Some models for contention resolution in cable networks

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Abstract

In this paper we consider some models for contention resolution in cable networks, in case the contention pertains to requests and is carried out by means of contention trees. More specifically, we study a number of variants of the standard machine repair model, that differ in the service order at the repair facility. Considered service orders are First Come First Served, Random Order of Service, and Gated Random Order of Service. For these variants, we study the sojourn time at the repair facility. In the case of the free access protocol for contention trees, the first two moments of the access delay in contention are accurately represented by those of the sojourn time at the repair facility under Random Order of Service. In the case of the blocked access protocol, Gated Random Order of Service is shown to be more appropriate.

keywords: machine repair model, contention tree, request-grant mechanism, cable networks

1 Introduction

Cable networks are currently being upgraded to support bidirectional data transport, see e.g. van Driel *et al.* [7]. The system is thus extended with an 'upstream' channel to complement the 'downstream' channel that is already present. This upstream channel is time slotted and shared among many stations so that contention resolution is essential for upstream data transport. An efficient way to carry out the upstream data transport is via a request-grant mechanism, like in Digital Video Broadcasting [8]: stations request data slots in contention with other stations via contention trees. After a successful request, data transfer follows in reserved slots, not in contention with other stations.

A tractable model for the access delay due to this request procedure is an essential step toward a better understanding of such a request-grant mechanism, and expressions for the first moments of the distribution of the access delay are particularly relevant. The performance analysis of contention trees has received considerable attention, see Mathys and Flajolet [12] or Tsybakov [17]. However, these analyses have been performed under the assumption of a Poisson source model, and do not easily lead to properties of the closed model in which a finite number of stations use contention trees for reservation: As the contention is for requests rather than data, there is at most a finite number of stations that can enter the contention procedure.

Therefore, we are looking for tractable finite-population models for contention resolution using contention trees. In this paper we propose to consider the machine repair model as a model for the access delay in contention resolution and analyse variants of the machine repair model to obtain the required approximations.

The machine repair model (e.g. Takács [15], Chapter 5), also known as the computer terminal model or as the time sharing system (e.g. Kleinrock [10], Section 4.11; Bertsekas and Gallager [1], Example 3.22), is one of the key performance models that assumes that the input population is finite. The basic model is illustrated in Figure 1. There are N machines working in parallel. After a working period a machine breaks down and joins the repair queue. At the repair facility, a single repairman repairs the machines according to some service discipline. Once repaired, a machine starts working again.

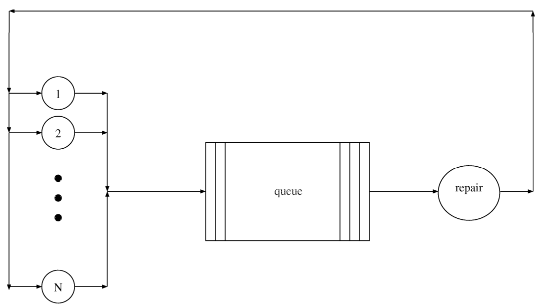

Figure 1: Machine repair model

In the basic model, the distribution of both the working time and the repair time of machines is assumed to be exponential and the service discipline at the repair facility is assumed to be First Come First Served (FCFS). For this model, the steady state distribution of the number of machines in repair is well-known. Furthermore, the arrival theorem (see Sevcik and Mitrani [14]) states that at the moment that a machine arrives at the repair facility, it sees the steady state distribution of the model with $N-1$ machines. Using the arrival theorem, also an expression for the steady-state sojourn time distribution at the repair facility can be found (e.g. Kobayashi [11], Section 3.9). From these results it easily follows that the variance of the sojourn time at the repair facility in the basic model is asymptotically linear in N, the number of machines.

In this paper, we show that the machine repair model can be an appropriate model for contention resolution in cable networks for the case that so-called Capetanakis-Tsybakov contention trees are used for reservation (see Capetanakis [3] and Tsybakov and Mikhailov [16]). It turns out that the average time spent in contention resolution, obtained via simulations, matches the average sojourn time at the repair facility in the basic machine repair model almost perfectly. However, the basic model fails to accurately predict the *variance* of the time spent in contention resolution.

Closer inspection of contention trees reveals a possible source for this mismatch. Contention trees operate by recursively splitting a group of stations into subgroups. Splitting stops as soon as each subgroup contains at most one station: a station is successful in transmitting its request as soon as it is the only contender in a group. The split is performed so that each station in a given group has the same probability of being successful, irrespective of the instant at which it became ready to transmit the request. Thus, contention trees deviate from queues with a first come first served service discipline. This suggests that variants of the basic machine repair model are needed to obtain a more appropriate model for the time spent in contention resolution, and that these variants should have some randomness built into their service discipline. In this paper, we consider two such variants.

