

# On the volume of the intersection of two Wiener sausages

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## Abstract

For  $a > 0$ , let  $W_1^a(t)$  and  $W_2^a(t)$  be the  $a$ -neighbourhoods of two independent standard Brownian motions in  $\mathbb{R}^d$  starting at 0 and observed until time  $t$ . We prove that, for  $d \geq 3$  and  $c > 0$ ,

$$\lim_{t \rightarrow \infty} \frac{1}{t^{(d-2)/d}} \log P\left(|W_1^a(ct) \cap W_2^a(ct)| \geq t\right) = -I_d^{\kappa_a}(c)$$

and derive a variational representation for the rate constant  $I_d^{\kappa_a}(c)$ . Here,  $\kappa_a$  is the Newtonian capacity of the ball with radius  $a$ . We show that the optimal strategy to realise the above large deviation is for  $W_1^a(ct)$  and  $W_2^a(ct)$  to “form a Swiss cheese”: the two Wiener sausages cover part of the space, leaving random holes whose sizes are of order 1 and whose density varies on scale  $t^{1/d}$  according to a certain optimal profile.

We study in detail the function  $c \mapsto I_d^{\kappa_a}(c)$ . It turns out that  $I_d^{\kappa_a}(c) = \Theta_d(\kappa_a c)/\kappa_a$ , where  $\Theta_d$  has the following properties: (1) For  $d \geq 3$ :  $\Theta_d(u) < \infty$  if and only if  $u \in (u_\diamond, \infty)$ , with  $u_\diamond$  a universal constant; (2) For  $d = 3$ :  $\Theta_d$  is strictly decreasing on  $(u_\diamond, \infty)$  with a zero limit; (3) For  $d = 4$ :  $\Theta_d$  is strictly decreasing on  $(u_\diamond, \infty)$  with a nonzero limit; (4) For  $d \geq 5$ :  $\Theta_d$  is strictly decreasing on  $(u_\diamond, u_d)$  and a nonzero constant on  $[u_d, \infty)$ , with  $u_d$  a constant depending on  $d$  that comes from a variational problem exhibiting “leakage”. This leakage is interpreted as saying that the two Wiener sausages form their intersection until time  $c^*t$ , with  $c^* = u_d/\kappa_a$ , and then wander off to infinity in different directions. Thus,  $c^*$  plays the role of a critical time horizon in  $d \geq 5$ .

We also derive the analogous result for  $d = 2$ , namely,

$$\lim_{t \rightarrow \infty} \frac{1}{\log t} \log P\left(|W_1^a(ct) \cap W_2^a(ct)| \geq t/\log t\right) = -I_2^{2\pi}(c),$$

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where the rate constant has the same variational representation as in  $d \geq 3$  after  $\kappa_a$  is replaced by  $2\pi$ . In this case  $I_2^{2\pi}(c) = \Theta_2(2\pi c)/2\pi$  with  $\Theta_2(u) < \infty$  if and only if  $u \in (u_\diamond, \infty)$  and  $\Theta_2$  is strictly decreasing on  $(u_\diamond, \infty)$  with a zero limit.

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## 1. Introduction and main results: Theorems 1–6

1.1. *Motivation.* In a paper that appeared in “The 1994 Dynkin Festschrift”, Khanin, Mazel, Shlosman and Sinai [9] considered the following problem. Let  $S(n), n \in \mathbb{N}_0$ , be the simple random walk on  $\mathbb{Z}^d$  and let

$$(1.1) \quad R = \{z \in \mathbb{Z}^d : S(n) = z \text{ for some } n \in \mathbb{N}_0\}$$

be its infinite-time range. Let  $R_1$  and  $R_2$  be two independent copies of  $R$  and let  $P$  denote their joint probability law. It is well known (see Erdős and Taylor [7]) that

$$(1.2) \quad P(|R_1 \cap R_2| < \infty) = \begin{cases} 0 & \text{if } 1 \leq d \leq 4, \\ 1 & \text{if } d \geq 5. \end{cases}$$

What is the tail of the distribution of  $|R_1 \cap R_2|$  in the high-dimensional case? In [9] it is shown that for every  $d \geq 5$  and  $\delta > 0$  there exists a  $t_0 = t_0(d, \delta)$  such that

$$(1.3) \quad \exp\left[-t^{\frac{d-2}{d}+\delta}\right] \leq P(|R_1 \cap R_2| \geq t) \leq \exp\left[-t^{\frac{d-2}{d}-\delta}\right] \quad \forall t \geq t_0.$$

Noteworthy about this result is the subexponential decay. The following problems remained open:

- (1) Close the  $\delta$ -gap and compute the rate constant.
- (2) Identify the “optimal strategy” behind the large deviation.
- (3) Explain where the exponent  $(d-2)/d$  comes from (which seems to suggest that  $d = 2$ , rather than  $d = 4$ , is a critical dimension).

In the present paper we solve these problems for the continuous space-time setting in which the simple random walks are replaced by Brownian motions and the ranges by Wiener sausages, but only after *restricting the time horizon to a multiple of  $t$* . Under this restriction we are able to fully describe the large deviations for  $d \geq 2$ . The large deviations beyond this time horizon will

remain open, although we will formulate a conjecture for  $d \geq 5$  (which we plan to address elsewhere).

Our results will draw heavily on some ideas and techniques that were developed in van den Berg, Bolthausen and den Hollander [3] to handle the large deviations for the volume of a single Wiener sausage. The present paper can be read independently.

Self-intersections of random walks and Brownian motions have been studied intensively over the past fifteen years (Lawler [10]). They play a key role e.g. in the description of polymer chains (Madras and Slade [13]) and in renormalisation group methods for quantum field theory (Fernández, Fröhlich and Sokal [8]).

1.2. *Wiener sausages.* Let  $\beta(t), t \geq 0$ , be the standard Brownian motion in  $\mathbb{R}^d$  – the Markov process with generator  $\Delta/2$  – starting at 0. The Wiener sausage with radius  $a > 0$  is the random process defined by

$$(1.4) \quad W^a(t) = \bigcup_{0 \leq s \leq t} B_a(\beta(s)), \quad t \geq 0,$$

where  $B_a(x)$  is the open ball with radius  $a$  around  $x \in \mathbb{R}^d$ .

Let  $W_1^a(t), t \geq 0$ , and  $W_2^a(t), t \geq 0$ , be two independent copies of (1.4), let  $P$  denote their joint probability law, let

$$(1.5) \quad V^a(t) = W_1^a(t) \cap W_2^a(t), \quad t \geq 0,$$

be their intersection up to time  $t$ , and let

$$(1.6) \quad V^a = \lim_{t \rightarrow \infty} V^a(t)$$

be their infinite-time intersection. It is well known (see e.g. Le Gall [11]) that

$$(1.7) \quad P(|V^a| < \infty) = \begin{cases} 0 & \text{if } 1 \leq d \leq 4, \\ 1 & \text{if } d \geq 5, \end{cases}$$

in complete analogy with (1.2). The aim of the present paper is to study the tail of the distribution of  $|V^a(ct)|$  for  $c > 0$  arbitrary. This is done in Sections 1.3 and 1.4 and applies for  $d \geq 2$ . We describe in detail the large deviation behaviour of  $|V^a(ct)|$ , including a precise analysis of the rate constant as a function of  $c$ . In Section 1.5 we formulate a conjecture about the large deviation behaviour of  $|V^a|$  for  $d \geq 5$ . In Section 1.6 we briefly look at the intersection volume of three or more Wiener sausages. In Section 1.7 we discuss the discrete space-time setting considered in [9]. In Section 1.8 we give the outline of the rest of the paper.

1.3. *Large deviations for finite-time intersection volume.* For  $d \geq 3$ , let  $\kappa_a = a^{d-2} 2\pi^{d/2} / \Gamma(\frac{d-2}{2})$  denote the Newtonian capacity of  $B_a(0)$  associated with the Green’s function of  $(-\Delta/2)^{-1}$ . Our main results for the intersection volume of two Wiener sausages over a finite time horizon read as follows:

THEOREM 1. *Let  $d \geq 3$  and  $a > 0$ . Then, for every  $c > 0$ ,*

$$(1.8) \quad \lim_{t \rightarrow \infty} \frac{1}{t^{(d-2)/d}} \log P(|V^a(ct)| \geq t) = -I_d^{\kappa_a}(c),$$

where

$$(1.9) \quad I_d^{\kappa_a}(c) = c \inf_{\phi \in \Phi_d^{\kappa_a}(c)} \left[ \int_{\mathbb{R}^d} |\nabla \phi|^2(x) dx \right]$$

with

$$(1.10) \quad \Phi_d^{\kappa_a}(c) = \left\{ \phi \in H^1(\mathbb{R}^d) : \int_{\mathbb{R}^d} \phi^2(x) dx = 1, \int_{\mathbb{R}^d} \left(1 - e^{-\kappa_a c \phi^2(x)}\right)^2 dx \geq 1 \right\}.$$

THEOREM 2. *Let  $d = 2$  and  $a > 0$ . Then, for every  $c > 0$ ,*

$$(1.11) \quad \lim_{t \rightarrow \infty} \frac{1}{\log t} \log P(|V^a(ct)| \geq t / \log t) = -I_2^{2\pi}(c),$$

where  $I_2^{2\pi}(c)$  is given by (1.9) and (1.10) with  $(d, \kappa_a)$  replaced by  $(2, 2\pi)$ .

Note that we are picking a time horizon of length  $ct$  and are letting  $t \rightarrow \infty$  for fixed  $c > 0$ . The sizes of the large deviation,  $t$  respectively  $t / \log t$ , come from the expected volume of a single Wiener sausage as  $t \rightarrow \infty$ , namely,

$$(1.12) \quad E|W^a(t)| \sim \begin{cases} \kappa_a t & \text{if } d \geq 3, \\ 2\pi t / \log t & \text{if } d = 2, \end{cases}$$

as shown in Spitzer [14]. So the two Wiener sausages in Theorems 1 and 2 are doing a large deviation *on the scale of their mean*.

The idea behind Theorem 1 is that the optimal strategy for the two Brownian motions to realise the large deviation event  $\{|V^a(ct)| \geq t\}$  is to behave like Brownian motions in a drift field  $xt^{1/d} \mapsto (\nabla \phi / \phi)(x)$  for some smooth  $\phi: \mathbb{R}^d \rightarrow [0, \infty)$  during the given time window  $[0, ct]$ . Conditioned on adopting this drift:

- Each Brownian motion spends time  $c\phi^2(x)$  per unit volume in the neighbourhood of  $xt^{1/d}$ , thus using up a total time  $t \int_{\mathbb{R}^d} c\phi^2(x) dx$ . This time must equal  $ct$ , hence the first constraint in (1.10).
- Each corresponding Wiener sausage covers a fraction  $1 - e^{-\kappa_a c \phi^2(x)}$  of the space in the neighbourhood of  $xt^{1/d}$ , thus making a total intersection volume  $t \int_{\mathbb{R}^d} (1 - e^{-\kappa_a c \phi^2(x)})^2 dx$ . This volume must exceed  $t$ , hence the second constraint in (1.10).

The cost for adopting the drift during time  $ct$  is  $t^{(d-2)/d} \int_{\mathbb{R}^d} c|\nabla \phi|^2(x) dx$ . The best choice of the drift field is therefore given by minimisers of the variational problem in (1.9) and (1.10), or by minimising sequences.

Note that the optimal strategy for the two Wiener sausages is to “form a Swiss cheese”: they cover only part of the space, leaving *random holes* whose sizes are of order 1 and whose density varies on space scale  $t^{1/d}$  (see [3]). The local structure of this Swiss cheese depends on  $a$ . Also note that the two Wiener sausages follow the optimal strategy *independently*. Apparently, under the joint optimal strategy the two Brownian motions are independent on space scales smaller than  $t^{1/d}$ .<sup>1</sup>

A similar optimal strategy applies for Theorem 2, except that the space scale is  $\sqrt{t/\log t}$ . This is only slightly below the diffusive scale, which explains why the large deviation event has a polynomial rather than an exponential cost. Clearly, the case  $d = 2$  is *critical* for a finite time horizon. Incidentally, note that  $I_2^{2\pi}(c)$  does not depend on  $a$ . This can be traced back to the recurrence of Brownian motion in  $d = 2$ . Apparently, the Swiss cheese has random holes that grow with time, washing out the dependence on  $a$  (see [3]).

There is no result analogous to Theorems 1 and 2 for  $d = 1$ : the variational problem in (1.9) and (1.10) certainly continues to make sense for  $d = 1$ , but it does not describe the Wiener sausages: holes are impossible in  $d = 1$ .

1.4. *Analysis of the variational problem.* We proceed with a closer analysis of (1.9) and (1.10). First we scale out the dependence on  $a$  and  $c$ . Recall from Theorem 2 that  $\kappa_a = 2\pi$  for  $d = 2$ .

THEOREM 3. *Let  $d \geq 2$  and  $a > 0$ .*

(i) *For every  $c > 0$ ,*

$$(1.13) \quad I_d^{\kappa_a}(c) = \frac{1}{\kappa_a} \Theta_d(\kappa_a c),$$

where  $\Theta_d: (0, \infty) \rightarrow [0, \infty]$  is given by

$$(1.14) \quad \Theta_d(u) = \inf \left\{ \|\nabla\psi\|_2^2 : \psi \in H^1(\mathbb{R}^d), \|\psi\|_2^2 = u, \int (1 - e^{-\psi^2})^2 \geq 1 \right\}.$$

(ii) *Define  $u_\diamond = \min_{\zeta>0} \zeta(1 - e^{-\zeta})^{-2} = 2.45541\dots$ . Then  $\Theta_d = \infty$  on  $(0, u_\diamond]$  and  $0 < \Theta_d < \infty$  on  $(u_\diamond, \infty)$ .*

(iii)  *$\Theta_d$  is nonincreasing on  $(u_\diamond, \infty)$ .*

(iv)  *$\Theta_d$  is continuous on  $(u_\diamond, \infty)$ .*

(v)  *$\Theta_d(u) \asymp (u - u_\diamond)^{-1}$  as  $u \downarrow u_\diamond$ .*

Next we exhibit the main quantitative properties of  $\Theta_d$ .

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<sup>1</sup>To prove that the Brownian motions *conditioned* on the large deviation event  $\{|V^a(ct)| \geq t\}$  actually follow the “Swiss cheese strategy” requires substantial extra work. We will not address this issue here.

THEOREM 4. Let  $2 \leq d \leq 4$ . Then  $u \mapsto u^{(4-d)/d} \Theta_d(u)$  is strictly decreasing on  $(u_\diamond, \infty)$  and

$$(1.15) \quad \lim_{u \rightarrow \infty} u^{(4-d)/d} \Theta_d(u) = \mu_d,$$

where

$$(1.16) \quad \mu_d = \begin{cases} \inf \left\{ \|\nabla \psi\|_2^2 : \psi \in H^1(\mathbb{R}^d), \|\psi\|_2 = 1, \|\psi\|_4 = 1 \right\} & \text{if } d = 2, 3, \\ \inf \left\{ \|\nabla \psi\|_2^2 : \psi \in D^1(\mathbb{R}^4), \|\psi\|_4 = 1 \right\} & \text{if } d = 4, \end{cases}$$

satisfying  $0 < \mu_d < \infty$ .<sup>2</sup>

THEOREM 5. Let  $d \geq 5$  and define

$$(1.17) \quad \eta_d = \inf \{ \|\nabla \psi\|_2^2 : \psi \in D^1(\mathbb{R}^d), \int (1 - e^{-\psi^2})^2 = 1 \}.$$

(i) There exists a radially symmetric, nonincreasing, strictly positive minimiser  $\psi_d$  of the variational problem in (1.17), which is unique up to translations. Moreover,  $\|\psi_d\|_2^2 < \infty$ .

(ii) Define  $u_d = \|\psi_d\|_2^2$ . Then  $u \mapsto \theta_d(u)$  is strictly decreasing on  $(u_\diamond, u_d)$  and

$$(1.18) \quad \Theta_d(u) = \eta_d \text{ on } [u_d, \infty).$$

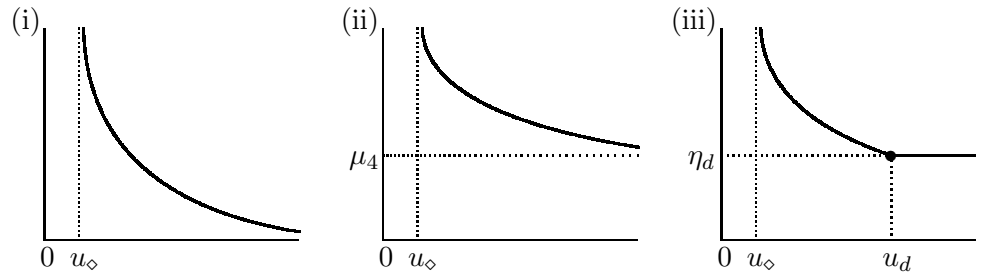


Figure 1 Qualitative picture of  $\Theta_d$  for: (i)  $d = 2, 3$ ; (ii)  $d = 4$ ; (iii)  $d \geq 5$ .

THEOREM 6. (i) Let  $2 \leq d \leq 4$  and  $u \in (u_\diamond, \infty)$  or  $d \geq 5$  and  $u \in (u_\diamond, u_d]$ . Then the variational problem in (1.14) has a minimiser that is strictly positive, radially symmetric (modulo translations) and strictly decreasing in the radial component. Any other minimiser is of the same type.

(ii) Let  $d \geq 5$  and  $u \in (u_d, \infty)$ . Then the variational problem in (1.14) does not have a minimiser.

<sup>2</sup>We will see in Section 5 that  $\mu_4 = S_4$ , the Sobolev constant in (4.3) and (4.4).

We expect that in case (i) the minimiser is unique (modulo translations). In case (ii) the critical point  $u_d$  is associated with “leakage” in (1.14); namely,  $L^2$ -mass  $u - u_d$  leaks away to infinity.

