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Abstract

In this paper we consider sums, and maxima of sequential sums, of random variables with regularly varying distribution tails of index $-\beta$, with $\beta \in (1,2)$. We establish first-order approximations and some refinements for the distributions of these sums and maxima, in a large deviations area. Similar results are obtained for probabilities of crossing arbitrary curvilinear boundaries in the large deviations range by random walks generated by such sequential sums.

Keywords: Random walks, regularly varying distribution tails, refinements of large deviation probabilities, asymptotic expansions

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1 Introduction and main results

In this paper we study the asymptotic behaviour of the large deviations probabilities for the following random walks. Let $X_1, X_2, \ldots$ be independent, identically distributed random variables with distribution function $F(\cdot)$ and with zero mean. In the sequel we shall assume one or more of the following conditions to hold.
Condition [**R**]:

$$1 - F(t) = V(t) := t^{-\beta} L(t), \quad t > 0, \quad 1 < \beta < 2, \tag{1}$$

where $L(t)$ is a slowly varying function as $t \to \infty$, cf. [1].
Condition [$\mathbf{R}^+$]:

$$1 - F(t) \leq V(t), \quad t > 0. \tag{2}$$

Condition [$\mathbf{R}^-$]:

$$F(-t) \leq W(t) := t^{-\alpha} L_W(t), \quad t \geq 0, \quad \alpha > 1, \tag{3}$$

where $L_W(t)$ is a slowly varying function at infinity.
Let $S_0 := 0$ and, for $n = 1, 2, \ldots$:

$$\begin{aligned} S_n &:= X_1 + \ldots + X_n, \\ \bar{S}_n &:= \max_{k \leq n} S_k, \\ \bar{S}_n(a) &:= \max_{k \leq n}(S_k - ak), \quad a \geq 0. \end{aligned} \tag{4}$$

The main subject of study will be asymptotics of

$$\begin{aligned} V_n(x) &:= \mathrm{P}(S_n > x), \\ \bar{V}_n(x) &:= \mathrm{P}(\bar{S}_n > x), \\ \bar{V}_{n,a}(x) &:= \mathrm{P}(\bar{S}_n(a) > x). \end{aligned} \tag{5}$$

In the case of $\mathrm{E}X_i = 0$, the quantity $\bar{S}_n(a)$ has the same distribution as the maximum of a random walk with drift $\mathrm{E}X_i = -a \leq 0$.

We shall also obtain some asymptotic results for more general boundary crossing probabilities. For $x > 0$ and integer $n \geq 1$, denote by $\mathcal{G}_{x,n}$ the class of all boundaries $g(k) = g(k; x)$, $k = 0, 1, \ldots, n$, such that

$$\min_{1 \leq k \leq n} g(k) \geq x.$$

For a given boundary g, introduce the event

$$G_n = G_n(g) := \{\max_{k \le n} (S_k - g(k)) > 0\}, \tag{6}$$

that the random walk S_n crosses the boundary at or before time n. It is evident that in this general setting of the problem, the assumptions concerning EX_i are not essential.

The choice $g(k) = \infty$, $k < n$, and $g(n) = x$ results in

$$P(G_n) = V_n(x) \equiv P(S_n > x), \tag{7}$$

and the choice $g(k) \equiv x$, $k = 0, 1, \dots, n$ results in

$$P(G_n) = \bar{V}_n(x) \equiv P(\bar{S}_n > x). \tag{8}$$

The choice of g depends on the application one may have in mind. Examples of such applications are statistical sequential ratio tests, and queueing or risk theory; notice that in queueing theory, $\bar{S}_n(a)$ has the same distribution as the waiting time of the n-th customer in a single server queue with the First-Come-First-Served service discipline, with $X_k - a$ representing the difference between the k-th service time and the preceding interarrival time, the first customer arriving in an empty system.

Related literature. This paper is in a sense a continuation of the paper [7] (see also [5]), where refinements of asymptotics for the distributions of the quantities in (4) and (6) were found for the case of regularly varying distribution tails of index $\beta > 2$. See Nagaev [20] and Rozovskii [26] for references regarding first-order asymptotics for that case. First-order approximations for $P(S_n > x)$ in the case $\beta < 2$ were obtained by Heyde [13, 14], under the condition that the scaled sums converge in distribution to a stable law, and by Nagaev [22] under broader conditions for negative tails but only for deviations $x \ge cn$, $c > 0$. Godovanchuk [12] presented asymptotics of $P(\bar{S}_n > x)$ and $P(G_n)$ for $\beta < 2$ in the special case where $L(t)$ and $L_W(t)$ approach constants when $t \to \infty$, and where $\alpha \le \beta$. A first-order approximation of $P(\bar{S}_n > x)$ follows from estimates in Borovkov [6] (see Corollary 6.2 in [6]). First-order asymptotics of $P(\bar{S}_n(a) > x)$ in the case $\beta > 1$ were derived by Korshunov [18]. For $n = \infty$ these were earlier obtained by Borovkov [4], Cohen [8] and Veraverbeke [27]. Concerning exponentially decreasing tails, see for instance [2, 3, 16, 20, 24].

Goal and motivation of the paper. Our goals are to find refinements of the above-mentioned asymptotics, and to obtain first-order asymptotics for $P(\bar{S}_n > x)$ and $P(G_n)$ under more general conditions than in [12]. In the present study the distribution tails of the random variables X_i are regularly varying of index $-\beta$, $\beta \in (1,2)$, and hence the X_i have infinite variance. This case is not only scientifically interesting, but also highly relevant in understanding newly observed phenomena in, e.g., modern communication networks. Several studies have indicated that the traffic in such networks often exhibits features like burstiness, self-similarity and long-range dependence (see, e.g., [19] for Ethernet traffic, [15, 17] for MPEG video streams and [9] for Internet traffic). One of the key explanations of these phenomena is that file sizes often have a regularly varying distribution tail of index $-\beta$ with $\beta \in (1,2)$. This has triggered theoretical developments in the modeling and queueing analysis of regularly varying traffic phenomena. The influence of regularly varying (and subexponential) service time distributions on waiting time and workload distributions of the single-server queue has been investigated in considerable detail; many recent results are gathered in the book [23].

For $\beta > 2$, the usefulness of refined asymptotics was demonstrated in [7]. If $\beta > 2$, $EX^2 < \infty$, then we have in the large deviations region, $x > c\sqrt{n \log n}$, where $c > \sqrt{\beta - 2}$ (cf. [20, 26, 7]):

$$V_n(x) \;=\; P(S_n > x) \;\sim nV(x), \qquad x \to \infty. \tag{9}$$

In this case we also have [7, 25]: for $x > c\sqrt{n \log n}$, where $c > \sqrt{\beta - 2}$,

$$\bar{V}_n(x) \;=\; P(\bar{S}_n > x) \;\sim nV(x), \qquad x \to \infty. \tag{10}$$

Introducing $\bar{X}_n := \max_{k \leq n} X_k$, one obviously has $P(\bar{X}_n > x) = nV(x) - \frac{n(n-1)}{2} V^2(x) + O((nV(x))^3)$, so

$$P(\bar{X}_n > x) \;\sim nV(x) + O((nV(x))^2), \qquad x \to \infty, \tag{11}$$

and we have the same first-order approximation as in (9) and (10). But in (11) apparently $nV(x)$ is a rather good first-order approximation of $P(\bar{X}_n > x)$ when $nV(x)$ is small. For the case $\beta > 2$, $EX^2 < \infty$, it has turned out [7] that $nV(x)$ is a much worse approximation for $V_n(x)$ and for $\bar{V}_n(x)$; the use of asymptotic expansions for $V_n(x)$ and $\bar{V}_n(x)$ there proved to be very efficient in improving approximations for $P(S_n > x)$ and $P(\bar{S}_n > x)$. Just like in [7], one of our goals is to derive more precise asymptotic representations of $V_n(x)$ and $\bar{V}_n(x)$. Interestingly, we shall see that, for $\beta < 2$, the approximation $nV(x)$ for $V_n(x)$ (not for $\bar{V}_n(x)$) sometimes becomes in a sense as good as it is for $\mathrm{P}(\bar{X}_n > x)$. For $\beta \in (1,2)$ the usefulness of the refined asymptotics for $\mathrm{P}(\bar{S}_n > x)$ will be illustrated by a numerical example (see the end of Section 2).

Organization of the paper. Section 2 contains our main results, viz., refined asymptotics of $V_n(x)$, $\bar{V}_n(x)$, $\bar{V}_{n,a}(x)$ and of the general $\mathrm{P}(G_n)$. Proofs are presented in Section 3. The present section is closed with some notational conventions.
Throughout the paper, the symbol c (with and without indices) will denote a constant. It may take different values at different places.
$f_1(x) \sim f_2(x)$ denotes $\lim_{x\to\infty} f_1(x)/f_2(x) = 1$.