Firstly, we consider the machine repair model as described above with a *random order of service* (ROS) discipline. Here, after a repair, the next machine to be repaired is chosen randomly from the machines in the repair queue. We analyse the sojourn time distribution at the repair queue for this model. In fact, it turns out that this model is closely related to the machine repair model considered in Mitra [13], in which the service discipline at the repair facility is *processor sharing* (PS). For PS, and also for ROS, the average sojourn time at the repair facility is identical to the average sojourn time in case of a first come first served service discipline. This is of course a direct consequence of a combination of Little's formula and the fact that the steady state distribution of the number of machines at the repair facility is the same for all work-conserving service disciplines that do not pay attention to the actual service requests of customers. The variance of the sojourn time under PS, however, dramatically differs from that for FCFS (which, as mentioned above, is asymptotically linear in N). Mitra [13] shows that the former variance is for large N proportional to N^2, and in Section 4 the same is seen to hold for ROS.

We shall see that the variance of the sojourn time under ROS gives an accurate prediction of the access delay of requests in contention, when the so-called *free access protocol* is used. However, the prediction is not accurate in case of the so-called *blocked access protocol.* For that protocol, we consider an extension of the machine repair model, as illustrated in Figure 2. In this extension, machines that broke down are first gathered in a waiting room before they are put in random order in the actual repair queue at the moments that this repair queue becomes empty. In the sequel this service discipline will be called *gated random order of service* (GROS). Again, also for the GROS service discipline, the average sojourn time at the repair facility is identical to the average sojourn time in case of a first come first served service discipline. Hence, the emphasis of our analysis will be on obtaining an (approximate) expression for the variance of the sojourn time at the repair facility.

It is appropriate to comment briefly on the relevance of the *variance* of the access delay in contention resolution. Firstly, low variability implies low jitter. As such, access variability is a key performance measure in itself. However, the main reason for studying the variance of the access delay is that it is needed in understanding the total *average* waiting time in cable networks. This follows from the request grant mechanism employed, as explained in the first paragraph of this introduction. Data transfer in cable networks consists of two stages. In the

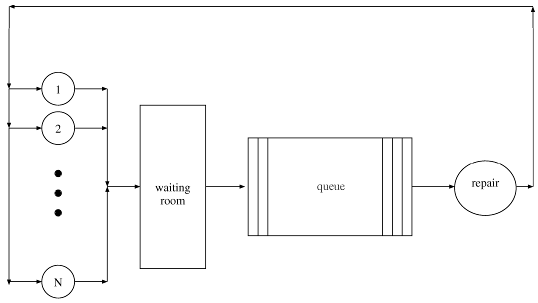

Figure 2: Extension of the machine repair model with a waiting room

first stage, bandwidth for data transfer is being requested via the contention procedure. Once successfully transmitted, the requests queue up in a second queue. In this queue, the service time distribution is given by the distribution of the number of packets for which transfer is being requested. Now, due to the phenomenon of request merging, which will be described in more detail in Section 2, the number of packets being requested depends on the time spent in contention so that the variance of the service time depends on the variance of the access delay in the contention resolution. Clearly, the variance of the service time is needed to estimate the average waiting time in this second queue.

The rest of the paper is organised as follows. In Section 2 we describe the contention resolution process using contention trees in more detail. Next, in Section 3, we review some of the properties of the basic machine repair model. Moreover, we derive expressions for the first two moments of the steady state sojourn time distribution. The machine repair model with ROS service discipline is considered in Section 4. Here, we first relate the model with ROS service discipline to the model with processor sharing service discipline. After that, we briefly review the main results from Mitra [13] for the model with the processor sharing service discipline. In Section 5, we give an approximate derivation of the moments of the sojourn time in the model with GROS service discipline. In Section 6 we present numerical results which show that the models of Section 4 and 5 can be used to approximate the sojourn time for contention resolution in cable networks using contention trees. Finally, Section 7 presents some conclusions.

2 Access via contention trees

Tree algorithms are a popular tool to provide access to a channel that is time slotted and shared among many stations. These algorithms and their many variants are also referred to as stack algorithms or splitting algorithms; we refer to Bertsekas and Gallager [1], Section 4.3, for a survey. In this paper, we will confine attention to the basic ternary tree, illustrated in Figure 3. The basic tree consists of nodes, and each of these nodes comprises three slots of the access channel. A collision occurs if more than one station attempts a transmission in a slot. These collisions are then resolved by recursively splitting the set of colliding stations, plus possible newcomers as explained below, into three disjoint subgroups. For this, usually, a random mechanism is employed. This splitting continues until all tree slots are either empty or contain a successful transmission. This splitting process can be thought of as a tree, but takes place in time slots of the communication channel devoted to the contention resolution, so that the nodes of the tree must be time ordered. For this, we will use the breadth first ordering, as illustrated in Figure 4.

