On a class of order pick strategies in paternosters

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Abstract

We study the travel time needed to pick n items in a paternoster, operating under the m-step strategy. This means that the paternoster chooses the shortest route among the ones that change direction only once, and after collecting at most m items. For random pick positions, we find the distribution and moments of the travel time, provided $n \geq 2m + 1$. It appears that, already for $m = 2$, the m-step strategy is very close to optimal, and better than the Nearest Item heuristic.

1 Introduction

A paternoster, or carousel system, is a computer controlled warehousing system consisting of a large number of shelves or drawers rotating in a closed loop in either direction. Such systems are mostly used for storage and retrieval of small and medium sized goods. The picker has a fixed position in front of the paternoster, which rotates the required items to the picker. The advantage of such systems is that the picker has time for sorting, packing, labeling *etc.*, while the paternoster is rotating. For a recent review of literature on paternosters, as part of a general overview of planning and control of warehousing systems, the reader is referred to Van den Berg [3].

An important performance characteristic is the total time needed to pick a list of items. It consists of the pure pick time and the rotation or travel time. Clearly, only the latter depends on the pick strategy. In this paper we study so-called m-step strategies: the paternoster chooses the shortest route among the ones that change direction only once, and only do so after collecting no more than m items. These strategies are closely related to the optimal strategy, i.e., the one minimizing the travel time. Bartoldi and Platzman [2] show that it is never optimal to turn more than once. Hence, if n denotes the number of items to be picked, then the optimal strategy is an $(n-1)$-step strategy.

For randomly distributed pick positions, Rouwenhorst *et al.* [10] analysed the m-step strategy for $m \leq 2$. Their results indicate that these strategies perform very well. In this paper we derive, for any $m \geq 0$, explicit expressions for the distribution and all moments of the travel time under the m-step strategy, provided $2m + 1 \leq n$. The analysis is based on probabilistic arguments, in particular on properties of exponentials.

The performance of m-step strategies will be compared with the performance of the optimal pick strategy. Numerical results show that, already for small values of m, the performance of the m-step strategy is very close to optimal. In fact, with high probability, the optimal strategy coincides with the 2-step strategy. Furthermore, m-step strategies are compared with the Nearest Item (NI) heuristic, where the next item to be picked is always the nearest one. The NI heuristic is frequently used in practice, and its statistical properties have been investigated by Litvak *et al.* [5, 6]. It appears that, already for $m = 2$, the m-step strategy performs better than the NI heuristic.

This paper is organized as follows. In the next section we describe the model and introduce some notation. The m-step strategy is analyzed in Section 3. In this section we first prove that the travel time under the m-step strategy can be expressed as the maximum of two sums of spacings, provided $2m+1 \leq n$. This representation is exploited in Section 4 to show that the travel time is distributed as a probabilistic mixture of sums of spacings. In Section 5 we derive closed-form expressions for the moments of the travel time. Then, in Section 6 we compare the performance of m-step strategies with the performance of the optimal strategy and the NI heuristic. Finally, Section 7 is devoted to comments and conclusions.

2 Paternoster model

Following Bartoldi and Platzman [2] and Rouwenhorst *et al.* [10] we represent a paternoster as a circle of length 1 (see Figure 1).

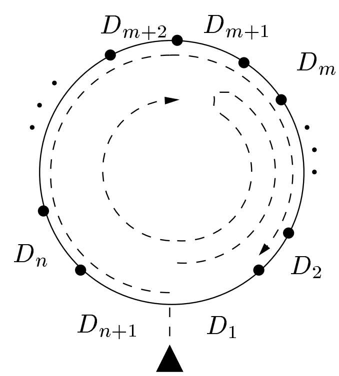

Figure 1: Paternoster model.

Let $U_0 = 0$ be the picker's starting point, and let the random variables U_i, where $i = 1, 2, \ldots, n$, denote the position of the ith item. We assume that the random variables U_i, $i = 1, 2, \ldots, n$, are independent and uniformly distributed on $[0, 1)$. Set $U_{n+1} = 1$. Let $U_{(0)}, U_{(1)}, \ldots, U_{(n+1)}$ be the order statistics of $U_0, U_1, \ldots, U_{n+1}$. Then the picker's starting point and the positions of the n items partition the circle into $n + 1$ uniform spacings

$$D_i = U_{(i)} - U_{(i-1)}, \quad 1 \leq i \leq n + 1. \tag{1}$$

Throughout this paper we will use the following relation between uniform spacings and exponentials (cf. Pyke [8, 9]). Let $X_1, X_2, \dots$ be i.i.d. exponentials with mean 1, and denote

$$S_0 = 0; \qquad S_i = \sum_{j=1}^{i} X_j, \quad i \geq 1.$$

Then

$$(D_1, D_2, \dots, D_{n+1}) \stackrel{d}{=} (X_1/S_{n+1}, X_2/S_{n+1}, \dots, X_{n+1}/S_{n+1}), \tag{2}$$

i.e., the spacings are distributed as normalized exponentials.

We assume that the paternoster rotates at unit speed, and that the acceleration and deceleration time of the paternoster is negligible. So the travel distance can be identified with the travel time. Further, for ease of presentation, we do as if the picker travels to the items, instead of the other way around. This completes the model description. In the next section we will analyze the travel time of the picker under the m-step strategy.

