite the right-hand side of (4.16) as

$$\left(\frac{\mathbb{E}^0_\sigma \left(\exp \left(\sum_{A \subset \Lambda} [\mathcal{U}^x(A,\omega) - \mathcal{U}^x(A,\bar{\sigma})] \right) \right)}{\mathbb{E}^0_\sigma \left(\exp \left(\sum_{A \subset \Lambda} [\mathcal{U}(A,\omega) - \mathcal{U}(A,\bar{\sigma})] \right) \right)} \right) \Psi_x(\sigma), \tag{4.21}$$

where

$$\Psi_x(\sigma) = \exp \left(\sum_{A \ni x} [\mathcal{U}(A,\bar{\sigma}) - \mathcal{U}(A,\bar{\sigma}^x)] \right) \tag{4.22}$$

is a continuous function of σ. In order to obtain the uniform convergence of (4.2), it suffices now to prove the uniform convergence of the expression between brackets in (4.21).

As in part I, we decompose the set of ω-active sites into disjoint nearest-neighbor connected sets $\gamma_1, \ldots, \gamma_k$ and rewrite, using the product character of $\mathbb{E}^0_\sigma$,

$$\begin{aligned} &\frac{\mathbb{E}^0_\sigma \left(\exp \left(\sum_{A \subset \Lambda} [\mathcal{U}^x(A,\omega) - \mathcal{U}^x(A,\bar{\sigma})] \right) \right)}{\mathbb{E}^0_\sigma \left(\exp \left(\sum_{A \subset \Lambda} [\mathcal{U}(A,\omega) - \mathcal{U}(A,\bar{\sigma})] \right) \right)} \\ &\qquad = \frac{\sum_{n=0}^\infty \frac{1}{n!} \sum_{\gamma_1,\ldots,\gamma_n \subset \Lambda, \gamma_i \cap \gamma_j = \emptyset} \prod_{i=1}^n w^x_\sigma(\gamma_i)}{\sum_{n=0}^\infty \frac{1}{n!} \sum_{\gamma_1,\ldots,\gamma_n \subset \Lambda, \gamma_i \cap \gamma_j = \emptyset} \prod_{i=1}^n w_\sigma(\gamma_i)}. \end{aligned} \tag{4.23}$$

The cluster weights are now given by

$$w_\sigma(\gamma) = e^{t|\gamma|} \mathbb{E}^0_\sigma \left(1\{J(\omega) \supset \gamma\} \exp \left(\sum_{A \cap \gamma \neq \emptyset} [\mathcal{U}(A, \omega_\gamma \bar{\sigma}_{\Lambda \setminus \gamma}) - \mathcal{U}(A,\bar{\sigma})] \right) \right), \tag{4.24}$$

and an analogous expression for w^x_σ after we replace $\mathcal{U}$ by $\mathcal{U}^x$. The factor $e^{t|\gamma|}$ arises from the probability

$$\mathbb{P}^0_\sigma \left(J(\omega)^c \supset \Lambda \setminus \cup_i \gamma_i \right) = e^{-t|\Lambda \setminus \cup_i \gamma_i|} = e^{-t|\Lambda|} \prod_i e^{t|\gamma_i|}. \tag{4.25}$$

Having arrived at this point, we can proceed as in the case of the independent spin-flip dynamics. Namely, we estimate the weights w_σ and prove that

$$w_\sigma(\gamma) \le e^{-\alpha_t |\gamma|} \tag{4.26}$$

with $\alpha_t \uparrow \infty$ as $t \downarrow 0$. To obtain this estimate, note that

$$\mathbb{P}^0_\sigma(J(\omega) \supset \gamma) \le (1 - e^{-t})^{|\gamma|}. \tag{4.27}$$

Then apply to (4.24) Cauchy-Schwarz, the bounds $\epsilon \le c_y \le M$ on the flip rates, and the estimate

$$C = \sup_\Lambda \sup_\sigma \frac{1}{|\Lambda|} \sum_{A \cap \Lambda \neq \emptyset} |U(A,\sigma)| < \infty, \tag{4.28}$$

to obtain

$$w_\sigma(\gamma) \le e^{Kt|\gamma|} (1 - e^{-t})^{-\frac{1}{2}|\gamma|} \qquad \text{for some } K = K(\epsilon, M, C). \tag{4.29}$$

This clearly implies (4.26).

The case $R_\nu, R_\mu < \infty$ is straightforward after redefining the γ_i's. ∎

5 Infinite-temperature dynamics

5.1 Set-up

In this section we consider the evolution of a Gibbs measure ν under a product dynamics, i.e., the flip rates $c(x,\sigma)$ depend only on $\sigma(x)$. The associated process $\{\sigma_t : t \geq 0\}$ is a product of independent Markov chains on $\{-1,+1\}$:

$$\mathbb{P}_\sigma = \otimes_{x\in\mathbb{Z}^d}\mathbb{P}_{\sigma(x)}, \tag{5.1}$$

where $\mathbb{P}_{\sigma(x)}$ is the Markov chain on $\{-1,+1\}$ with generator

$$L_x\varphi(\alpha) = c(x,\alpha)[\varphi(-\alpha)-\varphi(\alpha)]. \tag{5.2}$$

Let us denote by $p_t^x(\alpha,\beta)$ the probability for this Markov chain to go from α to β in time t. The Hamiltonian (3.32) of the joint distribution of (σ_0,σ_t) is then given by

$$H_t(\sigma,\eta) = H_\nu(\sigma) - \sum_x \log p_t^x(\sigma(x),\eta(x)). \tag{5.3}$$