2 Main results

In order to obtain refined representations for $V_n(x)$ and $\bar{V}_n(x)$ we shall need an additional smoothness condition:

Condition [$\mathbf{R_d}$]. In addition to Condition [**R**], there exists a function $U(t) \sim -\beta$, $t \to \infty$, such that, for the function $V(\cdot)$ given in (1) and for $t \to \infty$:

$$\begin{aligned} V(t(1+\Delta)) &= V(t)[1 + U(t)\Delta + \mathrm{O}(|\Delta|^d)], \quad 1 < d \le 2; \\ V(t(1+\Delta)) &= V(t)[1 + \mathrm{O}(|\Delta|^d)], \quad d \le 1. \end{aligned} \tag{12}$$

If $L(t)$ is differentiable and $\frac{\mathrm{d}L(t)}{\mathrm{d}t} = \mathrm{o}(\frac{L(t)}{t})$, then in (12) for $d > 1$ we can identify $U(t)$ with $tV'(t)/V(t)$ where $V'(t)$ is the derivative of $V(t)$.

Our main results for $\mathrm{P}(S_n > x)$ and $\mathrm{P}(\bar{S}_n > x)$ are contained in Theorems 2.1 and 2.2 below. Proofs are presented in Section 3. In order to write the statements of the theorems we need to introduce two more functions. Let $t = N(n)$ be the solution of the equation $nV(t) = 1$: $t = N(n) = V^{(-1)}(\frac{1}{n})$, where $V^{(-1)}$ is the inverse function of V. Using the definition of a slowly varying function, one can write $N(n) = n^{1/\beta} L_1(n)$, $n \to \infty$, where $L_1(n)$ is a slowly varying function at infinity. $N_W(n)$ is introduced in a similar way, as solution of the equation $nW(t) = 1$. For the sake of simplicity we assume that $V(t)$ is continuous for $t \geq t_0$ and some $t_0 > 0$. Then one can write $V(N(n)) = \frac{1}{n}$ for $n > \frac{1}{V(t_0)}$.

Theorem 2.1 *(i) If Conditions [R] and [R⁻] hold and* $W(t) \leq c_1 V(t)$, $nV(x) \to 0$, *then*

$$\mathrm{P}(S_n > x) \;=\; nV(x)[1 + \mathrm{o}(1)]. \tag{13}$$

If $W(t) \leq c_1 V(t)$ *does not hold, then (13) is still true if* $nW(\frac{x}{|\ln\, nV(x)|}) \to 0$ *as* $x \to \infty$.

(ii) If Conditions [$\mathbf{R_d}$] and [$\mathbf{R}^-$] hold, then

$$\mathrm{P}(S_n > x) \;=\; nV(x)[1 + r_{n,x}], \qquad x \to \infty, \tag{14}$$

where

$$\begin{aligned} r_{n,x} \; &= \mathrm{O}(nV(x) + nW(x)) && \text{if } d > \beta, \\ &= \mathrm{O}((\tfrac{N(n)}{x})^d) && \text{if } d < \beta,\; W(t) = \mathrm{O}(V(t)), \\ &= \mathrm{O}(nW(x) + (\tfrac{N(n)}{x})^d) && \text{if } \alpha < d < \beta, \\ &= \mathrm{O}((\tfrac{N_W(n)}{x})^d) && \text{if } d < \alpha < \beta; \end{aligned}$$

the $\mathrm{O}(\cdot)$ *behaviour being uniform in* n *and* x *in the domain* $nV(x) \leq \epsilon$, $nW(x) \leq \epsilon$, *for any fixed* $\epsilon > 0$.

Remark 2.1 The last assertion of the theorem means that, e.g., under the conditions $d > \beta$, $nV(x) \leq \epsilon$, $nW(x) \leq \epsilon$, we can write $r_{n,x} \leq c(nV(x) + nW(x))$. It should be noted that $N(n)/x \to 0$ if $nV(x) \to 0$. Indeed, if $N(n) = x/s$, with s fixed and large, then $nV(\frac{x}{s}) = 1$, hence $nV(x) \sim s^{-\beta} = (\frac{N(n)}{x})^{\beta}$, from which the statement follows. Similarly, $N_W(n)/x \to 0$ if $nW(x) \to 0$.

It is easy to see that, instead of the above four representations of $r_{n,x}$, we can write

$$r_{n,x} = \mathrm{O}(nV(x) + nW(x) + (\frac{N(n)}{x})^d + (\frac{N_W(n)}{x})^d), \tag{15}$$

to capture all four cases. For $d = 2$ the last three cases disappear, as well as the last two summands in (15). We refrain from a discussion of the more delicate boundary cases (like $d = \beta$, $d = \alpha$); they require more complicated considerations.

Remark 2.2 If $W(x) = \mathrm{O}(V(x))$ and $d > \beta$, then according to Theorem 2.1 one has for $\mathrm{P}(S_n > x)$ the approximation $nV(x)[1 + \mathrm{O}(nV(x))]$, which has the same form as an approximation for $\mathrm{P}(\bar{X}_n > x)$, cf. (11). This is different from the case $\beta > 2$, where (cf. [7]) we have $\mathrm{P}(S_n > x) = nV(x)[1 + \frac{cn}{x^2} + \ldots]$.

Theorem 2.2 *(i) If Conditions [R] and [R⁻] hold and* $W(t) \leq c_1 V(t)$, $nV(x) \to 0$, *then*

$$\mathrm{P}(\bar{S}_n > x) \quad = \quad nV(x)[1 + \mathrm{o}(1)]. \qquad (16)$$

If $W(t) \leq c_1 V(t)$ *does not hold, then (16) is still true if* $nW(\frac{x}{|\ln\, nV(x)|}) \to 0$ *as* $x \to \infty$.
(ii) If Conditions [$\mathbf{R_d}$*], with* $1 < d \leq 2$, *and [R⁻] hold, then*

$$\begin{aligned}\mathrm{P}(\bar{S}_n > x) &= nV(x)[1 - \frac{U(x)}{nx}\sum_{j=1}^{n-1} \mathrm{E}\bar{S}_j + r_{n,x}] \\ &= nV(x)[1 + \frac{1}{nx}(\beta + \mathrm{o}(1))\sum_{j=1}^{n-1} \mathrm{E}\bar{S}_j + r_{n,x}], \qquad x \to \infty; \ (17)\end{aligned}$$

$r_{n,x}$ *is as specified in Theorem 2.1, and the convergence is uniform in* n *and* x *in the domain* $nV(x) \leq \epsilon$, $nW(x) \leq \epsilon$, *for any fixed* $\epsilon > 0$.
(iii) If Condition [$\mathbf{R_d}$*], with* $d \leq 1$, *and [R⁻] hold, then*

$$\mathrm{P}(\bar{S}_n > x) = nV(x)[1 + \mathrm{O}(\frac{N_W(n)}{x})^d], \qquad x \to \infty. \qquad (18)$$

Corollary 2.1 *Under the conditions of Part (ii) of Theorem 2.2, and either* $W(t) = \mathrm{o}(V(t))$ *or* $W(t) \sim cV(t)$, $c > 0$,

$$\mathrm{P}(\bar{S}_n > x) = nV(x)[1 + \frac{\beta^2}{\beta+1}\frac{N(n)}{x}\mathrm{E}\bar{\zeta}(1) + \mathrm{o}(\frac{N(n)}{x})], \qquad x \to \infty, \qquad (19)$$

where $\bar{\zeta}(1) := \max_{u \leq 1}\zeta(u)$, *with* $\zeta(u)$ *the stable law limit process for* $S_{[nu]}/N(n)$. *The* $\mathrm{o}(\cdot)$ *behaviour is uniform in* n *and* x *in the domain* $nV(x) \leq \epsilon$, $nW(x) \leq \epsilon$, *for any fixed sequence* $\epsilon \to 0$.

In the statement of the corollary, $N(n)/x \to 0$ if $nV(x) \le \epsilon \to 0$; cf. Remark 2.1.

Consider now the asymptotics of $\bar{V}_{n,a}(x) = \mathrm{P}(\bar{S}_n(a) > x)$, with $a > 0$.

Theorem 2.3 *(i) If Condition [$\mathbf{R}$] holds then*

$$\mathrm{P}(\bar{S}_n(a) > x) \;=\; [\sum_{j=1}^{n} V(x + ja)][1 + \mathrm{o}(1)], \quad x \to \infty. \tag{20}$$

(ii) If Conditions [$\mathbf{R_d}$] with $d = 1$ and [$\mathbf{R}^-$] hold, then for $x \to \infty$,

$$\mathrm{P}(\bar{S}_n(a) > x) = \sum_{j=1}^{n} V(x+ja) + \mathrm{O}(\frac{mN(m)V(x)}{x}) + \mathrm{O}(\frac{mN_W(m)W(x)}{x}), \tag{21}$$

with $m := \min(n, x)$.