3 The m-step strategy

We assume that the picker operates under the m-step strategy: he chooses the shortest route among the routes that turn only once, and only turn when no more than m steps have been done. Clearly, there are $2(m+1)$ possible routes (see Figure 1); the ones that end in clockwise direction, i.e.,

$$\begin{aligned}
&D_2 + D_3 + \cdots + D_{n+1},\\
&2D_1 + D_3 + D_4 + \cdots + D_{n+1},\\
&\dots,\\
&2D_1 + 2D_2 + \cdots + 2D_{m-1} + D_{m+1} + D_{m+2} + \cdots + D_{n+1},\\
&2D_1 + 2D_2 + \cdots + 2D_{m-1} + 2D_m + D_{m+2} + D_{m+3} + \cdots + D_{n+1},
\end{aligned} \tag{3}$$

and, symmetrically, the other ones that end in counterclockwise direction,

$$\begin{aligned}
&D_n + D_{n-1} + \cdots + D_1,\\
&2D_{n+1} + D_{n-1} + D_{n-2} + \cdots + D_1,\\
&\dots,\\
&2D_{n+1} + 2D_n + \cdots + 2D_{n-m+3} + D_{n-m+1} + D_{n-m} + \cdots + D_1,\\
&2D_{n+1} + 2D_n + \cdots + 2D_{n-m+3} + 2D_{n-m+2} + D_{n-m} + D_{n-m-1} + \cdots + D_1.
\end{aligned} \tag{4}$$

Under the m-step strategy, the picker chooses the shortest of the $2(m+1)$ routes (3)–(4). Let the random variable $T_n^{(m)}$ denote the travel time under the m-step strategy, needed to pick n items. Then, by definition,

$$
\begin{aligned}
T_n^{(m)} &= \min\left\{ \min_{1\le j\le m+1}\left\{\sum_{l=1}^{j-1} 2D_l + \sum_{l=j+1}^{n+1} D_l\right\}, \min_{1\le j\le m+1}\left\{\sum_{l=1}^{j-1} 2D_{n+2-l} + \sum_{l=j+1}^{n+1} D_{n+2-l}\right\}\right\} \\
&= 1-\max\left\{ \max_{1\le j\le m+1}\left\{D_j - \sum_{l=1}^{j-1} D_l\right\}, \max_{1\le j\le m+1}\left\{D_{n+2-j} - \sum_{l=1}^{j-1} D_{n+2-l}\right\}\right\}, \qquad (5)
\end{aligned}
$$

where in the last expression we take $\sum_{l=1}^{n+1} D_l = 1$ outside the external minimum. This expression suggests an alternative interpretation for the m-step strategy. Clearly, $D_j - \sum_{l=1}^{j-1} D_l$ is a gain in travel time obtained by skipping the spacing D_j and going back instead. Under the m-step strategy the picker skips the spacing that provides the largest possible gain.

Bartoldi and Platzman [2] proved that the optimal route never allows more than one turn, and thus it is an m-step strategy with $m = n-1$. However, we only consider the case $2m+1 \le n$. In the analysis of $T_n^{(m)}$ it appears to be crucial that the spacings $D_1, D_2, \dots, D_{m+1}$, whose coefficients vary (-1, 0 or 1) in the first internal maximum of the last expression in (5), do not participate in the second internal maximum. This implies that $n-m+1 > m+1$, or $2m+1 \le n$.

Below we establish an elegant representation of the travel time. This representation will be used in the next section to derive the distribution of the travel time. Let us rewrite (5) using (2):

$$
T_n^{(m)} \stackrel{d}{=} 1 - \frac{1}{S_{n+1}} \max\left\{ \max_{1\le j\le m+1}\{X_j - S_{j-1}\}, \max_{1\le j\le m+1}\{X_{n+2-j} - (S_{n+1} - S_{n+2-j})\}\right\} \qquad (6)
$$

By exploiting properties of exponentials, we will reduce the two internal maxima in (6) to two sums of exponentials. First, we establish a preliminary result for the term

$$
\max_{1\le j\le m+1}\{X_j - S_{j-1}\}. \qquad (7)
$$

In fact, this is a special case of Theorem 3.4 in Litvak [4].

Lemma 3.1 *For any $m = 1, 2, \dots,$*

$$
\max_{1\le j\le m+1}\{X_j - S_{j-1}\} \stackrel{d}{=} \sum_{j=1}^{m+1} \frac{1}{2^j - 1} X_j.
$$

Proof. The proof is by induction. Let us assume that for some $i = 2, \dots, m+1$, expression (7) is distributed as

$$
\max\left\{ \sum_{j=1}^{i-1} \frac{1}{2^{i-j} - 1} X_j, \max_{i\le j\le m+1}\{X_j - S_{j-1}\}\right\}. \qquad (8)
$$

So the maximum of the first $i-1$ terms in (7) is distributed as a sum of exponentials. This trivially holds for $i = 2$. Now it suffices to show that if it holds for i, then it is also valid for $i+1$. In order to do it we rewrite (8) as

$$
\begin{aligned}
&\max\left\{\sum_{j=1}^{i-1}\frac{1}{2^{i-j}-1}X_j, \max_{i\le j\le m+1}\{X_j - S_{j-1}\}\right\}\\
&= \max\left\{\max\left\{\sum_{j=1}^{i-1}\frac{1}{2^{i-j}-1}X_j, X_i - S_{i-1}\right\}, \max_{i+1\le j\le m+1}\{X_j - S_{j-1}\}\right\}\\
&= \max\left\{\sum_{j=1}^{i-1}\frac{1}{2^{i-j}-1}X_j + \max\left\{0, X_i - \sum_{j=1}^{i-1}c_{i-j}X_j\right\}, \max_{i+1\le j\le m+1}\{X_j - S_{j-1}\}\right\}, \quad (9)
\end{aligned}
$$

where $c_j = 2^j/(2^j-1), \quad j \ge 1$. Then, as in [5, 4], we use conditioning on the random events

$$
\begin{aligned}
E_{i,k} &= [c_{i-1}X_1 + c_{i-2}X_2 + \cdots + c_{i-k+1}X_{k-1} \le X_i\\
&\qquad < c_{i-1}X_1 + c_{i-2}X_2 + \cdots + c_{i-k}X_k], \qquad 1 \le k \le i-1;\\
E_{i,i} &= [c_{i-1}X_1 + c_{i-2}X_2 + \cdots + c_1X_{i-1} \le X_i].
\end{aligned}
$$