This equation can be rewritten as

$$H_t(\sigma,\eta) = H_\nu(\sigma) - \sum_x h_1^x(t)\sigma(x) - \sum_x h_2^x(t)\eta(x) - \sum_x h_{12}^x(t)\sigma(x)\eta(x) \tag{5.4}$$

with

$$\begin{aligned} h_1^x(t) &= \frac{1}{4}\log\frac{p_t^x(+,+)p_t^x(+,-)}{p_t^x(-,+)p_t^x(-,-)} \\ h_2^x(t) &= \frac{1}{4}\log\frac{p_t^x(+,+)p_t^x(-,+)}{p_t^x(+,-)p_t^x(-,-)} \\ h_{12}^x(t) &= \frac{1}{4}\log\frac{p_t^x(+,+)p_t^x(-,-)}{p_t^x(+,-)p_t^x(-,+)}. \end{aligned} \tag{5.5}$$

The fields h_1^x resp. h_2^x tend to pull σ resp. η in their direction, while h_{12}^x is a *coupling* between σ and η that tends to align them. Indeed, note that $h_{12}^x(t)$ is positive because

$$p_t^x(+,+)p_t^x(-,-) - p_t^x(+,-)p_t^x(-,+) = \det(\exp(tL_x)) \geq 0. \tag{5.6}$$

In what follows we will consider the case where the single-site generators L_x are independent of x and are given by

$$L = \frac{1}{2}\begin{pmatrix} -1+\epsilon & 1-\epsilon \\ 1+\epsilon & -1-\epsilon \end{pmatrix} \qquad \text{for some } 0 \leq \epsilon < 1. \tag{5.7}$$

For $\epsilon > 0$ this means independent spin flips favoring plus spins, for $\epsilon = 0$ it means independent unbiased spin flips. The invariant measure of the single-site Markov chain is $(\nu(+),\nu(-)) = \frac{1}{2}(1+\epsilon, 1-\epsilon)$. The relevant parameter in what follows is

$$\delta = \frac{\nu(-)}{\nu(+)} = \frac{1-\epsilon}{1+\epsilon}. \tag{5.8}$$

In terms of this parameter the fields in (5.5) become

$$
\begin{aligned}
h_1(t) &= \frac{1}{4}\log\left(\frac{1+\delta e^{-t}}{1+\frac{1}{\delta}e^{-t}}\right) \\
h_2(t) &= -\frac{1}{2}\log\delta + h_1(t) \\
h_{12}(t) &= \frac{1}{4}\log\frac{(1+\delta e^{-t})(1+\frac{1}{\delta}e^{-t})}{(1-e^{-t})^2}.
\end{aligned} \tag{5.9}
$$

In particular, for $\delta = 1$ we get $h_1(t) = h_2(t) = 0$ and

$$
h_{12}(t) = \frac{1}{2}\log\frac{1+e^{-t}}{1-e^{-t}}. \tag{5.10}
$$

5.2 $1 << T_\nu \leq \infty$, $T_\mu = \infty$

Theorem 5.11 *Let ν be a high- or infinite-temperature Gibbs measure, i.e., its interaction U_ν satisfies (2.28). Let $S(t)$ be the semigroup of an arbitrary infinite-temperature dynamics. Then $\nu S(t)$ is a Gibbs measure for all $t \geq 0$.*

Proof. The joint distribution of (σ_0, σ_t) is Gibbs with Hamiltonian (recall (3.32) and (5.4))

$$
H_t(\sigma,\eta) = H_\nu(\sigma) + \sum_x [h_1(t) + h_{12}(t)\eta(x)]\sigma(x) + h_2(t)\sum_x \eta_x. \tag{5.12}
$$

For fixed η, the last term is constant in σ and can therefore be forgotten. Since $H_t(\cdot,\eta)$ differs from $H_\nu(\cdot)$ only in the single-site interaction, $H_t(\cdot,\eta)$ satisfies (2.28) if and only if $H_\nu(\cdot)$ satisfies (2.28). Hence $|\mathcal{G}(H_t(\cdot,\eta)| = 1$ for any η, and we conclude from Proposition 3.33 that $\nu S(t)$ is Gibbsian. ∎

Theorem 5.11 should not come as a surprise: the infinite-temperature dynamics act as a single-site Kadanoff transformation and in the Dobrushin uniqueness regime such renormalized measures stay Gibbsian [14], [18], [9].

5.3 $0 < T_\nu << 1$, $T_\mu = \infty$, $\delta = 1$

For the initial measure we choose the low-temperature plus-phase of the d-dimensional Ising model, $\nu = \nu_{\beta,h}$, i.e., the Hamiltonian H_ν is specified to be

$$
H_\nu(\sigma) = -\beta\sum_{<x,y>}\sigma(x)\sigma(y) - h\sum_x \sigma(x), \tag{5.13}
$$

where $\sum_{<x,y>}$ denotes the sum over nearest-neighbor pairs, and $\beta >> \beta_c$ with β_c the critical inverse temperature. The dynamics has generator

$$
Lf = \sum_x \nabla_x f, \tag{5.14}
$$

corresponding to the case $\delta = 1$. The joint measure has Hamiltonian as in (5.12), with $h_1(t) = h_2(t) = 0$ and $h_{12}(t) = h_t$:

$$
H_t(\sigma,\eta) = -\beta\sum_{<x,y>}\sigma(x)\sigma(y) - h\sum_x\sigma(x) - h_t\sum_x\sigma(x)\eta(x). \tag{5.15}
$$

The "dynamical field" is given by $h_t = -(1/2)\log[\tanh(t/2)]$.