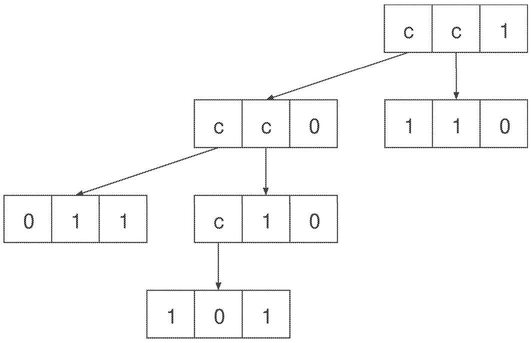

Figure 3: Basic tree algorithm: slots of the tree with a collision (c) are recursively split until all slots are empty (0) or have a successful transmission (1)

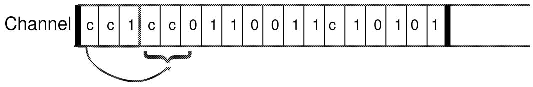

Figure 4: Same tree as in Figure 3, with a breadth first ordering of the nodes

This basic tree algorithm must be complemented with a 'channel access protocol' that describes the procedure to be followed by stations that have data to transmit and that are not already contending in the tree. We consider two such access protocols: free access and blocked access. In the former protocol, access to the tree is free and any station can transmit a request in the next node of the tree, as soon as it has data to transmit. In the latter protocol, the tree is blocked so that new stations can only transmit requests in the root node of the tree that is started as soon as the current tree has been completed.

The stations exhibit the following behaviour:

- A station becomes active in the contention process upon generation of a data packet. In case of free access it will then transmit a request in the next tree node, randomly choosing one of the three slots in this node. In case of blocked access it will wait for the next new tree to be started and transmit its request in one of the slots of the root node of this tree.
- The station stays active until its request has been successfully transmitted.
- While active, the station can update its request (*request merging*). Hence, packets that are generated at such an active station do not cause extra requests.
- After successful transmission of the request, the station becomes inactive, to become active again upon the generation of a new data packet.

Note that request merging implies that the number of stations that can be active in contention is bounded. Exactly this property makes results on the performance of contention trees in open models, as investigated in e.g. Mathys and Flajolet [12] or Tsybakov [17], less relevant to contention resolution in cable networks. This property also explains the approach in this paper, which approximates the access delay in transmitting a request by means of the sojourn time in a machine repair model.

3 Properties of the basic machine repair model

First we introduce some notation and quote some properties of the basic machine repair model; cf. Figure 1. The total number of machines in the system is denoted by N. The machines work in parallel and break down, independently, after an exponentially distributed working period with parameter λ. Machines that broke down queue up in the repair queue, where they are served first come first served by a single repairman. The repair times of machines are exponentially distributed with parameter μ.

With the random variables X and Y we denote the steady state number of machines that are in Q_W (i.e., are working) and that are in Q_R (i.e., are in repair), respectively. Clearly, the number of working machines and the number of machines in repair evolve as Markov processes. Their steady state distributions are equal to (cf. [10, 11])

$$\Pr(X = k) = \Pr(Y = N - k) = \frac{\rho^k/k!}{\sum_{i=0}^{N} \rho^i/i!}, \qquad k = 0, \ldots, N, \tag{1}$$

where

$$\rho := \mu/\lambda. \tag{2}$$

For the mean and variance of X and Y we have

$$\mathrm{E}(X) = \rho(1 - B_N(\rho)), \quad \mathrm{E}(Y) = N - \mathrm{E}(X), \tag{3}$$

$$\mathrm{var}(X) = \mathrm{var}(Y) = \mathrm{E}(X) - \rho B_N(\rho)[N - \mathrm{E}(X)], \tag{4}$$

where $B_N(\rho)$ denotes Erlang's loss probability, which is given by

$$B_N(\rho) = \frac{\rho^N/N!}{\sum_{i=0}^{N} \rho^i/i!}. \tag{5}$$

Indeed, it is well known that the number of operative machines has the same distribution as the number of busy lines in the classical Erlang loss model.