Given event $E_{i,k}$, where $k = 1, 2, \ldots, i$, the random variables $X_1, X_2, \ldots$ can be replaced in the following way:

$$
\begin{aligned}
&X_j = \frac{1}{c_{i-j}+1}Y_j + \mathbf{1}_{[k=j]}Y_{k+1}, \quad 1 \le j \le \min\{k, i-1\};\\
&X_j = Y_{j+1}, \quad k < j < i; \qquad\qquad (10)\\
&X_i = \sum_{j=1}^{\min\{k,i-1\}}\frac{c_{i-j}}{c_{i-j}+1}Y_j + \mathbf{1}_{[k=i]}Y_i; \qquad X_j = Y_j, \quad j > i,
\end{aligned}
$$

where $Y_1, Y_2, \ldots$ are independent exponentials with mean 1. This follows by observing that $\min\{X_i, c_{i-j}X_j\}$ is an exponential with mean $c_{i-j}/(c_{i-j}+1)$ and the overshoot of the bigger term is again an exponential with the same mean. For more detail see [5]. Under event $E_{i,k}$, where $k = 1, 2, \ldots, i$, we have

$$
\begin{aligned}
&\sum_{j=1}^{i-1}\frac{1}{2^{i-j}-1}X_j + \max\left\{0, X_i - \sum_{j=1}^{i-1}c_{i-j}X_j\right\} \qquad (11)\\
&= \sum_{j=1}^{\min\{k,i-1\}}\frac{1}{(2^{i-j}-1)(c_{i-j}+1)}Y_j + \sum_{j=k}^{i-1}\frac{1}{2^{i-j}-1}Y_{j+1} + \mathbf{1}_{[k=i]}Y_i = \sum_{j=1}^{i}\frac{1}{2^{i-j+1}-1}Y_j.
\end{aligned}
$$

Also, we obtain

$$
S_i = \sum_{j=1}^{k}\left(\frac{1}{c_{i-j}+1} + \frac{c_{i-j}}{c_{i-j}+1}\right)Y_j + \sum_{j=k}^{i-1}Y_{j+1} = Y_1 + Y_2 + \cdots + Y_i. \qquad (12)
$$

Here $c_0 = 0$. Substituting (10) into (9) and using (11) and (12) it follows that, under event $E_{i,k}$, expression (9) reduces to

$$\max\left\{\sum_{j=1}^{i}\frac{1}{2^{i-j+1}-1}Y_j, \max_{i+1\le j\le m+1}\{Y_j-(Y_1+Y_2+\cdots+Y_{j-1})\}\right\} \tag{13}$$

for any $k = 1, 2, \ldots, i$. Hence, the induction statement now immediately follows from the law of total probability and the identical joint distribution of X_j's and Y_j's. This completes the proof of the lemma. □

Clearly, given an event $E_{i,k}$ we know whether the ith term in (7) is bigger than the first $i-1$ terms or not. On the other hand, (7) is always distributed as (13) under any event $E_{i,k}$. So the events $E_{i,k}$ do not provide information on the distribution of the maximum of the first i, nor of all $m+1$ terms in (7). Hence, we may conclude that the events A_i defined as

$$A_i = \left[\arg\max_{1\le j\le i}\{X_j - S_{j-1}\} = i\right], \quad i = 1, 2, \ldots,$$

have the following properties; result *(iii)* in the corollary follows from

$$\Pr(A_i) = \Pr(E_{i,i}) = \prod_{j=1}^{i-1}\frac{2^j-1}{2^{j+1}-1} = \frac{1}{2^i-1}. \tag{14}$$

Corollary 3.2

(i) *The events $A_1, A_2, \ldots$ are independent;*

(ii) *The distribution of (7) is independent of the events A_i, $i = 1, \ldots, m+1$;*

(iii) $\Pr(A_i) = 1/(2^i - 1), \quad i = 1, 2, \ldots.$

We now proceed with (6). To reduce the first internal maximum in (6) to a sum of exponentials, we can repeat the arguments in the proof of Lemma 3.1. Note that in the induction step we only affect the random variables $X_1, \ldots, X_{m+1}$ by conditioning on the random events $E_{i,k}$ (see (10)). Their sum (see (12)) as well as the other random variables $X_{m+2}, \ldots, X_{n+1}$ remain unaltered. Hence, during the induction, we only change (7) and do not affect the 'structure' of the remaining terms in (6). Note that we will loose this property as soon as $m+1 > n-m$. In this case replacements (10) will change not only (7), but also the other internal maximum in (6).

Once we have reduced the first internal maximum to a sum of exponentials, we can use the same arguments to also reduce the second internal maximum in (6), finally yielding the following theorem.

Theorem 3.3 *For any $m = 0, 1, \ldots;\ n \geq 2m+1$,*

$$T_n^{(m)} \stackrel{d}{=} 1 - \frac{1}{S_{n+1}} \max\left\{\sum_{j=1}^{m+1} a_{m+2-j}^{-1} X_j, \sum_{j=1}^{m+1} a_{m+2-j}^{-1} X_{n+2-j}\right\}, \tag{15}$$

where

$$a_j = 2^j - 1, \quad j \geq 0.$$

In the remainder of this section we derive the distribution of the random variable $K_n^{(m)}$ defined as the number of steps before the picker turns, when collecting n items under the m-step strategy. By symmetry, the probability that the route under the m-step strategy ends in clockwise direction is equal to 1/2. From (6) we see that the event A_i means that, among the routes ending in clockwise direction, the route with $i-1$ steps before a turn (i.e., the route skipping the spacing D_i) is better than any of the routes with $j-1 < i-1$ steps before a turn. Since the events A_i, $i = 1, \ldots, m+1$, do not provide information on the distribution of the two internal maxima in the last expression of (6), they are independent of the event that the route under the m-step strategy ends in clockwise direction. Hence, we obtain