1.5. *Large deviations for infinite-time intersection volume.* Intuitively, by letting  $c \rightarrow \infty$  in (1.8) we might expect to be able to get the rate constant for an infinite time horizon. However, it is *not at all obvious* that the limits  $t \rightarrow \infty$  and  $c \rightarrow \infty$  can be interchanged. Indeed, the intersection volume might prefer to exceed the value  $t$  on a time scale of order larger than  $t$ , which is not seen by Theorems 1 and 2. The large deviations on this larger time scale are a whole new issue, which we will not address in the present paper.

Nevertheless, Figure 1(iii) clearly suggests that for  $d \geq 5$  the limits can be interchanged:

CONJECTURE. *Let  $d \geq 5$  and  $a > 0$ . Then*

$$(1.19) \quad \lim_{t \rightarrow \infty} \frac{1}{t^{(d-2)/d}} \log P(|V^a| \geq t) = -I_d^{\kappa_a},$$

where

$$(1.20) \quad I_d^{\kappa_a} = \inf_{c > 0} I_d^{\kappa_a}(c) = I_d^{\kappa_a}(c^*) = \frac{\eta d}{\kappa_a}$$

with  $c^* = u_d/\kappa_a$ .

The idea behind this conjecture is that the optimal strategy for the two Wiener sausages is *time-inhomogeneous*: they follow the Swiss cheese strategy until time  $c^*t$  and then wander off to infinity in different directions. The *critical time horizon*  $c^*$  comes from (1.13) and (1.18) as the value above which  $c \mapsto I_d^{\kappa_a}(c)$  is constant (see Fig. 1(iii)). During the time window  $[0, c^*t]$  the Wiener sausages make a Swiss cheese parametrised by the  $\psi_d$  in Theorem 5; namely, (1.9) and (1.10) have a minimising sequence  $(\phi_j)$  converging to  $\phi = (c^*\kappa_a)^{-1/2}\psi_d$  in  $L^2(\mathbb{R}^d)$ .

We see from Figure 1(ii) that  $d = 4$  is *critical* for an infinite time horizon. In this case the limits  $t \rightarrow \infty$  and  $c \rightarrow \infty$  apparently *cannot* be interchanged.

Theorem 4 shows that for  $2 \leq d \leq 4$  the time horizon in the optimal strategy is  $c = \infty$ , because  $c \mapsto I_d^{\kappa_a}(c)$  is strictly decreasing as soon as it is finite (see Fig. 1(i-ii)). Apparently, even though  $\lim_{t \rightarrow \infty} |V^a(t)| = \infty$   $P$ -almost surely (recall (1.7)), the rate of divergence is so small that a time of order larger than  $t$  is needed for the intersection volume to exceed the value  $t$  with a probability  $\exp[-o(t^{(d-2)/d})]$  respectively  $\exp[-o(\log t)]$ . So an even larger time is needed to exceed the value  $t$  with a probability of order 1.

1.6. *Three or more Wiener sausages.* Consider  $k \geq 3$  independent Wiener sausages, let  $V_k^a(t)$  denote their intersection up to time  $t$ , and let

$V_k^a = \lim_{t \rightarrow \infty} V_k^a(t)$ . Then the analogue of (1.7) reads (see e.g. Le Gall [11])

$$(1.21) \quad P(|V_k^a| < \infty) = \begin{cases} 0 & \text{if } 1 \leq d \leq \frac{2k}{k-1}, \\ 1 & \text{if } d > \frac{2k}{k-1}. \end{cases}$$

The critical dimension  $2k/(k - 1)$  comes from the following calculation:

$$(1.22) \quad E|V_k^a| = \int_{\mathbb{R}^d} P(\sigma_{B_a(x)} < \infty)^k dx = \int_{\mathbb{R}^d} \left[ 1 \wedge \left( \frac{a}{|x|} \right)^{d-2} \right]^k dx,$$

where  $\sigma_{B_a(x)} = \inf\{t \geq 0: \beta(t) \in B_a(x)\}$ . The integral converges if and only if  $(d - 2)k > d$ .

It is possible to extend the analysis in Sections 1.3 and 1.4 in a straightforward manner, leading to the following modifications (not proved in this paper):

(1) Theorems 1 and 2 carry over with:

- $V^a$  replaced by  $V_k^a$ ;
- $c$  replaced by  $kc/2$  in (1.9);
- $\int_{\mathbb{R}^d} (1 - e^{-\kappa_a c \phi^2(x)})^2 dx$  replaced by  $\int_{\mathbb{R}^d} (1 - e^{-\kappa_a c \phi^2(x)})^k dx$  in (1.10).

(2) Theorems 3, 4 and 5 carry over with:

- $\int (1 - e^{-\psi^2})^2$  replaced by  $\int (1 - e^{-\psi^2})^k$  in (1.14) and (1.17);
- $u_\diamond = \min_{\zeta > 0} \zeta (1 - e^{-\zeta})^{-k}$ ;
- $\|\psi\|_4$  replaced by  $\|\psi\|_{2k}$  in (1.16).

For  $k = 3$ , the critical dimension is  $d = 3$ , and a behaviour similar to that in Figure 1 shows up for: (i)  $d = 2$ ; (ii)  $d = 3$ ; (iii)  $d \geq 4$ , respectively. For  $k \geq 4$  the critical dimension lies strictly between 2 and 3, so that Figure 1(ii) drops out.

1.7. *Back to simple random walks.* We expect the results in Theorems 1 and 2 to carry over to the discrete space-time setting as introduced in Section 1.1. (A similar relation is proved in Donsker and Varadhan [6] for a single random walk, respectively, Brownian motion.) The only change should be that for  $d \geq 3$  the constant  $\kappa_a$  needs to be replaced by its analogue in discrete space and time:

$$(1.23) \quad \kappa = P(S(n) \neq 0 \ \forall n \in \mathbb{N}),$$

the escape probability of the simple random walk. The global structure of the Swiss cheese should be the same as before; the local structure should depend on the underlying lattice via the number  $\kappa$ .



1.8. *Outline.* Theorem 1 is proved in Section 2. The idea is to wrap the Wiener sausages around a torus of size  $Nt^{1/d}$ , to show that the error committed by doing so is negligible in the limit as  $t \rightarrow \infty$  followed by  $N \rightarrow \infty$ , and to use the results in [3] to compute the large deviations of the intersection volume on the torus as  $t \rightarrow \infty$  for fixed  $N$ . The wrapping is rather delicate because typically *the intersection volume neither increases nor decreases under the wrapping. Therefore we have to go through an elaborate clumping and reflection argument.* In contrast, the volume of a single Wiener sausage decreases under the wrapping, a fact that is very important to the analysis in [3].

Theorem 2 is proved in Section 3. The necessary modifications of the argument in Section 2 are minor and involve a change in scaling only.

Theorems 3–6 are proved in Sections 4–7. The tools used here are scaling and Sobolev inequalities. Here we also analyse the minimers of the variational problems in (1.14) and (1.17).

### 2. Proof of Theorem 1

By Brownian scaling,  $V^a(ct)$  has the same distribution as  $tV^{at^{-1/d}}(ct^{(d-2)/d})$ . Hence, putting

$$(2.1) \quad \tau = t^{(d-2)/d},$$

we have

$$(2.2) \quad P(|V^a(ct)| \geq t) = P(|V^{a\tau^{-1/(d-2)}}(c\tau)| \geq 1).$$

The right-hand side of (2.2) involves the Wiener sausages with a radius that shrinks with  $\tau$ . The claim in Theorem 1 is therefore equivalent to

$$(2.3) \quad \lim_{\tau \rightarrow \infty} \frac{1}{\tau} \log P(|V^{a\tau^{-1/(d-2)}}(c\tau)| \geq 1) = -I_d^{\kappa_a}(c).$$

We will prove (2.3) by deriving a lower bound (§2.2) and an upper bound (§2.3). To do so, we first deal with the problem on a finite torus (§2.1) and afterwards let the torus size tend to infinity. This is the standard compactification approach. On the torus we can use some results obtained in [3].

2.1. *Brownian motion wrapped around a torus.* Let  $\Lambda_N$  be the torus of size  $N > 0$ , i.e.,  $[-\frac{N}{2}, \frac{N}{2}]^d$  with periodic boundary conditions. Let  $\beta_N(s)$ ,  $s \geq 0$ , be the Brownian motion wrapped around  $\Lambda_N$ , and let  $W_N^{a\tau^{-1/(d-2)}}(s)$ ,  $s \geq 0$ , denote its Wiener sausage with radius  $a\tau^{-1/(d-2)}$ .

PROPOSITION 1.  $(|W_N^{a\tau^{-1/(d-2)}}(c\tau)|)_{\tau > 0}$  satisfies the large deviation principle on  $\mathbb{R}_+$  with rate  $\tau$  and with rate function

$$(2.4) \quad J_{d,N}^{\kappa_a}(b, c) = \frac{1}{2} c \inf_{\psi \in \Psi_{d,N}^{\kappa_a}(b,c)} \left[ \int_{\Lambda_N} |\nabla \psi|^2(x) dx \right],$$

where

$$(2.5) \quad \Psi_{d,N}^{\kappa_a}(b, c) = \left\{ \psi \in H^1(\Lambda_N) : \int_{\Lambda_N} \psi^2(x) dx = 1, \int_{\Lambda_N} \left( 1 - e^{-\kappa_a c \psi^2(x)} \right) dx \geq b \right\}.$$

*Proof.* See Proposition 3 in [3]. The function  $\psi$  parametrises the optimal strategy behind the large deviation:  $(\nabla\psi/\psi)(x)$  is the drift of the Brownian motion at site  $x$ ,  $c\psi^2(x)$  is the density for the time the Brownian motion spends at site  $x$ , while  $1 - e^{-\kappa_a c \psi^2(x)}$  is the density of the Wiener sausage at site  $x$ . The factor  $c$  enters (2.4) and (2.5) because the Wiener sausage is observed over a time  $c\tau$ . □

Proposition 1 gives us good control over the volume  $|W_N^{a\tau^{-1/(d-2)}}(\tau)|$ . In order to get good control over the intersection volume

$$(2.6) \quad \left| V_N^{a\tau^{-1/(d-2)}}(c\tau) \right| = \left| W_{1,N}^{a\tau^{-1/(d-2)}}(c\tau) \cap W_{2,N}^{a\tau^{-1/(d-2)}}(c\tau) \right|$$

of two independent shrinking Wiener sausages, observed until time  $c\tau$ , we need the analogue of Proposition 1 for this quantity, which reads as follows.

PROPOSITION 2.  $(|V_N^{a\tau^{-1/(d-2)}}(c\tau)|)_{\tau>0}$  satisfies the large deviation principle on  $\mathbb{R}_+$  with rate  $\tau$  and with rate function

$$(2.7) \quad \widehat{J}_{d,N}^{\kappa_a}(b, c) = c \inf_{\phi \in \widehat{\Phi}_{d,N}^{\kappa_a}(b,c)} \left[ \int_{\Lambda_N} |\nabla\phi|^2(x) dx \right],$$

where

$$(2.8) \quad \widehat{\Phi}_{d,N}^{\kappa_a}(b, c) = \left\{ \phi \in H^1(\Lambda_N) : \int_{\Lambda_N} \phi^2(x) dx = 1, \int_{\Lambda_N} \left( 1 - e^{-\kappa_a c \phi^2(x)} \right)^2 dx \geq b \right\}.$$

*Proof.* The extra power 2 in the second constraint (compare (2.5) with (2.8)) enters because  $(1 - e^{-\kappa_a c \phi^2(x)})^2$  is the density of the intersection of the two Wiener sausages at site  $x$ . The extra factor 2 in the rate function (compare (2.4) with (2.7)) comes from the fact that both Brownian motions have to follow the drift field  $\nabla\phi/\phi$ . The proof is a straightforward adaptation and generalization of the proof of Proposition 3 in [3]. We outline the main steps, while skipping the details.

*Step 1.* One of the basic ingredients in the proof in [3] is to approximate the volume of the Wiener sausage by its conditional expectation given a discrete skeleton. We do the same here. Abbreviate

$$(2.9) \quad \begin{aligned} W_i(c\tau) &= W_{i,N}^{a\tau^{-1/(d-2)}}(c\tau), \quad i = 1, 2, \\ V(c\tau) &= W_1(c\tau) \cap W_2(c\tau). \end{aligned}$$

Set

$$(2.10) \quad \mathbb{X}_{i,c\tau,\varepsilon} = \{\beta_i(j\varepsilon)\}_{1 \leq j \leq c\tau/\varepsilon}, \quad i = 1, 2,$$

where  $\beta_i(s)$ ,  $s \geq 0$ , is the Brownian motion on the torus  $\Lambda_N$  that generates the Wiener sausage  $W_i(c\tau)$ . Write  $\mathbb{E}_{c\tau,\varepsilon}$  for the conditional expectation given  $\mathbb{X}_{i,c\tau,\varepsilon}$ ,  $i = 1, 2$ . Then, analogously to Proposition 4 in [3], we have:

LEMMA 1. For all  $\delta > 0$ ,

$$(2.11) \quad \lim_{\varepsilon \downarrow 0} \limsup_{\tau \rightarrow \infty} \frac{1}{\tau} \log P\left(\left| |V(c\tau)| - \mathbb{E}_{c\tau,\varepsilon}(|V(c\tau)|) \right| \geq \delta\right) = -\infty.$$

*Proof.* The crucial step is to apply a concentration inequality of Talagrand twice, as follows. First note that, conditioned on  $\mathbb{X}_{i,c\tau,\varepsilon}$ ,  $W_i(c\tau)$  is a union of  $L = c\tau/\varepsilon$  independent random sets. Call these sets  $C_{i,k}$ ,  $1 \leq k \leq L$ , and write

$$(2.12) \quad V(c\tau) = \left(\bigcup_{k=1}^L C_{1,k}\right) \cap \left(\bigcup_{k=1}^L C_{2,k}\right).$$

Next note that, for any measurable set  $D \subset \Lambda_N$ , the function

$$(2.13) \quad \{C_k\}_{1 \leq k \leq L} \mapsto \left| \left(\bigcup_{k=1}^L C_k\right) \cap D \right|$$

is Lipschitz-continuous in the sense of equation (2.26) in [3], uniformly in  $D$ . From the proof of Proposition 4 in [3], we therefore get

$$(2.14) \quad \lim_{\varepsilon \downarrow 0} \limsup_{\tau \rightarrow \infty} \frac{1}{\tau} \log P\left(\left| |V(c\tau)| - E\left(|V(c\tau)| \mid \mathbb{X}_{1,c\tau,\varepsilon}, \beta_2\right) \right| \geq \delta \mid \beta_2\right) = -\infty,$$

uniformly in the realisation of  $\beta_2$ . On the other hand, the above holds true with  $\beta_1$  and  $\beta_2$  interchanged, and so we easily get

$$(2.15) \quad \lim_{\varepsilon \downarrow 0} \limsup_{\tau \rightarrow \infty} \frac{1}{\tau} \log P\left(\left| E\left(|V(c\tau)| \mid \mathbb{X}_{1,c\tau,\varepsilon}, \beta_2\right) - \mathbb{E}_{c\tau,\varepsilon}(|V(c\tau)|) \right| \geq \delta\right) = -\infty,$$

uniformly in the realisation of  $\beta_2$ . Clearly, (2.14) and (2.15) imply (2.11).  $\square$

*Step 2.* We fix  $\varepsilon > 0$  and prove an LDP for  $\mathbb{E}_{c\tau,\varepsilon}(|V(c\tau)|)$ , as follows. As in equation (2.43) in [3], define  $I_\varepsilon^{(2)}: \mathcal{M}_1^+(\Lambda_N \times \Lambda_N) \rightarrow [0, \infty]$  by

$$(2.16) \quad I_\varepsilon^{(2)}(\mu) = \begin{cases} h(\mu \mid \mu_1 \otimes \pi_\varepsilon) & \text{if } \mu_1 = \mu_2, \\ \infty & \text{otherwise,} \end{cases}$$

where  $h(\cdot|\cdot)$  denotes relative entropy between measures,  $\mu_1, \mu_2$  are the two marginals of  $\mu$  on  $\Lambda_N$ , and  $\pi_\varepsilon(x, dy) = p_\varepsilon(y-x)dy$  with  $p_\varepsilon$  the Brownian transi-

tion kernel on  $\Lambda_N$ . For  $\eta > 0$ , define  $\Phi_\eta: \mathcal{M}_1^+(\Lambda_N \times \Lambda_N) \times \mathcal{M}_1^+(\Lambda_N \times \Lambda_N) \mapsto [0, \infty)$  by

$$(2.17) \quad \begin{aligned} \Phi_\eta(\mu_1, \mu_2) &= \int_{\Lambda_N} dx \left\{ 1 - \exp \left[ -\eta \kappa_a \int_{\Lambda_N \times \Lambda_N} \varphi_\varepsilon(y-x, z-x) \mu_1(dy, dz) \right] \right\} \\ &\quad \times \left\{ 1 - \exp \left[ -\eta \kappa_a \int_{\Lambda_N \times \Lambda_N} \varphi_\varepsilon(y-x, z-x) \mu_2(dy, dz) \right] \right\}, \end{aligned}$$

where  $\varphi_\varepsilon$  is defined by

$$(2.18) \quad \varphi_\varepsilon(y, z) = \frac{\int_0^\varepsilon ds p_s(-y) p_{\varepsilon-s}(z)}{p_\varepsilon(z-y)}.$$

LEMMA 2.  $(\mathbb{E}_{c\tau, \varepsilon}(|V(c\tau)|))_{\tau > 0}$  satisfies the LDP on  $\mathbb{R}_+$  with rate  $\tau$  and with rate function

$$(2.19) \quad \begin{aligned} J_\varepsilon(b) &= \inf \left\{ \frac{c}{\varepsilon} \left( I_\varepsilon^{(2)}(\mu_1) + I_\varepsilon^{(2)}(\mu_2) \right) : \mu_1, \mu_2 \in \mathcal{M}_1^+(\Lambda_N \times \Lambda_N), \Phi_{c/\varepsilon}(\mu_1, \mu_2) = b \right\}. \end{aligned}$$

