

# CONVERGENCE OF THE STOCHASTIC MESH ESTIMATOR FOR PRICING AMERICAN OPTIONS

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## **Abstract**

Broadie and Glasserman (1997a) proposed a simulation-based method using a stochastic mesh for pricing high-dimensional American options. Based on simulated states of the assets underlying the option at each exercise opportunity, the stochastic mesh method produces an estimator of the option value at each sampled state. We derive an asymptotic bound on the probability of error of the mesh estimator, where both the error and the probability bound are decreasing to zero as the sample size increases, implying that the estimator converges in probability to the option price. We include the empirical performance of the mesh estimator for the test problems in Broadie and Glasserman (1997a) and find that it has large bias that decays very slowly with the sample size, suggesting that in applications it will most likely be necessary to employ variance-reduced variants of the mesh estimator.

## **1 Introduction**

In the financial markets, sophisticated, complex products are continuously offered and traded. The complexity of these instruments has been steadily increasing, and this trend seems likely to continue, as institutions wish to hedge against more refined and more numerous risks. For example, a portfolio manager who wishes to *hedge*, i.e., reduce the risk in a position in tech-

nology stocks, no longer has to buy a portfolio of options on individual stocks. He can instead buy a basket option, i.e., an option on an appropriate technology index or even a customized option tailored to his individual holdings, risk preferences, and time frame.

There are many financial products whose values depend on more than one underlying asset. Examples include basket options (options on the average of several underlying assets), out-performance options (options on the maximum of several assets), spread options (options on the difference between two assets), and quantos (options whose payoff is adjusted by some stochastic variable, typically an exchange rate). Even when there is a single underlying asset, there is trend towards models with multiple stochastic factors (sources of uncertainty), e.g., single-asset model with stochastic volatility. In addition, multi-factor models are gaining more acceptance and use for modeling interest rates, where models with two to four factors are common and models with up to ten factors are being tested (Broadie and Glasserman 1997c). As computing power is steadily increasing, multi-factor option-pricing models are likely to become more prevalent.

Pricing and hedging options (European or American) using multi-factor models is a difficult task. Especially for American options, which allow early exercise, analytical formulas for pricing are rarely available. Various deterministic numerical techniques are used, for example the numerical solution of the appropriate partial differential equation. numerical solution of the corresponding partial differential equation. However, such methods require work that grows exponentially in the number of state variables. This work requirement renders these methods ineffective when the state space dimension is higher than three or four.

Monte Carlo simulation techniques are conceptually simple, yet powerful in addressing option pricing problems of great complexity, whether the complexity arises from the stochastic process driving the assets, the structure of the payoff (path-dependent), or the early exercise features (American). Until recently, the prevailing opinion was that American options could

not be handled using Monte Carlo simulation. Recent developments, however, have started to pave the way for estimating American option prices via Monte Carlo methods.

Barraquand and Martineau (1995) proposed an algorithm that only approximately solves the American option pricing problem. They partition the state space of stochastic factors into a tractable number of cells and compute an approximately optimal exercise policy that is constant over each cell. Although this method is fast, it yields an estimate that does not necessarily converge to the true price as work increases. Broadie and Glasserman (1997b) were the first to develop a simulation procedure that yields provably convergent estimates for American option prices, clearly an attractive theoretical property. Their method is based on a simulated tree of the state variables. The main drawback of their method is that the work is exponential in the number of exercise opportunities. For a comprehensive review of the literature in Monte Carlo methods for Pricing American Options, see Broadie and Glasserman (1997c).

An important method developed recently for valuing American options via simulation is the stochastic mesh method (Broadie and Glasserman 1997a), henceforth referred to as BG1997a. The stochastic mesh method begins by generating a number  $b$  of randomly sampled states of the stochastic factors underlying the option at each exercise opportunity. Based on this sample, the *mesh estimator* of the option value at each sampled state is computed (a full description is deferred until Section 2.2). The authors also propose a *path estimator*, obtained by simulating paths of the stochastic factors underlying the options and estimating an approximate exercise policy based on the mesh values; see BG1997a for more details. It is shown that the mesh and path estimators are biased high and low, respectively. In addition, under certain technical assumptions, it is shown that both estimators converge (in a well-defined sense) to the true option value as the sample size, i.e. the number of sampled states, goes to infinity.

In this paper we derive an asymptotic bound on the probability of error of the mesh

estimator. Both the error and the bound on the probability of error are functions of the sample size  $b$ , and the probability bound is valid only asymptotically as  $b$  grows large. We also present empirical results on the estimator's behavior on the test problems in Broadie and Glasserman (1997a).

This paper is organized as follows. Section 2 contains brief background on the problem of pricing American options and a description of the stochastic mesh method. Section 3 contains our main theoretical result, namely a bound on the probability of error of the mesh estimator as the number  $b$  of states sampled at each stage grows large. In Section 4 we present computational results on the test problems in Broadie and Glasserman (1997a), and Section 5 contains a summary of our conclusions.

## 2 Background

### 2.1 American Option Pricing

Let  $S_t$  denote the vector of stochastic factors underlying the option, modeled as a Markov process on  $\mathbb{R}^d$  with discrete-time parameter  $t = 0, 1, 2, \dots, T$ . The argument  $t$  indexes the set of times when the option is exerciseable, also called exercise opportunities or simply stages. Let  $h(t, x)$  denote the payoff to the option holder from exercise at time  $t$  in state  $x$ , discounted to time 0 with the possibly stochastic discount factor recorded in  $x$ . This view of  $h(t, x)$  as the discounted-to-time-0 payoff is adopted to simplify the notation and does not reduce the generality of the method.