$$\begin{aligned}\Pr\left(K_n^{(m)} = k\right) &= 2 \cdot \frac{1}{2} \cdot \Pr\left(A_{k+1} \bigcap_{i=k+2}^{m+1} \bar{A}_i\right) = \Pr\left(A_{k+1}\right) \prod_{i=k+2}^{m+1} \Pr\left(\bar{A}_i\right) \\ &= \frac{1}{2^{k+1}-1} \prod_{i=k+2}^{m+1} \frac{2^i - 2}{2^i - 1} = \frac{2^{m-k}}{2^{m+1}-1} = \frac{1}{2^{k+1} - 2^{k-m}}, \quad 0 \leq k \leq m,\end{aligned}$$

where the factor 2 in (16) takes into account the completely symmetrical event that the route under the m-step strategy ends in counterclockwise direction. Our findings are summarized in the following theorem.

Theorem 3.4 *For any m satisfying $2m+1 \leq n$,*

$$\Pr(K_n^{(m)} = k) = \frac{1}{2^{k+1} - 2^{k-m}}, \quad k = 0, 1, \ldots, m.$$

4 Distribution of the travel time

We will now use Theorem 3.3 to prove that $T_n^{(m)}$ can be expressed as a probabilistic mixture of spacings. Let us first consider

$$\max\left\{\sum_{j=1}^{m+1} a_{m+2-j}^{-1} X_j, \sum_{j=1}^{m+1} a_{m+2-j}^{-1} X_{n+2-j}\right\}. \tag{16}$$

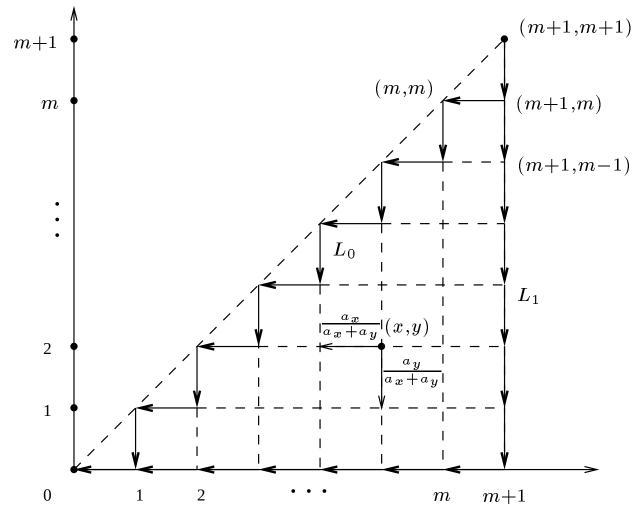

Figure 2: Markov chain interpretation of the maximum of two sums of exponentials.

Below we argue that this random variable is distributed as a probabilistic mixture of sums of $2(m+1)$ exponentials. This will be explained via the paths of the Markov chain in Figure 2. The states are the grid points (x, y) where $0 \le y \le x \le m+1$. From a state (x, y) with $0 < y < x$ it is possible to make a transition to $(x-1, y)$ with probability $a_x/(a_x + a_y)$, and to $(x, y-1)$ with probability $a_y/(a_x + a_y)$. From states on the horizontal axis and the diagonal, only one transition is possible. State $(0, 0)$ is absorbing. For this Markov chain we consider the paths, that start in $(m+1, m+1)$ and eventually end in $(0, 0)$, and show that these paths generate sums of exponentials representing maximum (16).

Starting from state $(m+1, m+1)$, we compare the terms $a_{m+1}^{-1}X_1$ and $a_{m+1}^{-1}X_{n+1}$ in (16). Without loss of generality we assume $[a_{m+1}^{-1}X_1 > a_{m+1}^{-1}X_{n+1}]$. Then $a_{m+1}^{-1}X_{n+1}$ is distributed as $(2a_{m+1})^{-1}X_{n+1}$, and we can take it outside the maximum, reducing the second sum by one term. Further, due to the memory-less property of exponentials, the overshoot of $a_{m+1}^{-1}X_1$ is independent of X_{n+1}, and it has the same distribution as $a_{m+1}^{-1}X_1$. Hence, the first sum in (16) remains the same, and therefore expression (16) is distributed as

$$(2a_{m+1})^{-1}X_{n+1} + \max\left\{\sum_{j=1}^{m+1} a_{m+2-j}^{-1}X_j, \sum_{j=2}^{m+1} a_{m+2-j}^{-1}X_{n+2-j}\right\}. \tag{17}$$

So, the transition from $(m+1, m+1)$ to $(m+1, m)$ can be interpreted as a transition from (16) to (17). By leaving state $(m+1, m+1)$ we have taken the term $(2a_{m+1})^{-1}X_{n+1}$ outside the maximum. Now we are at $(m+1, m)$, and we compare $a_{m+1}^{-1}X_1$ and $a_m^{-1}X_n$. If the event $[a_{m+1}^{-1}X_1 > a_m^{-1}X_n]$ takes place, then we take $(a_{m+1} + a_m)^{-1}X_n$ outside the maximum, and thus we reduce the second sum by one term again. Given $[a_{m+1}^{-1}X_1 < a_m^{-1}X_n]$ (the probability of this event is $a_{m+1}/(a_{m+1} + a_m)$), we take the term $(a_{m+1} + a_m)^{-1}X_1$ outside the maximum, and we reduce the first sum by one term. Hence, leaving $(m+1, m)$

we always get an exponential with mean $(a_{m+1}+a_m)^{-1}$ outside the maximum. With probability $a_m/(a_{m+1}+a_m)$ we make a transition to $(m+1, m-1)$, where the terms $a_{m+1}^{-1}X_1$ and $a_{m-1}^{-1}X_{n-1}$ are to be compared, and otherwise we move to (m,m), where we have to compare the terms $a_m^{-1}X_2$ and $a_m^{-1}X_n$.