For the general boundary crossing probability $\mathrm{P}(G_n)$, see (6), we have the following result:

Theorem 2.4 *(i) If Conditions [$\mathbf{R}$] and [$\mathbf{R}^-$] hold and $W(t) \le c_1 V(t)$, $nV(x) \to 0$, then for $g \in \mathcal{G}_{x,n}$,*

$$\mathrm{P}(G_n) \;=\; [\sum_{j=1}^{n} V(g_*(j))][1 + \mathrm{o}(1)], \tag{22}$$

where

$$g_*(j) \;:=\; \min_{j \le k \le n} g(k).$$

If $W(t) \le c_1 V(t)$ does not hold, then (22) is still true if $nW(\frac{x}{|\ln\, nV(x)|}) \to 0$ as $x \to \infty$.
(ii) If Conditions [$\mathbf{R_d}$], with $d = 1$, and [$\mathbf{R}^-$] hold, then

$$\mathrm{P}(G_n) \;=\; \sum_{j=1}^{n} V(g_*(j))[1 + \mathrm{O}(\frac{N(n)}{x}) + \mathrm{O}(\frac{N_W(n)}{x})], \quad x \to \infty; \tag{23}$$

the $\mathrm{O}(\cdot)$ estimates are uniform in $g \in \mathcal{G}_{x,n}$ for n and x in the domain $nV(x) \le \epsilon$, $nW(x) \le \epsilon$, for any fixed $\epsilon > 0$.

The case $d \neq 1$ can also be considered in Theorems 2.3 and 2.4, but the results won't be much better.

Below we present a numerical illustration of Theorem 2.2. We consider the case of symmetrically distributed X_j, with

$$\begin{aligned} P(X_1 > x) &= 0.5, \qquad 0 \leq x < \frac{4}{9}, \\ P(X_1 > x) &= \frac{4}{27} x^{-3/2}, \qquad x \geq \frac{4}{9}. \end{aligned}$$

Hence $\beta = 3/2$, $EX_1 = 0$. Conditions $[\mathbf{R}]$ and $[\mathbf{R_d}]$ (with $d = 2$) are satisfied. Figures 1 and 2 depict $P(\bar{S}_n > x)$ (the highest curve), the first-order approximation $nV(x)$ (the lowest curve) as well as the refined two-term expansion of Theorem 2.2, viz., $nV(x)[1 + \frac{3}{2nx} \sum_{j=1}^{n-1} E\bar{S}_j]$ (the middle curve). The values of $P(\bar{S}_n > x)$ and $E\bar{S}_j$ have been obtained via Monte Carlo simulation. We have generated 10^7 vectors $(X_1, \ldots, X_n)$ for $n = 10$ (Figure 1) and $n = 25$ (Figure 2). In both figures, x has been chosen in the range $(10, 40)$; this is a range in which $nV(x)$ is still not a very accurate approximation of $P(\bar{S}_n > x)$. The figures show that the two-term expansion of Theorem 2.2 may give a substantial improvement w.r.t. $nV(x)$ in this range.

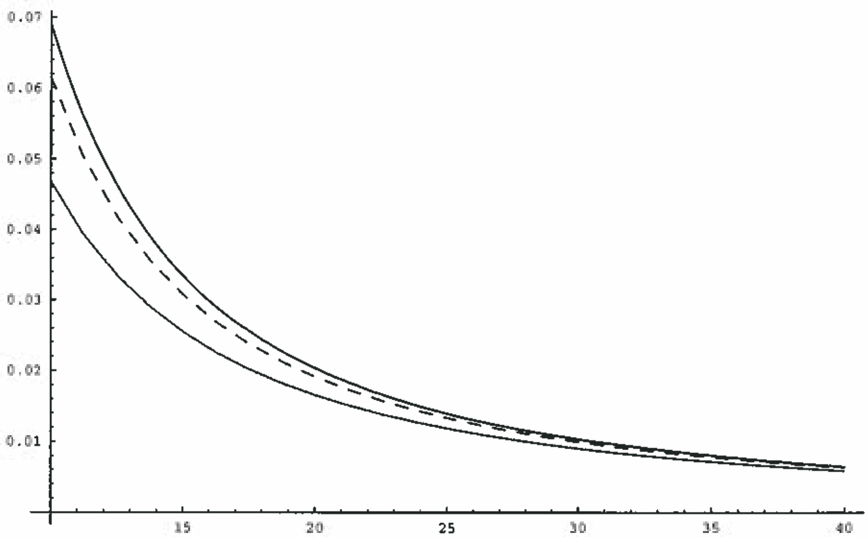

Figure 1: Figure with $n = 10$

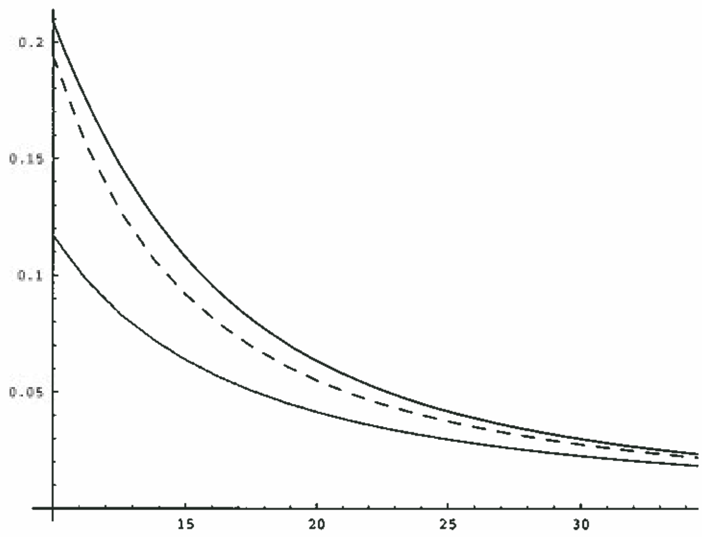

Figure 2: Figure with $n = 25$

3 Proofs

3.1. We start with the proof of Theorem 2.1.
Introduce $B_j := \{X_j \le y\}$, $B := \bigcap_{j=1}^{n} B_j$, and

$$P_n := \mathrm{P}(\bar{S}_n > x; B).$$

We shall choose $y \le x$ in such a way that the ratio $r := x/y$ is bounded. The following lemma gives bounds for P_n and $\mathrm{P}(\bar{S}_n > x)$ that will prove to be very useful.

Lemma 3.1 *Let Conditions [$\mathbf{R}^+$] and [$\mathbf{R}^-$] be fulfilled, and*

$$W(t) \le c_1 V(t). \tag{24}$$

Then for any sequence $\epsilon \downarrow 0$,

$$P_n \le (\frac{\mathrm{e}}{r})^r [nV(y)]^r (1 + \mathrm{o}(1)), \tag{25}$$

uniformly in n and y such that $nV(y) < \epsilon$. Besides,

$$\mathrm{P}(\bar{S}_n > x) \le nV(x)(1 + \mathrm{o}(1)), \tag{26}$$

uniformly in n and x such that $nV(x) < \epsilon \downarrow 0$.
If (24) does not hold, then (25), (26) still hold if

$$nW(\frac{y}{|\ln nV(y)|}) \to 0, \quad \text{as } y \to \infty. \tag{27}$$

The latter convergence will always take place if

$$nW(\frac{y}{\ln y}) \to 0, \quad \text{as } y \to \infty. \tag{28}$$

This lemma follows from Theorem 3.1 and Corollary 3.1 in [6].

Corollary 3.1 *If $n \leq x^\gamma$, $1 < \gamma < \min(\alpha, \beta)$, then (25), (26) are true without Conditions (24) or (28).*

The corollary is evident since $y^\gamma V(y) \to 0$, $y^\gamma W(\frac{y}{\ln y}) \to 0$ as $y \to \infty$, and (28) is fulfilled.