By the dynamic programming principle, the option value can be written as follows:

$$\begin{aligned} q(t, x) &= h(t, x), && \text{for } t = T, \text{ all } x \\ &= \max\{h(t, x), c(t, x)\} && \text{for } 0 \leq t \leq T - 1, \text{ all } x \end{aligned}$$

where

$$c(t, x) = E[q(t + 1, S_{t+1}) | S_t = x] \tag{1}$$

is called the *continuation value* at  $(t, x)$ , equal to the value of the option (discounted to time 0) when it is not exercised at (time, state) pair  $(t, x)$ . It is well-known from arbitrage pricing theory that the arbitrage-free price of the option is obtained when the conditional expectation in (1) is taken with respect to the risk-neutral measure, defined as the measure that makes the value of any tradeable security, discounted to time 0, a martingale. Given the known state of  $S_0$  at time 0, say  $x_0$ , the option-pricing problem is to compute  $q(0, x_0)$ .

## 2.2 The Stochastic Mesh Method

In reviewing the method, we follow BG1997a. The mesh method generates a stochastic *mesh* of sample states  $\{S_t^j\}$ ,  $j = 1, 2, \dots, b$  for each  $t = 1, \dots, T$ . For notational convenience, we define  $b$  nonrandom mesh points at stage 0,  $S_0^j = x_0$ ,  $j = 1, 2, \dots, b$ . For  $t = 1, 2, \dots, T$ , let  $g_t(\cdot)$  denote the probability density from which the points  $\{S_t^j\}_{j=1}^b$  are sampled (to be specified later), and let  $f_t(x, \cdot)$  denote the conditional risk-neutral density of  $S_{t+1}$  given  $S_t = x$ . (We assume throughout the paper the existence of such densities.) The Broadie-Glasserman mesh estimator is calculated as a backward recursion for  $t = T, T - 1, \dots, 0$ :

$$\widehat{q}_H(T, S_T^j) = h(T, S_T^j), \quad \text{for } j = 1, 2, \dots, b \quad (2)$$

$$\widehat{q}_H(t, S_t^j) = \max\{h(t, S_t^j), \widehat{c}(t, S_t^j)\} \text{ for } t = T - 1, T - 2, \dots, 0, \quad j = 1, 2, \dots, b, \quad (3)$$

where the estimate of the continuation value function  $\widehat{c}(t, x)$  is

$$\widehat{c}(t, x) := \sum_{j=1}^b \frac{\widehat{q}_H(t+1, S_{t+1}^j) \cdot f_t(x, S_{t+1}^j)}{g_{t+1}(S_{t+1}^j)} \quad (4)$$

Note that in (4), the point  $S_{t+1}^j$  is weighed by the likelihood ratio  $f_t(x, S_{t+1}^j)/g_{t+1}(S_{t+1}^j)$ .

In BG1997a, it is argued that the choice of sampling densities  $g_{t+1}(\cdot)$  is crucial to the success of the method; and the choice recommended by the authors is as follows. We simulate independently  $b$  paths of  $S_t$  starting from  $x_0$  at time 0 and let  $S_t^j$  denote the state of the  $j$ -th path at time  $t$ ; and then we "forget" the path to which a point belongs. This

is called by the authors the *stratified implementation*. For any  $t, j$ , we call the ordered pair  $(S_t^j, S_{t+1}^j)$  a *parent* and *child*, respectively.

We clarify some distributional properties of the stratified implementation. Let  $\pi$  be a random permutation of the integers in  $\{1, 2, \dots, b\}$  chosen with equal probability from all possible such permutations, and let  $\mathcal{F}_t$  be the  $\sigma$ -field  $\mathcal{F}_t = \sigma(S_t^1, S_t^2, \dots, S_t^b)$ . Then

$$\text{Conditional on } \mathcal{F}_t, \quad \{S_{t+1}^{\pi(1)}, S_{t+1}^{\pi(2)}, \dots, S_{t+1}^{\pi(b)}\} \stackrel{\text{i.d.}}{\sim} g_{t+1}(\cdot) := \frac{1}{b} \sum_{i=1}^b f_t(S_t^i, \cdot) \quad (5)$$

where  $\stackrel{\text{i.d.}}{\sim}$  means "are identically distributed with density...". Note that the density  $g_{t+1}(\cdot)$  is defined conditionally on  $\mathcal{F}_t$ . Also note that  $\{S_{t+1}^{\pi(1)}, S_{t+1}^{\pi(2)}, \dots, S_{t+1}^{\pi(b)}\}$  are conditionally dependent random vectors. On the other hand, the points  $\{S_{t+1}^1, S_{t+1}^2, \dots, S_{t+1}^b\}$  are conditionally independent but not identically distributed; they are unconditionally independent and identically distributed.

### 3 Convergence in Probability

Under an assumption on the finiteness of a certain moment, we will show that the estimator  $\hat{q}_H$  with the stratified implementation converges in probability to  $q$  when  $b \rightarrow \infty$ ; moreover, we provide a bound on the probability of error of  $\hat{q}_H$ , where both the error and the bound on the probability on error depend on the sample size  $b$ .