Theorem 5.16 *For $\beta >> \beta_c$:*

1. *There exists a $t_0 = t_0(\beta, h)$ such that $\nu_{\beta,h}S(t)$ is a Gibbs measure for all $0 \le t \le t_0$.*
2. *If $h > 0$, then there exists a $t_1 = t_1(\beta, h)$ such that $\nu_{\beta,h}S(t)$ is a Gibbs measure for all $t \ge t_1$.*
3. *If $h = 0$, then there exists a $t_2 = t_2(\beta)$ such that $\nu_{\beta,0}S(t)$ is not a Gibbs measure for all $t \ge t_2$.*
4. *For $d \ge 3$, if $h \le h(\beta)$ small enough, then there exist $t_3 = t_3(\beta, h)$ and $t_4 = t_4(\beta, h)$ such that $\nu_{\beta,h}S(t)$ is not a Gibbs measure for all $t_3 \le t \le t_4$.*

Proof. The proof uses (5.15).

1. For small t the dynamical field h_t is large and, for *given* η, forces σ in the direction of η. Rewrite the joint Hamiltonian in (5.15) as

$$\begin{aligned} H_t(\sigma,\eta) &= \sqrt{h_t}\left(-\frac{\beta}{\sqrt{h_t}}\sum_{<x,y>}\sigma(x)\sigma(y) - \frac{h}{\sqrt{h_t}}\sum_x \sigma(x) - \sqrt{h_t}\sum_x \sigma(x)\eta(x)\right) \\ &= \sqrt{h_t}\tilde{H}_t(\sigma,\eta). \end{aligned} \tag{5.17}$$

For $0 \le t \le t_0$ small enough, $\tilde{H}_t$ has the unique ground state η and so, for $\lambda \ge \lambda_0$ large enough, $\lambda\tilde{H}_t$ satisfies (2.28) (see [13], example 2, p. 147). Therefore, for $0 \le t \le t_1$ such that $\sqrt{h_t} \ge \lambda_0$, $H_t(\cdot,\eta)$ has a unique Gibbs measure for any η. Hence, $\nu S(t)$ is Gibbs by Proposition 3.33(1).

2. For large t the dynamical field h_t is small and cannot cancel the effect of the external field $h > 0$. Rewrite the joint Hamiltonian as

$$\begin{aligned} H_t(\sigma,\eta) &= \sqrt{\beta}\left(-\sqrt{\beta}\sum_{<x,y>}\sigma(x)\sigma(y) - \frac{h}{\sqrt{\beta}}\sum_x \sigma(x) - \frac{h_t}{\sqrt{\beta}}\sum_x \sigma(x)\eta(x)\right) \\ &= \sqrt{\beta}\tilde{H}_t(\sigma,\eta). \end{aligned} \tag{5.18}$$

For $t \ge t_1$ large enough (independently of β), $\tilde{H}_t(\cdot,\eta)$ has the unique ground state $\sigma = h/|h|$. Hence, for β large enough, $\sqrt{\beta}\tilde{H}_t(\cdot,\eta)$ has a unique Gibbs measure by (2.28) (again, see [13], example 2, p. 147). Hence, $\nu S(t)$ is Gibbs by Proposition 3.33(1).

3. This fact is a consequence of the results in [9], section 4.3.4, for the single-site Kadanoff transformation. Since the joint Hamiltonian in (5.15) is ferromagnetic, it suffices to show that there is a special configuration η_{spec} such that $|\mathcal{G}(H(\cdot,\eta_{spec})| \ge 2$. We choose η_{spec} to be the alternating configuration. For $t \ge t_2$ large enough, $H_t(\cdot,\eta_{spec})$ has two ground states, and by an application of Pirogov-Sinai theory (see [9] Appendix B), it follows that, for β large enough, $|\mathcal{G}(H_t(\cdot,\eta_{spec})| \ge 2$. Therefore η_{spec} is a bad configuration for $\nu S(t)$, implying that $\nu S(t)$ is not Gibbs by Proposition 3.33(2).

4. In this case we rewrite the Hamiltonian in (5.15) as

$$H_t(\sigma,\eta) = -\beta\sum_{<x,y>}\sigma(x)\sigma(y) - \sum_x[h + h_t\eta(x)]\sigma(x). \tag{5.19}$$

For "intermediate" t we have that h and h_t are of the same order. As explained in [9] section 4.3.6, we can find a bad configuration η_{spec} such that the term $\sum_x h_t\eta(x)\sigma(x)$ in the Hamiltonian "compensates" the effect of the homogeneous-field term $\sum_x h\sigma(x)$ and for which $H_t(\cdot,\eta_{spec})$ has two ground states which are predominatly plus and minus. Since the proof of existence of η_{spec} requires analysis of the random field Ising model, we have to restrict to the case $d \geq 3$ (unlike the previous case η_{spec} is not constructed, but chosen from a measure one set). Then for β large enough, by a Pirogov-Sinai argument (see appendix B, Theorem B 31 of [9]) $|\mathcal{G}(H_t(\cdot,\eta_{spec})| \geq 2$, implying that $\nu S(t)$ is not Gibbs by Proposition 3.33(2). ∎

Remark:

From the estimate (B89) in [9], Appendix B, we can conclude the following:

1. $t_0(\beta,h) \to 0$ as $\beta \to \infty$ and $t_0(\beta,h) \to \infty$ as $h \to \infty$.
2. $t_2(\beta) \to 0$ as $\beta \to \infty$.
3. $t_3(\beta,h) \to 0$ as $\beta \to \infty$.