We now turn to the moments of the sojourn time of an arbitrary machine at the repair facility. To this end, we consider the time epoch at which an arbitrary machine breaks down and jumps to the repair queue. Stochastic quantities related to this moment will be denoted by a subscript 1. Thus X_1 is the number of working machines at this moment, and Y_1 is the number of machines in repair at this moment. From the arrival theorem, see Sevcik and Mitrani [14], it follows that the distributions of X_1 and Y_1 are given by (1), but with N replaced by $N - 1$:

$$\Pr(X_1 = k) = \Pr(Y_1 = N - 1 - k) = \frac{\rho^k/k!}{\sum_{i=0}^{N-1} \rho^i/i!}, \qquad k = 0, \ldots, N - 1. \tag{6}$$

The sojourn time of an arbitrary machine at the repair facility equals its own repair time plus the sum of the repair times of the machines already present at the repair facility. Thus, denoting this sojourn time by S, we have that

$$S = \sum_{i=1}^{Y_1+1} B_i, \tag{7}$$

with $B_i, i = 1, 2, \ldots$, a sequence of independent, exponentially distributed random variables with parameter μ. Equation (7) enables us to obtain the Laplace-Stieltjes transform (LST) of the sojourn time at the repair facility (see also [11]):

$$\mathrm{E}(e^{-\omega S}) = \sum_{j=0}^{N-1} \frac{\rho^{N-1-j}/(N-1-j)!}{\sum_{i=0}^{N-1} \rho^i/i!} \left(\frac{\mu}{\mu + \omega}\right)^{j+1}. \tag{8}$$

Here, we are mainly interested in the first two moments of the sojourn time. These can be obtained by consideration of the moments of the random sum, i.e.,

$$\begin{aligned}
\mathrm{E}(S) &= \mathrm{E}(Y_1+1)\mathrm{E}(B_i) \\
&= \frac{1}{\mu}(N-\rho(1-B_{N-1}(\rho))), \qquad (9) \\
\mathrm{var}(S) &= \mathrm{E}(Y_1+1)\mathrm{var}(B_i)+\mathrm{var}(Y_1+1)\mathrm{E}(B_i)^2 \\
&= \frac{1}{\mu^2}\left(N-\rho B_{N-1}(\rho)[N-1-\rho(1-B_{N-1}(\rho))]\right). \qquad (10)
\end{aligned}$$

Now, for N large and $N >> \mu/\lambda$, $B_N(\rho)$ goes to zero like $\rho^N/N!$. Hence, in that case, the following are extremely sharp approximations:

$$\mathrm{E}(S) \approx \frac{N}{\mu}-\frac{1}{\lambda}, \qquad (11)$$

$$\mathrm{var}(S) \approx \frac{N}{\mu^2}. \qquad (12)$$

In Sections 4 and 5 we shall study the sojourn time distribution at Q_R under the assumption that the service discipline at that queue is Random Order of Service (ROS) and Gated Random Order of Service (GROS), respectively. As was briefly indicated in Section 1, the *mean* sojourn time in Q_R is the same under FCFS, ROS and GROS; this is a direct consequence of Little's formula and the fact that the distribution of the number of customers in Q_R is the same for any work-conserving service discipline that does not pay attention to the actual service requests of customers. We therefore focus in particular on the *variance* of the sojourn time in Q_R. Formula (12) shows that for the FCFS discipline, asymptotically, this variance is linear in the number of machines and does not depend on λ, the parameter of the distribution of the working times.

Remark 3.1. It is well known that the machine repair model has an insensitivity property: The steady state distribution of the number of machines in repair only depends on the *mean* working time of machines and not on the actual *distribution* of these working times. This implies that the results of this section remain valid for arbitrarily distributed working times of machines.

4 The model with ROS service discipline

Again we consider the model of Figure 1, but now the service discipline at Q_R is *random order of service*. For reasons that will soon become clear, we assume that the system contains not N but $N+1$ machines. The main goals of this section are: (i) to determine the LST of the waiting time distribution at Q_R, (ii) to relate this distribution to the sojourn time distribution at Q_R in case the service discipline is PS instead of ROS, and (iii) to determine the asymptotic behaviour of the variance of the waiting (and sojourn) time at Q_R under the ROS discipline.

Consider a tagged machine, C, at the moment it arrives at Q_R. Let S_{ROS} (W_{ROS}) denote the steady state sojourn (waiting) time of C at Q_R. S_{ROS} is the sum of W_{ROS} and a service time that is independent of W_{ROS}, and hence we can concentrate on W_{ROS}. We denote by $Y_1^{(N+1)}$ the number of machines in Q_R, as seen by C upon arrival in Q_R. Introduce

$$\phi_j(\omega) := \mathrm{E}[\mathrm{e}^{-\omega W_{ROS}}|Y_1^{(N+1)}=j+1], \quad \mathrm{Re}\ \omega \geq 0, \quad j=0,\ldots,N-1.$$

We can write, for Re $\omega \geq 0$,

$$\mathrm{E}[\mathrm{e}^{-\omega W_{ROS}}|W_{ROS}>0] = \sum_{j=0}^{N-1} \mathrm{P}(Y_1^{(N+1)}=j+1|Y_1^{(N+1)}>0)\phi_j(\omega). \qquad (13)$$