For the stratified implementation, we observe that

$$\hat{c}(t, x) := \frac{1}{b} \sum_{j=1}^b \frac{\hat{q}_H(t+1, S_{t+1}^{\pi(j)}) \cdot f_t(x, S_{t+1}^{\pi(j)})}{g_{t+1}(S_{t+1}^{\pi(j)})} = \frac{1}{b} \sum_{j=1}^b \frac{\hat{q}_H(t+1, S_{t+1}^j) \cdot f_t(x, S_{t+1}^j)}{g_{t+1}(S_{t+1}^j)}$$

The first equality is the definition of the continuation value function in the stratified implementation. The second equality follows from the invariance of the sum over permutations of the  $\{S_{t+1}^1, S_{t+1}^2, \dots, S_{t+1}^b\}$ , and this observation will be essential in proving our convergence result.

We require the following moment assumptions, where  $S_t^1, S_t^2, S_t^3$  denote paths which are independent of each other and have the distribution of  $S_t$  conditioned under  $S_0 = x_0$ , and

where  $C$  is a constant that will appear on the probability bound.

$$E \left[ \max_{t+1 \leq r \leq T} \{h^4(r, S_r^1)\} \right] \leq C/8 \quad \text{for each } t \in \{0, 1, 2, \dots, T-1\} \quad (6)$$

$$E \left[ \max_{t+1 \leq r \leq T} \{h^4(r, S_r^2)\} \cdot \frac{f_t^4(S_t^1, S_{t+1}^2)}{f_t^4(S_t^3, S_{t+1}^2)} \right] \leq C/8 \quad \text{for each } t \in \{0, 1, 2, \dots, T-1\} \quad (7)$$

$$E \left[ \max_{t+1 \leq r \leq T} \{h^4(r, S_r^1)\} \cdot \frac{f_t^4(S_t^1, S_{t+1}^1)}{f_t^4(S_t^3, S_{t+1}^1)} \right] < \infty \quad \text{for each } t \in \{0, 1, 2, \dots, T-1\} \quad (8)$$

$$E \left[ \frac{f_t^4(S_t^1, S_{t+1}^2)}{f_t^4(S_t^3, S_{t+1}^2)} \right] \leq C/8 \quad \text{for each } t \in \{0, 1, 2, \dots, T-1\} \quad (9)$$

$$E \left[ \frac{f_t^4(S_t^1, S_{t+1}^1)}{f_t^4(S_t^3, S_{t+1}^1)} \right] < \infty \quad \text{for each } t \in \{0, 1, 2, \dots, T-1\} \quad (10)$$

**Theorem 1.** *Suppose the mesh paths  $\{S_t^j\}_{j=1}^b$  are generated independently with  $S_0^j = x_0$  for all  $j \in \{1, 2, \dots, b\}$ , where  $x_0 \in \mathbb{R}^d$  is known at time 0. Under assumptions (6)-(10),*

$$P \left\{ |\widehat{q}_H(0, x_0) - q(0, x_0)| \geq (1 + \frac{\delta}{b^\gamma})^T - 1 \right\} \leq \frac{6CT}{\delta^4 b^{1-4\gamma}} + O(b^{-3}) \quad \text{for any } \delta > 0 \text{ and } 0 < \gamma < 1/4.$$

**Proof.** We start with a few definitions. Let

$$\bar{c}(t, x) := \frac{1}{b} \sum_{j=1}^b \frac{q(t+1, S_{t+1}^j) f(x, S_{t+1}^j)}{g_{t+1}(S_{t+1}^j)}$$

In other words  $\bar{c}_1(t, x)$  is the natural estimate we would make of  $c(t, x)$  if  $q(t+1, \cdot)$  was known (which of course is not the case). Fix  $\delta > 0$  and  $0 < \gamma < 1/4$ , and define the events

$$E_I(t) = \left\{ \omega : |\bar{c}(t, S_t^j)(\omega) - c(t, S_t^j)(\omega)| \leq \frac{\delta}{b^\gamma} \text{ for all } j \in \{1, 2, \dots, b\} \right\} \quad (11)$$

and

$$E_{II}(t) = \left\{ \omega : \left| \left( \frac{1}{b} \sum_{j=1}^b \frac{f_t(S_t^j, S_{t+1}^j)}{g_{t+1}(S_{t+1}^j)}(\omega) \right) - 1 \right| \leq \frac{\delta}{b^\gamma} \text{ for all } j \in \{1, 2, \dots, b\} \right\}$$

where  $\omega$  denotes a generic point in the sample space. Finally, let  $E_I$  be the event that  $E_{1,I}(t)$  holds for each  $t \in \{0, 1, 2, \dots, T-1\}$ , i.e.,  $E_I := \cap_{t \in \{0, 1, 2, \dots, T-1\}} E_I(t)$ . Similarly, define  $E_{II} = \cap_{t \in \{0, 1, 2, \dots, T-1\}} E_{II}(t)$ .

**Claim 1:** If events  $E_I$  and  $E_{II}$  both hold, then  $|\widehat{q}_H(0, x_0) - q(0, x_0)| \leq (1 + \frac{\delta}{b^\gamma})^T - 1$ .