*Proof.* The proof is a straightforward extension of the proof of Proposition 5 in [3]. The basis is the observation that

$$(2.20) \quad \begin{aligned} &\mathbb{E}_{c\tau, \varepsilon}(|V(c\tau)|) \\ &= \int_{\Lambda_N} dx \mathbb{P}_{c\tau, \varepsilon}(x \in W_1(c\tau)) \mathbb{P}_{c\tau, \varepsilon}(x \in W_2(c\tau)) \\ &= \int_{\Lambda_N} dx \left\{ 1 - \exp \left[ \frac{c\tau}{\varepsilon} \int_{\Lambda_N \times \Lambda_N} \log(1 - q_{\tau, \varepsilon}(y-x, z-x)) L_{1, c\tau, \varepsilon}(dy, dz) \right] \right\} \\ &\quad \times \left\{ 1 - \exp \left[ \frac{c\tau}{\varepsilon} \int_{\Lambda_N \times \Lambda_N} \log(1 - q_{\tau, \varepsilon}(y-x, z-x)) L_{2, c\tau, \varepsilon}(dy, dz) \right] \right\}, \end{aligned}$$

where

$$(2.21) \quad q_{\tau, \varepsilon}(y, z) = P_y(\exists 0 \leq s \leq \varepsilon \text{ with } \beta_s \in B_{a\tau^{-1}/(d-2)}(0) \mid \beta_\varepsilon = z),$$

and  $L_{i, c\tau, \varepsilon}$  is the bivariate empirical measure

$$(2.22) \quad L_{i, c\tau, \varepsilon} = \frac{\varepsilon}{c\tau} \sum_{k=1}^{c\tau/\varepsilon} \delta_{(\beta_i((k-1)\varepsilon), \beta_i(k\varepsilon))}, \quad i = 1, 2.$$

Through a number of approximation steps we prove that

$$(2.23) \quad \lim_{\tau \rightarrow \infty} \left\| \mathbb{E}_{c\tau, \varepsilon}(|V(c\tau)|) - \Phi_{c/\varepsilon}(L_{1, c\tau, \varepsilon}, L_{2, c\tau, \varepsilon}) \right\|_\infty = 0 \quad \forall \varepsilon > 0.$$

This then proves our claim, since we can apply a standard LDP for  $\Phi_{c/\varepsilon}(L_{1,c\tau,\varepsilon}, L_{2,c\tau,\varepsilon})$ . The proof of (2.23) runs as in the proof of Proposition 5 in [3] via the following telescoping. Set

$$(2.24) \quad f_i(x) = \exp \left[ \frac{c\tau}{\varepsilon} \int_{\Lambda_N \times \Lambda_N} \log \left( 1 - q_{\tau,\varepsilon}(y-x, z-x) \right) L_{i,c\tau,\varepsilon}(dy, dz) \right],$$

$$g_i(x) = \exp \left[ -\frac{c\kappa_a}{\varepsilon} \int_{\Lambda_N \times \Lambda_N} \varphi_\varepsilon(y-x, z-x) L_{i,c\tau,\varepsilon}(dy, dz) \right].$$

Then

$$(2.25) \quad \mathbb{E}_{c\tau,\varepsilon}(|V(c\tau)|) - \Phi_{c/\varepsilon}(L_{1,c\tau,\varepsilon}, L_{2,c\tau,\varepsilon})$$

$$= \int_{\Lambda_N} dx [1 - f_1(x)][1 - f_2(x)] - \int_{\Lambda_N} dx [1 - g_1(x)][1 - g_2(x)]$$

$$= \int_{\Lambda_N} dx [g_1(x) - f_1(x)][1 - f_2(x)] + \int_{\Lambda_N} dx [1 - g_1(x)][g_2(x) - f_2(x)],$$

and hence

$$(2.26) \quad \left| \mathbb{E}_{c\tau,\varepsilon}(|V(c\tau)|) - \Phi_{c/\varepsilon}(L_{1,c\tau,\varepsilon}, L_{2,c\tau,\varepsilon}) \right|$$

$$\leq \int_{\Lambda_N} dx |g_1(x) - f_1(x)| + \int_{\Lambda_N} dx |g_2(x) - f_2(x)|.$$

We can therefore do the approximations on  $L_{1,c\tau,\varepsilon}$  and  $L_{2,c\tau,\varepsilon}$  separately, which is exactly what is done in [3]. In fact, the various approximations on pp. 371–377 in [3] have all been done by taking absolute values under the integral sign, and so the argument carries over. □

*Step 3.* The last step is a combination of the two previous steps to obtain the limit  $\varepsilon \downarrow 0$  in the LDP. If  $f: \mathbb{R}_+ \rightarrow \mathbb{R}$  is bounded and continuous, then from the two previous steps we get

$$(2.27) \quad \lim_{\tau \rightarrow \infty} \frac{1}{\tau} \log E(\exp[\tau|V(c\tau)|])$$

$$= \limsup_{\varepsilon \downarrow 0} \sup_{\mu_1, \mu_2} \left\{ f(\Phi_{c/\varepsilon}(\mu_1, \mu_2)) - \frac{c}{\varepsilon} \left( I_\varepsilon^{(2)}(\mu_1) + I_\varepsilon^{(2)}(\mu_2) \right) \right\}.$$

Now set, for  $\nu_1, \nu_2 \in \mathcal{M}_1(\Lambda_N)$ ,

$$(2.28) \quad \Psi_{c/\varepsilon}(\nu_1, \nu_2) = \int_{\Lambda_N} dx \left\{ 1 - \exp \left[ -\frac{c\kappa_a}{\varepsilon} \int_0^\varepsilon ds \int_{\Lambda_N} p_s(x-y) \nu_1(dy) \right] \right\}$$

$$\times \left\{ 1 - \exp \left[ -\frac{c\kappa_a}{\varepsilon} \int_0^\varepsilon ds \int_{\Lambda_N} p_s(x-y) \nu_2(dy) \right] \right\},$$

and, for  $f_1, f_2 \in L_1^+(\Lambda_N)$ ,

$$(2.29) \quad \Gamma(f_1, f_2) = \int_{\Lambda_N} dx \left(1 - e^{-c\kappa_a f_1(x)}\right) \left(1 - e^{-c\kappa_a f_2(x)}\right).$$

Then, repeating the approximation arguments on pp. 379–381 in [3], we get from (2.27) that

$$(2.30) \quad \lim_{\tau \rightarrow \infty} \frac{1}{\tau} \log E(\exp[\tau|V(c\tau)|]) = \lim_{K \rightarrow \infty} \lim_{\varepsilon \downarrow 0} \\ \times \sup_{\nu_1, \nu_2: \frac{1}{\varepsilon} I_\varepsilon(\nu_1) \leq K, \frac{1}{\varepsilon} I_\varepsilon(\nu_2) \leq K} \left\{ f(\Psi_{c/\varepsilon}(\nu_1, \nu_2)) - \frac{c}{\varepsilon} (I_\varepsilon(\nu_1) + I_\varepsilon(\nu_2)) \right\},$$

where  $I_\varepsilon$  is the rate function of the discrete-time Markov chain on  $\Lambda_N$  with transition kernel  $p_\varepsilon$ , i.e.,

$$(2.31) \quad I_\varepsilon(\nu) = \inf \left\{ I_\varepsilon^{(2)}(\mu) : \mu_1 = \nu \right\}.$$

The right-hand side of (2.30) equals (see equation (2.96) in [3])

$$(2.32) \quad \sup_{i=1,2: \phi_i \in H^1(\Lambda_N), \|\phi_i\|_2^2=1} \left\{ f(\Gamma(\phi_1^2, \phi_2^2)) - \frac{c}{2} (\|\nabla\phi_1\|_2^2 + \|\nabla\phi_2\|_2^2) \right\}.$$

(Line 3 on p. 381 in [3] contains a typo:  $f(\Gamma(\phi^2))$  should appear instead of  $f(\phi^2)$ .) Using the lemma by Bryc [5], we see from (2.30) and (2.32) that  $(V(c\tau))_{\tau>0}$  satisfies the LDP with rate  $\tau$  and with rate function

$$(2.33) \quad \hat{J}(b) = \inf \left\{ \frac{c}{2} (\|\nabla\phi_1\|_2^2 + \|\nabla\phi_2\|_2^2) : \|\phi_1\|_2^2 = \|\phi_2\|_2^2 = 1, \int_{\Lambda_N} dx \left(1 - e^{-c\kappa_a \phi_1^2(x)}\right) \left(1 - e^{-c\kappa_a \phi_2^2(x)}\right) \geq b \right\} \\ = \inf \left\{ c \|\nabla\phi\|_2^2 : \|\phi\|_2^2 = 1, \int_{\Lambda_N} dx \left(1 - e^{-c\kappa_a \phi^2(x)}\right)^2 \geq b \right\}.$$

The last equality, showing that the variational problem reduces to the diagonal  $\phi_1 = \phi_2$ , holds because if  $\phi^2 = \frac{1}{2}(\phi_1^2 + \phi_2^2)$ , then

$$(2.34) \quad 2|\nabla\phi|^2 \leq |\nabla\phi_1|_2^2 + |\nabla\phi_2|_2^2, \quad (1 - e^{-c\kappa_a \phi_1^2})(1 - e^{-c\kappa_a \phi_2^2}) \leq (1 - e^{-c\kappa_a \phi^2})^2.$$

This completes the proof of Proposition 2. □

2.2. *The lower bound in Theorem 1.* In this section we prove:

PROPOSITION 3. *Let  $d \geq 3$  and  $a > 0$ . Then, for every  $c > 0$ ,*

$$(2.35) \quad \liminf_{\tau \rightarrow \infty} \frac{1}{\tau} \log P\left(|V^{a\tau^{-1/(d-2)}}(c\tau)| \geq 1\right) \geq -I_d^{\kappa_a}(c),$$

where  $I_d^{\kappa_a}(c)$  is given by (1.9) and (1.10).

*Proof.* Let  $C_N(c\tau)$  denote the event that neither of the two Brownian motions comes within a distance  $a\tau^{-1/(d-2)}$  of the boundary of  $[-\frac{N}{2}, \frac{N}{2}]^d$  until time  $c\tau$ . Clearly,

$$(2.36) \quad P\left(|V^{a\tau^{-1/(d-2)}}(c\tau)| \geq 1\right) \geq P\left(|V_N^{a\tau^{-1/(d-2)}}(c\tau)| \geq 1, C_N(c\tau)\right) \quad \forall N > 0.$$

We can now simply repeat the argument that led to Proposition 2, but restricted to the event  $C_N(c\tau)$ . The result is that

$$(2.37) \quad \lim_{\tau \rightarrow \infty} \frac{1}{\tau} \log P\left(|V_N^{a\tau^{-1/(d-2)}}(c\tau)| \geq 1 \mid C_N(c\tau)\right) = -\tilde{J}_{d,N}^{\kappa_a}(1, c),$$

where  $\tilde{J}_{d,N}^{\kappa_a}(1, c)$  is given by the same formulas as in (2.7) and (2.8), except that  $\phi$  satisfies the extra restriction  $\text{supp}(\phi) \cap \partial\{[-\frac{N}{2}, \frac{N}{2}]^d\} = \emptyset$  (and  $b = 1$ ). We have

$$(2.38) \quad \lim_{\tau \rightarrow \infty} \frac{1}{\tau} \log P(C_N(c\tau)) = -2c\lambda_N$$

with  $\lambda_N$  the principal Dirichlet eigenvalue of  $-\Delta/2$  on  $[-\frac{N}{2}, \frac{N}{2}]^d$ . Hence (2.36)–(2.38) give

$$(2.39) \quad \liminf_{\tau \rightarrow \infty} \frac{1}{\tau} \log P\left(|V^{a\tau^{-1/(d-2)}}(c\tau)| \geq 1\right) \geq -\tilde{J}_{d,N}^{\kappa_a}(1, c) - 2c\lambda_N \quad \forall N > 0.$$

Let  $N \rightarrow \infty$  and use that  $\lim_{N \rightarrow \infty} \lambda_N = 0$  and

$$(2.40) \quad \lim_{N \rightarrow \infty} \tilde{J}_{d,N}^{\kappa_a}(1, c) = \hat{J}_d^{\kappa_a}(1, c) = I_d^{\kappa_a}(c),$$

to complete the proof. Here,  $\hat{J}_d^{\kappa_a}(1, c)$  is given by the same formulas as in (2.7) and (2.8), except that  $\phi$  lives on  $\mathbb{R}^d$  (and  $b = 1$ ). The convergence in (2.40) can be proved by the same argument as in [3, §2.6]. □

2.3. *The upper bound in Theorem 1.* In this section we prove:

PROPOSITION 4. *Let  $d \geq 3$  and  $a > 0$ . Then, for every  $c > 0$ ,*

$$(2.41) \quad \limsup_{\tau \rightarrow \infty} \frac{1}{\tau} \log P\left(|V^{a\tau^{-1/(d-2)}}(c\tau)| \geq 1\right) \leq -I_d^{\kappa_a}(c),$$

where  $I_d^{\kappa_a}(c)$  is as given by (1.9) and (1.10).

Propositions 3 and 4 combine to yield Theorem 1 by means of (2.1) and (2.2).

The proof of Proposition 4 will require quite a bit of work. The hard part is to show that the intersection volume of the Wiener sausages on  $\mathbb{R}^d$  is close to the intersection volume of the Wiener sausages wrapped around  $\Lambda_N$  when  $N$  is large. Note that the intersection volume may either increase or decrease when the Wiener sausages are wrapped around  $\Lambda_N$ , so there is no simple comparison available.

*Proof.* The proof is based on a clumping and reflection argument, which we decompose into 14 steps. Throughout the proof  $a > 0$  and  $c > 0$  are fixed.

1. Partition  $\mathbb{R}^d$  into  $N$ -boxes as

$$(2.42) \quad \mathbb{R}^d = \bigcup_{z \in \mathbb{Z}^d} \Lambda_N(z),$$

where  $\Lambda_N(z) = \Lambda_N + Nz$ . For  $0 < \eta < \frac{N}{2}$ , let  $S_{\eta,N}$  denote the  $\frac{1}{2}\eta$ -neighborhood of the faces of the boxes, i.e., the set that when wrapped around  $\Lambda_N$  becomes  $\Lambda_N \setminus \Lambda_{N-\eta}$ . For convenience let us take  $N/\eta$  as an integer. If we shift  $S_{\eta,N}$  by  $\eta$  exactly  $N/\eta$  times in each of the  $d$  directions, then we obtain  $dN/\eta$  copies of  $S_{\eta,N}$ :

$$(2.43) \quad S_{\eta,N}^j, \quad j = 1, \dots, \frac{dN}{\eta},$$

and each point of  $\mathbb{R}^d$  is contained in exactly  $d$  copies.

2. We are going to look at how often the two Brownian motions cross the slices of width  $\eta$  that make up all of the  $S_{\eta,N}^j$ 's. To that end, consider all the hyperplanes that lie at the center of these slices and all the hyperplanes that lie at a distance  $\frac{1}{2}\eta$  from the center (making up the boundary of the slices). Define an  $\eta$ -crossing to be a piece of the Brownian motion path that crosses a slice and lies fully inside this slice. Define the *entrance-point* (*exit-point*) of an  $\eta$ -crossing to be the point at which the crossing hits the central hyperplane for the first (last) time. We are going to reflect the Brownian motion paths in various central hyperplanes with the objective of moving them inside a large box. We will do the reflections only on those excursions that begin with an exit-point at a given central hyperplane and end with the next entrance-point at the same central hyperplane, thus leaving unreflected those parts of the path that begin with an entrance-point and end with the next exit-point. This is done because the latter cross the central hyperplane too often and therefore would give rise to an entropy associated with the reflection that is too large. In order to control the entropy we need the estimates in Lemmas 3–5 below.

3. Abbreviate

$$(2.44) \quad \mathcal{O}_{c\tau} = \left\{ \beta_i(s) \in [-\tau^2, \tau^2]^d \forall s \in [0, c\tau], i = 1, 2 \right\}.$$

LEMMA 3.  $\lim_{\tau \rightarrow \infty} \frac{1}{\tau} \log P([\mathcal{O}_{c\tau}]^c) = -\infty$ .



*Proof.* The claim is an elementary large deviation estimate. □

4. Let  $C_{c\tau}^k(\eta)$ ,  $k = 1, \dots, d$ , be the total number of  $\eta$ -crossings made by the two Brownian motions up to time  $c\tau$  across those slices that are perpendicular to direction  $k$ , and let  $C_{c\tau}(\eta) = \sum_{k=1}^d C_{c\tau}^k(\eta)$ . (These random variables do not depend on  $N$  because we consider crossings of all the slices.) We begin by deriving a large deviation upper bound showing that the latter sum cannot be too large.

LEMMA 4. For every  $M > 0$ ,

$$(2.45) \quad \limsup_{\eta \rightarrow \infty} \limsup_{\tau \rightarrow \infty} \frac{1}{\tau} \log P\left(C_{c\tau}(\eta) > \frac{dM}{\eta}c\tau\right) \leq -C(M),$$

with  $\lim_{M \rightarrow \infty} C(M) = \infty$ .

*Proof.* Since

$$(2.46) \quad P\left(C_{c\tau}(\eta) > \frac{dM}{\eta}c\tau\right) \leq dP\left(C_{c\tau}^1(\eta) > \frac{M}{\eta}c\tau\right),$$

it suffices to estimate the  $\eta$ -crossings perpendicular to direction 1. Let  $T_1, T_2, \dots$  denote the independent and identically distributed times taken by these  $\eta$ -crossings for the first Brownian motion. Since for both Brownian motions the crossings must occur prior to time  $c\tau$ , we have

$$(2.47) \quad P\left(C_{c\tau}^1(\eta) > \frac{M}{\eta}c\tau\right) \leq 2P\left(\sum_{i=1}^{\frac{M}{2n}c\tau} T_i < c\tau\right).$$

By Brownian scaling,  $T_1$  has the same distribution as  $\eta^2\sigma_1$  with  $\sigma_1$  the crossing time of a slice of width 1. Moreover, by a standard large deviation estimate for  $\sigma_1, \sigma_2, \dots$  corresponding to  $T_1, T_2, \dots$ , we have

$$(2.48) \quad \lim_{n \rightarrow \infty} \frac{1}{n} \log P\left(\sum_{i=1}^n \sigma_i < \zeta n\right) = -I(\zeta)$$

with

$$(2.49) \quad I(\zeta) > 0 \text{ for } 0 < \zeta < E(\sigma_1), \quad \lim_{\zeta \downarrow 0} \zeta I(\zeta) = \frac{1}{2},$$

where the limit  $1/2$  comes from the fact that  $P(\sigma_1 \in dt) = \exp\{-\frac{1}{2t}[1+o(1)]\} dt$  as  $t \downarrow 0$ . It follows from (2.46)–(2.48) that

$$(2.50) \quad \limsup_{\tau \rightarrow \infty} \frac{1}{\tau} \log P\left(C_{c\tau}(\eta) > \frac{M}{\eta}c\tau\right) \leq -c\frac{2M}{\eta}I\left(\frac{1}{2M\eta}\right).$$

By (2.49), as  $\eta \rightarrow \infty$  the right-hand side of (2.50) tends to  $-2cM^2$ . Hence we get the claim in (2.45) with  $C(M) = 2cM^2$ . □

Abbreviate

$$(2.51) \quad \mathcal{C}_{c\tau, M, \eta} = \left\{ C_{c\tau}(\eta) \leq \frac{dM}{\eta} c\tau \right\}.$$

5. We next derive a large deviation estimate showing that the total intersection volume cannot be too large.

LEMMA 5.  $\lim_{\tau \rightarrow \infty} \frac{1}{\tau} \log P\left(|V^{a\tau^{-1/(d-2)}}(c\tau)| > 2c\kappa_a\right) = -\infty$  for all  $c > 0$ .