The following set of N equations for the N unknown functions $\phi_0(\omega), \ldots, \phi_{N-1}(\omega)$ holds:

$$\begin{aligned}\phi_j(\omega) &= \frac{\mu+(N-j-1)\lambda}{\mu+(N-j-1)\lambda+\omega}[\frac{(N-j-1)\lambda}{\mu+(N-j-1)\lambda}\phi_{j+1}(\omega) \\ &+ \frac{\mu}{\mu+(N-j-1)\lambda}(\frac{1}{j+1}+\frac{j}{j+1}\phi_{j-1}(\omega))]. \end{aligned} \tag{14}$$

Notice that the pre-factors of $\phi_{-1}(\omega)$ and $\phi_N(\omega)$ equal zero. Formula (14) can be understood in the following way. The pre-factor $(\mu+(N-j-1)\lambda)/(\mu+(N-j-1)\lambda+\omega)$ is the LST of the time until the first 'event': Either an arrival at Q_R or a departure from Q_R. An arrival occurs first with probability $(N-j-1)\lambda/(\mu+(N-j-1)\lambda)$. In this event, the memoryless property of the exponential working and repair times implies that the tagged machine C sees the system as if it only now arrives at Q_R, meeting $j+2$ other machines there. A departure occurs first with probability $\mu/(\mu+(N-j-1)\lambda)$. In this event, C is with probability $1/(j+1)$ the one to leave the waiting room and enter the service position; if it does not leave, it sees Q_R as if it only now arrives, meeting j other machines there.

We can use (14) to obtain numerical values of $\mathrm{E}(W_{ROS}|W_{ROS}>0)$ and $\mathrm{var}(W_{ROS}|W_{ROS}>0)$. This will be exploited in Section 6. Formula (14) can also be used to study this mean and variance asymptotically, for $N \to \infty$. In fact, for this purpose we can also use the analysis given by Mitra [13] for a strongly related model: The machine-repair model with *processor sharing* at Q_R and with N (instead of $N+1$) machines. Denote the LST of the *sojourn* time distribution of a machine meeting j machines at Q_R, in the case of processor sharing, by $\psi_j(\omega)$. A careful study of Formula (14) and the explanation following it reveals that *exactly* the same set of equations holds for $\psi_j(\omega)$, if in the PS case there are not $N+1$ but N machines in the system. If C meets j machines in the PS node Q_R, then he leaves $N-j-1$ machines behind in Q_W. Now observe that the time until either an arrival at or a departure from Q_R occurs is exponentially distributed with parameter $\mu+(N-j-1)\lambda$, leading to the same pre-factor as in (14). And if an event occurs, it is a departure from Q_R with probability $\mu/(\mu+(N-j-1)\lambda)$. If a departure from Q_R occurs, C is with probability $1/(j+1)$ the machine to leave. If it does not leave, it sees Q_R as if it only now arrives, meeting $j-1$ machines there. Not only do we have $\phi_j(\omega)=\psi_j(\omega)$, $j=0,\ldots,N-1$, but it also follows from (6) that $\mathrm{P}(Y_1^{(N+1)}=j+1|Y_1^{(N+1)}>0)=\mathrm{P}(Y_1^{(N)}=j)$, $j=0,\ldots,N-1$. The above equalities, combined with (13), imply that W_{ROS}, conditionally upon it being positive, in the machine-repair system with $N+1$ machines, has the same distribution as the sojourn time under processor sharing in the corresponding system with N machines. Adding a superscript (N) for the case of a machine-repair system with N machines, we can write:

$$\mathrm{P}(S_{PS}^{(N)}>t)=\mathrm{P}(W_{ROS}^{(N+1)}>t|W_{ROS}^{(N+1)}>0). \tag{15}$$

This equivalence result between ROS and PS may be viewed as a special case of a more general result in [2] (see [5] for another special case). In the $G/M/1$ queue, the *sojourn* time under PS is equal in distribution to the *waiting* time under ROS of a customer arriving to a non-empty system. This equivalence is in [2] extended to a class of closed product-form networks (notice that the two-queue network in the present paper indeed is a closed product-form network). In particular, again adding a superscript (N) for the case of a machine repair model with N machines, it follows from [2] that

$$\begin{aligned}\mathrm{P}(S_{PS}^{(N)}>t) &= \mathrm{P}(W_{ROS}^{(N+1)}>t|W_{ROS}^{(N+1)}>0) \\ &= \frac{\mathrm{P}(W_{ROS}^{(N+1)}>t)}{\mathrm{P}(W_{ROS}^{(N+1)}>0)}, \qquad t\geq 0, \end{aligned} \tag{16}$$

with

$$\mathrm{P}(W_{ROS}^{(N+1)} > 0) \; = \; \frac{\sum_{i=0}^{N-1} \frac{\rho^i}{i!}}{\sum_{i=0}^{N} \frac{\rho^i}{i!}}. \tag{17}$$