**Proof of Claim 1.** The proof is by a recursive argument going backwards in time. We start by showing how an error bound that holds uniformly over all estimates at time  $t+1$  can be iterated backwards in time. Fix  $\varepsilon > 0$  and suppose that for some  $t$  ( $0 < t \leq T-1$ ) the error of the estimates at the forward points satisfies

$$|\widehat{q}_H(t+1, S_{t+1}^j) - q(t+1, S_{t+1}^j)| \leq \varepsilon \text{ for all } j \in \{1, 2, \dots, b\}. \quad (12)$$

Then

$$\begin{aligned} |\widehat{c}(t, x) - \bar{c}(t, x)| &= \frac{1}{b} \left| \sum_{j=1}^b \frac{\widehat{q}_H(t+1, S_{t+1}^j) \cdot f_t(x, S_{t+1}^j)}{g_{t+1}(S_{t+1}^j)} - \sum_{j=1}^b \frac{q(t+1, S_{t+1}^j) \cdot f_t(x, S_{t+1}^j)}{g_{t+1}(S_{t+1}^j)} \right| \\ &= \frac{1}{b} \left| \sum_{j=1}^b \frac{(\widehat{q}_H(t+1, S_{t+1}^j) - q(t+1, S_{t+1}^j)) \cdot f_t(x, S_{t+1}^j)}{g_{t+1}(S_{t+1}^j)} \right| \\ &\leq \frac{\varepsilon}{b} \sum_{j=1}^b \frac{f_t(x, S_{t+1}^j)}{g_{t+1}(S_{t+1}^j)} \\ &\leq \varepsilon \left(1 + \frac{\delta}{b^\gamma}\right) \text{ for all } x \in \{S_t^1, S_t^2, \dots, S_t^b\}, \end{aligned} \quad (13)$$

where the last inequality follows since  $E_{II}$  holds. So if (12) holds, then the error of  $\widehat{q}_H$  at stage  $t$  ( $0 \leq t \leq T-1$ ) can be bound uniformly as follows:

$$\begin{aligned} |\widehat{q}_H(t, S_t^j) - q(t, S_t^j)| &= |\max\{h(t, S_t^j), \widehat{c}(t, S_t^j)\} - \max\{h(t, S_t^j), c(t, S_t^j)\}| \\ &\leq |\widehat{c}(t, S_t^j) - c(t, S_t^j)| \\ &\leq |\widehat{c}(t, S_t^j) - \bar{c}(t, S_t^j)| + |\bar{c}(t, S_t^j) - c(t, S_t^j)| \\ &\leq \varepsilon \left(1 + \frac{\delta}{b^\gamma}\right) + \frac{\delta}{b^\gamma} \text{ for all } j \in \{1, 2, \dots, b\} \end{aligned} \quad (14)$$



where in the last inequality we used the bound (13) and the definition (11).

Now the recursive bounding is as follows. We start the error bounding with the special case  $t = T - 1$ , where we observe that  $\widehat{c}(T - 1, S_{T-1}^j) - \bar{c}(T - 1, S_{T-1}^j) = 0$  for all  $j$ , and so the definition of the event  $E_I(T - 1)$  implies that (14) holds for  $t = T - 1$  with  $\varepsilon = 0$ . Iterating the bounding argument in (14) with  $t = T - 2, T - 3, \dots, 0$ , we get

$$\begin{aligned} |\widehat{q}_H(0, x_0) - q(0, x_0)| &\leq \frac{\delta}{b^\gamma} \sum_{j=0}^{T-1} \left(1 + \frac{\delta}{b^\gamma}\right)^j \\ &= \frac{\delta}{b^\gamma} \frac{\left(1 + \frac{\delta}{b^\gamma}\right)^T - 1}{1 + \frac{\delta}{b^\gamma} - 1} \\ &= \left(1 + \frac{\delta}{b^\gamma}\right)^T - 1 \end{aligned}$$

which completes the proof of Claim 1. ■

Letting  $E$  be the event that  $|\widehat{q}_H(0, x_0) - q(0, x_0)| \leq \left(1 + \frac{\delta}{b^\gamma}\right)^T - 1$ , we have just proven that  $E \supset E_I \cap E_{II}$ . Letting  $A^c$  denote the complement of the event  $A$ , we have  $P(E^c) \leq P(E_I^c) + P(E_{II}^c)$ . To complete the proof, we need to show that  $P(E_I^c) \leq \frac{3CT}{\delta^4 b^{1-4\gamma}} + O(b^{-3})$  and  $P(E_{II}^c) \leq \frac{3CT}{\delta^4 b^{1-4\gamma}} + O(b^{-3})$ .

We first obtain the upper bound for  $P(E_I^c)$ . Define the event

$$E_I(t, i) = \left\{ \omega : |\bar{c}_1(t, S_t^i)(\omega) - c(t, S_t^i)(\omega)| \leq \frac{\delta}{b^\gamma} \right\}$$

Recall that  $E_I = \bigcap_{t=0}^{T-1} E_I(t) = \bigcap_{t=0}^{T-1} \bigcap_{i=1}^b E_I(t, i)$ , so

$$P(E_I^c) \leq \sum_{t=0}^{T-1} \sum_{i=1}^b P(E_I^c(t, i)) = \sum_{t=0}^{T-1} b P(E_I^c(t, 1)),$$

since  $\{\{S_t^i, \{S_t^j\}_{j=1}^b\}_{i=1}^b\}$  are identically distributed. We will show that

$$P(E_I^c(t, 1)) \leq \frac{3C}{\delta^4 b^{2-4\gamma}} + O(b^{-3}) \text{ for all } t \in \{0, 1, \dots, T-1\} \quad (15)$$

which then proves that  $P(E_I^c) \leq \frac{3CT}{\delta^4 b^{1-4\gamma}}$ .