We proceed in this way traveling from $(m+1, m+1)$ to $(0,0)$, without crossing the diagonal. Every time we leave a state (x,y), we get an exponential with mean $(a_x+a_y)^{-1}$ outside the maximum. A transition to $(x, y-1)$ means that the second sum in the maximum has been reduced by the first term; a transition to $(x-1, y)$ means the same for the first sum.

Let L denote the set of states visited along a path from $(m+1, m+1)$ to $(1,0)$, and let $\Pr(L)$ be the probability of this path, i.e., the product of the probabilities of each transition in path L. Then it is clear, from the exposition above, that along path L, maximum (16) becomes a linear combination of exponentials with coefficients $(a_x+a_y)^{-1}$, $(x,y)\in L$. For example, path L_1 in Figure 2 generates the sum

$$\sum_{j=1}^{m+1} a_{m+2-j}^{-1}X_j + \sum_{j=1}^{m+1}(a_{m+1}+a_{m+2-j})^{-1}X_{n+2-j}.$$

The probability that maximum (16) is distributed as the sum above is the product of transition probabilities

$$\frac{a_m}{a_{m+1}+a_m}\cdot\frac{a_{m-1}}{a_{m+1}+a_{m-1}}\cdots\frac{1}{a_{m+1}+1}.$$

One can also say that path L_1 corresponds to the event

$$\left[a_{m+1}^{-1}X_1 > \sum_{j=1}^{m+1} a_{m+2-j}^{-1}X_{n+2-j}\right].$$

We can conclude that, with probability $\Pr(L)$, maximum (16) is distributed as the sum

$$S(L) = \sum_{(x,y)\in L}(a_x+a_y)^{-1}X_{x+y}.$$

Note that each path goes through the states $(m+1, m+1)$, $(m+1, m)$ and $(1,0)$. Hence, $S(L)$ always includes exponentials with coefficients $(2a_{m+1})^{-1}$, $(a_{m+1}+a_m)^{-1}$ and 1.

It is readily verified that, just as in the derivation of Theorem 3.3, conditioning on path L does not alter the sum S_{n+1}. For example, after the first transition from $(m+1, m+1)$ to $(m+1, m)$, thus under event $[a_{m+1}^{-1}X_1 > a_{m+1}^{-1}X_{n+1}]$, we replace the X_j's by Y_j's as follows:

$$a_{m+1}^{-1}X_{n+1} = (2a_{m+1})^{-1}Y_{n+1}; \qquad a_{m+1}^{-1}X_1 = (2a_{m+1})^{-1}Y_{n+1} + a_{m+1}^{-1}Y_1;$$
$$X_j = Y_j, \quad j \neq 1, n+1,$$

where $Y_1, Y_2, \ldots$ are i.i.d. exponentials with mean 1. Since $X_1 + X_{n+1} = Y_1 + Y_{n+1}$ (cf. (12)), the sum S_{n+1} remains $Y_1 + Y_2 + \cdots + Y_{n+1}$. Renaming again the Y_j's by X_j's, we can repeat this procedure in the second transition, and so on. Hence, from Theorem 3.3 and (2), we obtain that, with probability $\Pr(L)$, the random variable $1-T_n^{(m)}$ is distributed as

$$T(L) = \sum_{(x,y)\in L} (a_x + a_y)^{-1} D_{x+y}.$$

This is summarized in the following theorem, where $\mathcal{L}(m)$ is the set of all paths from $(m+1, m+1)$ to $(1,0)$.

Theorem 4.1 *For any $m \geq 0$ and any $n \geq 2m+1$,*

$$\Pr(T_n^{(m)} < t) = \sum_{L\in\mathcal{L}(m)} \Pr(L) \Pr(1 - T(L) < t). \tag{18}$$

Remark 4.2 It is well-known (see, for example, Yaglom and Yaglom [11], problem 83a) that the cardinality of $\mathcal{L}(m)$ is a Catalan number, i.e.,

$$|\mathcal{L}(m)| = \frac{1}{m+2}\binom{2m+2}{m+1}. \tag{19}$$

For more detail on lattice path counting and various applications we refer to the book of Mohanty [7].

Remark 4.3 The probability $\Pr(L)$ is maximal for the path passing through all states on the diagonal, i.e., path L_0 in Figure 2. To prove this, we consider two possible ways to reach state (x,y) from $(x+1, y+1)$; see Figure 3. For $x = y$ route (a) is not possible. For

Figure 3: Two possible ways from $(x+1, y+1)$ to (x, y).

$x > y$ the probability of the (a)-route is

$$\frac{2^{x+1}-1}{2^{x+1}+2^{y+1}-2} \cdot \frac{2^{y+1}-1}{2^{x}+2^{y+1}-2}, \quad x > y+1; \qquad \frac{2^{x+1}-1}{2^{x+1}+2^{y+1}-2} \cdot 1, \quad x = y+1,$$

which is obviously larger than the probability of the (b)-route, given by

$$\frac{2^{y+1}-1}{2^{x+1}+2^{y+1}-2} \cdot \frac{2^{x+1}-1}{2^{x+1}+2^{y}-2}.$$

Hence, for $x > y$, replacing the (b)-route by the (a)-route always gives a more likely path. Thus, the probability of path L_1 in Figure 2 is the smallest, and the probability of the path L_0 is the biggest.