It is easily verified that, for the machine repair model with N machines, $\mathrm{E}S_{ROS} = \mathrm{E}S_{PS} = \mathrm{E}S_{FCFS}$, just as indicated in Section 1, the latter quantity equalling $\frac{\mathrm{E}(Y_1+1)}{\mu}$ (cf. (9)). For example, the first equality follows after some calculation from the following relation, that is obtained from (16) by integration over t:

$$\mathrm{E}S_{PS}^{(N)} = \frac{\mathrm{E}W_{ROS}^{(N+1)}}{\mathrm{P}(W_{ROS}^{(N+1)} > 0)}. \tag{18}$$

Multiplication by t and integration over t in (16) yields similarly:

$$\mathrm{var}(S_{PS}^{(N)}) = \frac{\mathrm{var}(W_{ROS}^{(N+1)})}{\mathrm{P}(W_{ROS}^{(N+1)} > 0)}. \tag{19}$$

When N is large and $N > \mu/\lambda$, then $\mathrm{P}(W_{ROS}^{(N+1)} = 0)$ is negligibly small. The previous formula hence implies that, for $N \to \infty$, $\mathrm{var}(S_{PS}^{(N)}) \sim \mathrm{var}(W_{ROS}^{(N)})$ – and hence also $\mathrm{var}(S_{PS}^{(N)}) \sim \mathrm{var}(S_{ROS}^{(N)})$.

For an asymptotic analysis of $\mathrm{E}W_{ROS}^{(N)}$ and $\mathrm{var}(W_{ROS}^{(N)})$ we can thus immediately apply corresponding asymptotics of Mitra [13] for the PS-variant. Mitra [13] derives a similar set of equations as (14), albeit for $\mathrm{P}(S_{PS} > t|Y_1 = j)$ rather than for its LST. He writes his set of N equations in matrix form. He shows that the corresponding matrix has N real and negative eigenvalues $\mu_N \leq \mu_{N-1} \leq \ldots \leq \mu_1$. Using the equivalent of (13) for PS, he finally shows that

$$\mathrm{P}(S_{PS} > u) \; = \; \sum_{i=1}^{N} \alpha_i \mathrm{e}^{\mu_i u}, \qquad u \geq 0, \tag{20}$$

with $\alpha_i > 0$ for $i = 1, \ldots, N$ and $\sum_{i=1}^{N} \alpha_i = 1$. Hence,

$$\mathrm{P}(S_{PS} > u) \; \leq \; \mathrm{e}^{\mu_1 u}, \tag{21}$$

$$\mathrm{P}(S_{PS} > u) \; \sim \; \alpha_1 \mathrm{e}^{\mu_1 u}, \qquad u \to \infty. \tag{22}$$

The fact that S_{PS} is hyper-exponentially distributed immediately implies that (see Proposition 12 in [13]),

$$\mathrm{var}(S_{PS}) \geq (\mathrm{E}S_{PS})^2. \tag{23}$$

Hence $\mathrm{var}(S_{PS}) = \mathrm{O}(N^2)$ for $N \to \infty$, which sharply contrasts with the $\mathrm{O}(N)$ behavior for FCFS (cf. (12)).

5 The model with GROS service discipline

In this section, we consider the model with GROS service discipline, as illustrated in Figure 2 and described in Section 1. Again, we let Y denote the number of machines in the total waiting area (i.e. waiting room plus waiting queue). Obviously the distribution of Y equals the distribution of the number of machines in the repair queue in the standard model described in Section 3, and is given by (1).

We will now consider the sojourn time until repair, S_{GROS}, of an arbitrary (tagged) machine for the model with GROS service discipline. Observe that this sojourn time consists of two components:

$$S_{GROS} = \sum_{i=1}^{Y_1^{(1)}} B_i^{(1)} + \sum_{i=1}^{Y_1^{(2)}+1} B_i^{(2)}. \tag{24}$$

Here, the random variables $B_i^{(1)}$ and $B_i^{(2)}$ are independent, exponentially distributed service times with parameter μ. The random variable $Y_1^{(1)}$ is the number of machines in the waiting queue (including the one in repair) at the instant that the tagged machine breaks down. The random variable $Y_1^{(2)} + 1$ equals the random position allocated to the tagged machine in the waiting queue at the instant it is moved from the waiting room to the waiting queue.

This model is not a closed product-form network, so that an exact analysis of the sojourn time is considerably more difficult than the analysis for the models considered above. However, a particularly easy approximation of the first moments can be obtained, if one makes the following two assumptions:

- The two components of S_{GROS} in (24) are uncorrelated.
- The random variables $Y_1^{(1)}$ and $Y_1^{(2)}$ are uniformly distributed on $0, 1, \cdots, Y_1$, where the random variable Y_1 is as defined in Section 3.