The key for proving that  $\bar{c}_1(t, S_t^1) - c(t, S_t^1)$  is small with high probability as  $b \rightarrow \infty$  is that it can be written as the sum of  $b$  random variables which conditionally have mean 0 and are independent.

**Claim 2:**  $\bar{c}(t, S_t^1) - c(t, S_t^1) = \frac{1}{b} \sum_{j=1}^b Z^j(t)$ , where

$$Z^j(t) := \frac{q(t+1, S_{t+1}^j) \cdot f(S_t^1, S_{t+1}^j)}{g_{t+1}(S_{t+1}^j)} - E\left[\frac{q(t+1, S_{t+1}^j) \cdot f(S_t^1, S_{t+1}^j)}{g_{t+1}(S_{t+1}^j)} \middle| \mathcal{F}_t\right], \quad j = 1, 2, \dots, b,$$

where  $\mathcal{F}_t$  denotes the  $\sigma$ -field  $\mathcal{F}_t = \sigma(S_s^i | i \in \{1, 2, \dots, b\}; s \in \{0, 1, \dots, t\})$ .

**Proof of Claim 2.**

$$\begin{aligned} \frac{1}{b} \sum_{j=1}^b Z^j(t) &= \frac{1}{b} \sum_{j=1}^b \left( \frac{q(t+1, S_{t+1}^j) \cdot f(S_t^1, S_{t+1}^j)}{g_{t+1}(S_{t+1}^j)} - E\left[\frac{q(t+1, S_{t+1}^j) \cdot f(S_t^1, S_{t+1}^j)}{g_{t+1}(S_{t+1}^j)} \middle| \mathcal{F}_t\right] \right) \\ &= \bar{c}(t, S_t^1) - E\left[\frac{1}{b} \sum_{j=1}^b \frac{q(t+1, S_{t+1}^j) \cdot f(S_t^1, S_{t+1}^j)}{g_{t+1}(S_{t+1}^j)} \middle| \mathcal{F}_t\right] \\ &= \bar{c}(t, S_t^1) - E\left[\frac{1}{b} \sum_{j=1}^b \frac{q(t+1, S_{t+1}^{\pi(j)}) \cdot f(S_t^1, S_{t+1}^{\pi(j)})}{g_{t+1}(S_{t+1}^{\pi(j)})} \middle| \mathcal{F}_t\right] \\ &= \bar{c}(t, S_t^1) - E\left[\frac{q(t+1, X) \cdot f(S_t^1, X)}{g_{t+1}(X)} \middle| \mathcal{F}_t\right] \end{aligned}$$

where  $X$  represents a random variable which is obtained by choosing one of the points  $S_{t+1}^1, S_{t+1}^2, \dots, S_{t+1}^b$  at random with equal probability. The key behind the third step is the invariance of the sum inside the expectation with respect to permutations of the  $\{S_{t+1}^j\}_{j=1}^b$ . The conditional distribution of  $X$  when conditioned under  $\mathcal{F}_t$  has the density  $g_{t+1}(\cdot)$  in (5), so the second term in the last expression is simply a measure transformed expectation, and thus

$$E\left[\frac{q(t+1, X) \cdot f(S_t^1, X)}{g_{t+1}(X)} \middle| \mathcal{F}_t\right] = E[q(t+1, S_{t+1}^1) | \mathcal{F}_t] = c(t, S_t^1)$$

which completes the proof of Claim 2. ■

Conditioned under  $\mathcal{F}_t$  the  $\{Z^j(t)\}_{j=1}^b$  have mean 0 and are independent. We will exploit this observation to obtain a sufficient probability bound on their average. First, we need two lemmas.

**Lemma 1.** *Suppose  $Y$  is a nonnegative random variable with  $E[Y^4] < \infty$ . Then  $E[(Y - E[Y|\mathcal{F}])^4] \leq 8E[Y^4]$ , where  $\mathcal{F}$  is an arbitrary  $\sigma$ -field.*

**Proof.**

$$\begin{aligned}
E[(Y - E[Y|\mathcal{F}])^4] &= E(Y^4 - 4Y^3E[Y|\mathcal{F}] + 6Y^2E^2[Y|\mathcal{F}] - 4YE^3[Y|\mathcal{F}] + E^4[Y|\mathcal{F}]) \\
&\leq E[Y^4] + 6E(Y^2E^2[Y|\mathcal{F}]) + E(E^4[Y|\mathcal{F}]) \\
&\leq E[Y^4] + 6\sqrt{E[Y^4]}\sqrt{E(E^4[Y|\mathcal{F}])} + E(E[Y^4|\mathcal{F}]) \\
&\leq 2E[Y^4] + 6\sqrt{E[Y^4]}\sqrt{E(Y^4)} \\
&= 8E[Y^4].
\end{aligned}$$

In the second step, we dropped nonpositive random variables from the expectation. In the third step, we used the Cauchy-Schwartz inequality for the second term and Jensen's inequality for the third term, and in the fourth step we used again Jensen's inequality inside the second square root. ■