To obtain a tractable expression for (18), first note that

$$1 - T(L) = D_1 + D_2 + \cdots + D_{n+1} - \sum_{(x,y)\in L} a^{-1}_{(x,y)} D_{x+y},$$

where $a_{(x,y)} = a_x + a_y$. So, $1-T(L)$ is a linear combination of $n+1$ spacings; actually, only of n spacings, because $a_{(1,0)} = 1$, and thus D_1 vanishes. Then a closed-form expression for the right-hand side of (18) straightforwardly follows from Theorem 2 of Ali and Obaidullah [1]. This yields, for any $L \in \mathcal{L}(m)$,

$$\begin{aligned} \Pr\left(1 - T(L) < t\right) &= \sum_{(x,y)\in L} \left(a_{(x,y)}t - a_{(x,y)} + 1\right)^n_+ \prod_{\substack{(x',y')\in L \\ (x',y')\neq(x,y)}} \frac{a_{(x',y')}}{a_{(x',y')} - a_{(x,y)}}, \\ & \quad t < 1; \\ \Pr\left(1 - T(L) < t\right) &= 1, \quad t \geq 1. \end{aligned} \tag{20}$$

Here $t_+ = t$, if $t > 0$, and $t_+ = 0$, otherwise.

Example 4.4 We will derive the distributions of the travel time for the 0-, 1- and 2-step strategies using Figure 4, where we display $a_{(x,y)}$ at every state (x, y), $x, y = 0, 1, 2, 3$, and the transition probabilities at the arrows.

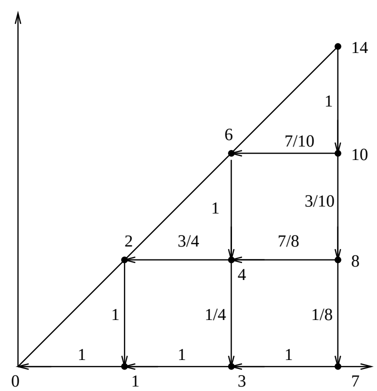

Figure 4: Illustration for the analysis of 0-, 1- and 2-step strategies.

Let us first consider the 0-step strategy, also known as the Shorter Direction (SD) heuristic described in [2, 10, 6]. Under the SD heuristic the picker is not allowed to turn; he chooses the shortest of two possible routes. In Figure 4 there is only one possible path from $(1, 1)$ to $(0, 0)$. Hence, the travel time under the SD heuristic satisfies

$$T_n^{(0)} \stackrel{d}{=} 1 - \max\{D_1, D_{n+1}\} \stackrel{d}{=} 1 - D_1 - \frac{1}{2}D_2 \tag{21}$$

with

$$\Pr\left(T_n^{(0)} < t\right) = 2t^n - (2t-1)^n_+, \quad 0 \leq t \leq 1.$$

Under the 1-step strategy the picker chooses the best of 4 routes. As we can see in Figure 4, there are two possible paths from $(2,2)$ to $(0,0)$, thus for $n \geq 3$ the travel time $T_n^{(1)}$ is distributed as:

$$\begin{array}{ll} 1 - D_1 - \frac{1}{2}D_2 - \frac{1}{4}D_3 - \frac{1}{6}D_4 & \text{with probability (w.p.) } 3/4, \\ 1 - D_1 - \frac{1}{3}D_2 - \frac{1}{4}D_3 - \frac{1}{6}D_4 & \text{w.p. } 1/4. \end{array} \tag{22}$$

Then

$$\begin{aligned} \Pr\left(T_n^{(1)} < t\right) &= 3t_+^n - \frac{9}{4}(2t-1)_+^n - (3t-2)_+^n + \frac{3}{2}(4t-3)_+^n - \frac{1}{4}(6t-5)_+^n, \\ & \quad 0 \leq t \leq 1. \end{aligned}$$

Finally, for $n \geq 5$, the travel time under the 2-step strategy is distributed as a mixture of 5 sums of spacings, corresponding to the 5 paths from $(3,3)$ to $(0,0)$. From Figure 4 it is clear that $T_n^{(2)}$ is distributed as:

$$\begin{array}{ll} 1 - D_1 - \frac{1}{2}D_2 - \frac{1}{4}D_3 - \frac{1}{6}D_4 - \frac{1}{10}D_5 - \frac{1}{14}D_6 & \text{w.p. } \frac{3}{4}\cdot\frac{7}{10}, \\ 1 - D_1 - \frac{1}{3}D_2 - \frac{1}{4}D_3 - \frac{1}{6}D_4 - \frac{1}{10}D_5 - \frac{1}{14}D_6 & \text{w.p. } \frac{1}{4}\cdot\frac{7}{10}, \\ 1 - D_1 - \frac{1}{2}D_2 - \frac{1}{4}D_3 - \frac{1}{8}D_4 - \frac{1}{10}D_5 - \frac{1}{14}D_6 & \text{w.p. } \frac{3}{4}\cdot\frac{7}{8}\cdot\frac{3}{10}, \\ 1 - D_1 - \frac{1}{3}D_2 - \frac{1}{4}D_3 - \frac{1}{8}D_4 - \frac{1}{10}D_5 - \frac{1}{14}D_6 & \text{w.p. } \frac{1}{4}\cdot\frac{7}{8}\cdot\frac{3}{10}, \\ 1 - D_1 - \frac{1}{3}D_2 - \frac{1}{7}D_3 - \frac{1}{8}D_4 - \frac{1}{10}D_5 - \frac{1}{14}D_6 & \text{w.p. } \frac{1}{8}\cdot\frac{3}{10}. \end{array} \tag{23}$$

It then follows that, for $0 \leq t \leq 1$,

$$\begin{aligned} \Pr(T_n^{(2)} < t) &= \frac{7}{2}t_+^n - \frac{49}{16}(2t-1)_+^n - \frac{7}{4}(3t-2)_+^n + \frac{49}{16}(4t-3)_+^n - \frac{49}{64}(6t-5)_+^n \\ &+ \frac{1}{4}(7t-6)_+^n - \frac{7}{16}(8t-7)_+^n + \frac{7}{32}(10t-9)_+^n - \frac{1}{64}(14t-13)_+^n. \end{aligned}$$