We now turn to the proof of Theorems 2.1-2.4. The main ideas of the proofs are similar to ideas in [7].
Proof of Theorem 2.1. Denote $G_n := \{S_n > x\}$. In the beginning of this section we have introduced $r = x/y$. In the proof we take $r = 2$. It follows from Lemma 3.1 and Corollary 3.1 for $y = x/2$ that

$$\begin{aligned} V_n(x) &= \mathrm{P}(G_n) = \mathrm{P}(G_n\bar{B}) + \mathrm{P}(G_n B) \\ &= \mathrm{P}(G_n\bar{B}) + \mathrm{O}((nV(x))^2). \end{aligned} \tag{29}$$

Here

$$\begin{aligned} \sum_{j=1}^{n} \mathrm{P}(G_n\bar{B}_j) \ &\geq \ \mathrm{P}(G_n\bar{B}) \ \geq \ \sum_{j=1}^{n} \mathrm{P}(G_n\bar{B}_j) \ - \ \sum_{i<j} \mathrm{P}(G_n\bar{B}_i\bar{B}_j) \\ &= \ \sum_{j=1}^{n} \mathrm{P}(G_n\bar{B}_j) \ + \ \mathrm{O}((nV(x))^2). \end{aligned} \tag{30}$$

So

$$V_n(x) = \sum_{j=1}^{n} \mathrm{P}(G_n\bar{B}_j) + \mathrm{O}((nV(x))^2). \tag{31}$$

Consider now, for $y = x/2$,

$$\begin{aligned} \mathrm{P}(G_n\bar{B}_j) &= \mathrm{P}(G_n\bar{B}_n) = \mathrm{P}(S_n > x, X_n > y) \\ &= \mathrm{P}(S_{n-1} + X_n > x, X_n > y) \\ &= \mathrm{P}(X_n > y, S_{n-1} \geq x - y) \\ &+ \mathrm{P}(S_{n-1} < x - y, S_{n-1} + X_n > x). \end{aligned} \tag{32}$$

Since X_n and S_{n-1} are independent, the first summand in the r.h.s. is seen to be bounded by $cnV^2(x)$ (use Lemma 3.1). The second summand can be written in the form

$$P \equiv \mathrm{E}[V(x - S_{n-1}); S_{n-1} < x/2] \equiv P_1 + P_2, \tag{33}$$

where

$$P_1 := \mathrm{E}[V(x - S_{n-1}); S_{n-1} \leq -x/2], \tag{34}$$

$$P_2 := \mathrm{E}[V(x - S_{n-1}); |S_{n-1}| < x/2]. \tag{35}$$

Using Lemma 3.1 for negative tails,

$$P_1 = \mathrm{E}[V(x - S_{n-1}); S_{n-1} \leq -x/2] \leq V(\frac{3x}{2})nW(\frac{x}{2}) = \mathrm{O}(nV(x)W(x)). \tag{36}$$

Let us now first prove Part (i). For this it suffices to show that, under the conditions of (i), it holds that $P_2 = V(x)(1 + \mathrm{o}(1))$.

Since $nV(x) \to 0$ we can always find $M = o(x)$ such that still $nV(M) \to 0$, $nW(\frac{M}{|\ln\ nV(M)|}) \to 0$. Then by Lemma 3.1

$$\mathrm{P}(|S_{n-1}| > M) \to 0, \tag{37}$$

and

$$P_2 = \mathrm{E}[V(x - S_{n-1}); |S_{n-1}| < M] + o(V(x)). \tag{38}$$

But $V(x - S) \sim V(x)$ for all S, $|S| < M$. Hence

$$P_2 = V(x)(1 + o(1)),$$

and statement (i) follows.

If smoothness condition $[\mathbf{R_d}]$ holds, then we are able to obtain more than these first-order asymptotics for $\mathrm{P}(S_n > x)$. Again consider P_2. First take

$d > \beta$. By Condition $[\mathbf{R_d}]$ we have for $d > \beta$ (without loss of generality we can take $\alpha \leq \beta$):

$$P_2 = V(x)\mathrm{E}[1 - U(x)\frac{S_{n-1}}{x} + \mathrm{O}(|\frac{S_{n-1}}{x}|^d); |S_{n-1}| < x/2]. \tag{39}$$

The following inequalities, to be proven below, will play a key role in the sequel – in fact, very similar inequalities are used in the proofs of each of the Theorems 2.1 – 2.4. For $k = 0, 1$:

$$T_n^{k,+} := \mathrm{E}[S_n^k; S_n \geq x/2] \leq cx^k nV(x); \tag{40}$$

$$T_n^{k,-} := \mathrm{E}[|S_n|^k; S_n \leq -x/2] \leq cx^k nW(x); \tag{41}$$

and for $d > \beta$:

$$I_n^{d,+} := \mathrm{E}[S_n^d; 0 < S_n < x/2] \leq cx^d nV(x); \tag{42}$$

$$I_n^{d,-} := \mathrm{E}[|S_n|^d; 0 > S_n > -x/2] \leq cx^d nW(x). \tag{43}$$

To prove (40) and (41), first do one partial integration (this is not needed for $k = 0$); e.g., in (40),

$$\mathrm{E}[S_{n-1}^k; S_{n-1} \geq x/2] = -\int_{x/2}^{\infty} u^k \, \mathrm{dP}(S_{n-1} > u)$$

$$= (\frac{x}{2})^k \mathrm{P}(S_{n-1} > \frac{x}{2}) + \int_{x/2}^{\infty} ku^{k-1}\mathrm{P}(S_{n-1} > u)\mathrm{d}u. \tag{44}$$

Subsequently use $S_{n-1} \leq \bar{S}_{n-1}$ and Lemma 3.1 in both resulting terms. It is then readily seen that both terms in the righthand side of (44) are bounded by a constant times $x^k nV(x)$. To obtain the bounds (42) and (43), first do one partial integration yielding $I_n^{d,+} \leq \int_0^{x/2} du^{d-1}\mathrm{P}(S_n > u)\mathrm{d}u$; similarly for $I_n^{d,-}$. In the above integral, distinguish between the cases $x/2 < N(n)$ and $x/2 \geq N(n)$. In the former case, bounding $\mathrm{P}(S_n > u)$ by 1,

$$I_n^{d,+} \leq \int_0^{N(n)} du^{d-1}\mathrm{P}(S_{n-1} > u)\mathrm{d}u \leq N^d(n). \tag{45}$$

As it follows from Remark 2.1, if $nV(x) \to 0$,

$$(\frac{N(n)}{x})^d = \mathrm{O}((nV(x))^{d/\beta}), \tag{46}$$

and (42) follows since $d > \beta$. In the case $x/2 \geq N(n)$,

$$I_n^{d,+} \leq \int_0^{N(n)} du^{d-1}\mathrm{P}(S_{n-1} > u)\mathrm{d}u + \int_{N(n)}^{x/2} du^{d-1}\mathrm{P}(S_{n-1} > u)\mathrm{d}u. \qquad (47)$$

The first integral is bounded as above. Using Lemma 3.1, the second integral is seen to be bounded by $c^* n(x/2)^{d-\beta}$. Finally (42) follows again. The inequality in (43) is proved in a similar way, replacing $N(n)$ by $N_W(n)$ and $V(x)$ by $W(x)$.

Using $\mathrm{E}S_{n-1} = 0$ and combining (39) and the four bounds (40)–(43) yields for $d > \beta$:

$$P_2 = V(x)[1 + \mathrm{O}(nV(x)) + \mathrm{O}(nW(x))]. \qquad (48)$$

Combining this with (31) – (36) proves Theorem 2.1 for $d > \beta$.

If $d < \beta$, then we still define $I_n^{d,+}$ and $I_n^{d,-}$ as in (42) and (43), but at least one of the two inequalities in (42) or (43) is no longer true. In this case we consider three subcases:
(1) $W(t) = \mathrm{O}(V(t))$;
(2) $\alpha < d < \beta$;
(3) $d < \alpha < \beta$.
If $d \leq 1$ then only subcase (3) holds.

Subcase (1). Now $x^d V(x) \to 0$ and $x^d W(x) \to 0$, and indeed both (42) and (43) are not true. However, we can to a large extent proceed as above for the case $d > \beta$. E.g., (45) still holds for $x/2 < N(n)$; for $x/2 \geq N(n)$, the only change in (47) is that the second integral is bounded from above by $c^{**}n(N(n))^{d-\beta}$. Observing that $N(n) > c_1 n^{1/\beta-\delta}$, $\forall \delta > 0$, it is seen that $x^{-d}I_n^{d,+} \leq c(N(n))^d$. Using $W(t) = \mathrm{O}(V(t))$, the same bound holds for $x^{-d}I_n^{d,-}$. Hence

$$P_2 = V(x)[1 + \mathrm{O}((\frac{N(n)}{x})^d)]. \qquad (49)$$

Combining this with (31) – (36), and using that $nV(x)$ and $nW(x)$ are $\mathrm{o}((\frac{N(n)}{x})^d)$ if $d < \beta$, it is seen that $r_{n,x} = \mathrm{O}((\frac{N(n)}{x})^d)$ in Subcase (1).

Subcase (2). Now (43) is true, leading to a term $\mathrm{O}(nW(x))$ in $r_{n,x}$. Instead of (42) we again have $I_n^{d,+} \leq c(N(n))^d$, leading to a term $\mathrm{O}((\frac{N(n)}{x})^d)$ in $r_{n,x}$.