**Lemma 2.** *Let  $\mathcal{F}$  denote an arbitrary  $\sigma$ -field, and let  $Z_1, Z_2, \dots, Z_b$  be random variables which, conditional on  $\mathcal{F}$  have mean 0, are conditionally independent of each other, and such that  $E[Z_1^4] < \infty$  and  $E[Z_j^4] \leq C$  for each  $j \neq 1$ , where all expectations are unconditional, and  $C$  is a constant. Then*

$$P\left(\frac{1}{b}|Z_1 + Z_2 + \dots + Z_b| \geq \varepsilon\right) \leq \frac{3C}{b^2\varepsilon^4} + O(b^{-3}) \quad \text{uniformly in } \varepsilon > 0.$$

**Proof.**

$$\begin{aligned}
P\left(\frac{1}{b}|Z_1 + Z_2 + \dots + Z_b| \geq \varepsilon\right) &= P\left(\frac{1}{b^4}|Z_1 + Z_2 + \dots + Z_b|^4 \geq \varepsilon^4\right) \\
&\leq \frac{E[(Z_1 + Z_2 + \dots + Z_b)^4]}{b^4\varepsilon^4} \tag{16}
\end{aligned}$$

where we used Markov's inequality. Now  $E[(Z^1+Z^2+\dots+Z^b)^4] = \Sigma E[E[Z_{j_1} Z_{j_2} Z_{j_3} Z_{j_4}|\mathcal{F}]]$ , where the four indices are ranging independently from 1 to  $b$ . Since  $E[Z_{j_1}|\mathcal{F}] = 0$ , the conditional independence of the  $Z$ 's implies that the summand vanishes if there is one index different from the three others. This leaves terms of the form  $E[E[Z_{j_1}^4|\mathcal{F}]]$ , of which there are  $b$ , and terms of the form  $E[E[Z_{j_1}^2 Z_{j_2}^2|\mathcal{F}]]$  for  $j_1 \neq j_2$ , of which there are  $3b(b-1)$ . For each of the two

different forms, the number of terms with any index equal to 1 is  $O(b^{-1})$  of the total number of such terms, and so the finiteness of  $E[Z_1^4]$  implies that the relative contribution of these terms to the total is  $O(b^{-1})$ . Now focusing on terms where all indices are different than 1, we have  $E[E[Z_{j_1}^4 | \mathcal{F}]] = E[Z_{j_1}^4] \leq C$ , and  $E[E[Z_{j_1}^2 Z_{j_2}^2 | \mathcal{F}]] = E[Z_{j_1}^2 Z_{j_2}^2] \leq \sqrt{E[Z_{j_1}^4]} \sqrt{E[Z_{j_2}^4]} \leq C$ . Hence

$$\begin{aligned} E[(Z_1 + Z_2 + \dots + Z_b)^4] &\leq bC[1 + O(b^{-1})] + 3b(b-1)C[1 + O(b^{-1})] \\ &\leq 3b^2C + O(b), \end{aligned}$$

which completes the proof of Lemma 2. ■

Applying Lemma 1 with  $Y = \frac{q^{(t+1, S_{t+1}^j)} \cdot f(S_t^1, S_{t+1}^j)}{g_{t+1}(S_{t+1}^j)}$  and  $\mathcal{F} = \mathcal{F}_t$ , we get

$$\begin{aligned} E[(Z^j(t))^4] &\leq 8E \left[ \frac{q^4(t+1, S_{t+1}^j) \cdot f^4(S_t^1, S_{t+1}^j)}{g_{t+1}^4(S_{t+1}^j)} \right] \\ &\leq 8E \left[ \frac{\max_{t+1 \leq r \leq T} \{h^4(r, S_r^j)\} \cdot f^4(S_t^1, S_{t+1}^j)}{g_{t+1}^4(S_{t+1}^j)} \right] \text{ for all } j \in \{1, 2, \dots, b\}. \end{aligned}$$

Now by Jensen's inequality, for any  $x_1, x_2, \dots, x_b > 0$  we have that

$$\frac{1}{\left(\frac{x_1 + \dots + x_b}{b}\right)^4} \leq \frac{1}{b} \left( \frac{1}{x_1^4} + \dots + \frac{1}{x_b^4} \right)$$

The  $\{Z^j(t)\}_{j=2}^b$  are identically distributed, and we have

$$\begin{aligned} E[(Z^2(t))^4] &\leq 8E \left[ \max_{t+1 \leq r \leq T} \{h^4(r, S_r^2)\} \cdot \frac{1}{b} \left( \frac{f^4(S_t^1, S_{t+1}^2)}{f^4(S_t^1, S_{t+1}^2)} + \sum_{s \neq 1} \frac{f^4(S_t^1, S_{t+1}^2)}{f^4(S_t^s, S_{t+1}^2)} \right) \right] \\ &= 8 \left\{ \frac{1}{b} E \left[ \max_{t+1 \leq r \leq T} \{h^4(r, S_r^2)\} \right] + \frac{b-1}{b} E \left[ \max_{t+1 \leq r \leq T} \{h^4(r, S_r^j)\} \cdot \left( \frac{f^4(S_t^1, S_{t+1}^2)}{f^4(S_t^3, S_{t+1}^2)} \right) \right] \right\} \\ &\leq 8 \left\{ \frac{1}{b} C + \frac{b-1}{b} \frac{C}{8} \right\} \\ &= C \text{ for all } t \in \{0, 1, 2, \dots, T-1\}. \end{aligned}$$