5 Moments of the travel time

In this section we shall calculate, for any path L, the moments of $T(L)$. From these moments we can obtain, by virtue of Theorem 4.1, the corresponding moments for the travel time $T_n^{(m)}$. For the kth moment of $T(L)$ we obtain

$$E\left([T(L)]^k\right) = E\left(\left(\sum_{(x,y)\in L} a_{(x,y)}^{-1} D_{x+y}\right)^k\right) = \binom{n+k}{k}^{-1} \sum_{\substack{k_1,k_2,\cdots,k_{2m+2}\geq 0 \\ k_1+k_2+\cdots+k_{2m+2}=k}} \prod_{(x,y)\in L} a_{(x,y)}^{-k_{x+y}},$$

where we used a well-known formula

$$E\left(D_1^{k_1} D_2^{k_2} \dots D_{2m+2}^{k_{2m+2}}\right) = \frac{k_1! k_2! \dots k_{2m+2}! n!}{(n + k_1 + k_2 + \cdots + k_{2m+2})!}.$$

The first two moments of $T(L)$ are given by

$$\begin{aligned} E(T(L)) &= \frac{1}{n+1} \sum_{(x,y)\in L} a_{(x,y)}^{-1}, \\ E\left([T(L)]^2\right) &= \frac{1}{(n+1)(n+2)} \left(\sum_{(x,y)\in L} a_{(x,y)}^{-2} + \left(\sum_{(x,y)\in L} a_{(x,y)}^{-1} \right)^2 \right). \end{aligned}$$

Example 5.1 The mean and variance of the travel time for the 0-, 1- and 2-step strategies can readily be derived from (21)–(23). For the 0-step strategy we obtain

$$\begin{aligned} E\left(T_n^{(0)}\right) &= 1 - \frac{3}{2(n+1)}, \qquad E\left(\left(T_n^{(0)}\right)^2\right) = \frac{n^2 - 1/2}{(n+1)(n+2)}, \\ Var\left(T_n^{(0)}\right) &= \frac{1}{4} \cdot \frac{5n-4}{(n+1)^2(n+2)}, \end{aligned}$$

and the 1-step strategy gives

$$\begin{aligned} E\left(T_n^{(1)}\right) &= 1 - \frac{15}{8(n+1)}, \qquad E\left(\left(T_n^{(1)}\right)^2\right) = \frac{144n^2 - 108n - 97}{144(n+1)(n+2)}, \\ Var\left(T_n^{(1)}\right) &= \frac{5}{576} \cdot \frac{151n - 254}{(n+1)^2(n+2)}, \end{aligned}$$

which is valid for $n \geq 3$. For the 2-step strategy we confine ourselves to the mean travel time only, yielding

$$E\left(T_n^{(2)}\right) = 1 - \frac{9073}{4480(n+1)}, \qquad n \geq 5. \tag{24}$$

Of course, it holds that $E(T_n^{(0)}) > E(T_n^{(1)}) > E(T_n^{(2)})$, $n \geq 5$.

6 Performance evaluation

In this section we present numerical results on the performance of the m-step strategy, and we compare it with the performance of the optimal pick strategy and the NI heuristic.

In Table 1 we list the mean and standard deviation of the travel time under the m-step strategy for various values of m and n, and we compare them with the ones for the optimal pick strategy and the NI heuristic. The random variables T_n^{OPT} and T_n^{NI} denote the travel time under the optimal strategy and the NI heuristic, respectively. For each n, the results for the optimal strategy have been obtained from a simulation of 10^6 trials; for the NI heuristic we have (see [6, 5])

$$E\left(T_n^{NI}\right) = 1 - \frac{2}{n+1} + \frac{1}{(n+1)2^n},$$
$$Var\left(T_n^{NI}\right) = \frac{1}{(n+1)(n+2)}\left(\frac{4n}{3} - \frac{8}{3} + \frac{1}{2^{n-2}} - \frac{n}{3\cdot 4^n} - \frac{1}{3\cdot 4^{n-1}}\right).$$

Hence, from (24), we can immediately conclude that

$$E\left(T_n^{(2)}\right) < E\left(T_n^{NI}\right), \qquad n \geq 5.$$

Thus, already for $m = 2$, the m-step strategy outperforms the NI heuristic.

n	m	$E(T_n^{(m)})$	$\sigma(T_n^{(m)})$	$E(T_n^{OPT})$	$\sigma(T_n^{OPT})$	$E(T_n^{NI})$	$\sigma(T_n^{NI})$
5	0	0.750	0.144	0.659	0.123	0.672	0.128
	1	0.688	0.131				
	2	0.663	0.123				
10	0	0.864	0.089	0.805	0.083	0.818	0.086
	1	0.830	0.087				
	2	0.816	0.085				
	3	0.810	0.084				
	4	0.807	0.083				
20	0	0.929	0.050	0.897	0.049	0.905	0.050
	1	0.911	0.050				
	2	0.904	0.049				
	3	0.900	0.049				
	4	0.899	0.049				
	5	0.898	0.049				
	6	0.898	0.049				

Table 1: Mean and standard deviation of the travel time.

The results in Table 1 show that, indeed, already for small values of m the performance of the m-step strategy is very close to optimal. This is not only valid for the mean and standard deviation of the travel time; it is also true for the distribution. This is demonstrated in Figure 5, where we display for $n = 10$ the complementary distribution function of the travel time for the optimal and NI strategy, and the 0-, 1-, 2- and 4-step strategy. The distribution function for the optimal strategy has been obtained from a simulation of 10^6 trials; the one for the NI strategy has been calculated exactly (see Theorem 3 in [5]).