*Proof.* After undoing the scaling we did in (2.1), we get

$$(2.52) \quad P\left(|V^{a\tau^{-1/(d-2)}}(c\tau)| > 2c\kappa_a\right) = P(|V^a(ct)| > 2c\kappa_a t).$$

We have  $|V^a(ct)| \leq |W_1^a(ct)|$ . It is known that  $E|W_1^a(ct)| \sim c\kappa_a t$  as  $t \rightarrow \infty$  (recall (1.12)) and that  $P(|W_1^a(ct)| > 2c\kappa_a t)$  decays exponentially fast in  $t = \tau^{d/(d-2)} \gg \tau$  (see van den Berg and Tóth [4] or van den Berg and Bolthausen [2]). □

Abbreviate

$$(2.53) \quad \mathcal{V}_{c\tau} = \{|V^{a\tau^{-1/(d-2)}}(c\tau)| \leq 2c\kappa_a\}.$$

6. For  $j = 1, \dots, \frac{dN}{\eta}$ , define

$$(2.54) \quad C_{c\tau}(S_{\eta, N}^j) = \text{number of } \eta\text{-crossings in } S_{\eta, N}^j \text{ up to time } c\tau,$$

$$V_{c\tau}(S_{\eta, N}^j) = V^{a\tau^{-1/(d-2)}}(c\tau) \cap S_{\eta, N}^j.$$

Because the copies in (2.43) cover  $\mathbb{R}^d$  exactly  $d$  times, on the event  $\mathcal{C}_{c\tau, M, \eta} \cap \mathcal{V}_{c\tau}$  defined by (2.51) and (2.53) we have

$$(2.55) \quad \sum_{j=1}^{\frac{dN}{\eta}} C_{c\tau}(S_{\eta, N}^j) \leq \frac{d^2 M}{\eta} c\tau,$$

$$\sum_{j=1}^{\frac{dN}{\eta}} |V_{c\tau}(S_{\eta, N}^j)| \leq 2dc\kappa_a.$$

Hence there exists a  $J$  (which depends on the two Brownian motions) such that

$$(2.56) \quad \begin{aligned} C_{c\tau}(S_{\eta, N}^J) &\leq \frac{2dM}{N} c\tau, \\ |V_{c\tau}(S_{\eta, N}^J)| &\leq 4c\kappa_a \frac{\eta}{N}. \end{aligned}$$

These two bounds will play a crucial role in the sequel. We will pick  $\eta = \sqrt{N}$  and  $M = \log N$ , do our reflections with respect to the central hyperplanes in

$S^J_{\sqrt{N},N}$ , and use the fact that for large  $N$  both the number of crossings and the intersection volume in  $S^J_{\sqrt{N},N}$  are small because of (2.56). This fact will allow us to control both the entropy associated with the reflections and the change in the intersection volume caused by the reflections.

7. Let  $x^J_{\sqrt{N},N}$  denote the shift through which  $S^J_{\sqrt{N},N}$  is obtained from  $S_{\sqrt{N},N}$  (recall (2.43)). For  $z \in \mathbb{Z}^d$ , we define

$$(2.57) \quad \begin{aligned} V_{c\tau,N}^J(z) &= V^{a\tau^{-1/(d-2)}}(c\tau) \cap \Lambda_N^J(z), \\ V_{c\tau,\sqrt{N},N,\text{out}}^J(z) &= V^{a\tau^{-1/(d-2)}}(c\tau) \cap S^J_{\sqrt{N},N}(z), \\ V_{c\tau,\sqrt{N},N,\text{in}}^J(z) &= V^{a\tau^{-1/(d-2)}}(c\tau) \cap [\Lambda_N^J(z) \setminus S^J_{\sqrt{N},N}(z)], \end{aligned}$$

where  $\Lambda_N^J(z) = \Lambda_N + x^J_{\sqrt{N},N}$  and  $S^J_{\sqrt{N},N}(z) = (\Lambda_N \setminus \Lambda_{N-\sqrt{N}}) + Nz + x^J_{\sqrt{N},N}$ . The rest of the proof of Proposition 4 will be based on Propositions 5 and 6 below. Proposition 5 states that the intersection volume has a tendency to clump: the blocks where the intersection volume is below a certain threshold have a negligible total contribution as this threshold tends to zero. Proposition 6 states that, at a negligible cost as  $N \rightarrow \infty$ , the Brownian motions can be reflected in the central hyperplanes of  $S^J_{\sqrt{N},N}$  and then be wrapped around the torus  $\Lambda_{2^{4c\kappa_a/\epsilon}N}$  in such a way that almost no intersection volume is gained nor lost.

Define

$$(2.58) \quad Z_{\epsilon,N}^J = \left\{ z \in \mathbb{Z}^d : |W_{1,c\tau,N}^J(z)| > \epsilon \text{ or } |W_{2,c\tau,N}^J(z)| > \epsilon \right\},$$

where

$$(2.59) \quad W_{i,c\tau,N}^J(z) = W_i^{a\tau^{-1/(d-2)}}(c\tau) \cap \Lambda_N^J(z), \quad i = 1, 2.$$

Abbreviate

$$(2.60) \quad \mathcal{W}_{c\tau} = \left\{ |W_1^{a\tau^{-1/(d-2)}}(c\tau)| \leq 2c\kappa_a, |W_2^{a\tau^{-1/(d-2)}}(c\tau)| \leq 2c\kappa_a \right\} \subset \mathcal{V}_{c\tau}.$$

Note that on the event  $\mathcal{W}_{c\tau}$  we have  $|Z_{\epsilon,N}^J| \leq 4c\kappa_a/\epsilon$ , while

$$(2.61) \quad \lim_{\tau \rightarrow \infty} \frac{1}{\tau} \log P([\mathcal{W}_{c\tau}]^c) = -\infty \quad \forall c > 0,$$

as shown in the proof of Lemma 5 above.

PROPOSITION 5. *There exists an  $N_0$  such that for every  $0 < \epsilon \leq 1$  and  $\delta > 0$ ,*

$$(2.62) \quad \limsup_{\tau \rightarrow \infty} \sup_{N \geq N_0} \frac{1}{\tau} \log P \left( \left\{ \sum_{z \in \mathbb{Z}^d \setminus Z_{\epsilon,N}^J} |V_{c\tau,N}^J(z)| > \delta \right\} \cup \left\{ \sum_{z \in Z_{\epsilon,N}^J} |V_{c\tau,\sqrt{N},N,\text{out}}^J(z)| > \delta \right\} \right) \leq -K(\epsilon, \delta),$$

with  $\lim_{\epsilon \downarrow 0} K(\epsilon, \delta) = \infty$  for any  $\delta > 0$ .

PROPOSITION 6. Fix  $N \geq 1$  and  $\epsilon, \delta > 0$ .

- (i) After at most  $|Z_{\epsilon,N}^J| - 1$  reflections in the central hyperplanes of  $S_{\sqrt{N},N}^J$  the Brownian motions are such that, when wrapped around the torus  $\Lambda_{2^{|z_{\epsilon,N}^J|}N}$ , all the intersection volumes  $|V_{c\tau, \sqrt{N}, N, \text{in}}^J(z)|$ ,  $z \in Z_{\epsilon,N}^J$ , end up in disjoint  $N$ -boxes inside  $\Lambda_{2^{|z_{\epsilon,N}^J|}N}$ .
- (ii) On the event  $\mathcal{O}_{c\tau} \cap \mathcal{C}_{c\tau, \log N, \sqrt{N}} \cap \mathcal{W}_{c\tau}$ , the reflections have a probabilistic cost at most  $\exp[\gamma_N \tau + O(\log \tau)]$  as  $\tau \rightarrow \infty$ , with  $\lim_{N \rightarrow \infty} \gamma_N = 0$ .<sup>3</sup>

An important point to note is that on the complement of the event on the left-hand side of (2.62) we have

$$(2.63) \quad 0 \leq |V^{a\tau^{-1/(d-2)}}(c\tau)| - \sum_{z \in Z_{\epsilon,N}^J} |V_{c\tau, \sqrt{N}, N, \text{in}}^J(z)| \leq 2\delta.$$

The sum on the right-hand side is invariant under the reflections (because the  $|V_{c\tau, \sqrt{N}, N, \text{in}}^J(z)|$  with  $z \in Z_{\epsilon,N}^J$  end up in disjoint  $N$ -boxes), and therefore the estimate in (2.63) implies that most of the intersection volume is unaffected by the reflections.

8. Before giving the proof of Propositions 5 and 6, we complete the proof of Proposition 4. By (2.61), (2.63), Lemmas 3 and 4 and Proposition 5 we have, for  $\tau, N$  large enough,  $0 < \epsilon \leq 1$  and  $\delta > 0$ ,

$$(2.64) \quad P\left(|V^{a\tau^{-1/(d-2)}}(c\tau)| \geq 1\right) \leq e^{-\frac{1}{2}K(\epsilon, \delta)\tau} + P\left(\sum_{z \in Z_{\epsilon,N}^J} |V_{c\tau, \sqrt{N}, N, \text{in}}^J(z)| \geq 1 - 2\delta, \mathcal{O}_{c\tau} \cap \mathcal{C}_{c\tau, \log N, \sqrt{N}} \cap \mathcal{W}_{c\tau}\right),$$

while by Proposition 6 we have, for any  $N \geq 1$ ,  $0 < \epsilon \leq 1$  and  $\delta > 0$ ,

$$(2.65) \quad P\left(\sum_{z \in Z_{\epsilon,N}^J} |V_{c\tau, \sqrt{N}, N, \text{in}}^J(z)| \geq 1 - 2\delta, \mathcal{O}_{c\tau} \cap \mathcal{C}_{c\tau, \log N, \sqrt{N}} \cap \mathcal{W}_{c\tau}\right) \leq e^{\gamma_N \tau + O(\log \tau)} \times P\left(\sum_{z \in Z_{\epsilon,N}^J} |V_{c\tau, \sqrt{N}, N, \text{in}}^J(z)| \geq 1 - 2\delta, \mathcal{O}_{c\tau} \cap \mathcal{C}_{c\tau, \log N, \sqrt{N}} \cap \mathcal{W}_{c\tau} \cap \mathcal{D}\right)$$

with  $\mathcal{D}$  the disjointness property stated in Proposition 6(i). However, subject to this disjointness property we have

$$(2.66) \quad |V_{2^{4c\kappa_a/\epsilon}N}^{a\tau^{-1/(d-2)}}(c\tau)| \geq \sum_{z \in Z_{\epsilon,N}^J} |V_{c\tau, \sqrt{N}, N, \text{in}}^J(z)|,$$

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<sup>3</sup>This statement means that if  $\mathcal{R}$  denotes the reflection transformation, then  $d\tilde{P}/dP \leq \exp[\gamma_N \tau + O(\log \tau)]$  with  $\tilde{P}$  the path measure for the two Brownian motions defined by  $\tilde{P}(A) = P(\mathcal{R}^{-1}A)$  for any event  $A$ .

where we use the fact that  $|Z_{\epsilon, N}^J| \leq 4c\kappa_a/\epsilon$  on  $\mathcal{W}_{c\tau}$ , and the left-hand side is the intersection volume after the two Brownian motions are wrapped around the  $2^{4c\kappa_a/\epsilon}N$ -torus. Combining (2.64)–(2.66) we obtain that, for  $\tau, N$  large enough,  $0 < \epsilon \leq 1$  and  $\delta > 0$ ,

$$(2.67) \quad P\left(|V^{a\tau^{-1/(d-2)}}(c\tau)| \geq 1\right) \leq e^{-\frac{1}{2}K(\epsilon, \delta)\tau} + e^{\gamma_N\tau + O(\log \tau)} P\left(|V_{2^{4c\kappa_a/\epsilon}N}^{a\tau^{-1/(d-2)}}(c\tau)| \geq 1 - 2\delta\right).$$

We are now in a position to invoke Proposition 2 to obtain that, for  $N$  large enough,  $0 < \epsilon \leq 1$  and  $\delta > 0$ ,

$$(2.68) \quad \limsup_{\tau \rightarrow \infty} \frac{1}{\tau} \log P\left(|V^{a\tau^{-1/(d-2)}}(c\tau)| \geq 1\right) \leq \max\left\{-\frac{1}{2}K(\epsilon, \delta), \gamma_N - \widehat{J}_{d, 2^{4c\kappa_a/\epsilon}N}^{\kappa_a}(1 - 2\delta, c)\right\}.$$

Next, let  $N \rightarrow \infty$  and use the facts that  $\gamma_N \rightarrow 0$  and  $\widehat{J}_{d, 2^{4c\kappa_a/\epsilon}N}^{\kappa_a}(1 - 2\delta, c) \rightarrow \widehat{J}_d^{\kappa_a}(1 - 2\delta, c)$  (similarly as in (2.40)), to obtain that, for any  $0 < \epsilon \leq 1$  and  $\delta > 0$ ,

$$(2.69) \quad \limsup_{\tau \rightarrow \infty} \frac{1}{\tau} \log P\left(|V^{a\tau^{-1/(d-2)}}(c\tau)| \geq 1\right) \leq \max\left\{-\frac{1}{2}K(\epsilon, \delta), -\widehat{J}_d^{\kappa_a}(1 - 2\delta, c)\right\}.$$

Next, let  $\epsilon \downarrow 0$  and hence  $K(\epsilon, \delta) \rightarrow \infty$ , to obtain that, for any  $\delta > 0$ ,

$$(2.70) \quad \limsup_{\tau \rightarrow \infty} \frac{1}{\tau} \log P\left(|V^{a\tau^{-1/(d-2)}}(c\tau)| \geq 1\right) \leq -\widehat{J}_d^{\kappa_a}(1 - 2\delta, c).$$

Finally, note from (2.7) and (2.8) that

$$(2.71) \quad \widehat{J}_d^{\kappa_a}(1 - 2\delta, c) = (1 - 2\delta)^{\frac{d-2}{d}} \widehat{J}_d^{\kappa_a}\left(1, \frac{c}{1 - 2\delta}\right) = (1 - 2\delta)^{\frac{d-2}{d}} I_d^{\kappa_a}\left(\frac{c}{1 - 2\delta}\right),$$

where the first equality uses scaling (see also (4.1) and (4.2)). Let  $\delta \downarrow 0$  and use Theorems 3(i) and (iv), to see that the right-hand side converges to  $I_d^{\kappa_a}(c)$ . This proves the claim in Proposition 4. In the remaining six steps we prove Propositions 5 and 6.

9. We proceed with the proof of Proposition 6(i).

*Proof.* For  $k = 1, \dots, d$  carry out the following reflection procedure. A  $k$ -slice consists of all boxes  $\Lambda_N^J(z)$ ,  $z = (z_1, \dots, z_d)$ , for which  $z_k$  is fixed and the  $z_l$ 's with  $l \neq k$  are running. Label all the  $k$ -slices in  $\mathbb{Z}^d$  that contain one or more elements of  $Z$ . The number  $R$  of such slices is at most  $|Z|$ . Now:

- (1) Look for the right-most central hyperplane  $H_k$  (perpendicular to the direction  $k$ ) such that all the labelled  $k$ -slices lie to the right of  $H_k$ . Number the labelled  $k$ -slices to the right of  $H_k$  by  $1, \dots, R$  and let  $d_1N, \dots, d_{R-1}N$  denote the successive distances between them.

- (2) If  $d_1 \geq 1$ , then look for the left-most central hyperplane  $H'_k$  to the right of slice 1 such that, when the two Brownian motions are reflected in  $H'_k$ , slice 2 lands to the left of  $H_k$  at a distance either 0 or  $N$  (depending on whether  $d_1$  is odd, respectively, even). If  $d_1 = 0$ , then do not reflect. (As already emphasized in part 2, we reflect only those excursions moving a distance  $\frac{1}{2}\sqrt{N}$  away from  $H'_k$  that begin with an exit-point at  $H'_k$  and end with the next entrance-point at  $H'_k$ . After the reflection, both Brownian motions lie entirely on one side of the hyperplane at distance  $\frac{1}{2}\sqrt{N}$  from  $H'_k$ .)