An analogous argument combined with assumption (8) shows that  $E[(Z^1(t))^4] < \infty$  for all

$t$ . Now applying Lemma 2 with  $Z_j = Z^j(t)$ ,  $\mathcal{F} = \mathcal{F}_t$  and  $\varepsilon = \frac{\delta}{b^\gamma}$ , we have

$$\begin{aligned} P(E_I^c(t, 1)) &= P(|\bar{c}(t, S_t^1) - c(t, S_t^1)| \geq \frac{\delta}{b^\gamma}) \\ &= P\left(\left|\sum_{j=1}^b Z^j(t)\right| \geq \frac{\delta}{b^\gamma}\right) \\ &\leq \frac{3C}{\delta^4 b^{2-4\gamma}} + O(b^{-3}) \text{ for each } t \in \{0, 1, \dots, T-1\} \end{aligned}$$

as claimed in (15), which completes the proof that  $P(E_I^c) \leq \frac{3CT}{\delta^4 b^{1-4\gamma}} + O(b^{-3})$ .

The probability bound  $P(E_{II}^c) \leq \frac{3CT}{\delta^4 b^{1-4\gamma}} + O(b^{-3})$  is proved by noting that  $E_{II}^c$  can be written as an event of the form  $E_I^c$  for the special function  $q(\cdot, \cdot) = 1$ , and assumptions (9) and (10) will serve in place of (7) and (8), respectively. This completes the proof of Theorem 1. ■

The following result shows that the rate of convergence may be sharpened using moments of order higher than 4 as we did in assumptions (6)-(10).

**Theorem 2.** *Suppose the mesh paths  $\{S_t^j\}_{j=1}^b$  are generated independently with  $S_0^j = x_0$  for all  $j \in \{1, 2, \dots, b\}$ , where  $x_0 \in \mathbb{R}^d$  is the known state at time 0. Under assumptions (6)-(10) where we replace the power 4 by the power 8 and let  $C_1$  be the corresponding constant,*

$$P\left\{|\widehat{q}_H(0, x_0) - q(0, x_0)| \geq (1 + \frac{\delta}{b^\gamma})^T - 1\right\} \leq \frac{2520C_1T}{\delta^8 b^{5-8\gamma}} + O(b^{-5}) \quad \text{for any } \delta > 0 \text{ and } 0 < \gamma < 5/8.$$

**Sketch of Proof.** One can show that  $P(E_I^c(t, 1)) \leq \frac{1260C_1}{\delta^8 b^{5-8\gamma}} + O(b^{-5})$  using Markov's inequality with power 8 and a result analogous to Lemma 2 using the 8th power for bounding. The other steps in the proof are as in Theorem 1. ■

## 4 Computational Results

We report empirical results on the performance of the mesh estimator on the test problems in BG1997a. Under the risk-neutral measure, the  $n$  assets are independent, and each follows

a geometric Brownian motion process:

$$dS_t(k) = S_t(k)[(r - \delta)dt + \sigma dW_t(k)], \quad k = 1, \dots, n,$$

where  $W_t(k), k = 1, \dots, n$  are independent Brownian motions,  $r$  is the riskless interest rate,  $\delta$  is the dividend rate, and  $\sigma$  is a volatility parameter. Exercise opportunities occur at the set of calendar times  $t_i = iT/d, i = 0, \dots, d$ , where  $T$  is the calendar option expiration time, so that  $i$  is the equivalent of  $t$  of the previous sections, and  $d$  is the equivalent of  $T$  of the previous sections. Under the risk-neutral measure, the random variables  $\log(S_{t_i}(k)/S_{t_{i-1}}(k))$  for  $k = 1, \dots, n$  are independent and normally distributed with mean  $(r - \delta - \sigma^2/2)(t_i - t_{i-1})$  and variance  $\sigma^2(t_i - t_{i-1})$ .

Tables 1-3 contain results for a call option on the maximum of the assets with payoff equal to  $\max\{\max_{1 \leq k \leq n} S_T(k) - K, 0\}$  and parameters  $n = 5, r = 0.05, \delta = 0.1, \sigma = 0.2, K = 100, T = 3$ , and  $d = 3, 6$ , and  $9$ , respectively. Tables 4-5 contain results for a call option on the geometric average of the assets with payoff equal to  $\max\left\{\left(\prod_{k=1}^n S_T(k)\right)^{\frac{1}{n}} - K, 0\right\}$  and parameters  $n = 5$  and  $7$  assets respectively,  $r = 0.03, \delta = 0.05, \sigma = 0.4, K = 100, T = 1$ , and  $d = 10$ . Within each table, the two panels contain results for out-of-the-money and in-the-money cases, specifically with  $S_o(k) = x_0, k = 1, \dots, n$ , where  $x_0 = 90$  and  $110$ , respectively. Within each panel, we set the mesh size  $b$  to the values  $200, 400, 800$ , and  $1600$ . The column labeled ‘‘CPU’’ measures CPU time in seconds per replication of  $\widehat{q}_H$  on a SUN Ultra 5 workstation. Our performance measures are the relative bias (RB), relative standard error (RSE), and relative root mean square error (RRMSE) of  $\widehat{q}_H$ , defined as the bias, standard error, and root mean square error (RMSE) divided by the true option value, respectively. We approximated the true option values using the results in BG1997a as follows. For the max option, we used the most accurate estimates in that paper, which have a relative error less than 0.35% with 99% confidence. For the geometric average option, the values are calculated from a single-asset binomial tree, presumably with negligible error. These approximated ‘‘true’’ option values are listed in the bottom of each table. The estimates  $\widehat{\text{RB}}, \widehat{\text{RSE}}$ , and

Table 1: Max Option on Five Assets,  $d = 3$ .