The results suggest that, if the picker turns under the optimal strategy, then it is very likely that he does so after collecting a small number of items. In other words, already for small values of m, the optimal strategy will coincide with the m-step strategy with high probability. This is also confirmed by the results listed in Table 2. For various values of n, we estimated from a simulation of 10^6 trials, the probability that the picker, operating under the optimal strategy, will turn after collecting m items, $m = 0, 1, \ldots, 5$. Here $m = 0$ means that the picker does not turn.

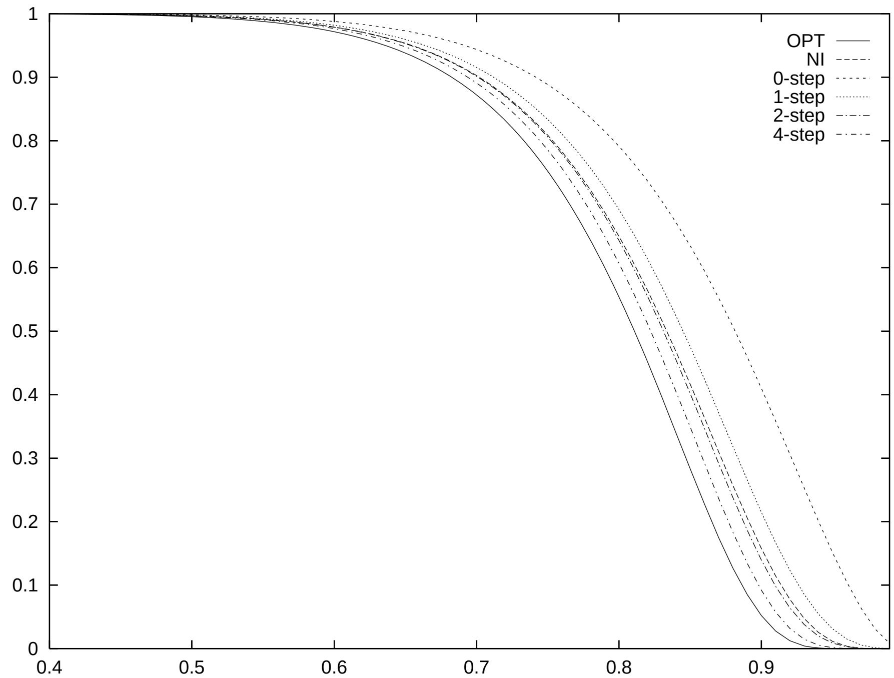

Figure 5: The complementary distribution function of the travel time.

From Table 2 one can see that the probability that the optimal route turns after k steps converges to $1/2^{k+1}$ when n goes to infinity. This is proved below. Let K_n^{OPT} be the number of steps before the picker turns under the optimal strategy for n items. Then the following assertion holds.

Theorem 6.1 *For any fixed k,*

$$\lim_{n\to\infty} \Pr\left(K_n^{OPT} = k\right) = \frac{1}{2^{k+1}}.$$

Proof. First note that under event $[K_n^{OPT} \leq m]$ the optimal streategy and the m-step strategy prescribe the same picking sequence. Hence, for any fixed $k \leq m$, event

n	m					
	0	1	2	3	4	5
3	0.646	0.291	0.062			
5	0.558	0.277	0.124	0.037	0.004	
8	0.516	0.259	0.129	0.062	0.026	0.008
10	0.506	0.254	0.127	0.063	0.030	0.013
15	0.501	0.251	0.126	0.062	0.031	0.016
20	0.499	0.250	0.125	0.062	0.031	0.016

Table 2: Probability that the picker, collecting a list of n items under the optimal strategy, will turn after m steps.

$[K_n^{OPT} = k]$ occurs if and only if (i) the optimal route turns after at most m steps, and (ii) the route under the m-step strategy turns after exactly k steps. Hence, for $0 \leq k \leq m$; $2m + 1 \leq n$,

$$\Pr\left(K_n^{(m)} = k\right) - \Pr\left(K_n^{OPT} > m\right) \leq \Pr\left(K_n^{OPT} = k\right) \leq \Pr\left(K_n^{(m)} = k\right). \tag{25}$$

By letting both m and n go to infinity such that the inequality $2m + 1 \leq n$ is always satisfied, we obtain from Theorem 3.4 that

$$\lim_{\substack{m,n\to\infty \\ 2m+1\leq n}} \Pr\left(K_n^{(m)} = k\right) = \frac{1}{2^{k+1}}. \tag{26}$$

Further, by using (14), we get

$$\begin{aligned} \Pr\left(K_n^{OPT} > m\right) &< 2 \cdot \Pr\left(\bigcup_{i=m+2}^{n} A_i\right) \\ &= 2 \cdot \sum_{i=m+2}^{n} \Pr\left(A_i\right) \prod_{j=m+2}^{i-1} \Pr\left(\bar{A}_j\right) < \frac{4}{2^{m+2} - 1}, \end{aligned}$$

yielding

$$\lim_{\substack{m,n\to\infty \\ 2m+1\leq n}} \Pr\left(K_n^{OPT} > m\right) = 0. \tag{27}$$

Now the statement of the theorem directly follows from (25)-(27). □

7 Conclusion

In this paper we studied the performance of so-called m-step strategies for order picking in paternosters. For uniformly distributed pick positions we found the distribution and the moments of the travel time needed to pick n items. The method presented in this paper is only applied to the case $2m + 1 \leq n$. In principle the method also works for larger values of m, but then the resulting expressions will become essentially more complicated.

We have seen that, already for small values of m, the performance of m-step strategies is very close to optimal. In practice, the NI heuristic is frequently used for order picking. Our analysis showed that the 2-step strategy on average performs better than the NI heuristic, and it may be even easier to implement.

For the optimal route we derived the probability of turning after k steps, as the number of items to be picked tends to infinity. However, the complete characterization of the distribution or the moments of the travel time under the optimal strategy remains a challenging open problem.

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