Subcase (3). As in Subcase (1), both (42) and (43) are not true. Derivations similar to (45) and (47) yield $I_n^{d,-} \leq c(N_W(n))^d$. In this subcase the left tail of $F(\cdot)$ is heavier, and accordingly $I_n^{d,+} \leq c(N_W(n))^d$. Combining these bounds with (31) – (36), and using that $nW(x)$ and $nV(x)$ are $o((\frac{N_W(n)}{x})^d)$, it is seen that $r_{n,x} = O((\frac{N_W(n)}{x})^d)$. This completes the proof of Subcase (3).

If Condition $[\mathbf{R_d}]$ holds with $d \leq 1$, then Subcase (2) does not occur, and the analysis and statements of Subcases (1) and (3) remain unchanged. Theorem 2.1 is proved.

3.2 Proof of Theorem 2.2.

Write $G_n := P(\bar{S}_n > x)$. The proof begins just like the proof of Theorem 2.1, leading to the counterpart of (31):

$$\bar{V}_n(x) = \sum_{j=1}^{n} P(G_n \bar{B}_j) + O((nV(x))^2). \tag{50}$$

We can again choose $r = 2$, so that for $y = x/2$,

$$\begin{aligned} P(G_n \bar{B}_j) &= P(\bar{S}_n > x, X_j > y) \\ &= P(\bar{S}_n > x, X_j > y, \bar{S}_{j-1} > x) + P(\bar{S}_n > x, X_j > y, \bar{S}_{j-1} \leq x) \\ &= P(\bar{S}_n > x, X_j > y, \bar{S}_{j-1} \leq x) + O(jV^2(x)). \end{aligned} \tag{51}$$

The last step follows from Lemma 3.1. Note that $\{\bar{S}_n > x, X_j > y, \bar{S}_{j-1} \leq x\}$ $= \{S_j + \bar{S}^{(j)}_{n-j} > x, X_j > y, \bar{S}_{j-1} \leq x\}$, where $\bar{S}^{(j)}_{n-j} := \max_{0 \leq m \leq n-j}(S_{m+j} - S_j)$. Hence

$$\begin{aligned} P(G_n \bar{B}_j) &= P(S_j + \bar{S}^{(j)}_{n-j} > x, X_j > y, \bar{S}_{j-1} \leq x) + O(jV^2(x)) \\ &= P(S_j + \bar{S}^{(j)}_{n-j} > x, X_j > y) + O(jV^2(x)) \\ &= P(X_j + S_{j-1} + \bar{S}^{(j)}_{n-j} > x, X_j > y, S_{j-1} + \bar{S}^{(j)}_{n-j} \leq x/2) \\ &+ P(X_j + S_{j-1} + \bar{S}^{(j)}_{n-j} > x, X_j > y, S_{j-1} + \bar{S}^{(j)}_{n-j} > x/2) + O(jV^2(x)) \\ &= P(X_j + S_{j-1} + \bar{S}^{(j)}_{n-j} > x, S_{j-1} + \bar{S}^{(j)}_{n-j} \leq x/2) + O(nV^2(x)). \end{aligned} \tag{52}$$

In the second equality we have used the independence of X_j and $\bar{S}_{j-1}$, and Lemma 3.1. The last step follows since $\{X_j > y\}$ is redundant, while $P(X_j > y, S_{j-1} + \bar{S}^{(j)}_{n-j} > x/2) \leq P(X_j > y)P(\bar{S}_{n-1} > x/2) = O(nV^2(x))$. Denote

$$Z_{j,n} := S_{j-1} + \bar{S}^{(j)}_{n-j}. \tag{53}$$

Then

$$
\begin{aligned}
& \mathrm{P}(X_j + S_{j-1} + \bar{S}^{(j)}_{n-j} > x, S_{j-1} + \bar{S}^{(j)}_{n-j} \le x/2) \\
= \ & \mathrm{E}[\mathrm{E}[I(X_j > x - Z_{j,n}) | S_{j-1}, X_{j+1}, \ldots, X_n]; Z_{j,n} \le x/2] \\
= \ & \mathrm{E}[V(x - Z_{j,n}); Z_{j,n} \le x/2]. \qquad (54)
\end{aligned}
$$

It follows that

$$
\mathrm{P}(G_n) = \sum_{j=1}^{n} \mathrm{E}[V(x - Z_{j,n}); Z_{j,n} \le x/2] + \mathrm{O}((nV(x))^2). \qquad (55)
$$

Now note that

$$
S_{j-1} \le Z_{j,n} \overset{d}{\le} \bar{S}_{n-1}.
$$

Similar to (33), we divide the above summand into the terms $P_{j,1}$ and $P_{j,2}$, where (use Lemma 3.1 for negative tails):

$$
P_{j,1} := \mathrm{E}[V(x - Z_{j,n}); Z_{j,n} < -x/2] = \mathrm{O}(nV(x)W(x)). \qquad (56)
$$

The proof of Part (i) now proceeds similarly as the proof of Part (i) of Theorem 2.1, and is here further omitted. Now consider Part (ii). We shall prove that the first equality in the theorem holds; the second equality then immediately follows from $U(x) \sim -\beta$, $x \to \infty$. Consider

$$
P_{j,2} := \mathrm{E}[V(x - Z_{j,n}); |Z_{j,n}| \le x/2]. \qquad (57)
$$

We use $\mathrm{E}S_j = 0$ and Condition $[\mathbf{R_d}]$ for $1 < d \le 2$, which allows us to write (remember that $U(x) \sim -\beta$):

$$
\begin{aligned}
P_{j,2} &= V(x)\mathrm{E}[1 - U(x)\frac{Z_{j,n}}{x} + \mathrm{O}(|\frac{Z_{j,n}}{x}|^d); |Z_{j,n}| \le x/2] \\
&= V(x)\{1 - \mathrm{E}[Z^0_{j,n}; |Z_{j,n}| > x/2] - U(x)\frac{\mathrm{E}\bar{S}^{(j)}_{n-j}}{x} \\
&+ U(x)\mathrm{E}[\frac{Z_{j,n}}{x}; |Z_{j,n}| > x/2] + \mathrm{O}(\mathrm{E}[|\frac{Z_{j,n}}{x}|^d; |Z_{j,n}| \le x/2])\}. \qquad (58)
\end{aligned}
$$

First consider, for $k = 0, 1$:

$$
T^{k,+}_{j,n} := \mathrm{E}[Z^k_{j,n}; Z_{j,n} > x/2] \qquad (59)
$$

and

$$
T^{k,-}_{j,n} := \mathrm{E}[|Z_{j,n}|^k; Z_{j,n} < -x/2]. \qquad (60)
$$

Proceeding similarly as in (40) and (41), using $S_{n-1} \leq^d Z_{j,n} = S_{j-1} + \bar{S}^{(j)}_{n-j}$ for $T^{k,-}_{j,n}$ and $Z_{j,n} \leq^d \bar{S}_{n-1}$ for $T^{k,+}_{j,n}$, we find:

$$T^{k,+}_{j,n} = \mathrm{O}(x^k nV(x)), \tag{61}$$

$$T^{k,-}_{j,n} = \mathrm{O}(x^k nW(x)). \tag{62}$$

Finally consider

$$I^{d,+}_{j,n} := \mathrm{E}[|Z_{j,n}|^d; 0 < Z_{j,n} \leq x/2], \tag{63}$$

$$I^{d,-}_{j,n} := \mathrm{E}[|Z_{j,n}|^d; 0 > Z_{j,n} \geq -x/2]. \tag{64}$$

Distinguish between the same cases as in the proof of Theorem 2.1. First the case $d > \beta$. Then the expectation of $|Z_{j,n}|^d$ is not finite; however, using $Z_{j,n} \leq^d \bar{S}_{n-1}$ it is easily seen as in (42) that

$$x^{-d} I^{d,+}_{j,n} = x^{-d}\mathrm{O}(nx^d V(x)) = \mathrm{O}(nV(x)), \tag{65}$$

while (use $Z_{j,n} \geq^d S_{n-1}$ and (43)):

$$x^{-d} I^{d,-}_{j,n} = \mathrm{O}(nW(x)). \tag{66}$$

This proves Theorem 2.2 for $d > \beta$.

If $d < \beta$, then we consider the same three subcases as in the proof of Theorem 2.1.

Subcase (1): $W(t) = \mathrm{O}(V(t))$.