$x_0$	$b$	CPU	$\widehat{\text{RB}}$	$\widehat{\text{RSE}}$	$\widehat{\text{RRMSE}}$
90	200	3.3	0.175	0.093	0.198
	400	8.4	0.127	0.052	0.137
	800	24.1	0.089	0.038	0.097
	1600	78.1	0.064	0.023	0.068
110	200	3.3	0.149	0.044	0.155
	400	8.4	0.115	0.036	0.121
	800	24.3	0.074	0.021	0.077
	1600	78.0	0.054	0.015	0.056

The true option values for the cases  $x_0 = 90$  and 110 are 16.006 and 35.695, respectively.

$\widehat{\text{RRMSE}}$  in these tables are based on 64 independent replications of  $\widehat{q}_H$ .

It is obvious that the mesh estimator is highly positively biased, with bias being the dominant factor in the estimator's overall error, as measured by RRMSE. The bias decays quite slowly in the range of sample sizes tested here. As expected from our theoretical result, the RRMSE is increasing fast with the number of exercise opportunities. This is not surprising in view of Theorem 1, which shows a geometric growth of the estimator's error bound with the number of exercise opportunities.

## 5 Conclusion

We have derived a bound on the probability of error of the mesh estimator of Broadie and Glasserman (1997a) for pricing American options as the number  $b$  of states sampled at each stage grows. Both the estimate's error and the bound on the probability of error are decreasing to 0 as  $b$  grows. The constant  $C$  appearing in the probability of error involves the fourth moment of the likelihood ratio of 1-step transition densities between a parent and

Table 2: Max Option on Five Assets,  $d = 6$ .

$x_0$	$b$	CPU	$\widehat{\text{RB}}$	$\widehat{\text{RSE}}$	$\widehat{\text{RRMSE}}$
90	200	6.6	0.402	0.098	0.414
	400	17.0	0.337	0.066	0.343
	800	49.0	0.288	0.043	0.291
	1600	158.5	0.231	0.029	0.233
110	200	6.6	0.370	0.066	0.376
	400	16.9	0.331	0.038	0.333
	800	48.7	0.256	0.023	0.257
	1600	158.5	0.203	0.018	0.204

The true option values for the cases  $x_0 = 90$  and 110 are 16.474 and 36.497, respectively.

Table 3: Max Option on Five Assets,  $d = 9$ .

$x_0$	$b$	CPU	$\widehat{\text{RB}}$	$\widehat{\text{RSE}}$	$\widehat{\text{RRMSE}}$
90	200	9.9	0.557	0.096	0.566
	400	25.6	0.521	0.064	0.525
	800	73.2	0.466	0.042	0.468
	1600	238.4	0.402	0.032	0.403
110	200	9.8	0.556	0.061	0.559
	400	25.5	0.503	0.040	0.505
	800	73.2	0.445	0.026	0.446
	1600	239.4	0.368	0.021	0.368

The true option values for the cases  $x_0 = 90$  and 110 are 16.659 and 36.782, respectively.



Table 4: Geometric Average Option on Five Assets,  $d = 10$ .

$x_0$	$b$	CPU	$\widehat{\text{RB}}$	$\widehat{\text{RSE}}$	$\widehat{\text{RRMSE}}$
90	200	10.9	0.621	0.320	0.699
	400	28.4	0.610	0.218	0.647
	800	80.7	0.584	0.139	0.601
	1600	260.3	0.493	0.090	0.502
110	200	11.0	0.533	0.101	0.542
	400	28.6	0.460	0.061	0.464
	800	81.7	0.367	0.042	0.370
	1600	260.4	0.277	0.032	0.279

The true option values for the cases  $x_0 = 90$  and 110 are 1.362 and 10.211, respectively.

Table 5: Geometric Average Option on Seven Assets,  $d = 10$ .

$x_0$	$b$	CPU	$\widehat{\text{RB}}$	$\widehat{\text{RSE}}$	$\widehat{\text{RRMSE}}$
90	200	15.4	0.628	0.336	0.712
	400	39.5	0.635	0.269	0.690
	800	112.9	0.605	0.198	0.636
	1600	362.9	0.610	0.141	0.626
110	200	15.4	0.477	0.100	0.488
	400	39.3	0.455	0.061	0.459
	800	112.6	0.396	0.041	0.398
	1600	365.3	0.338	0.029	0.340

The true option values for the cases  $x_0 = 90$  and 110 are 1.362 and 10.211, respectively.

a non-child to another non-parent and the same child multiplied by the maximum future payoff over a path that starts at the child. Our computational experience with the mesh estimator shows very poor behavior, specifically very large positive bias. In view of our theoretical result, we conclude that for the specific problems studied, the constant  $C$  is very large. This observation is consistent with the experience of many researchers that likelihood ratios are often highly variable random variables.

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