5.4 $0 < T_\nu << 1$, $T_\mu = \infty$, $\delta < 1$

Let us now consider a biased dynamics. At first sight one might expect this case to be analogous to the case of an unbiased dynamics with an initial measure having $h > 0$. However, this intuition is false.

Theorem 5.20 *The same results as in Theorem 5.16 hold, but with the t_i's also depending on δ. For item 4 we need the restrictions $d \geq 3$ and $|h + \frac{1}{4}\log\delta|$ small enough.*

Proof. The last term in (5.4) being irrelevant, we can drop it and study the Hamiltonian

$$\hat{H}_t(\sigma,\eta) = -\beta \sum_{<x,y>} \sigma(x)\sigma(y) - \sum_x \sigma(x)\left[(h + h_1(t)) + h_{12}(t)\eta(x)\right]. \tag{5.21}$$

This Hamiltonian is of the same form as (5.15), but with h becoming t-dependent. We have $\lim_{t\uparrow\infty} h_1(t) = 0$ and $\lim_{t\uparrow\infty} h_{12}(t) = 0$ with

$$\lim_{t\uparrow\infty} \frac{h_{12}(t)}{h_1(t)} = \frac{1+\delta}{1-\delta} > 1, \tag{5.22}$$

so that, in the regime where $\beta >> \beta_c$, $h = 0$, $t >> 1$, we find that the effect of $h_{12}(t)$ dominates. Hence we can find a special configuration that compensates the effect of the field $h_1(t)$ and for which the Hamiltonian (5.21) has two ground states, implying that $\nu S(t) \notin \mathcal{G}$. Similarly, when $h > 0$ we can find t intermediate such that $\sum_x (h+h_1(t))\sigma(x)$ is "compensated" by $\sum h_{12}(t)\sigma(x)\eta(x)$. ∎

Remark:

Note that if $T_\nu = 0$, $T_\mu = \infty$, then $\nu S(t)$ is a product measure for all $t > 0$ and hence is Gibbs.

6 High-temperature dynamics

6.1 Set-up

In this section we generalize our results in Section 5 for the infinite-temperature dynamics to the case of a high-temperature dynamics. The key technical tool is a cluster expansion that allows us to obtain Gibbsianness of the joint distribution of (σ_0, σ_t) with a Hamiltonian of the form (3.32). The main difficulty is to give meaning to the term $\log p_t(\sigma, \eta)$, i.e., to obtain Gibbsianness of the measure $\delta_\sigma S(t)$ for any σ. In the whole of this section we will assume that the rates $c(x, \sigma)$ satisfy the conditions in Section 2.2 and, in addition,

$$c(x, \sigma) = 1 + \epsilon(x, \sigma) \tag{6.1}$$

with

$$\begin{aligned} &\sup_{\sigma,x} |\epsilon(x, \sigma)| = \delta << 1 \\ &\epsilon(x, \sigma) = \epsilon(x, -\sigma). \end{aligned} \tag{6.2}$$

The latter corresponds to a high-temperature unbiased dynamics, i.e., a small unbiased perturbation of the unbiased independent spin-flip process. For the initial measure we consider two cases:

1. A high- or infinite-temperature Gibbs measure ν. In that case we will find that $\nu S(t)$ stays Gibbsian for all $t > 0$.
2. The plus-phase of the low-non-zero-temperature d-dimensional Ising model, $\nu_{\beta,h}$, corresponding to the Hamiltonian in (5.13). In that case we will find the same transitions as for the infinite-temperature dynamics.

6.2 Representation of the joint Hamiltonian

In this section we formulate the main result of the space-time cluster expansion in [28] and [25]. We indicate the line of proof of this result, and refer the reader to [25] for the complete details.

Theorem 6.3 *Let ν be a Gibbs measure with Hamiltonian H_ν, and let the dynamics be governed by rates satisfying (6.1–6.2). Then the joint distribution of (σ_0, σ_t), when σ_0 is distributed according to ν, is a Gibbs measure with Hamiltonian*

$$H_t(\sigma, \eta) = H_\nu(\sigma) + H^t_{dyn}(\sigma, \eta). \tag{6.4}$$

The Hamiltonian $H^t_{dyn}(\sigma, \eta)$ corresponds to an interaction $U^t_{dyn}(A, \sigma, \eta)$, $A \in \mathcal{S}$, that has the following properties:

1. *The interaction splits into two terms*

$$U^t_{dyn} = U^t_0 + U^t_\delta, \tag{6.5}$$

where U^t_0 is the single-site potential corresponding to the Kadanoff transformation:

$$\begin{aligned} U^t_0(\{x\}, \sigma, \eta) &= -\tfrac{1}{2}\log[\tanh(t/2)]\sigma(x)\eta(x) \quad && x \in \mathbb{Z}^d, \\ U^t_0(A, \sigma, \eta) &= 0 && \textit{if } |A| \neq 1. \end{aligned} \tag{6.6}$$

2. The term $U^t_\delta = U^t_\delta(A,\sigma,\eta)$ decays exponentially in the diameter of A, i.e., there exists $\alpha(\delta) > 0$ such that

$$\sup_{t\geq 0}\sup_x \sum_{A\ni x}\sup_{\sigma,\eta} e^{\alpha(\delta)\mathrm{diam}(A)}|U^t_\delta(A,\sigma,\eta)| < \infty. \tag{6.7}$$

and $\alpha(\delta)\uparrow\infty$ as $\delta\downarrow 0$.