The effect of (1) and (2) is that slices 1 and 2 fall inside a  $3N$ -box. Repeat. If  $d_2 \geq 3$ , then again reflect, this time making slice 3 land to the right of slices 1 and 2 at a distance either 0 or  $N$ . If  $d_2 \leq 2$ , then do not reflect. The effect is that slices 1, 2 and 3 fall inside a  $6N$ -box, etc. After we are through, the  $R$  slices fit inside a box of size  $3 \times 2^{R-2}N (\leq 2^R N)$ . After we have done the reflections in all the directions  $k = 1, \dots, d$ , all the labelled slices fit inside a box of size  $2^{|Z'_{\epsilon,N}|}N$ .  $\square$

10. Next we proceed with the proof of Proposition 6(ii).

*Proof* . Each reflection of an excursion beginning with an exit-point and ending with an entrance-point costs a factor 2 in probability. On the event  $\mathcal{C}_{c\tau, \log N, \sqrt{N}}$ , the total number of excursions of the two Brownian motions is bounded above by  $d \frac{\log N}{\sqrt{N}} c\tau$ . Moreover, on the event  $\mathcal{O}_{c\tau}$  the number of central hyperplanes available for the reflection is bounded above by  $2\tau^2/N$ , on the event  $\mathcal{W}_{c\tau}$  the total number of reflections is bounded above by  $|Z^J_{\epsilon,N}| \leq 4\kappa_a c/\epsilon$ , while the total number of shifted copies of  $S_{\sqrt{N},N}$  available is  $d\sqrt{N}$ . Therefore we indeed get Proposition 6(ii) with  $\gamma_N$  given by  $2^{d \log N/\sqrt{N}} = e^{\gamma_N}$  and the error term given by  $(2\tau^2/N)^{4\kappa_a c/\epsilon} d\sqrt{N} = e^{O(\log \tau)}$ . Note that the reflections preserve the intersection volume in the  $N$ -boxes without the  $\sqrt{N}$ -slices, i.e., the  $|V^J_{c\tau, \sqrt{N}, \log N, \text{in}}(z)|$  with  $z \in Z^J_{\epsilon,N}$  (recall the remark below (2.63)).  $\square$

11. Finally, we prove Proposition 5, which requires four more steps.

*Proof* . First note that the second event on the left-hand side of (2.62) is redundant for  $N \geq N_0 = (4c\kappa_a/\delta)^2$  because of (2.56) with  $\eta = \sqrt{N}$  and  $M = \log N$ . Indeed, recall that

$$(2.72) \quad |V_{c\tau}(S^J_{\sqrt{N},N})| = \sum_{z \in \mathbb{Z}^d} |V^J_{c\tau, \sqrt{N}, N, \text{out}}(z)|.$$

Thus, we need to show that there exists an  $N_0$  such that for every  $0 < \epsilon \leq 1$  and  $\delta > 0$ ,

$$(2.73) \quad \limsup_{\tau \rightarrow \infty} \sup_{N \geq N_0} \frac{1}{\tau} \log P \left( \sum_{z \in \mathbb{Z}^d \setminus Z^J_{\epsilon,N}} |V^J_{c\tau,N}(z)| > \delta \right) \leq -K(\epsilon, \delta),$$

with  $\lim_{\epsilon \downarrow 0} K(\epsilon, \delta) = \infty$  for any  $\delta > 0$ . To that end, for  $N \geq 1$  and  $\epsilon > 0$ , let

$$(2.74) \quad \mathcal{A}_{\epsilon, N} = \left\{ A \subset \mathbb{R}^d \text{ Borel} : \inf_{x \in \mathbb{R}^d} \sup_{z \in \mathbb{Z}^d} |(A+x) \cap \Lambda_N(z)| \leq \epsilon \right\}.$$

This class of sets is closed under translations and its elements become ever more sparse as  $\epsilon \downarrow 0$ . The key to the proof of Proposition 5 is the following clumping property for a single Wiener sausage. Recall that

$$(2.75) \quad W_{c\tau} = W^{a\tau^{-1/(d-2)}}(c\tau).$$

LEMMA 6. For every  $0 < \epsilon \leq 1$  and  $\delta > 0$ ,

$$(2.76) \quad \lim_{\epsilon \downarrow 0} \limsup_{\tau \rightarrow \infty} \frac{1}{\tau} \log \sup_{N \geq 1} \sup_{A \in \mathcal{A}_{\epsilon, N}} P(|A \cap W_{c\tau}| > \delta) = -K(\epsilon, \delta),$$

with  $\lim_{\epsilon \downarrow 0} K(\epsilon, \delta) = \infty$  for any  $\delta > 0$ .

Let us see how to get Proposition 5 from Lemma 6. Consider the random set

$$(2.77) \quad A^* = \bigcup_{z \in \mathbb{Z}^d: |W_{1,c\tau} \cap \Lambda_N^J(z)| \leq \epsilon} \{W_{1,c\tau} \cap \Lambda_N^J(z)\}.$$

Clearly,  $A^* \in \mathcal{A}_{\epsilon, N}$  and (recall (2.57) and (2.58))

$$(2.78) \quad \begin{aligned} \sum_{z \in \mathbb{Z}^d \setminus Z_{\epsilon, N}^J} |V_{c\tau, N}^J(z)| &= \sum_{z \in \mathbb{Z}^d \setminus Z_{\epsilon, N}^J} |W_{1,c\tau} \cap W_{2,c\tau} \cap \Lambda_N^J(z)| \\ &\leq \sum_{z \in \mathbb{Z}^d} |A^* \cap W_{2,c\tau} \cap \Lambda_N^J(z)| \\ &= |A^* \cap W_{2,c\tau}|. \end{aligned}$$

Therefore

$$(2.79) \quad P\left(\sum_{z \in \mathbb{Z}^d \setminus Z_{\epsilon, N}^J} |V_{c\tau, N}^J(z)| > \delta\right) \leq \sup_{A \in \mathcal{A}_{\epsilon, N}} P(|A \cap W_{2,c\tau}| > \delta).$$

This bound together with Lemma 6 yields (2.73) and completes the proof of Proposition 5. □

12. Thus it remains to prove Lemma 6.

*Proof.* We will show that

$$(2.80) \quad \lim_{\epsilon \downarrow 0} \limsup_{\tau \rightarrow \infty} \frac{1}{\tau} \log \sup_{N \geq 1} \sup_{A \in \mathcal{A}_{\epsilon, N}} E\left(\exp\left[\epsilon^{-1/3d} \tau |A \cap W_{c\tau}|\right]\right) = 0.$$

Together with the exponential Chebyshev inequality

$$(2.81) \quad P(|A \cap W_{c\tau}| > \delta) \leq e^{-\delta \epsilon^{-1/3d} \tau} E\left(\exp\left[\epsilon^{-1/3d} \tau |A \cap W_{c\tau}|\right]\right) \quad \forall A \subset \mathbb{R}^d,$$

(2.80) will prove Lemma 6.

13. To prove (2.80), we use the subadditivity of  $s \mapsto |A \cap W^{a\tau^{-1/(d-2)}}(s)|$  in the following estimate:

$$\begin{aligned}
 (2.82) \sup_{A \in \mathcal{A}_{\epsilon, N}} E \left( \exp \left[ \epsilon^{-1/3d} \tau |A \cap W_{c\tau}| \right] \right) \\
 = \sup_{A \in \mathcal{A}_{\epsilon, N}} E \left( \exp \left[ \epsilon^{-1/3d} \tau |A \cap W^{a\tau^{-1/(d-2)}}(c\tau)| \right] \right) \\
 \leq \left\{ \sup_{A \in \mathcal{A}_{\epsilon, N}} \sup_{x \in \mathbb{R}^d} E_x \left( \exp \left[ \epsilon^{-1/3d} \tau |A \cap W^{a\tau^{-1/(d-2)}}(\epsilon^{1/d})| \right] \right) \right\}^{\epsilon^{-1/d} c\tau}.
 \end{aligned}$$

Here, the lower index  $x$  refers to the starting point of the Brownian motion ( $E = E_0$ ), and we use the Markov property at times  $j\epsilon^{1/d}$ ,  $j = 1, \dots, \epsilon^{-1/d} c\tau$ , together with the fact that  $\mathcal{A}_{\epsilon, N}$  is closed under translations. Next, scale space by  $\tau^{1/(d-2)}$  and time by  $\tau^{2/(d-2)}$ , and put  $T = \epsilon^{1/d} \tau^{2/(d-2)}$ , to get

$$\begin{aligned}
 (2.83) \\
 E_x \left( \exp \left[ \epsilon^{-1/3d} \tau |A \cap W^{a\tau^{-1/(d-2)}}(\epsilon^{1/d})| \right] \right) \\
 = E_{(\epsilon^{-1/2d} \sqrt{T})x} \left( \exp \left[ \epsilon^{2/3d} \frac{1}{T} |(\epsilon^{-1/2d} \sqrt{T})A \cap W^a(T)| \right] \right) \quad \forall A \subset \mathbb{R}^d, x \in \mathbb{R}^d.
 \end{aligned}$$

14. Abbreviate

$$(2.84) \quad T_\epsilon = \epsilon^{-1/2d} \sqrt{T}.$$

Use the inequality  $e^u \leq 1 + u + \frac{1}{2}u^2 e^u$ ,  $u \geq 0$ , in combination with Cauchy-Schwarz, to obtain

$$\begin{aligned}
 (2.85) \\
 (2.83) \leq 1 + \epsilon^{2/3d} \frac{1}{T} E_{T_\epsilon x} |T_\epsilon A \cap W^a(T)| + \frac{1}{2} \epsilon^{4/3d} \sqrt{\frac{1}{T^4} E_{T_\epsilon x} |W^a(T)|^4} \\
 \times \sqrt{E_{T_\epsilon x} \left( \exp \left[ 2\epsilon^{2/3d} \frac{1}{T} |W^a(T)| \right] \right)} \quad \forall A \subset \mathbb{R}^d, x \in \mathbb{R}^d,
 \end{aligned}$$

where in the last term we overestimate by removing the intersection with  $T_\epsilon A$ . The two expectations under the square roots are independent of  $x$  and are bounded uniformly in  $T \geq 1$  and  $0 < \epsilon \leq 1$  (see van den Berg and Tóth [4] or van den Berg and Bolthausen [2]). Hence

$$\begin{aligned}
 (2.86) \quad (2.83) \leq 1 + \epsilon^{2/3d} \frac{1}{T} E_{T_\epsilon x} |T_\epsilon A \cap W^a(T)| + C_1 \epsilon^{4/3d} \\
 \forall A \subset \mathbb{R}^d, x \in \mathbb{R}^d, T \geq 1, 0 < \epsilon \leq 1.
 \end{aligned}$$



The remaining expectation can be estimated as follows. First write

$$\begin{aligned}
 (2.87) \quad & E_{T_\epsilon x} |T_\epsilon A \cap W^a(T)| \\
 &= \sum_{z \in \mathbb{Z}^d} E_{T_\epsilon x} |T_\epsilon A \cap W^a(T) \cap \Lambda_{T_\epsilon N}(z)| \\
 &= \sum_{z \in \mathbb{Z}^d} P_{T_\epsilon x} \left( W^a(T) \cap \Lambda_{T_\epsilon N}(z) \neq \emptyset \right) \\
 &\quad \times E_{T_\epsilon x} \left( |T_\epsilon A \cap W^a(T) \cap \Lambda_{T_\epsilon N}(z)| \mid W^a(T) \cap \Lambda_{T_\epsilon N}(z) \neq \emptyset \right).
 \end{aligned}$$

Then note that

$$\begin{aligned}
 (2.88) \quad & \sup_{A \in \mathcal{A}_{\epsilon, N}} \sup_{x \in \mathbb{R}^d} \sup_{z \in \mathbb{Z}^d} E_x \left( |T_\epsilon A \cap W^a(T) \cap \Lambda_{T_\epsilon N}(z)| \mid W^a(T) \cap \Lambda_{T_\epsilon N}(z) \neq \emptyset \right) \\
 &\leq \sup_{A \in \mathbb{R}^d: |A \cap \Lambda_N(0)| \leq \epsilon} \sup_{x \in \mathbb{R}^d} E_x |T_\epsilon A \cap W^a(T) \cap \Lambda_{T_\epsilon N}(0)| \\
 &= \sup_{A \in \mathbb{R}^d: |A \cap \Lambda_N(0)| \leq \epsilon} \sup_{x \in \mathbb{R}^d} \int_{T_\epsilon A \cap \Lambda_{T_\epsilon N}(0)} P_x(\sigma_{B_a(y)} \leq T) dy,
 \end{aligned}$$

with  $\sigma_{B_a(y)}$  the first hitting time of  $B_a(y)$ . Since the integrand is a decreasing function of  $|y - x|$ , the integral on the right-hand side is bounded above, uniformly in  $A \in \mathcal{A}_{\epsilon, N}$  and  $x \in \mathbb{R}^d$ , by

$$(2.89) \quad \int_{T_\epsilon B_{(\epsilon/\omega_d)^{1/d}}(0)} P(\sigma_{B_a(y)} \leq T) dy$$

( $\omega_d$  is the volume of the ball with unit radius). Since

$$(2.90) \quad P(\sigma_{B_a(y)} \leq T) \leq P(\sigma_{B_a(y)} < \infty) = 1 \wedge \left( \frac{a}{|y|} \right)^{d-2} \leq \left( \frac{a}{|y|} \right)^{d-2},$$

we find that (recall (2.84))

$$(2.91) \quad (2.89) \leq C_2 \epsilon^{1/d} T.$$

Consequently,

$$(2.92) \quad (2.87) \leq C_2 \epsilon^{1/d} T E_x |\{z \in \mathbb{Z}^d : W^a(T) \cap \Lambda_{T_\epsilon N}(z) \neq \emptyset\}|.$$

But the last expectation is bounded above by  $C_3$  uniformly in  $x \in \mathbb{R}^d$ ,  $N \geq 1$  and  $0 < \epsilon \leq 1$ . Hence (recall (2.86))

$$(2.93) \quad \sup_{x \in \mathbb{R}^d} \sup_{T \geq 1} (2.83) \leq 1 + C_2 C_3 \epsilon^{5/3d} + C_1 \epsilon^{4/3d} \quad \forall 0 < \epsilon \leq 1.$$

Substitution into (2.82) yields the claim in (2.80). This completes the proof of Lemma 6. □

This completes the proof of Proposition 4 and hence of Theorem 1. □

### 3. Proof of Theorem 2

In this section we indicate how the arguments given in Section 2 for the Wiener sausages in  $d \geq 3$  can be carried over to  $d = 2$ . The necessary modifications are minor and only involve a change in the choice of the scaling parameters.

By Brownian scaling,  $V^a(ct)$  has the same distribution as  $\frac{t}{\log t} V^a \sqrt{\frac{\log t}{t}} \times (c \log t)$ ,  $t > 1$ . Hence, putting

$$(3.1) \quad \tau = \log t,$$

we have

$$(3.2) \quad P(|V^a(ct)| \geq t/\log t) = P(|V^{a\sqrt{\tau e^{-\tau}}}(c\tau)| \geq 1).$$

The claim in Theorem 2 is therefore equivalent to

$$(3.3) \quad \lim_{\tau \rightarrow \infty} \frac{1}{\tau} \log P(|V^{a\sqrt{\tau e^{-\tau}}}(c\tau)| \geq 1) = -I_2^{2\pi}(c).$$

Both the argument for the lower bound (§2.2) and for the upper bound (§2.3) carry over, with the shrinking rate  $\sqrt{\tau e^{-\tau}}$  for  $d = 2$  replacing the shrinking rate  $\tau^{-1/(d-2)}$  for  $d \geq 3$  (and  $2\pi$  for  $d = 2$  replacing  $\kappa_a$  for  $d \geq 3$ ). The necessary ingredients can be found in [3, §4].

The only part that needs some consideration is the proof of Lemma 6. After the scaling we find that in (2.83) the factor  $1/T$  gets replaced by  $(\log T)/T$ . This can be accommodated in (2.85). The analogue of (2.89) for  $d = 2$  reads

$$(3.4) \quad \int_{(\epsilon^{-1/4}\sqrt{T})B_{(\epsilon/\pi)^{1/2}}(0)} P(\sigma_{B_a(y)} \leq T) dy.$$

To estimate this integral, we argue as follows. According to Spitzer [14], equation (3.3),

$$(3.5) \quad \lambda \int_0^\infty e^{-\lambda t} P(\sigma_{B_a(y)} \leq t) dt = 1 \wedge \frac{K_0(\sqrt{2\lambda}|y|)}{K_0(\sqrt{2\lambda}a)} \quad \forall y \in \mathbb{R}^d, \lambda > 0,$$

where  $K_0$  is the Bessel function of the second kind with imaginary argument of order 0. Consequently,

$$(3.6) \quad P(\sigma_{B_a(y)} \leq 1/\lambda) \leq e \frac{K_0(\sqrt{2\lambda}|y|)}{K_0(\sqrt{2\lambda}a)} \quad \forall y \in \mathbb{R}^d, \lambda > 0.$$

Hence

$$(3.7) \quad \begin{aligned} \int_{y \in \mathbb{R}^d: |y| \leq \rho} P(\sigma_{B_a(y)} \leq 1/\lambda) dy &\leq \frac{2\pi e}{K_0(\sqrt{2\lambda}a)} \int_0^\rho r K_0(\sqrt{2\lambda}r) dr \\ &= \frac{\pi e}{\lambda K_0(\sqrt{2\lambda}a)} \int_0^{\sqrt{2\lambda}\rho} r K_0(r) dr \quad \forall \rho > 0. \end{aligned}$$

Put  $\lambda = 1/T$  and  $\rho = \epsilon^{1/4}\sqrt{T/\pi}$ . Then

$$(3.8) \quad (3.4) \leq \frac{\pi e T}{K_0(\sqrt{2/T} a)} \int_0^{\epsilon^{1/4}\sqrt{2/\pi}} r K_0(r) dr.$$

Since  $K_0(r) = (1 + o(1)) \log(1/r)$  as  $r \downarrow 0$ , we obtain from (3.8) that

$$(3.9) \quad \limsup_{T \rightarrow \infty} \epsilon^{1/3} \frac{\log T}{T} (3.4) \leq \epsilon^{1/3} 2\pi e \int_0^{\epsilon^{1/4}\sqrt{2/\pi}} r K_0(r) dr.$$

Here we multiply by  $\epsilon^{1/3}(\log T)/T$ , which is the factor in the second term on the right-hand side of the analogue of (2.86). The integral on the right-hand side of (3.9) is of order  $\epsilon^{1/2} \log(1/\epsilon)$ . Hence we get  $C\epsilon^{5/6} \log(1/\epsilon)$  for the second term on the right-hand side of the analogue of (2.93).

### 4. Proof of Theorem 3

In Sections 4–6 we prove Theorems 3–5. The proof follows the same line of reasoning as in [3, §5], but there are some subtle differences.