Neither assumption is strictly valid; however, for the case considered in which $N\lambda > \mu$ and N large, they appear to be good approximations.

It is now straightforward to show that

$$\begin{aligned} E(S_{GROS}) &= \left(E(Y_1^{(1)}) + E(Y_1^{(2)}) + 1\right)/\mu \\ &= E(Y_1)/\mu + 1/\mu \\ &\approx (N - \mu/\lambda)/\mu, \end{aligned} \tag{25}$$

and that

$$\begin{aligned} \mathrm{var}(S_{GROS}) &\approx 2\,\mathrm{var}(\sum_{i=1}^{Y_1^{(1)}} B_i^{(1)}) \\ &= 2(E(Y_1^{(1)}) + \mathrm{var}(Y_1^{(1)}))/\mu^2 \\ &\approx \left((N - \mu/\lambda)^2/6 + (4N - 2\mu/\lambda)/3\right)\left(\frac{1}{\mu}\right)^2. \end{aligned} \tag{26}$$

Thus the average sojourn time for the model with GROS service discipline is identical to the average sojourn time in the models previously considered. The variance of the sojourn time, however, has an intermediate magnitude. For large N, it is on one hand much larger than the variance in the machine repair model with the FCFS service discipline. However, on the other hand, it is considerably smaller than the variance in the machine repair model with the ROS service discipline.

6 A comparison

We now turn to a comparison of the access delay due to contention resolution and the sojourn time in the variants of the machine repair model. In this comparison, we will confine ourselves to the first two moments of the various distributions: we consider first moments in Section 6.1 and standard deviations in Section 6.2.

The procedures for contention resolution were described in Section 2, and the access delay due to contention resolution is the delay experienced by stations that use contention trees for reservation. More formally, it is defined as the number of tree slots elapsed from the moment a station becomes active until the moment its request is successfully transmitted. As already indicated in Section 2, there are no closed form expressions for the moments of the distribution of the access delay. Hence, these are obtained via simulation. In these simulations, the stations execute the procedure outlined in Section 2: they become active after an exponentially distributed inactive period with parameter λ. Then they enter the contention resolution at the earliest possible moment, as defined by the channel access protocol. Thus, we use a source model in which each of a finite number, N, of stations generates packets according to a Poisson process with rate λ, independently of the other stations.

The average delays thus obtained are denoted $\widehat{ES_F}$ and $\widehat{ES_B}$, for the 'free' and 'blocked' channel access protocol respectively. Likewise, the estimated standard deviations are denoted by $\widehat{\sigma_F}$ and $\widehat{\sigma_B}$. The 'hat' serves as a reminder that the moments are estimated from a simulation. We use 1000 trees in each simulation.

The moments of the sojourn time of the various machine repair models have been obtained in Sections 3 to 5. In utilizing the results from these sections, we will use $\mu = \log(3)$ for the rate of the service time distribution. The motivation behind this value is in Janssen and de Jong ([9], Eq. 26-27). They show that the average number of nodes to complete a tree with n contenders is well approximated by $n/\log(3)$. Hence, the rate at which the contenders are served can be approximated by $\log(3)$.

6.1 First moments

The average access delays for the tree models and the expected sojourn time for the machine repair model are given in Table 1. There is only one entry in the table corresponding to the expected sojourn time, as it is the same for all variants of the machine repair model considered. In the table, we have varied the number of stations, N, and the total traffic intensity $\Lambda := N\lambda$. The primary purpose of this table is to compare average access delay with expected sojourn time. Whence, the intensities are chosen so that Λ is well above μ, which is the case most relevant to access in cable networks.

	$N=100$			$N=200$			$N=1000$		
Λ	$\widehat{ES_F}$	$\widehat{ES_B}$	ES	$\widehat{ES_F}$	$\widehat{ES_B}$	ES	$\widehat{ES_F}$	$\widehat{ES_B}$	ES
2.5	43.0	50.1	51.0	86.0	101.3	102.0	429.1	509.4	510.0
5.0	63.0	70.5	71.0	125.9	141.5	142.0	629.9	710.8	710.0
10.0	73.1	80.5	81.0	146.0	161.6	162.0	729.7	811.2	810.0
16.5	77.1	84.5	84.9	154.5	169.5	169.9	824.9	848.5	850.0

Table 1: Average access delay for reservation with free tree, $\widehat{ES_F}$, with blocked tree, $\widehat{ES_B}$, and expected sojourn time for the machine repair model, ES, for number of stations N, and total traffic intensity Λ

From the figures we can draw various conclusions. Firstly, and most importantly, we observe that the expected sojourn time in the machine repair model provides an excellent approximation to the average access delay for reservation with contention trees. The agreement with the figures obtained via simulations with blocked access is almost perfect; the agreement with the results for free access is somewhat less good. The former result is closely related to a result in Denteneer and Pronk ([6]) on the average number of contenders in a contention tree.