3. The potential U^t_{dyn} converges exponentially fast to the potential U_μ of the high-temperature reversible Gibbs measure:

$$\lim_{t\uparrow\infty}\sup_x \sum_{A\ni x}\sup_{\sigma,\eta} e^{\alpha(\delta)\mathrm{diam}(A)}|U^t_\delta(A,\sigma,\eta) - U_\mu(A,\eta)| = 0. \tag{6.8}$$

4. The term U^t_δ is a perturbation of the term U^t_0, i.e.,

$$\lim_{\delta\downarrow 0}\sup_{t\geq 0}\frac{\sup_x\sum_{A\ni x}\sup_{\sigma,\sigma',\eta}|U^t_\delta(A,\sigma,\eta) - U^t_\delta(A,\sigma',\eta)|}{\log[\tanh(t/2)]} = 0. \tag{6.9}$$

Remarks:

1. Equation (6.6) corresponds to the infinite-temperature dynamics (i.e., $c\equiv 1$).
2. Equation (6.9) expresses that the potential as a function of the rates c is continuous at the point $c\equiv 1$, and that the Kadanoff term is dominant for $\delta << 1$.

Main steps in the proof of Theorem 6.3 in [25]:

- **Discretization:** The semigroup $S(t)$ can be approximated in a strong sense by discrete-time probabilistic cellular automata with transition operator of the form $P_n(\sigma'|\sigma) = \prod_x P_n(\sigma'(x)|\sigma)$, where

$$P_n(\sigma'(x)|\sigma) = \left[1 - \frac{1}{n}c(x,\sigma)\right]\delta_{\sigma'(x),\sigma(x)} + \frac{1}{n}c(x,\sigma)\delta_{\sigma'(x),-\sigma(x)}. \tag{6.10}$$

- **Space-time cluster expansion for fixed discretization n:** For n fixed the quantity

$$\Psi^x_n(\sigma,\eta) = \log\frac{(d\delta_\sigma P_n^{\lfloor nt\rfloor})^x}{(d\delta_\sigma P_n^{\lfloor nt\rfloor})} \tag{6.11}$$

  is defined by the convergent cluster expansion

$$\Psi^x_n(\sigma,\eta) = \sum_{\Gamma\ni x,\Gamma\in\mathcal{C}} w^{x,n}_{\sigma,\eta}(\Gamma), \tag{6.12}$$

  where $\mathcal{C}$ is an appropriate set of clusters on $\mathbb{Z}^{d+1}$.

- **Uniformity in the discretization n:** The functions Ψ^x_n converge uniformly as $n\uparrow\infty$ to a continuous function Ψ^x (which defines a continuous version of $d\mu^x/d\mu$). This is shown in two steps:

1. **Uniform boundedness:**

$$\sup_n \sup_x \sup_{\sigma,\eta} |\Psi_n^x(\sigma,\eta)| < \infty. \tag{6.13}$$

2. **Uniform continuity:**

$$\lim_{\Lambda\uparrow\mathbb{Z}^d} \sup_{\zeta,\xi} \sup_x \sup_n |\Psi_n^x(\sigma_\Lambda\zeta_{\Lambda^c},\eta_\Lambda\xi_{\Lambda^c}) - \Psi_n^x(\sigma,\eta)| = 0 \qquad \forall\sigma,\eta\in\Omega. \tag{6.14}$$

Equations (6.13) and (6.14) imply that Ψ_n^x as a function of n contains a uniformly convergent subsequence. The limiting Ψ^x is independent of the subsequence, since it is a continuous version of $d\mu^x/d\mu$.

6.3 $1 << T_\nu \leq \infty$, $1 << T_\mu < \infty$

Given the result of Theorem 6.3, the case of a high- or infinite-temperature initial measure is dealt with via Dobrushin's uniqueness criterion (recall Theorem 5.11 in Section 5.2).

Theorem 6.15 *Let ν be a high-temperature Gibbs measure, i.e., its interaction U_ν satisfies (2.28). Let the rates satisfy (6.1–6.2). Then, for δ small enough, $\nu S(t)$ is a Gibbs measure for all $t \geq 0$.*

Proof. For fixed η, the Hamiltonian $H_t(\cdot,\eta)$ of (6.4) corresponds to an interaction $U_t^{\eta,\delta}$. By (6.7) and (6.9), this interaction satisfies

$$\begin{aligned}\lim_{\delta\downarrow 0} \sup_t \sup_x \sum_{A\ni x} (|A|-1) \sup_{\sigma,\sigma'} |U_t^{\eta,\delta}(\sigma) - U_t^{\eta,\delta}(\sigma')| \\ = \sum_{A\ni x}(|A|-1)|U_\nu(\sigma) - U_\nu(\sigma')| < 2.\end{aligned} \tag{6.16}$$

Therefore, for δ small enough, (2.28) is satisfied for the interaction $U_t^{\eta,\delta}$ for all $t\geq 0$ and all η. Hence $|\mathcal{G}(H_t(\cdot,\eta))| = 1$, and we conclude from Proposition 3.33(1) that $\nu S(t) \in \mathcal{G}$. ∎

6.4 $0 < T_\nu << 1$, $1 << T_\mu < \infty$.