As with $I^{d,+}_n$ in the proof of the previous theorem, do one partial integration to obtain $I^{d,+}_{j,n} \leq \int_0^{x/2} du^{d-1}\mathrm{P}(Z_{j,n} > u)du$. Now use that $Z_{j,n} \leq^d \bar{S}_{n-1}$, bounding $\mathrm{P}(Z_{j,n} > u)$ by $\mathrm{P}(\bar{S}_{n-1} > u)$. From here on we can treat the above integral just like we handled $I^{d,+}_n$ in the proof of Subcase (1) of the previous theorem. And in handling $I^{d,-}_{j,n}$ we use that $Z_{j,n} \geq^d S_{n-1}$, so for $Z_{j,n} < 0$: $|Z_{j,n}| \leq |S_{n-1}|$. Again, we can from here on treat $I^{d,-}_{j,n}$ just like we handled $I^{d,-}_n$ in the proof of Subcase (1) of the previous theorem. This gives the result for Subcase (1).

Subcase (2): $\alpha < d < \beta$.

We have, as in (43), that

$$x^{-d} I^{d,-}_{j,n} = \mathrm{O}(nW(x)).$$

In handling $I_{j,n}^{d,+}$ again use $Z_{j,n} \leq^d \bar{S}_{n-1}$, and now proceed similar to Subcase (1) to show that

$$x^{-d} I_{j,n}^{d,+} = \mathrm{O}((\frac{N(n)}{x})^d).$$

This gives the result of Subcase (2).

Subcase (3): $d < \alpha < \beta$.
The left tail of $F(\cdot)$ now is the heavier one. Similar reasonings as in Subcase (1) yield:

$$x^{-d}(I_{j,n}^{d,+} + I_{j,n}^{d,-}) = \mathrm{O}((\frac{N_W(n)}{x})^d), \tag{67}$$

and accordingly $r_{n,x} = \mathrm{O}((\frac{N_W(n)}{x})^d)$.

Part (iii) of the theorem concerns the case $d \leq 1$. Condition $[\mathbf{R_d}]$ reduces to the last part of (12), resulting in small changes in (58) but not in (55)-(57). Formula (67) remains valid. The latter order term dominates $\mathrm{O}(nV(x) + nW(x))$ for $d \leq 1$, resulting in (18). This completes the proof.

Proof of Corollary 2.1. Corollary 2.1 follows from Formula (17) in Theorem 2.2. First observe that, under the conditions of Corollary 2.1, $S_j/N(j)$ converges weakly in distribution as $j \to \infty$ to the stable law F_β with parameter β, cf. [11]. Furthermore, $(\frac{|S_j|}{N(j)})^d$ is uniformly integrable. The uniform integrability follows from Lemma 3.1:

$$\begin{aligned} \mathrm{P}(|\frac{S_j}{N(j)}| > v) \leq cjV(vN(j)) &= cjv^{-\beta}N^{-\beta}(j)L(N(j))\frac{L(vN(j))}{L(N(j))} \\ &= cv^{-\beta}\frac{L(vN(j))}{L(N(j))} \leq v^{-\beta+\delta},\ \forall\delta > 0. \end{aligned} \tag{68}$$

It follows that $\mathrm{E}\bar{S}_j/N(j) \to \mathrm{E}\bar{\zeta}(1)$ for $j \to \infty$, see also Heyde [13]; here $\bar{\zeta}(1) := \max_{u\leq 1}\zeta(u)$, with $\zeta(u)$ the stable law limit process for $S_{[nu]}/N(n)$.

Now write $\sum_{j=1}^{n-1}\mathrm{E}\bar{S}_j = \sum_{j=1}^{n-1}\frac{\mathrm{E}\bar{S}_j}{N(j)}N(j)$, and remember that $N(j) = j^{1/\beta}L_1(j)$, $j \to \infty$. The corollary follows by writing, for $n \to \infty$:

$$\sum_{j=1}^{n-1} N(j) = \sum_{j=1}^{n-1} j^{1/\beta}L_1(j) \sim \int_1^n x^{1/\beta}L_1(x)\mathrm{d}x \sim \frac{1}{\frac{1}{\beta}+1}n^{\frac{1}{\beta}+1}L_1(n) = \frac{\beta}{\beta+1}nN(n).$$

3.3. We now turn to the tail asymptotics of the distribution of $\bar{S}_n(a) = \max_{k\le n}(S_k - ak)$, $a > 0$. Put $B_j(v) := \{X_j \le y + jv\}$, $B(v) := \bigcap_{j=1}^n B_j(v)$. To prove Theorem 2.3, we need the following lemma that takes the place of Lemma 3.1.

Lemma 3.2 *Let Condition [$\mathbf{R}^+$] be fulfilled. Then for all n, x and $v < a/2r$,*

$$\mathrm{P}(\bar{S}_n(a) > x; B(v)) \le c[mV(x)]^{r_1}, \tag{69}$$

$$\mathrm{P}(\bar{S}_n(a) > x) \le cmV(x), \tag{70}$$

where $r = x/y$, $m := \min(n, x)$, $r_1 := \frac{r}{1+vr}$.

The lemma follows from Theorem 3.2 and Corollary 3.3 in [6].

In order to prove Theorem 2.3, we need one more lemma. Put here

$$G_n := \{\bar{S}_n(a) > x\}, \qquad m := \min(n, x).$$

Lemma 3.3 *Let Condition [$\mathbf{R}^+$] be fulfilled. For all n and x such that $mV(x) \to 0$ and for $v \le \min(\frac{a}{2r}, \frac{r-2}{2r})$, $r > 2$,*

$$\mathrm{P}(\bar{S}_n(a) > x) = \sum_{j=1}^n \mathrm{P}(G_n \bar{B}_j(v)) + \mathrm{O}((mV(x))^2). \tag{71}$$

Proof. Somewhat similar to (30), we can write

$$\begin{aligned}\sum_{j=1}^n \mathrm{P}(G_n \bar{B}_j(v)) \quad &\ge \quad \mathrm{P}(G_n \bar{B}(v)) \;\ge\; \sum_{j=1}^n \mathrm{P}(G_n \bar{B}_j(v)) \\ &- \quad \sum_{i<j} \mathrm{P}(G_n \bar{B}_i(v) \bar{B}_j(v)) \\ &= \quad \sum_{j=1}^n \mathrm{P}(G_n \bar{B}_j(v)) \;+\; \mathrm{O}((mV(x))^2);\end{aligned} \tag{72}$$

here we have used that $\sum_{i<j} \mathrm{P}(G_n \bar{B}_i(v) \bar{B}_j(v)) \le (\sum_{j=1}^n \mathrm{P}(\bar{B}_j(v)))^2$ and that $\sum_{j=1}^n \mathrm{P}(\bar{B}_j(v)) \le cmV(x)$. Hence, using Lemma 3.2 for $v \le \min(\frac{a}{2r}, \frac{r-2}{2r})$ (in this case $r_1 \ge 2$), the lemma is proven.

We are now ready to prove Theorem 2.3. We shall choose $r = x/y$ and v such that $r_1 = 2$ (see Lemma 3.2).

Proof of Theorem 2.3. In view of the previous lemma, we start by studying $\mathrm{P}(G_n\bar{B}_j(v))$:

$$\begin{aligned}\mathrm{P}(G_n\bar{B}_j(v)) &= \mathrm{P}(\bar{S}_n(a) > x, X_j > y + jv) \\ &= \mathrm{P}(\bar{S}_n(a) > x, X_j > y + jv, \bar{S}_{j-1}(a) \le x) \\ &+ \mathrm{P}(\bar{S}_n(a) > x, X_j > y + jv, \bar{S}_{j-1}(a) > x) \\ &= \mathrm{P}(\bar{S}_n(a) > x, X_j > y + jv, \bar{S}_{j-1}(a) \le x) \\ &+ \rho_{n,j,x},\end{aligned} \tag{73}$$

where (use Lemma 3.2),

$$\rho_{n,j,x} \le cV(y + jv)\min(j, x)V(x).$$

It is evident that

$$\begin{aligned}&\{\bar{S}_n(a) > x, X_j > y + jv, \bar{S}_{j-1}(a) \le x\} \\ = \ &\{S_j(a) + \bar{S}^{(j)}_{n-j}(a) > x, X_j > y + jv, \bar{S}_{j-1}(a) \le x\},\end{aligned}$$

where $S_j(a) := S_j - ja$ and $\bar{S}^{(j)}_{n-j}(a) := \max(0, X_{j+1} - a, X_{j+1} + X_{j+2} - 2a, \ldots, S_n - S_j - (n-j)a)$. So putting $Z_{j,n}(a) := S_{j-1} + \bar{S}^{(j)}_{n-j}(a)$,

$$\begin{aligned}&\mathrm{P}(\bar{S}_n(a) > x, X_j > y + jv, \bar{S}_{j-1}(a) \le x) \\ = \ &\mathrm{P}(S_j(a) + \bar{S}^{(j)}_{n-j}(a) > x, X_j > y + jv, \bar{S}_{j-1}(a) \le x) \\ = \ &\mathrm{P}(S_{j-1} + X_j + \bar{S}^{(j)}_{n-j}(a) > x + ja, X_j > y + jv) + \rho_{n,j,x} \\ = \ &\mathrm{P}(X_j + Z_{j,n}(a) > x + ja, X_j > y + jv, \\ &Z_{j,n}(a) \le x + ja - y - jv) + \rho_{n,j,x}.\end{aligned} \tag{74}$$