We will repeatedly make use of the following scaling relations. Let  $\phi \in H^1(\mathbb{R}^d)$ . For  $p, q > 0$ , define  $\psi \in H^1(\mathbb{R}^d)$  by

$$(4.1) \quad \psi(x) = q\phi(x/p).$$

Then

$$(4.2) \quad \|\nabla\psi\|_2^2 = q^2 p^{d-2} \|\nabla\phi\|_2^2, \quad \|\psi\|_2^2 = q^2 p^d \|\phi\|_2^2, \quad \|\psi\|_4^4 = q^4 p^d \|\phi\|_4^4, \\ \int (1 - e^{-\psi^2})^2 = p^d \int (1 - e^{-q^2\phi^2})^2.$$

We will also repeatedly make use of the following Sobolev inequalities (see Lieb and Loss [12, pp. 186 and 190]):

$$(4.3) \quad S_d \|f\|_r^2 \leq \|\nabla f\|_2^2, \quad d \geq 3, f \in D^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d),$$

with

$$(4.4) \quad r = \frac{2d}{d-2}, \quad S_d = d(d-2)2^{-2(d-1)/d} \pi^{(d+1)/d} \left[ \Gamma\left(\frac{d+1}{2}\right) \right]^{-2/d},$$

and

$$(4.5) \quad \|f\|_4 \leq S_{2,4} (\|\nabla f\|_2^2 + \|f\|_2^2)^{1/2}, \quad d = 2, f \in H^1(\mathbb{R}^2),$$

with  $S_{2,4} = (4/27\pi)^{1/4}$ .

Finally, we will use the fact that the variational problem in (1.14) reduces to radially symmetric nonincreasing (RSNI) functions (see [3, Lemma 10 and its proof]).

We now start the proof of Theorem 3, numbered in parts (i–v).

(i) Picking  $p = 1$  and  $q = (c\kappa_a)^{-1/2}$  in (4.1) and (4.2) and inserting this into (1.9) and (1.10), we see that (1.9) and (1.10) transform into (1.13) and (1.14).

(ii) Let  $K = \max_{\zeta > 0} \zeta^{-1}(1 - e^{-\zeta})^2$ . The maximum is attained at  $\zeta = \zeta_\diamond = 1.25643\dots$ . We have, for any  $\psi \in H^1(\mathbb{R}^d)$ ,

$$(4.6) \quad \int (1 - e^{-\psi^2})^2 \leq K \int \psi^2.$$

Therefore the set over which the infimum in (1.14) is taken is empty when  $Ku < 1$ , implying that  $\Theta_d(u) = \infty$  for  $u \in (0, 1/K)$ . Next, let  $\psi_\diamond$  be defined by

$$(4.7) \quad \psi_\diamond = \sqrt{\zeta_\diamond} 1_{B[u/\zeta_\diamond]},$$

where  $B[u/\zeta_\diamond]$  is the ball with volume  $u/\zeta_\diamond$ . Then

$$(4.8) \quad \int \psi_\diamond^2 = \zeta_\diamond \frac{u}{\zeta_\diamond} = u, \quad \int (1 - e^{-\psi_\diamond^2})^2 = (1 - e^{-\zeta_\diamond})^2 \frac{u}{\zeta_\diamond} = Ku.$$

Therefore when  $Ku > 1$  there exists a  $\psi \in H^1(\mathbb{R}^d)$ , playing the role of a smooth approximation of  $\psi_\diamond$ , such that

$$(4.9) \quad \|\psi\|_2^2 = u, \quad \int (1 - e^{-\psi^2})^2 \geq 1,$$

implying that  $\Theta_d(u) < \infty$  for  $u \in (1/K, \infty)$ . Finally,  $\Theta_d(1/K) = \infty$  because  $\psi_\diamond \notin H^1(\mathbb{R}^d)$  and any smooth perturbation of  $\psi_\diamond$  violates (4.9) when  $u = 1/K$ . This proves the claim with  $u_\diamond = 1/K$ . The fact that  $\Theta_d$  is strictly positive everywhere follows from part (iii) in combination with the asymptotics in Theorems 4 and 5.

(iii) To prove that  $\Theta_d$  is nonincreasing, we need the following identity.

LEMMA 7. *Let*

$$(4.10) \quad \begin{aligned} \widehat{\Theta}_d(u) &= \inf\{\|\nabla\psi\|_2^2 : \|\psi\|_2^2 = u, \int (1 - e^{-\psi^2})^2 = 1\}, \\ \widetilde{\Theta}_d(u) &= \inf\{\|\nabla\psi\|_2^2 : \|\psi\|_2^2 \leq u, \int (1 - e^{-\psi^2})^2 = 1\}. \end{aligned}$$

Then

$$(4.11) \quad \widehat{\Theta}_d(u) = \widetilde{\Theta}_d(u) = \Theta_d(u) \quad \forall u > u_\diamond.$$

Since  $\widetilde{\Theta}_d$  is obviously nonincreasing, the claim follows from (4.11). Thus, it remains to prove Lemma 7.

*Proof.* The proof proceeds in four steps.

1. It is clear that  $\widehat{\Theta}_d(u) \geq \Theta_d(u)$ . To prove the converse, let  $(\psi_j)$  be a minimising sequence of  $\Theta_d(u)$ , i.e.,  $\|\psi_j\|_2^2 = u$  and  $\int (1 - e^{-\psi_j^2})^2 \geq 1$  for all  $j$  and  $\|\nabla\psi_j\|_2^2 \rightarrow \Theta_d(u)$  as  $j \rightarrow \infty$ . Define

$$(4.12) \quad g_\psi(a) = a^d \int \left(1 - e^{-a^{-d}\psi^2}\right)^2, \quad a > 0.$$

Then  $g_\psi(1) \geq 1$ . In part 2 we will prove that  $\lim_{a \rightarrow \infty} g_\psi(a) = 0$ . Hence, by the intermediate value theorem, there exists a sequence  $(a_j)$  such that  $a_j \geq 1$  and  $g_\psi(a_j) = 1$  for all  $j$ . Let  $\phi_j \in H^1(\mathbb{R}^d)$  be defined by  $\phi_j(x) = a_j^{-d/2} \psi_j(x/a_j)$ . Then, by (4.1) and (4.2), we have

$$(4.13) \quad \|\nabla \phi_j\|_2^2 = a_j^{-2} \|\nabla \psi_j\|_2^2, \quad \|\phi_j\|_2^2 = \|\psi_j\|_2^2 = u,$$

$$\int (1 - e^{-\phi_j^2})^2 = g_\psi(a_j) = 1 \quad \forall j.$$

Recalling (4.10), we therefore have

$$(4.14) \quad \widehat{\Theta}_d(u) \leq \|\nabla \phi_j\|_2^2 = a_j^{-2} \|\nabla \psi_j\|_2^2 \leq \|\nabla \psi_j\|_2^2 \quad \forall j.$$

Let  $j \rightarrow \infty$  and use the fact that  $\|\nabla \psi_j\|_2^2 \rightarrow \Theta_d(u)$ , to get  $\widehat{\Theta}_d(u) \leq \Theta_d(u)$ .

2. Next we prove that  $\lim_{a \rightarrow \infty} g_\psi(a) = 0$ . For  $d \geq 4$  we have  $e^{-x} \geq 1 - x^{d/2(d-2)}$ ,  $x \geq 0$ , so it follows from (4.3) and (4.4) that

$$(4.15) \quad g_\psi(a) \leq a^{-2d/(d-2)} \int \psi^{2d/(d-2)} \leq a^{-2d/(d-2)} S_d^{-1} \|\nabla \psi\|_2^{2d/(d-2)}.$$

For  $d = 3$  we have, by Cauchy-Schwarz and (4.3) and (4.4), that

$$(4.16) \quad g_\psi(a) \leq a^{-3} \int \psi^4 \leq a^{-3} \|\psi\|_2 \|\psi\|_6^3 \leq a^{-3} u^{1/2} S_3^{-1/2} \|\nabla \psi\|_2^3.$$

For  $d = 2$  we have, by (4.5), that

$$(4.17) \quad g_\psi(a) = a^{-2} \int \psi^4 \leq a^{-4} S_{2,4}^4 (\|\nabla \psi\|_2^2 + u)^2.$$

3. It is clear that  $\widetilde{\Theta}_d(u) \leq \widehat{\Theta}_d(u)$ . To prove the converse, we begin with the following observation.

LEMMA 8. *The set*

$$(4.18) \quad \left\{ \psi \in H^1(\mathbb{R}^d) : \psi \text{ RSNI, } \|\nabla \psi\|_2^2 \leq C, \|\psi\|_2^2 \leq u, \int (1 - e^{-\psi^2})^2 = 1 \right\}$$

*is compact for all  $u > u_\diamond$  and  $C < \infty$ .*

Before proving Lemma 8 we first complete the proof of Lemma 7. Since  $\psi \mapsto \|\nabla \psi\|_2$  is lower semi-continuous, it follows from Lemma 8 that the variational problem for  $\widetilde{\Theta}_d(u)$  has a minimiser, say  $\psi^*$ . Define

$$(4.19) \quad p_n(x) = \frac{1}{\pi^{d/2} n^d} e^{-|x|^2/n^2}, \quad x \in \mathbb{R}^d, n \in \mathbb{N},$$

and note that  $\int p_n = 1$  and  $\|\nabla \sqrt{p_n}\|_2^2 = 2d/n^2$  for all  $n$ . Define  $\psi_n^*$  by

$$(4.20) \quad \psi_n^{*2} = \psi^{*2} + (u - \|\psi^*\|_2^2) p_n, \quad n \in \mathbb{N}.$$

Then  $\|\psi_n^*\|_2^2 = u$  for all  $n$ . Moreover, since  $x \mapsto (1 - e^{-x})^2$  is nondecreasing on  $[0, \infty)$ , we have

$$(4.21) \quad \int (1 - e^{-\psi_n^{*2}})^2 \geq \int (1 - e^{-\psi^{*2}})^2 \geq 1 \quad \forall n.$$

So  $\psi_n^*$  satisfies the constraints in the variational problem for  $\Theta_d(u)$ . Hence

$$(4.22) \quad \Theta_d(u) \leq \|\nabla \psi_n^*\|_2^2 \quad \forall n.$$

By the convexity inequality for gradients (Lieb and Loss [12, Theorem 7.8]), we have

$$(4.23) \quad \|\nabla \psi_n^*\|_2^2 \leq \|\nabla \psi^*\|_2^2 + (u - \|\psi^*\|_2^2) \|\nabla \sqrt{p_n}\|_2^2 = \tilde{\Theta}_d(u) + (u - \|\psi^*\|_2^2) \frac{2d}{n^2}.$$

Let  $n \rightarrow \infty$  to conclude that  $\Theta_d(u) \leq \tilde{\Theta}_d(u)$ . But we already know from part 1 that  $\Theta_d(u) = \hat{\Theta}_d(u)$ , and so  $\hat{\Theta}_d(u) \leq \tilde{\Theta}_d(u)$ . This completes the proof of Lemma 7.  $\square$

4. It remains to prove Lemma 8.

*Proof.* The key point is to show that the contribution to the integral in (4.18) coming from large  $x$  and from small  $x$  is uniformly small. Indeed, since  $\psi$  is RSNI, we have

$$(4.24) \quad u \geq \|\psi\|_2^2 \geq \int_{B_R} \psi^2 \geq \omega_d R^d \psi^2(x) \quad \forall |x| \geq R > 0,$$

and so

$$(4.25) \quad \int_{B_R^c} (1 - e^{-\psi^2})^2 \leq \int_{B_R^c} \psi^4 \leq \frac{u}{\omega_d R^d} \int_{B_R^c} \psi^2 \leq \frac{u^2}{\omega_d R^d} \quad \forall R > 0,$$

while

$$(4.26) \quad \int_{B_r} (1 - e^{-\psi^2})^2 \leq \int_{B_r} 1 = \omega_d r^d \quad \forall r > 0.$$

So the last two integrals tend to zero when  $R \rightarrow \infty$ , respectively,  $r \downarrow 0$ . Next we note that any sequence  $(\psi_j)$  in  $H^1(\mathbb{R}^d)$  has a subsequence that converges to some  $\psi \in H^1(\mathbb{R}^d)$  such that the convergence is uniform on every annulus  $B_R \setminus B_r$  (since  $\psi_j$  is RSNI and  $\|\psi_j\|_2^2 \leq u$  for all  $j$ ). Clearly,  $\psi$  inherits the first three constraints in (4.18) from  $\psi_j$ . Moreover, since

$$(4.27) \quad \int (1 - e^{-\psi_j^2})^2 = 1 \quad \forall j,$$

$$\lim_{j \rightarrow \infty} \int_{B_R \setminus B_r} (1 - e^{-\psi_j^2})^2 = \int_{B_R \setminus B_r} (1 - e^{-\psi^2})^2,$$

$$\lim_{R \rightarrow \infty, r \downarrow 0} \int_{B_R \setminus B_r} (1 - e^{-\psi^2})^2 = \int (1 - e^{-\psi^2})^2,$$

we have that  $\psi$  also inherits the fourth constraint in (4.18) from  $\psi_j$ . Therefore  $\psi$  belongs to the set. This completes the proof of Lemma 8 and hence of part (iii).  $\square$

(iv) To prove that  $\Theta_d$  is continuous on  $(u_\diamond, \infty)$ , we argue as follows.

1. Suppose that the variational problem for  $\widehat{\Theta}_d(u)$  in (4.10) does not have a minimiser. By Lemma 8, the variational problem for  $\widetilde{\Theta}_d(u)$  does have a minimiser  $\psi$ . Therefore  $\|\psi\|_2^2 < u$ , otherwise  $\psi$  would also be a minimiser for  $\widehat{\Theta}_d(u)$ . Let  $v = \|\psi\|_2^2$ . Then  $\psi$  is a minimiser for both  $\widehat{\Theta}_d(v)$  and  $\widetilde{\Theta}_d(v)$ . So, by Lemma 7, we have  $\widehat{\Theta}_d(v) = \widetilde{\Theta}_d(v) = \widetilde{\Theta}_d(u) = \widehat{\Theta}_d(u)$ . Since  $\widetilde{\Theta}_d$  is right-continuous, it follows that it is continuous at  $u$ , and therefore so is  $\Theta_d$ . Suppose next that the variational problem for  $\widehat{\Theta}_d(u)$  does have a minimiser  $\psi$ . Then  $\psi$  is radially symmetric, continuous and strictly decreasing (see Lemma 10 in Section 5). Without loss of generality we may assume that  $\psi$  is centered at 0. Fix  $d, u$  and let  $\delta > 0$  be arbitrary. We will construct a  $\bar{\psi}$  (depending on  $d, u, \psi, \delta$ ) that is radially symmetric and nonincreasing such that

$$(4.28) \quad \|\nabla \bar{\psi}\|_2^2 \leq \Theta_d(u) + \delta, \quad \|\bar{\psi}\|_2^2 = \bar{u} < u, \quad \int (1 - e^{-\bar{\psi}^2})^2 \geq 1.$$

Since  $\Theta_d$  is nonincreasing, it follows from (4.28) that

$$(4.29) \quad \Theta_d(u) \leq \Theta_d(v) \leq \Theta_d(\bar{u}) \leq \Theta_d(u) + \delta \quad \forall v \in (\bar{u}, u).$$

Since  $\delta$  is arbitrary, this implies the claim.

2. For  $\epsilon > 0$ , define

$$(4.30) \quad r_\epsilon = \min\{|x| : \psi(x) < \epsilon\}.$$

For  $0 < \alpha < 1$ , define

$$(4.31) \quad q_\epsilon(x) = \begin{cases} 1 & \text{if } |x| \leq r_\epsilon, \\ 1 - \alpha \frac{|x| - r_\epsilon}{r_\epsilon} & \text{if } r_\epsilon < |x| \leq 2r_\epsilon, \\ 1 - \alpha & \text{if } |x| > 2r_\epsilon. \end{cases}$$

Then

$$(4.32) \quad 0 < q_\epsilon \leq 1, \quad \|\nabla q_\epsilon\|_2^2 = \omega_d \alpha^2 (2^d - 1) r_\epsilon^{d-2}.$$

Let

$$(4.33) \quad \tilde{\psi}(x) = q_\epsilon(x)\psi(x), \quad x \in \mathbb{R}^d.$$

Then

$$\begin{aligned}
 (4.34) \quad \int (1 - e^{-\tilde{\psi}^2})^2 &= 1 + \int [(1 - e^{-\tilde{\psi}^2})^2 - (1 - e^{-\psi^2})^2] \\
 &= 1 + \int (e^{-\psi^2} - e^{-q_\epsilon^2 \psi^2})(2 - e^{-\psi^2} - e^{-q_\epsilon^2 \psi^2}) \\
 &\geq 1 - \int (1 - q_\epsilon^2) \psi^2 (1 + q_\epsilon^2) \psi^2 \\
 &\geq 1 - 2\epsilon^2 \int (1 - q_\epsilon^2) \psi^2,
 \end{aligned}$$

where we use that  $\psi^2 < \epsilon$  on the set  $\{x \in \mathbb{R}^d : q_\epsilon^2(x) < 1\}$ . Moreover,

$$(4.35) \quad \|\tilde{\psi}\|_2^2 = u - \int (1 - q_\epsilon^2) \psi^2$$

and

$$\begin{aligned}
 (4.36) \quad \|\nabla \tilde{\psi}\|_2^2 &= \int |\psi \nabla q_\epsilon + q_\epsilon \nabla \psi|^2 \\
 &\leq \int \psi^2 |\nabla q_\epsilon|^2 + \int q_\epsilon^2 |\nabla \psi|^2 + 2 \int q_\epsilon \psi |\nabla q_\epsilon \cdot \nabla \psi| \\
 &= \int_{\{r_\epsilon \leq |x| \leq 2r_\epsilon\}} \psi^2 |\nabla q_\epsilon|^2 + \int q_\epsilon^2 |\nabla \psi|^2 + 2 \int_{\{r_\epsilon \leq |x| \leq 2r_\epsilon\}} q_\epsilon \psi |\nabla q_\epsilon \cdot \nabla \psi| \\
 &\leq \epsilon^2 \int |\nabla q_\epsilon|^2 + \int |\nabla \psi|^2 + 2\epsilon \int |\nabla q_\epsilon| |\nabla \psi| \\
 &\leq \epsilon^2 \|\nabla q_\epsilon\|_2^2 + \Theta_d(u) + 2\epsilon \|\nabla q_\epsilon\|_2 \|\nabla \psi\|_2.
 \end{aligned}$$

3. Next choose

$$(4.37) \quad \alpha = \min\left\{\frac{1}{2}, (2^d - 1)^{-1/2} \omega_d^{-1/2} r_\epsilon^{1-d/2}\right\}.$$

Then, by (4.32), (4.36) and (4.37),

$$(4.38) \quad \|\nabla \tilde{\psi}\|_2^2 \leq (\sqrt{\Theta_d(u)} + \epsilon)^2.$$

Finally, let

$$(4.39) \quad \bar{\psi}(x) = q \tilde{\psi}(x/p)$$

with

$$(4.40) \quad p = \left(1 - 2\epsilon^2 \int (1 - q_\epsilon^2) \psi^2\right)^{-1/d}, \quad q = 1,$$



and  $\epsilon$  small enough so that the right-hand side of (4.34) is strictly positive. Then (4.1) and (4.2) in combination with (4.34), (4.35) and (4.38) imply that  $\bar{\psi}$  satisfies

$$(4.41) \quad \begin{aligned} \|\nabla \bar{\psi}\|_2^2 &\leq \left(1 - 2\epsilon^2 \int (1 - q_\epsilon^2)\psi^2\right)^{1-d/2} (\sqrt{\Theta_d(u)} + \epsilon)^2, \\ \|\bar{\psi}\|_2^2 &= \frac{u - \int (1 - q_\epsilon^2)\psi^2}{1 - 2\epsilon^2 \int (1 - q_\epsilon^2)\psi^2}, \\ \int (1 - e^{-\bar{\psi}^2})^2 &\geq 1. \end{aligned}$$

It follows from (4.41) that for any  $\delta > 0$  there exists an  $\epsilon > 0$  such that  $\bar{\psi}$  satisfies (4.28).