We consider as the initial measure the plus-phase of the low-temperature Ising model $\nu_{\beta,h}$, introduced in Section 5.3. The joint distribution of (σ_0,σ_t) has the Hamiltonian

$$H_t(\sigma,\eta) = -\beta \sum_{<x,y>} \sigma(x)\sigma(y) - h\sum_x \sigma(x) - \frac{1}{2}\log[\tanh(t/2)]\sum_x \sigma(x)\eta(x) + H_t^\delta(\sigma,\eta), \tag{6.17}$$

where H_t^δ corresponds to the interaction U_{dyn}^δ introduced in (6.5). The following is the analogue of Theorem 5.16

Theorem 6.18 *For $\beta >> \beta_c$ and $0 < \delta << 1$:*

1. *There exists $t_0 = t_0(\beta,h,\delta)$ such that $\nu_{\beta,h}S(t)$ is a Gibbs measure for all $0 \leq t \leq t_0$.*

2. *If $h > 0$, then there exists $t_1 = t_1(\beta, h, \delta)$ such that $\nu_{\beta,h}S(t)$ is a Gibbs measure for all $t \geq t_1$.*

3. *If $h = 0$, then there exists $t_2 = t_2(\beta, \delta)$ such that $\nu_{\beta,0}S(t)$ is not a Gibbs measure for all $t > t_2$.*

4. *For $d \geq 3$, if $0 < h < h(\beta)$ and $0 < \delta < \delta(\beta, h)$, then there exists $t_3(\beta, h, \delta), t_4(\beta, h, \delta)$ such that $\nu_{\beta,h}S(t)$ is not a Gibbs measure for all $t \in [t_3, t_4]$.*

Proof.

1. This a consequence of Theorem 4.1.

2. This is proved in exactly the same way as the corresponding point in Theorem 5.16.

3. Here we cannot rely on monotonicity as was the case in Theorem 5.16. It is therefore not sufficient to show that for the fully alternating configuration η^a, the Hamiltonian $H(\cdot, \eta^a)$ exhibits a phase transition. We have to show the following slightly stronger fact: if $m_\Lambda^+(d\sigma)$ is any Gibbs measure corresponding to the interaction $H(\cdot, \eta_\Lambda^a +_{\Lambda^c})$, then

$$\int m_\Lambda^+(d\sigma)\sigma(0) > \gamma > 0. \tag{6.19}$$

This proof of this fact relies on Pirogov-Sinai theory for the Hamiltonian $H_t(\cdot, \eta_\Lambda^a +_{\Lambda^c})$. The first step is to prove that the all-plus-configuration is the unique ground state of this Hamiltonian. Since the Ising Hamiltonian satisfies the Peierls condition, we conclude from [9] Proposition B.24 that the set of ground states of $H_t(\cdot, \eta_\Lambda^a +_\Lambda^c)$ is a subset of $\{+, -\}$. If we drop the term $H_t^\delta(\cdot, \eta_\Lambda^a +_{\Lambda^c})$ (i.e., if $\delta = 0$), then the remaining Hamiltonian has as the unique ground state the all-plus-configuration and satisfies the Peierls condition. Therefore, for δ small enough, we conclude from [9] Proposition B.24 that $H_t(\cdot, \eta_\Lambda^a +_{\Lambda^c})$ has the all-plus-configuration as the only possible ground state. From (6.17) it is easy to verify that the all-plus-configuration is actually a ground state for δ small enough. In order to conclude that for β large enough, the unique phase of $H_t(\cdot, \eta_\Lambda^a +_{\Lambda^c})$ is a weak perturbation of the all plus configuration (uniformly in Λ), we can rely on the theory developed in [3], or [6] which allows exponentially decaying perturbations of a finite range interaction satisfying the Peierls condition (see e.g. equations (1.3),(2.2) of [3]). Similarly, $H_t(\cdot, \eta_\Lambda^a -_{\Lambda^c})$ has a unique phase which is a weak perturbation of the all minus configuration. This is sufficient to conclude that no version of the conditional probabilities is continuous at η^a, see the discussion [9] p. 980-981.

4. We can use the same argument as developed in [9], section 4.3.6, introducing a random perturbation of the alternating configuration to "compensate the uniform magnetic field" (since this requires analysis of the random field Ising model, we have the restriction $d \geq 3$). The only complication is the extra term in the Hamiltonian arising from $\delta \neq 0$. This requires Pirogov Sinai theory for the interaction $H_t(\sigma, \eta)$, where $\eta = \eta^\epsilon$ is a random modification of the fully alternating configuration obtained by flipping the spins in the alternating configuration with probability $\epsilon/2\beta$ for a flip from + to $-$. Since the couplings between η and σ are not finite range, we cannot apply directly Theorem B31 of [9] for the random Hamiltonian $H_t(\sigma, \eta^\epsilon)$. However, as the interaction decays exponentially fast and Pirogov-Sinai analyses do not distinguish between finite range and exponentially decaying interactions, similar arguments as those developed in [40]

still work in our case and yield the analogue of Theorem B31 of [9]. However we have not written out the details.