Similar as before, observe that $\{X_j > y + jv\}$ is redundant, and rewrite the last probability as an expectation, yielding:

$$\begin{aligned}&\mathrm{P}(G_n\bar{B}_j(v)) = \mathrm{P}(\bar{S}_n(a) > x, X_j > y + jv) \\ = \ &\mathrm{E}[V(x + ja - Z_{j,n}(a)); Z_{j,n}(a) \le x + ja - y - jv] \\ + \ &\rho_{n,j,x}.\end{aligned} \tag{75}$$

In the above expectation, y and v can be chosen such that $x+ja-y-jv$ can be replaced by $x(j) := c(x+ja)$, with constant $c \in (0,1)$. Divide the resulting expectation, as before, into terms $P_{j,1}$ and $P_{j,2}$, where (use $S_{j-1} \leq Z_{j,n}(a)$ and Lemma 3.1 for negative tails):

$$\begin{aligned} P_{j,1} &:= \mathrm{E}[V(x+ja-Z_{j,n}(a)); Z_{j,n}(a) < -x(j)] \\ &= \mathrm{O}(jV(x+ja)W(x+ja)). \end{aligned} \tag{76}$$

The proof of Part (i) proceeds similarly as the proof of Part (i) of Theorem 2.1. One can prove that

$$P_{j,2} := \mathrm{E}[V(x+ja-Z_{j,n}(a)); |Z_{j,n}(a)| \leq x(j)] = V(x+ja)(1+\mathrm{o}(1)), \quad x \to \infty.$$

The details are here further omitted. Now consider Part (ii). Using Condition $[\mathbf{R_d}]$ with $d=1$, we can write:

$$\begin{aligned} P_{j,2} &= \mathrm{E}[V(x+ja-Z_{j,n}(a)); |Z_{j,n}(a)| \leq x(j)] \\ &= V(x+ja)\mathrm{E}[1-\mathrm{O}(|\frac{Z_{j,n}(a)}{x+ja}|); |Z_{j,n}(a)| \leq x(j)]. \end{aligned} \tag{77}$$

It should be noticed that if we had assumed Condition $[\mathbf{R_d}]$ to hold with $1 < d \leq 2$, then we would have had a contribution $\mathrm{E}\bar{S}^{(j)}_{n-j}(a)/(x+ja)$ in (77), but this expectation goes to infinity for $n-j \to \infty$. To emphasize the similarity with the proof of Theorem 2.2, let us define, similar to (59) – (60) and (63) – (64):

$$T^{0,+}_{j,n} := \mathrm{P}(Z_{j,n}(a) > x(j)), \tag{78}$$

$$T^{0,-}_{j,n} := \mathrm{P}(Z_{j,n}(a) < -x(j)), \tag{79}$$

$$I^{1,+}_{j,n} := \mathrm{E}[Z_{j,n}(a); 0 < Z_{j,n}(a) \leq x(j)], \tag{80}$$

$$I^{1,-}_{j,n} := \mathrm{E}[|Z_{j,n}(a)|; 0 > Z_{j,n}(a) \geq -x(j)]. \tag{81}$$

Since $Z_{j,n}(a) \geq S_{j-1}$,

$$T^{0,-}_{j,n} \leq cjW(x(j)). \tag{82}$$

Since

$$\mathrm{P}(Z_{j,n}(a) > x(j)) \leq \mathrm{P}(S_{j-1} > x(j)/2) + \mathrm{P}(\bar{S}^{(j)}_{n-j}(a) > x(j)/2), \tag{83}$$

we have

$$T^{0,+}_{j,n} \leq c_1 jV(x(j)) + c_2 m(x(j))V(x(j)), \tag{84}$$

with $m(u) := \min(n, u)$. It follows, with $m = \min(n, x)$, that:

$$\sum_{j=1}^{n} V(x+ja)(T_{j,n}^{0,+} + T_{j,n}^{0,-}) \leq c_3 \int_1^n tV^2(x+at)\ \mathrm{d}t$$

$$+ \quad c_4 \int_1^n m(x+at)V^2(x+at)\ \mathrm{d}t$$

$$+ \quad c_5 \int_1^n tV(x+at)W(x+at)\ \mathrm{d}t. \tag{85}$$

If $n \leq x$, then the sum of the three integrals is bounded by $c_6 n^2 V^2(x) + c_7 n^2 V(x)W(x)$. If $n > x$, then split each of these three integrals into a part $\int_1^x$ and a part $\int_x^n$. The former parts, summed, are bounded by $c_6 x^2 V^2(x) + c_7 x^2 V(x)W(x)$. The latter parts are readily seen to be bounded by $\int_x^\infty c_8 t^{1-2\beta}\mathrm{d}t = c_9 x^{2-2\beta} \leq c_{10} x^2 V^2(x)$. Hence, both for $n \leq x$ and $n > x$,

$$\sum_{j=1}^{n} V(x+ja)\mathrm{P}(|Z_{j,n}(a)| > x(j)) \leq cm^2V^2(x) + c^* m^2 V(x)W(x). \tag{86}$$

It remains to consider $I_{j,n}^{1,+}$ and $I_{j,n}^{1,-}$. First assume that $W(t) = \mathrm{O}(V(t))$. In $I_{j,n}^{1,+}$, we do one partial integration and use (83) to obtain:

$$\frac{I_{j,n}^{1,+}}{x+ja} \leq \frac{1}{x+ja}\int_0^{x(j)} \mathrm{P}(S_{j-1} > u/2)\ \mathrm{d}u + \frac{1}{x+ja}\int_0^{x(j)} \mathrm{P}(\bar{S}_{n-j}^{(j)}(a) > u/2)\ \mathrm{d}u. \tag{87}$$

Distinguish the cases $x(j) < N(j)$ and $x(j) \geq N(j)$. If $x(j) < N(j)$, then clearly $\frac{I_{j,n}^{1,+}}{x+ja} \leq \frac{N(j)}{x+ja}$. If $x(j) \geq N(j)$, then split the integral into a part $\int_0^{N(j)}$ – that is bounded by $N(j)$ – and a part $\int_{N(j)}^{x(j)}$. In a similar way as we bounded the sums above (cf. also (84)), we bound the latter integral by $cjx(j)V(x(j))$ $+cm(x(j))x(j)V(x(j))$. The analysis of $I_{j,n}^{1,-}$ is easier:

$$\frac{I_{j,n}^{1,-}}{x+ja} \leq \frac{1}{x+ja}\int_0^{x(j)} \mathrm{P}(S_{j-1} < -u)\ \mathrm{d}u \leq cjW(x(j)). \tag{88}$$

It follows that

$$\sum_{j=1}^{n} V(x+ja)\left(\frac{I_{j,n}^{1,+}}{x+ja} + \frac{I_{j,n}^{1,-}}{x+ja}\right)$$

$$\leq \quad c_1 \int_1^n V(x+at)\frac{N(t)}{x+at}\ \mathrm{d}t + c_2 \int_1^n tV^2(x+at)\ \mathrm{d}t$$

$$+ \quad c_3 m \int_1^n V^2(x+at)\ \mathrm{d}t + c_4 \int_1^n tV(x+at)W(x+at)\ \mathrm{d}t$$
$$\leq \quad c_5 \frac{V(x)}{x} \min(nN(n), xN(x)) + c_6 m^2 V^2(x) + c_7 m^2 V(x) W(x). \quad (89)$$

Since $N(k)$ is a nondecreasing function of k, we can write $\min(nN(n), xN(x)) = mN(m)$. Now notice that $mV(x) = \mathrm{o}(\frac{N(m)}{x})$ and that $mW(x) = \mathrm{o}(\frac{N(m)}{x})$. Finally use Lemma 3.3, (75), (76) and (77).

Now assume that $d = 1 < \alpha < \beta$, so that $W(t)$ dominates $V(t)$.
The analysis proceeds similarly as above, but now the term with $I_{j,n}^{1,-}$ is the dominant one, resulting in a leading term that is $\mathrm{O}(\frac{mN_W(m)W(x)}{x})$. Part (ii) of the theorem is proven.