(v) The divergence of  $\Theta_d(u)$  as  $u \downarrow u_\diamond$  comes from the following bounds.

LEMMA 9. *There exist constants  $c_1, c_2$  (depending only on  $d$ ) such that*

$$(4.42) \quad c_1 \leq (u - u_\diamond)\Theta_d(u) \leq c_2 \quad \text{for } u_\diamond < u \leq u_\diamond \min\{\frac{9}{8}, 1 + 2^{-d}\}.$$

*Proof.* Recall the definition of  $K = 1/u_\diamond$  and  $\zeta_\diamond$  from part (ii). The variational problem in (1.14) may be rewritten as

$$(4.43) \quad \Theta_d(u) = \inf \left\{ \|\nabla \psi\|_2^2 : \|\psi\|_2^2 = u, \int F(\psi^2) \leq Ku - 1, \psi \text{ RSNI} \right\},$$

where

$$(4.44) \quad F(t) = Kt - (1 - e^{-t})^2, \quad t \geq 0.$$

1. First we derive the lower bound. By the definition of  $K$  and the inequality  $e^{-t} \geq 1 - t, t \geq 0$ , we have  $F(t) \geq \max\{0, Kt - t^2\}$ . Let

$$(4.45) \quad \mu(t) = |\{x : \psi^2(x) \geq t\}|, \quad t \geq 0.$$

Suppose that  $\psi$  satisfies the constraints in (4.43). Then we have

$$(4.46) \quad \begin{aligned} Ku - 1 &\geq \int_{\{K/3 \leq \psi^2 < 2K/3\}} F(\psi^2) \\ &\geq \int_{\{K/3 \leq \psi^2 < 2K/3\}} (K\psi^2 - \psi^4) \geq \frac{2K^2}{9} [\mu(K/3) - \mu(2K/3)]. \end{aligned}$$

Moreover,

$$(4.47) \quad \begin{aligned} 1 &\leq \int (1 - e^{-\psi^2})^2 \leq \mu(2K/3) + \int_{\{\psi^2 < 2K/3\}} (1 - e^{-\psi^2})^2 \\ &\leq \mu(2K/3) + \int_{\{\psi^2 < 2K/3\}} \psi^4 \leq \mu(2K/3) + 2Ku/3, \end{aligned}$$

implying that  $\mu(2K/3) \geq 1/4$  for  $u_\diamond < u \leq \frac{9}{8}u_\diamond$ . This in turn implies that

$$(4.48) \quad R_0 = \min\{|x|: \psi^2(x) < 2K/3\} \geq (4\omega_d)^{-1/d}.$$

Using (4.48) and Cauchy-Schwarz, we get

$$(4.49) \quad \begin{aligned} K/6 &= - \int_{\{K/3 \leq \psi^2 < 2K/3\}} \psi(r) \frac{d\psi}{dr}(r) dr \\ &\leq \frac{R_0^{1-d}}{d\omega_d} \int_{\{K/3 \leq \psi^2 < 2K/3\}} \left[ \psi(r) r^{(d-1)/2} (d\omega_d)^{1/2} \right] \\ &\quad \times \left[ - \frac{d\psi}{dr}(r) r^{(d-1)/2} (d\omega_d)^{1/2} \right] dr \\ &\leq \frac{R_0^{1-d}}{d\omega_d} \left( \int_{\{K/3 \leq \psi^2 < 2K/3\}} \psi^2 \right)^{1/2} \|\nabla\psi\|_2 \\ &\leq \frac{R_0^{1-d}}{d\omega_d} (2K/3)^{1/2} [\mu(K/3) - \mu(2K/3)]^{1/2} \|\nabla\psi\|_2. \end{aligned}$$

It follows from (4.46), (4.48) and (4.49) that the lower bound in (4.42) holds with  $c_1 = 2^{-8+4/d} 3d^2 \omega_d^{2/d}$ .

2. Next we derive the upper bound. For  $R_2 \geq R_1 \geq 0$ , consider the test function  $\psi_{R_1, R_2}$  defined by

$$(4.50) \quad \psi_{R_1, R_2}(x) = \begin{cases} \zeta_\diamond^{1/2} & \text{if } 0 \leq |x| < R_1, \\ \zeta_\diamond^{1/2} \frac{R_2 - |x|}{R_2 - R_1} & \text{if } R_1 \leq |x| \leq R_2, \\ 0 & \text{if } |x| > R_2. \end{cases}$$

A straightforward calculation gives

$$(4.51) \quad \begin{aligned} \|\nabla\psi_{R_1, R_2}\|_2^2 &\leq \omega_d R_2^d (R_2 - R_1)^{-1} \zeta_\diamond, \\ \|\psi_{R_1, R_2}\|_2^2 &= \omega_d R_1^d \zeta_\diamond + d\omega_d \zeta_\diamond (R_2 - R_1) \int_0^1 [R_1 + (R_2 - R_1)v]^{d-1} v^2 dv. \end{aligned}$$

From (4.44) we have

$$(4.52) \quad F''(t) = \frac{1}{4} - 4\left(e^{-t} - \frac{1}{4}\right)^2 \leq \frac{1}{4}, \quad t \geq 0.$$

Since  $F'(\zeta_\diamond) = 0$ , it follows that

$$(4.53) \quad F(t) \leq \frac{1}{4}(t - \zeta_\diamond)^2, \quad t \geq 0.$$

By the definition of  $\zeta_\diamond$ ,

$$(4.54) \quad \begin{aligned} \int F(\psi_{R_1, R_2}^2) &= \int_{\{R_1 \leq |x| \leq R_2\}} F(\psi_{R_1, R_2}^2) \\ &\leq \frac{1}{4} \int_{\{R_1 \leq |x| \leq R_2\}} (\psi_{R_1, R_2}^2 - \zeta_\diamond)^2 \leq \frac{1}{4} d\omega_d \zeta_\diamond^2 (R_2 - R_1) R_2^{d-1}. \end{aligned}$$

Let  $R_2 = R_2(R_1)$  be the unique positive root of

$$(4.55) \quad R = R_1 + \frac{4(Ku - 1)}{d\omega_d R^{d-1} \zeta_\diamond^2}.$$

It follows from (4.54) that  $\psi_{R_1, R_2}$  satisfies the second constraint in (4.43). The choice  $R_1 = (u/\omega_d \zeta_\diamond)^{1/d}$  yields that the second expression in (4.51) is strictly larger than  $u$ . On the other hand, the choice  $R_1 = 0$  gives  $R_2^d = 4(Ku - 1)/d\omega_d \zeta_\diamond^2$  and yields, by the constraint on  $u$  in (4.42),

$$(4.56) \quad \|\psi_{0, R_2(0)}\|_2^2 = \frac{4(Ku - 1)}{\zeta_\diamond} \int_0^1 (1+v)^{d-1} (1-v)^2 dv \leq u.$$

Hence there exists a pair  $(R_1(u), R_2(u))$  such that  $\psi_{R_1(u), R_2(u)}$  satisfies the constraints in (4.43). Finally, from the first expression in (4.51) in combination with (4.55) we get

$$(4.57) \quad \Theta_d(u) \leq \|\nabla \psi_{R_1(u), R_2(u)}\|_2^2 \leq \frac{1}{4} \zeta_\diamond^3 d\omega_d^2 (Ku - 1)^{-1} R_2(u)^{2d-1}.$$

3. It remains to find an upper bound on  $R_2(u)$ . Since  $R_1(u) \leq (u/\omega_d \zeta_\diamond)^{1/d}$  we have, by (4.55),

$$(4.58) \quad R_2(u) \leq R_1(u) + \frac{4(Ku - 1)}{d\omega_d R_2(u)^{d-1} \zeta_\diamond^2} \leq \left(\frac{u}{\omega_d \zeta_\diamond}\right)^{1/d} + \frac{4(Ku - 1)}{d\omega_d R_2(u)^{d-1} \zeta_\diamond^2}.$$

Moreover, by the expression in (4.51) we have

$$(4.59) \quad u \leq \omega_d R_2(u)^d \zeta_\diamond + d\omega_d \zeta_\diamond R_2(u)^d / 3 \leq 2d\omega_d \zeta_\diamond R_2(u)^d.$$

Combining (4.58) and (4.59), we obtain the required upper bound on  $R_2(u)$ :

$$(4.60) \quad R_2(u) \leq \left(\frac{u}{\omega_d \zeta_\diamond}\right)^{1/d} + \frac{4(Ku - 1)}{d\omega_d^2 \zeta_\diamond^2} \left(\frac{u}{2d\omega_d \zeta_\diamond}\right)^{(1-d)/d}.$$

For  $u$  satisfying the constraint in (4.42), there exists a constant  $c_3$  (depending only on  $d$ ) such that  $R_2(u) \leq c_3$ . Hence the upper bound in (4.42) holds with  $c_2 = \zeta_\diamond^3 d\omega_d^2 K c_3^{2d-1} / 4$ . □

### 5. Proof of Theorem 4

1. Let  $\phi \in H^1(\mathbb{R}^d)$  and pick  $p = u^{2/d}$ ,  $q = u^{-1/2}$  in (4.1) and (4.2). Then (recall (1.14))

$$(5.1) \quad u^{(4-d)/d} \Theta_d(u) = \inf \left\{ \|\nabla \phi\|_2^2 : \phi \in H^1(\mathbb{R}^d), \|\phi\|_2^2 = 1, \int u^2 (1 - e^{-u^{-1}\phi^2})^2 \geq 1 \right\}.$$

Since  $u \mapsto u^2(1 - e^{-u^{-1}\phi^2})^2$  is nondecreasing on  $[0, \infty)$ , we see that  $u \mapsto u^{(4-d)/d} \Theta_d(u)$  is nonincreasing on  $[0, \infty)$ . In part 4 we will prove that it is strictly decreasing on  $(u_\diamond, \infty)$ .

2. Since  $u^2(1 - e^{-u^{-1}\phi^2})^2 \leq \phi^4$ , we have

$$(5.2) \quad u^{(4-d)/d} \Theta_d(u) \geq \mu_d, \quad d = 2, 3, 4.$$

Note that the constraint  $\|\phi\|_4 = 1$  in (1.16) may be replaced by  $\|\phi\|_4 \geq 1$  via an argument similar to the one in the proof of Lemma 7. On the other hand,  $u^2(1 - e^{-u^{-1}\phi^2})^2 \geq \phi^4 - \phi^6/2u$ , and hence

$$(5.3) \quad u^{(4-d)/d} \Theta_d(u) \leq \inf\{\|\nabla\phi\|_2^2 : \phi \in H^1(\mathbb{R}^d), \|\phi\|_2^2 = 1, \|\phi\|_4^4 \geq 1 + \frac{1}{2u}\|\phi\|_6^6\}.$$

In [3, §5.5] it was shown that

$$(5.4) \quad \limsup_{\delta \downarrow 0} \inf\{\|\nabla\phi\|_2^2 : \phi \in H^1(\mathbb{R}^d), \|\phi\|_2^2 = 1, \|\phi\|_4^4 \geq 1 + \frac{2\delta}{3}\|\phi\|_6^6\} \leq \mu_d.$$

Replacing  $\delta$  by  $3/4u$  in (5.4), we get from (5.3) that

$$(5.5) \quad \limsup_{u \rightarrow \infty} u^{(4-d)/d} \Theta_d(u) \leq \mu_d.$$

3. To settle the claim made in part 1 we need the following fact.

LEMMA 10. *Let  $2 \leq d \leq 4$ . Then for every  $u > u_\diamond$  the variational problem for  $\widehat{\Theta}_d(u)$  has a minimiser.*

*Proof.* By Lemma 8, the variational problem for  $\widetilde{\Theta}_d(u)$  has a minimiser, say  $\psi^*$ . There are two cases. Either  $\|\psi^*\|_2^2 = u$ , in which case  $\psi^*$  is also a minimiser of  $\widehat{\Theta}_d(u)$  and we are done, or  $\|\psi^*\|_2^2 < u$ . We will show that the latter is impossible. This goes as follows.

$d = 2, 3$ : Suppose that  $\|\psi^*\|_2^2 = u' < u$ . Then

$$(5.6) \quad \widetilde{\Theta}_d(u) = \|\nabla\psi^*\|_2^2 \geq \Theta_d(u').$$

But it follows from part 1 that  $u \mapsto \widetilde{\Theta}_d(u)$  is strictly decreasing on  $(u_\diamond, \infty)$  for  $d = 2, 3$ . Hence we have a contradiction.

$d = 3, 4$ : Perturbing  $\psi^*$  inside the set  $\{\psi \in D^1(\mathbb{R}^d) : \|\psi\|_2^2 \leq u, \int(1 - e^{-\psi^2})^2 = 1\}$  with smooth perturbations, we have that  $\psi^*$  must satisfy the Euler-Lagrange equation associated with the variational problem

$$(5.7) \quad \inf\left\{\|\nabla\psi\|_2^2 : \psi \in D^1(\mathbb{R}^d), \int(1 - e^{-\psi^2})^2 = 1\right\},$$

which reads

$$(5.8) \quad \Delta\psi = -\lambda_d\psi e^{-\psi^2}(1 - e^{-\psi^2}),$$

where  $\lambda_d > 0$  is a Lagrange multiplier. The formal derivation of (5.8) uses the results in Berestycki and Lions [1, §5b]. Without loss of generality we may consider only RSNI-solutions of (5.8):

$$(5.9) \quad \frac{d}{dr}\left(r^{d-1}\frac{d\psi}{dr}\right) = -\lambda_d\psi e^{-\psi^2}(1 - e^{-\psi^2})r^{d-1}.$$

We have  $\frac{d\psi}{dr}(0) = 0$  (see the same reference). Integrating (5.9) over  $[0, r]$ , we get

$$(5.10) \quad r^{d-1} \frac{d\psi}{dr} = -\lambda_d \int_0^r \psi e^{-\psi^2} (1 - e^{-\psi^2}) r^{d-1} dr$$

$$\leq -\lambda_d \int_0^1 \psi e^{-\psi^2} (1 - e^{-\psi^2}) r^{d-1} dr = -c_\psi \quad \forall r \geq 1 \exists c_\psi > 0.$$

Hence  $\frac{d\psi}{dr}(r) \leq -c_\psi r^{1-d}$ . Integrating this inequality over  $[r, \infty)$  and using the fact that  $\lim_{r \rightarrow \infty} \psi(r) = 0$ , we get

$$(5.11) \quad \psi(r) \geq \frac{c_\psi}{d-2} r^{2-d} \quad \forall r \geq 1.$$

Hence  $\psi \notin L^2(\mathbb{R}^d)$  for  $d = 3, 4$ . □

4. We are now ready to prove the strict monotonicity of  $u \mapsto u^{(4-d)/d} \Theta_d(u)$  on  $(u_\infty, \infty)$ . Pick  $u > u_\infty$ . Lemma 10 guarantees the existence of a minimiser  $\phi^*$  for the variational problem

$$(5.12) \quad u^{(4-d)/d} \widehat{\Theta}_d(u) = \inf \left\{ \|\nabla \phi\|_2^2 : \phi \in H^1(\mathbb{R}^d), \|\phi\|_2^2 = 1, \int u^2 (1 - e^{-u^{-1}\phi^2})^2 = 1 \right\}$$

(recall (5.1) and Lemma 7). Pick  $v > u$ . Since  $u \mapsto u^2 (1 - e^{-u^{-1}\phi^{*2}})^2$  is strictly increasing on  $[0, \infty)$  when  $\phi^* > 0$ , there exists a  $\delta_{u,v} > 0$  such that

$$(5.13) \quad \int v^2 (1 - e^{-v^{-1}\phi^{*2}})^2 = 1 + \delta_{u,v}.$$

Hence we have

$$(5.14) \quad u^{(4-d)/d} \widehat{\Theta}_d(u) = \|\nabla \phi^*\|_2^2$$

$$\geq \inf \{ \|\nabla \phi\|_2^2 : \|\phi\|_2^2 = 1, \int v^2 (1 - e^{-v^{-1}\phi^2})^2 = 1 + \delta_{u,v} \}$$

$$= (1 + \delta_{u,v})^{(d-2)/d} v^{(4-d)/d} \inf \{ \|\nabla \phi\|_2^2 : \|\phi\|_2^2 = \frac{v}{1 + \delta_{u,v}}, \int (1 - e^{-\phi^2})^2 = 1 \}$$

$$= (1 + \delta_{u,v})^{(d-2)/d} v^{(4-d)/d} \widehat{\Theta}_d\left(\frac{v}{1 + \delta_{u,v}}\right)$$

$$= (1 + \delta_{u,v})^{2/d} \left[ \left(\frac{v}{1 + \delta_{u,v}}\right)^{(4-d)/d} \widehat{\Theta}_d\left(\frac{v}{1 + \delta_{u,v}}\right) \right],$$

where the second equality uses (4.1) and (4.2). But, by part 1 and Lemma 7, the right-hand side is  $\geq (1 + \delta_{u,v})^{2/d} v^{(4-d)/d} \widehat{\Theta}_d(v)$ , and so  $u^{(4-d)/d} \widehat{\Theta}_d(u) > v^{(4-d)/d} \widehat{\Theta}_d(v)$  because  $\delta_{u,v} > 0$ .

