On the volume of the intersection of two Wiener sausages

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Abstract

For $a > 0$, let $W^a_1(t)$ and $W^a_2(t)$ be the $a$-neighbourhoods of two independent standard Brownian motions in $\mathbb{R}^d$ starting at 0 and observed until time $t$. We prove that, for $d \geq 3$ and $c > 0$,

$$
\lim_{t \to \infty} \frac{1}{t^{(d-2)/d}} \log P\left(|W^a_1(ct) \cap W^a_2(ct)| \geq t\right) = -I^a_d(c)
$$

and derive a variational representation for the rate constant $I^a_d(c)$. Here, $\kappa_a$ is the Newtonian capacity of the ball with radius $a$. We show that the optimal strategy to realise the above large deviation is for $W^a_1(ct)$ and $W^a_2(ct)$ to “form a Swiss cheese”: the two Wiener sausages cover part of the space, leaving random holes whose sizes are of order 1 and whose density varies on scale $t^{1/d}$ according to a certain optimal profile.

We study in detail the function $c \mapsto I^a_d(c)$. It turns out that $I^a_d(c) = \Theta_d(\kappa_a c)/\kappa_a$, where $\Theta_d$ has the following properties: (1) For $d \geq 3$: $\Theta_d(u) < \infty$ if and only if $u \in (u_0, \infty)$, with $u_0$ a universal constant; (2) For $d = 3$: $\Theta_d$ is strictly decreasing on $(u_0, \infty)$ with a zero limit; (3) For $d = 4$: $\Theta_d$ is strictly decreasing on $(u_0, \infty)$ with a nonzero limit; (4) For $d \geq 5$: $\Theta_d$ is strictly decreasing on $(u_0, u_d)$ and a nonzero constant on $[u_d, \infty)$, with $u_d$ a constant depending on $d$ that comes from a variational problem exhibiting “leakage”. This leakage is interpreted as saying that the two Wiener sausages form their intersection until time $c^* t$, with $c^* = u_d/\kappa_a$, and then wander off to infinity in different directions. Thus, $c^*$ plays the role of a critical time horizon in $d \geq 5$.

We also derive the analogous result for $d = 2$, namely,

$$
\lim_{t \to \infty} \frac{1}{\log t} \log P\left(|W^a_1(ct) \cap W^a_2(ct)| \geq t/\log t\right) = -I^a_2(c),
$$

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