∎

Remark:

A result related to Theorem 6.15 was obtained in [28]. Although the abstract of that paper is formulated in a somewhat ambiguous manner, its results apply only to initial measures which are product measures (in particular Dirac measures) . In particular this includes the case $T_\nu = 0$ and $1 << T_\mu < \infty$. The results of [28] (or [25]) then imply that the measure is Gibbs for all $t > 0$. This seems surprising, because $t_2(\beta, \delta) \downarrow 0$ as $\beta \uparrow \infty$. It is therefore better for the intuition to imagine a Dirac-measure as a product measure than to view it as a limit of low-temperature measures.

7 Discussion

7.1 Dynamical interpretation

In the case of renormalization-group pathologies, the interpretation of non-Gibbsianness is usually the presence of a *hidden* phase transition in the original system conditioned on the image spins (the constrained system). In the context of the present paper, we would like to view the phenomenon of transition from Gibbs to non-Gibbs as a *change in the choice of most probable history of an improbable configuration at time* $t > 0$.

To that end, let us consider the case of the low-temperature plus-phase of the Ising model in zero magnetic field ($\beta >> \beta_c$, $h = 0$) with an unbiased ($\delta = 1$) infinite-temperature dynamics. Consider the spin at the origin at time t conditioned on a neutral (say alternating) configuration in a *sufficiently large annulus* Λ around it. For small times the occurrence of such an improbable configuration indicates that with overwhelming probability a configuration very similar was present already at time 0. As the initial measure is an Ising Gibbs measure, the distribution at time 0 of the spin at the origin is determined by its local environment only and does not depend on what happens outside the annulus Λ. As all spins flip independently, no such dependence can appear within small times.

However, after a sufficient amount of time (larger than the transition time t_2), if the same improbable configuration is observed, then it has much more chance of being recently created (due to atypical fluctuations in the spin-flip processes) than of being the survivor of an initial state of affairs. Indeed, to have been there at time 0 is improbable, but to have survived for a large time is even more improbable. Suppose now that outside the annulus Λ we observe an *enormous annulus* Γ in which the magnetization is more negative than $-m^*(t)/2$, where $m^*(t)$ is the value of the evolved magnetization (which starts from $m^*(0)$ and decays exponentially fast to zero). Because a large droplet of the minus-phase shrinks only at finite speed and typically carries a magnetization characteristic of the evolved minus-phase, with large probability there was an *enormous droplet* of the minus phase (even a bit larger than Γ) at time 0, which the spin at the origin remembers. Indeed, the probability of this happening is governed by the size of the *surface* of Γ. In contrast, the probability of a large negatively magnetized droplet, arising through a large fluctuation in the spin-flip process starting from a typical plus-phase configuration, is governed by the *volume* of Γ. Therefore, this second scenario can safely be forgotten. Although for any size of the initial droplet of the minus-phase there is a time after which it has shrunk away, for each fixed time t we can choose an initial droplet size such that at time t it has shrunk no more than to size Γ. Since we want the shrinkage until time t to be negligible with respect to the linear size of Γ, we need to choose Γ larger when t is larger.

Thus, the transition reflects a changeover between two improbable histories for seeing an improbable (alternating) annulus configuration. It can be viewed as a kind of large deviation phenomenon for a time-inhomogeneous system. One could alternatively describe it by saying that for small times a large alternating droplet must have occurred at time 0, while after the transition time t_2 a large alternating droplet must have been created by the random spin-flips: a *"nature to nurture" transition* [35]. The mathematical analysis of this interpretation would rely on finding the (constrained) minimum of an entropy function on the space of trajectories. Alternatively, one could try to study the large deviation rate function for the magnetization of the measure at time 0 conditioned on an alternating configuration at time t. This rate function should exhibit a unique minimum for $0 \leq t < t_2$ and two minima for $t > t_2$.

7.2 Large deviations

A measure can be non-Gibbsian for different reasons (see [9], section 4.5.5) One of the possibilities is having "wrong large deviations", i.e., the probability

$$\nu S(t)\left(\sum_{x\in\Lambda} \tau_x f(\sigma) \simeq \alpha\right) \tag{7.1}$$

for fixed t and $\alpha \neq \int S(t) f d\nu$ does not decay exponentially in $|\Lambda|$, i.e., not as $\exp[-|\Lambda| I_f(\alpha) + o(|\Lambda|)]$, or equivalently, there exists a function $f \in \mathcal{L}$, $f \geq 0$, $f \neq 0$ such that

$$\lim_{\Lambda\uparrow\mathbb{Z}^d} \frac{1}{|\Lambda|} \log \int \nu S(t)(d\sigma) \exp\left[\sum_{x\in\Lambda} \tau_x f(\sigma)\right] = 0. \tag{7.2}$$