5. The proof that  $\mu_d > 0$  for  $2 \leq d \leq 4$  is given in [3, Lemma 15]. There it is also shown that  $\mu_4 = S_4$ , the Sobolev constant in (4.3). It is easy to see that  $\mu_d < \infty$  for  $2 \leq d \leq 4$ .

**6. Proof of Theorem 5**

(i) The proof again relies on the Sobolev inequality in (4.3).

1. Since  $e^{-\psi^2} \geq 1 - \psi^{d/(d-2)}$  for  $d \geq 5$ , we have

$$\begin{aligned} (6.1) \quad \eta_d &= \inf\{\|\nabla\psi\|_2^2 : \psi \in D^1(\mathbb{R}^d), \int (1 - e^{-\psi^2})^2 = 1\} \\ &\geq \inf\{\|\nabla\psi\|_2^2 : \psi \in D^1(\mathbb{R}^d), \int \psi^{2d/(d-2)} \geq 1\} \\ &= S_d, \end{aligned}$$

by the Sobolev inequality (4.3). To prove that  $\eta_d < \infty$ , we simply note that  $\psi_a$  given by  $\psi_a(x) = e^{-a^{-1}|x|^2}$  is in  $D^1(\mathbb{R}^d)$ . Adjusting  $a$  such that  $\psi_a$  satisfies the integral constraint in (1.17), we see that  $\eta_d \leq \|\nabla\psi_a\|_2^2$ .

2. To prove that (1.17) has a minimiser, let  $(\psi_j)$  be a minimising sequence for  $\eta_d$ , i.e.,  $\psi_j$  is RSNI and  $\int(1 - e^{-\psi_j^2})^2 = 1$  for all  $j$  and  $\lim_{j \rightarrow \infty} \|\nabla\psi_j\|_2^2 = \eta_d$ . We can extract a subsequence, again denoted by  $(\psi_j)$ , such that  $\psi_j \rightharpoonup \psi^*$  weakly in  $D^1(\mathbb{R}^d)$  and almost everywhere in  $\mathbb{R}^d$  as  $j \rightarrow \infty$  for some  $\psi^* \in D^1(\mathbb{R}^d)$ . Clearly,  $\psi^*$  is RSNI and  $\eta_d \geq \|\nabla\psi^*\|_2^2$ . It therefore suffices to show that  $\psi^*$  satisfies the integral constraint in (1.17), since this implies that  $\eta_d \leq \|\nabla\psi^*\|_2^2$  and hence that  $\psi^*$  is a minimiser.

3. Estimate

$$(6.2) \quad 0 \leq \int (1 - e^{-\psi^{*2}})^2 \leq \int \psi^{*2d/(d-2)} \leq \left(\frac{\eta_d}{S_d}\right)^{(d-2)/d}.$$

Fix  $\epsilon > 0$ . Then there exists an  $R_1(\epsilon) > 0$  such that

$$(6.3) \quad 0 \leq \int_{B_{R_1(\epsilon)}^c} (1 - e^{-\psi^{*2}})^2 \leq \epsilon.$$

Let  $C = \sup_j \|\nabla\psi_j\|_2^2$  and define  $R_2(\epsilon)$  by

$$(6.4) \quad \left(\frac{C}{S_d}\right)^2 \omega_d^{(4-d)/d} \frac{d}{d-4} R_2(\epsilon)^{4-d} = \epsilon.$$

Since  $\psi_j$  is RSNI,  $\psi_j \in D^1(\mathbb{R}^d)$  and  $\|\nabla\psi_j\|_2^2 \leq C$  for all  $j$ , it follows from the Sobolev inequality in (4.3) that

$$\begin{aligned} (6.5) \quad C &\geq S_d \|\psi_j\|_{2d/(d-2)}^2 \geq S_d \|\psi_j 1_{B_r(0)}\|_{2d/(d-2)}^2 \\ &\geq S_d \psi_j(r)^2 |B_r(0)|^{(d-2)/d} = S_d \psi_j(r)^2 \omega_d^{(d-2)/d} r^{d-2} \quad \forall r > 0 \forall j. \end{aligned}$$

Combining (6.4) and (6.5), we find

$$(6.6) \quad \int_{B_{R_2(\epsilon)}^c} (1 - e^{-\psi_j^2})^2 \leq \int_{B_{R_2(\epsilon)}^c} \psi_j^4 \leq \left(\frac{C}{S_d}\right)^2 \omega_d^{(4-2d)/d} \int_{B_{R_2(\epsilon)}^c} |x|^{4-2d} dx = \epsilon \quad \forall j.$$

Now put  $R(\epsilon) = \max\{R_1(\epsilon), R_2(\epsilon)\}$ . Then, since  $\int (1 - e^{-\psi_j^2})^2 = 1$  for all  $j$ , we get from (6.6) that

$$(6.7) \quad 1 - \epsilon \leq \int_{B_{R(\epsilon)}} (1 - e^{-\psi_j^2})^2 \leq 1 \quad \forall j.$$

Since  $\psi_j \rightarrow \psi^*$  almost everywhere, it follows from the dominated convergence theorem that

$$(6.8) \quad 1 - \epsilon \leq \int_{B_{R(\epsilon)}} (1 - e^{-\psi^{*2}})^2 \leq 1.$$

Combining this inequality with (6.3), we obtain

$$(6.9) \quad 1 - \epsilon \leq \int (1 - e^{-\psi^{*2}})^2 \leq 1 + \epsilon.$$

Since  $\epsilon$  was arbitrary, we conclude that  $\psi^*$  satisfies the integral constraint in (1.17) and therefore is a minimiser of (1.17).

4. It remains to show that  $\psi^*$  is unique up to translations and that  $\|\psi^*\|_2 < \infty$ . Once this is done, we can identify  $\psi_d$  in Theorem 5 with  $\psi^*$ . To prove uniqueness, we recall that  $\psi^*$  satisfies (5.8), the Euler-Lagrange equation associated with (1.17):

$$(6.10) \quad \Delta\psi^* = -\lambda_d\psi^*e^{-\psi^{*2}}(1 - e^{-\psi^{*2}}),$$

where the Lagrange multiplier  $\lambda_d$  is uniquely determined by Pohozaev’s identity (see Berestycki and Lions [1, §5b]). The resolvent  $(-\Delta)^{-1}$  has  $K(x, y) = \frac{1}{4\pi^{d/2}}\Gamma(\frac{d}{2} - 1)|x - y|^{-(d-2)}$  as integral kernel. Hence

$$(6.11) \quad \psi^*(x) = \lambda_d \int K(x, y)\psi^*(y)e^{-\psi^{*2}(y)}(1 - e^{-\psi^{*2}(y)}) dy.$$

It follows from the arguments in [3, §5.6(III)2], that

$$(6.12) \quad \psi^*(x) \leq \min\{\psi^*(0), C|x|^{-(d-2)}\} \quad \text{for some } C < \infty.$$

Combining (6.11) and (6.12), we obtain

$$(6.13) \quad \lim_{|x| \rightarrow \infty} |x|^{d-2}\psi^*(x) = \lambda_d \frac{1}{4\pi^{d/2}}\Gamma\left(\frac{d}{2} - 1\right) \int \psi^*(y)e^{-\psi^{*2}(y)}(1 - e^{-\psi^{*2}(y)}) dy,$$

where the right-hand side is strictly positive and finite. Thus, we see that  $\psi^*$  is a “fast decay solution” of (6.10). We can now apply Tang [15, Theorem 2] to conclude that  $\psi^*$  is the unique minimiser of (1.17) up to translations.

Hypotheses (H1) and (H2) in [15] (with  $p = 1$ ,  $q = 3$  and  $m = 2$ ), which need to be satisfied by the right-hand side of (6.10), are easily verified. Finally, (6.12) implies that  $\|\psi^*\|_2 < \infty$ .

(ii) The proof is based on leakage of  $L^2(\mathbb{R}^d)$ -mass.

1. By dropping the constraint  $\|\psi\|_2^2 = u$  in (1.14) we obtain the lower bound  $\Theta_d(u) \geq \eta_d$  in (1.18). To prove the upper bound, let  $u \in (u_d, \infty)$  and define

$$(6.14) \quad \psi_{n,u}^{*2} = \psi_d^2 + (u - \|\psi_d\|_2^2)p_n$$

with  $p_n$  given by (4.19). Then  $\|\psi_{n,u}^*\|_2^2 = u$  and

$$(6.15) \quad \int (1 - e^{-\psi_{n,u}^{*2}})^2 \geq \int (1 - e^{-\psi_d^2})^2 = 1.$$

Hence  $\psi_{n,u}^*$  satisfies the constraints in (1.14). Moreover, by the convexity inequality for gradients we have

$$(6.16) \quad \|\nabla \psi_{n,u}^*\|_2^2 \leq \|\nabla \psi_d\|_2^2 + (u - \|\psi_d\|_2^2)\|\nabla \sqrt{p_n}\|_2^2 = \eta_d + \frac{2d(u - u_d)}{n^2}.$$

Let  $n \rightarrow \infty$  to obtain  $\Theta_d(u) \leq \eta_d$  for  $u \in (u_d, \infty)$ , which proves the upper bound in (1.18).

2. To prove that  $u \mapsto \Theta_d(u)$  is strictly decreasing on  $(u_\diamond, u_d)$  we argue as follows. The following result, which is an analogue of Lemma 10, is valid for  $d \geq 2$ , though we will need it only for  $d \geq 5$ .

LEMMA 11. *Let  $d \geq 2$ . Suppose that  $u_- > u_\diamond$  and that  $\Theta_d(v) > \Theta_d(u_-)$  for all  $v \in (u_\diamond, u_-)$ . Then the variational problem for  $\Theta_d(u_-)$  has a minimiser.*

*Proof.* By Lemma 8,  $\tilde{\Theta}_d(u_-)$  has a minimiser, say  $\tilde{\psi}$ . Let  $\|\tilde{\psi}\|_2^2 = v$ . Then, by (4.10),  $v \leq u_-$  and  $\tilde{\psi}$  is a minimiser also for  $\tilde{\Theta}_d(v)$ . Hence,  $\tilde{\Theta}_d(v) = \tilde{\Theta}_d(u_-)$ . Therefore, by Lemma 7,  $\Theta_d(v) = \Theta_d(u_-)$ . Hence  $v = u_-$  (by the assumption in the lemma), so that  $\|\tilde{\psi}\|_2^2 = u_-$ . Consequently,  $\tilde{\psi}$  is a minimiser also for  $\Theta_d(u_-)$ .  $\square$

The rest of the proof is via contradiction. Suppose that  $u_- \in (u_\diamond, u_d)$  is such that  $\Theta_d(v) > \Theta_d(u_-)$  for  $v$  in a left neighbourhood of  $u_-$  and  $\Theta_d(v) = \Theta_d(u_-)$  for  $v$  in a right neighbourhood of  $u_-$ . By Lemma 8,  $\tilde{\Theta}_d(v)$  has a minimiser for  $v$  in a right neighbourhood of  $u_-$ . By taking smooth variations of this minimiser under the constraint  $u_- \leq \|\psi\|_2^2 \leq u_- + \epsilon$  for some  $\epsilon > 0$ , we obtain that  $\Theta_d(v)$  has a minimiser  $\psi$  satisfying the Euler-Lagrange equation (recall (5.8))

$$(6.17) \quad \Delta \psi = -\lambda_- \psi e^{-\psi^2} (1 - e^{-\psi^2}),$$

where  $\lambda_- = \frac{d-2}{2d} \Theta_d(u_-)$  by Pohozaev’s identity. The minimiser  $\psi_-$  for  $\Theta_d(u_-)$ , which exists by Lemma 11, also satisfies (6.17). Now, let  $\psi_d$  be the unique “fast



decay solution” of the variational problem for  $\Theta_d(u_d)$  (according to Tang [15]). Then  $\psi_d(\cdot(\lambda_-/\lambda_d)^{1/2})$  is a fast decay solution of (6.17), which by uniqueness equals  $\psi_-(\cdot)$ . By scaling we have

$$(6.18) \quad u_- = \|\psi_-\|_2^2 = \left(\frac{\lambda_d}{\lambda_-}\right)^{d/2} u_d$$

and

$$(6.19) \quad \int (1 - e^{-\psi^2})^2 = \left(\frac{\lambda_d}{\lambda_-}\right)^{d/2} \int (1 - e^{-\psi_d^2})^2 = \left(\frac{\lambda_d}{\lambda_-}\right)^{d/2}.$$

Since  $\psi_-$  is a minimiser for  $\Theta_d(u_-)$ , we have  $\lambda_- = \lambda_d$  by (6.19). Hence  $u_- = u_d$  by (6.18), leading to a contradiction. Consequently,  $u \mapsto \Theta_d(u)$  is strictly decreasing on  $(u_\diamond, u_d)$ .

### 7. Proof of Theorem 6

(i) By Theorems 4 and 5(ii), we have that  $u \mapsto \Theta_d(u)$  is strictly decreasing for  $2 \leq d \leq 4$  and  $u \in (u_\diamond, \infty)$  or  $d \geq 5$  and  $u \in (u_\diamond, u_d]$ . Hence  $\Theta_d(u)$  has a minimiser by Lemma 11. The proof that this minimiser is RSNI is similar to the proof of Theorem 5(i) (see also the proof of Theorems 4 and 5 in [3]).

(ii) The proof runs via contradiction. Let  $d \geq 5$  and  $u \in (u_d, \infty)$ . Suppose that  $\Theta_d(u)$  has a minimiser, say  $\psi$ . Let

$$(7.1) \quad \bar{\eta}_d = \inf\{\|\nabla\psi\|_2^2 : \psi \in D^1(\mathbb{R}^d), \int (1 - e^{-\psi^2})^2 \geq 1\}.$$

Then, clearly,  $\psi$  is a minimiser of  $\bar{\eta}_d$  as well. It is easy to see that  $\bar{\eta}_d = \eta_d$  (compare (1.17) and (7.1)). Moreover, by Theorem 5(ii),  $\eta_d = \Theta_d(u)$  for  $u \in (u_d, \infty)$ . Hence  $\psi$  is a minimiser of  $\eta_d$  also. By Theorem 5(i), all minimisers of  $\eta_d$  have  $L^2$ -norm  $u_d$ . This contradicts the constraint  $\|\psi\|_2^2 = u$  in the variational problem for  $\Theta_d(u)$  for  $u \in (u_d, \infty)$ . Hence  $\Theta_d(u)$  does not have a minimiser for  $u \in (u_d, \infty)$ .

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### REFERENCES

[1] H. BERESTYCKI and P. L. LIONS, Nonlinear scalar field equations, I. Existence of a ground state, II. Existence of infinitely many solutions, *Arch. Rational Mech. Anal.* **82** (1983), 313–345 and 347–375.

- [2] M. VAN DEN BERG and E. BOLTHAUSEN, Asymptotics of the generating function for the volume of the Wiener sausage, *Probab. Theory Related Fields* **99** (1994), 389–397.
- [3] M. VAN DEN BERG, E. BOLTHAUSEN, and F. DEN HOLLANDER, Moderate deviations for the volume of the Wiener sausage, *Ann. of Math.* **153** (2001), 355–406.
- [4] M. VAN DEN BERG and B. TÓTH, Exponential estimates for the Wiener sausage, *Probab. Theory Related Fields* **88** (1991), 249–259.
- [5] W. BRYC, Large deviations by the asymptotic value method, in *Diffusion Processes and Related Problems in Analysis* (M. Pinsky, ed.), Vol. 1, 447–472, Birkhäuser, Boston, 1990.
- [6] M. D. DONSKER and S. R. S. VARADHAN, On the number of distinct sites visited by a random walk, *Comm. Pure Appl. Math.* **32** (1979), 721–747.
- [7] P. ERDÖS and S. J. TAYLOR, Some intersection properties of random walk paths, *Acta Math. Acad. Sci. Hung.* **11** (1960), 231–248.
- [8] R. FERNÁNDEZ, J. FRÖHLICH, and A. D. SOKAL, *Random Walks, Critical Phenomena, and Triviality in Quantum Field Theory, Texts and Monographs in Physics*, Springer-Verlag, New York, 1992.
- [9] K. M. KHANIN, A. E. MAZEL, S. B. SHLOSMAN, and YA. G. SINAI, Loop condensation effects in the behavior of random walks, in *The Dynkin Festschrift, Markov Processes and their Applications* (M. Freidlin, ed.), *Progr. Probab.* **34**, Birkhäuser, Boston, 1994, pp. 167–184.
- [10] G. LAWLER, *Intersections of Random Walks, Probability and its Applications*, Birkhäuser, Boston, 1991.
- [11] J.-F. LE GALL, Propriétés d’intersection des marches aléatoires, II. Etude des cas critiques, *Comm. Math. Phys.* **104** (1986), 509–528.
- [12] E. H. LIEB and M. LOSS, *Analysis, Graduate Studies in Math.*, Vol. 14, A. M. S., Providence, RI, 1997.
- [13] N. MADRAS and G. SLADE, *The Self-Avoiding Walk*, Birkhäuser, Boston, 1993.
- [14] F. SPITZER, Electrostatic capacity, heat flow and Brownian motion, *Z. Wahrscheinlichkeitstheor. Verw. Geb.* **3** (1964), 110–121.
- [15] M. TANG, Uniqueness and global structure of positive radial solutions for quasilinear elliptic equations, *Comm. Partial Differential Equations* **26** (2001), 909–938.

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