3.4. Proof of Theorem 2.4.
Put here $G_n := \{\max_{k \leq n}(S_k - g(k)) > 0\}$, $B_j := \{X_j \leq y\}$, $B := \bigcap_{j=1}^n B_j$ and $P_n := \mathrm{P}(\bar{S}_n > x; B)$. Since $\min_{k \leq n} g(k) \geq x$, we have

$$\mathrm{P}(G_n B) \leq P_n.$$

According to Lemma 3.1, $P_n \leq c(nV(y))^r$. Hence

$$\mathrm{P}(G_n) = \mathrm{P}(G_n \bar{B}) + \mathrm{O}((nV(y))^r).$$

We also have (cf. the proof of Theorem 2.2):

$$\sum_{j=1}^n \mathrm{P}(G_n \bar{B}_j) \quad \geq \quad \mathrm{P}(G_n \bar{B}) \ \geq \ \sum_{j=1}^n \mathrm{P}(G_n \bar{B}_j) \ - \ \sum_{i<j} \mathrm{P}(G_n \bar{B}_i \bar{B}_j) \quad (90)$$
$$= \quad \sum_{j=1}^n \mathrm{P}(G_n \bar{B}_j) \ + \ \mathrm{O}((nV(x))^2).$$

Taking $r \geq 2$, it follows that

$$\mathrm{P}(G_n) = \sum_{j=1}^n \mathrm{P}(G_n, X_j > y) + \mathrm{O}((nV(y))^2). \quad (91)$$

Now consider $\mathrm{P}(G_n, X_j > y)$, proceeding very similarly as in the proof of Theorem 2.2. First write

$$\begin{aligned} & \mathrm{P}(G_n, X_j > y) \\ = \quad & \mathrm{P}(G_n, X_j > y, \bar{S}_{j-1} \leq x) + \mathrm{P}(G_n, X_j > y, \bar{S}_{j-1} > x) \\ = \quad & \mathrm{P}(G_n, X_j > y, \bar{S}_{j-1} \leq x) + \mathrm{O}(jV^2(x)). \end{aligned} \quad (92)$$

The last step follows from the independence of X_j and $\bar{S}_{j-1}$ and Lemma 3.1. Note that, if G_n holds and $\bar{S}_{j-1} \le x$ (hence G_{j-1} does not hold), then $\max_{j\le k\le n}(S_k - g(k)) > 0$. Now introduce

$$\begin{aligned} M_{j,n} &:= \max\nolimits_{0\le h\le n-j}(S_{h+j} - S_j - g(h+j)) + g_*(j) \\ &= \max\nolimits_{0\le h\le n-j}(S_{h+j} - g(h+j)) + g_*(j) - S_j, \end{aligned} \tag{93}$$

and write:

$$\begin{aligned} \mathrm{P}(G_n, X_j > y) &= \mathrm{P}(\max\nolimits_{j\le k\le n}(S_k - g(k)) > 0, X_j > y, \bar{S}_{j-1} \le x) + \mathrm{O}(jV^2(x)) \\ &= \mathrm{P}(\max\nolimits_{j\le k\le n}(S_k - g(k)) > 0, X_j > y) + \mathrm{O}(jV^2(x)) \\ &= \mathrm{P}(S_j + M_{j,n} > g_*(j), X_j > y, S_{j-1} + M_{j,n} \le x/2) \\ &+ \mathrm{P}(S_j + M_{j,n} > g_*(j), X_j > y, S_{j-1} + M_{j,n} > x/2) + \mathrm{O}(jV^2(x)) \\ &= \mathrm{P}(X_j + S_{j-1} + M_{j,n} > g_*(j), S_{j-1} + M_{j,n} \le x/2) \\ &+ \mathrm{O}(nV^2(x)). \end{aligned} \tag{94}$$

In the second equality we have used that $\mathrm{P}(X_j > y, \bar{S}_{j-1} > x) = \mathrm{O}(jV^2(x))$, cf. Lemma 3.1. In the last step we have used that the probability of the simultaneous occurrence of $\{X_j > y\}$ and $\{S_{j-1} + M_{j,n} > x/2\}$ is again $\mathrm{O}(nV^2(x))$ (notice that $M_{j,n} \le \max_{0\le h\le n-j}(S_{h+j} - S_j)$ and hence

$$S_{j-1} + M_{j,n} \le^d \bar{S}_{n-1}; \tag{95}$$

now apply Lemma 3.1 once more). Furthermore, since $g_*(j) \ge x$, one cannot simultaneously have $S_j + M_{j,n} > g_*(j)$ and also $X_j \le y = x/2$ and $S_{j-1} + M_{j,n} \le x/2$.

With a slight abuse of notation (but thus emphasizing the similarity with the proofs of Theorems 2.2 and 2.3), we again introduce quantities $Z_{j,n}$, now defined as

$$Z_{j,n} := S_{j-1} + M_{j,n}, \qquad j = 1, \dots, n. \tag{96}$$

Write

$$\begin{aligned} & \mathrm{P}(X_j + Z_{j,n} > g_*(j), Z_{j,n} \le x/2) \\ = \ & \mathrm{E}[\mathrm{E}[I(X_j > g_*(j) - Z_{j,n}) | S_{j-1}, X_{j+1}, \dots, X_n]; Z_{j,n} \le x/2] \\ = \ & \mathrm{E}[V(g_*(j) - Z_{j,n}); Z_{j,n} \le x/2]. \end{aligned} \tag{97}$$

It follows that

$$\mathrm{P}(G_n) = \sum_{j=1}^{n} \mathrm{E}[V(g_*(j) - Z_{j,n}); Z_{j,n} \le x/2] + \mathrm{O}((nV(x))^2). \tag{98}$$

As in the proofs of the previous theorems, we divide the above expectation into $P_{j,1}$ and $P_{j,2}$, where

$$P_{j,1} := \mathrm{E}[V(g_*(j) - Z_{j,n}); Z_{j,n} < -x/2] = \mathrm{O}(nV(x)W(x)). \qquad (99)$$

Now consider

$$P_{j,2} := \mathrm{E}[V(g_*(j) - Z_{j,n}); |Z_{j,n}| \leq x/2]. \qquad (100)$$

Use Condition $[\mathbf{R_d}]$ with $d = 1$, which allows us to write:

$$\begin{aligned} & \mathrm{E}[V(g_*(j) - Z_{j,n}); |Z_{j,n}| \leq x/2] \\ = \quad & V(g_*(j))\mathrm{E}[1 - \mathrm{O}(|\frac{Z_{j,n}}{g_*(j)}|); |Z_{j,n}| \leq x/2]. \end{aligned} \qquad (101)$$

Under Condition $[\mathbf{R_d}]$ with $d > 1$ there would have been a term $\mathrm{E}M_{j,n}$, but for general boundaries $g(k)$ it is difficult to make explicit statements about this expectation. $\mathrm{P}(|Z_{j,n}| \leq x/2) = 1 - \mathrm{P}(Z_{j,n} > x/2) - \mathrm{P}(Z_{j,n} < -x/2)$ in the righthand side of (101) is handled by using (95) and Lemma 3.1:

$$\mathrm{P}(Z_{j,n} > x/2) \leq \mathrm{P}(\bar{S}_{n-1} > x/2) = \mathrm{O}(nV(x)),$$

and similarly,

$$\mathrm{P}(Z_{j,n} < -x/2) \leq \mathrm{P}(S_{n-1} < -x/2) = \mathrm{O}(nW(x)).$$

Just like in Theorem 2.3, the case $d > \beta$ does not occur since $d = 1$, and we consider two cases for $d = 1 < \beta$:
(1) $W(t) = \mathrm{O}(V(t))$;
(2) $d = 1 < \alpha < \beta$.

Case (1): $W(t) = \mathrm{O}(V(t))$.
Remembering that $\bar{S}_h^{(j)} := \max_{0 \leq l \leq h} S_{l+j} - S_j$ and also introducing

$$\underline{S}_h^{(j)} := \min_{0 \leq l \leq h} S_{l+j} - S_j,$$

we can write:

$$\underline{S}_{n-j}^{(j)} \leq M_{j,n} \leq \bar{S}_{n-j}^{(j)}, \qquad (102)$$

since $\min x_n \leq \max (x_n - y_n) + \min y_n \leq \max x_n$ for any sequences x_n, y_n. Hence $\bar{S}_{n-j} \geq^d M_{j,n} \geq^d \min_{k \leq n-j} S_k$. In handling the $\mathrm{O}(\cdot)$ term in (101) we can now proceed similarly as in Subcase (1) in the proof of Theorem 2.2, using the above stochastic ordering. There results an estimate of $O(\frac{N(n)}{x})$ (notice that $g_*(j) \geq x$). The statement of the theorem for Case (1) now follows from (98), (99) and the fact that $nV(x)$ and $nW(x)$ are $o(\frac{N(n)}{x})$ since $\beta > 1$.

Case (2): $d = 1 < \alpha < \beta$. This case is handled by again using the above stochastic ordering, and then proceeding similarly as in Subcase (3) of Theorem 2.2. Since the left tail is now heavier, there results an estimate of $O(\frac{N_W(n)}{x})$. The theorem is proved.

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