An example where this phenomenon of "wrong large deviations" occurs is the stationary measure of the voter model (see e.g. [22]). However, it does not occur in our setting. Namely, if the scale of the large deviations of the random measure $L_\Lambda = \sum_{x\in\Lambda} \delta_{\tau_x \sigma}$ under ν is the volume $|\Lambda|$, then the same holds under $\nu S(t)$ for any $t > 0$. Indeed, by Jensen's inequality and by the translation invariance of the dynamics we have, for $f \in \mathcal{L}$, $f \geq 0$, $f \neq 0$,

$$\begin{aligned}
&\limsup_{\Lambda\uparrow\mathbb{Z}^d} \frac{1}{|\Lambda|} \log \int \nu S(t)(d\sigma) \exp\left[\sum_{x\in\Lambda} \tau_x f(\sigma)\right] \\
&\quad \geq \limsup_{\Lambda\uparrow\mathbb{Z}^d} \frac{1}{|\Lambda|} \log \int \nu(d\sigma) \exp\left[\sum_{x\in\Lambda} \tau_x S(t) f(\sigma)\right] \\
&\quad = \sup_{\mu} \left[\int S(t) f d\mu - h(\mu|\nu)\right] \\
&\quad > 0
\end{aligned} \tag{7.3}$$

with $h(\cdot|\cdot)$ denoting relative entropy density. The equality follows from the volume-scale large deviations of ν, and the last inequality follows from the fact that $S(t)f \in \mathcal{C}(\Omega)$, $S(t)f \geq 0$, $S(t)f \neq 0$ imply $\int S(t) f d\nu > 0$.

7.3 Reversibility

Throughout the whole paper, we have assumed the stationary measure μ to be reversible. However, this is a condition that only serves to make formulas nicer. It is not at all a necessary condition: if we consider any high-temperature spin-flip dynamics, then we know that the stationary measure μ is a high-temperature Gibbs-measure. Equation (3.2) can be rewritten in the general situation: we have to replace $S_\Lambda(t)$ in the right-hand side by $S^*_\Lambda(t)$, where $S^*(t)$ is the semigroup corresponding to the rates of the reversed process, i.e., the rates

$$c^*(x,\sigma) = c(x,\sigma^x)\frac{d\mu^x}{d\mu}. \tag{7.4}$$

In all the formulas of Section 2, we then have to replace $\mathbb{E}_\sigma$ by $\mathbb{E}^*_\sigma$, referring to expectation in the process with semigroup $S^*(t)$.

7.4 Open problems

1. **Infinite-range interactions.** How much can we save when relaxing the condition that the interactions be finite-range?

2. **Trajectory of the interaction.** In the regime $1 << T_\nu \leq \infty$, $1 << T_\mu \leq \infty$, what can we say about the trajectory $t \mapsto U_t$? It is not hard to prove that it is analytic in $\mathcal{B}_{ti}$ and converges to U_μ. But can we say something about the rate of convergence? Note that we can view the *curve* $\{U_{\nu_t} : t \geq 0\}$ as a continuous trajectory in the space $\mathcal{B}$, interpolating between U_ν and U_μ, which implies that $\mathcal{G}$ contains an arc-connected subset. Other topological characteristics of $\mathcal{G}$ are discussed in [9], section 4.5.6.

3. **Uniqueness of the transitions.** Even in the case $T_\mu = \infty$ we have not proved that the transition from Gibbs to non-Gibbs is unique e.g. that $t_0(\beta, 0) = t_2(\beta)$ in Theorem 5.16. However, we expect that when $h = 0$ the alternating configuration is "the worst configuration", i.e., the transition is sharp and occurs at the first time at which the alternating configuration is bad.

4. **Estimates for the transition times.** Can we find good estimates for the t_i's as a function of e.g. the temperatures, the magnetic fields and the ranges of the interaction in ν and μ.

5. **Weak Gibbsianness.** In the regimes where $\nu S(t)$ is not a Gibbs measure we expect that we can still define a $\nu S(t)$-a.s. converging interaction U_t for which $\nu S(t)$ is a "weakly Gibbsian measure" (see [8], [27]). This interaction U_t can e.g. be constructed along similar lines as are followed in the proof of Kozlov's theorem (see [21],[26]) and its convergence is to be controlled by the decay of "quenched correlations", i.e., the decay of correlations in the measure at time 0 conditioned on having a fixed configuration η at time t. These correlations are expected to decay exponentially for $\nu S(t)$-a.e. η, which would lead to $\nu S(t)$-a.s. convergence of the Kozlov-potential.

6. **Low-temperature dynamics.** The main problem of analyzing the regime $0 < T_\mu << 1$ for large t is the impossibility of a perturbative representation of $-\log p_t(\sigma, \eta)$. If we still continue to work with the picture of the joint Hamiltonian in (3.32), then the term $-\log p_t(\sigma, \eta)$ will not converge to a σ-independent Hamiltonian as $t \uparrow \infty$. Therefore we

cannot argue that for large t the Gibbsianness of the measure $\nu S(t)$ depends only on the presence or absence of a phase transition in the Hamiltonian H_ν of the initial measure ν. The dynamical part of the joint Hamiltonian can induce a phase transition. The regime $0 < T_\mu << 1$ is very delicate and there is no reason to expect a robust result for general models. Metastability will enter.

7. **Zero-temperature dynamics.** What happens when $T_\mu = 0$? In this case there is only nature, no nurture. We therefore expect the behavior to be different from $0 < T_\mu << 1$. Trapping phenomena will enter.

8. **Other dynamics.** Do similar phenomena occur under spin-exchange dynamics, like Kawasaki dynamics ? In particular, how do conservation laws influence the picture (see [16], [17], [1])

Acknowledgments: We thank C. Maes and K. Netocny for fruitful discussions. A.C.D.v.E. thanks H. van Beijeren for pointing out reference [1] to him. Part of this collaboration was made possible by the "Samenwerkingsverband Mathematische Fysica". R. F. thanks the Department of Theoretical Physics at Groningen for kind hospitality.

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