Secondly, we see that free access is a more efficient access protocol than blocked access in that the average access delay with the former is smaller than the average delay with the latter. This result parallels the result for the open model and the Poisson source model, as graphically illustrated in Figure 16 of Mathys and Flajolet [12]. The considered variants of the machine repair model all lead to the same expected sojourn time and are apparently not sufficiently detailed as models to capture the first moment differences between the blocked and the free access protocols.

Finally, we observe that all quantities investigated in Table 1 approximately display a linear dependence on the number of stations (for the cases with $N >> \mu/\lambda$).

6.2 Standard deviations

We next turn to a numerical comparison of the standard deviations in the various models. These are given in table 2, again for different N and Λ.

Several conclusions can be drawn from the table. Firstly, we observe that the standard deviation in either tree model changes with traffic intensity and grows approximately linearly with the number of stations. Neither of these properties is captured by the basic machine repair model; there, the standard deviation of the sojourn time is independent of the traffic intensity and grows only with the square root of the number of stations in the model.

Secondly, the standard deviation of the access delay in the *blocked tree* model corresponds closely to the corresponding figure for the GROS machine repair model. The difference between the two standard deviations is approximately 15%. The results for the GROS model capture both the dependence on the traffic intensity and the dependence on the number of machines that is observed in the tree simulations. Similarly, the standard deviation of the access delay in the *free tree* model corresponds closely to the corresponding figure for the ROS machine repair model.

Looking more closely at the results, we see that the standard deviations obtained for the GROS machine repair model are always larger than those obtained in the blocked tree simulations. Of course, the analysis of the GROS model was completely heuristic so that it is possible to explain the differences by the approximations involved. Our feeling here, however, is that a fundamental limitation of the machine repair model as an approximation shows up. The batch nature of the contention trees implies that it takes some initial time before the first successful request is transmitted. Statistical analysis suggests that, after this initial period, successful transmissions occur fairly uniformly over the length of the trees. Thus the variability of the waiting period is somewhat reduced as compared to the proposed model in which the successful transmissions occur uniformly over the full length of the tree. This also suggests that there is an even more appropriate extension of the basic machine repair model, i.e. one in which the transfer from the waiting room to the queue takes some time and in which the server operates at a slightly larger speed.

Thirdly, the standard deviations with the free access protocol far exceed those with the blocked access protocol. This result has no parallel in the open model. In fact, Figure 17 in Mathys and Flajolet [12] shows that the standard deviation of the delay with free access protocol is *below* the corresponding value with blocked access for most traffic intensities. However, for large traffic intensities just below the stability bound the order reverses and blocked access then results in smaller standard deviations. Of course, our simulations operate at total traffic

$N = 100$

Λ	Tree $\widehat{\sigma_F}$	$\widehat{\sigma_B}$	σ	Repair σ_{ROS}	σ_{GROS}
2.5	46.1	19.5	9.1	50.45	22.8
5.0	68.0	26.7	9.1	70.18	30.6
10.0	78.4	30.4	9.1	80.13	34.6
16.5	83.2	31.5	9.1	84.06	36.2

$N = 200$

Λ	Tree $\widehat{\sigma_F}$	$\widehat{\sigma_B}$	σ	Repair σ_{ROS}	σ_{GROS}
2.5	92.7	37.9	12.9	101.46	43.7
5.0	135.3	52.7	12.9	141.20	59.6
10.0	158.2	60.3	12.9	161.15	67.7
16.5	167.2	62.5	12.9	169.02	70.9

$N = 1000$

Λ	Tree $\widehat{\sigma_F}$	$\widehat{\sigma_B}$	σ	Repair σ_{ROS}	σ_{GROS}
2.5	429.1	185.1	28.8	509.64	210.4
5.0	629.9	261.6	28.8	709.39	291.7
10.0	729.7	299.2	28.8	809.34	332.4
16.5	786.6	310.6	28.8	848.73	348.4

Table 2: Standard deviations of the access delay for reservation with free tree, $\widehat{\sigma_F}$, with blocked tree, $\widehat{\sigma_B}$, and standard deviations for the basic machine repair model, σ, the ROS machine repair model, σ_{ROS}, and the GROS machine repair model, σ_{GROS} for number of stations N, and total traffic intensity Λ

intensities that exceed the stability bound for the open system.

In Table 3 we have no longer kept $\mu = \log(3)$. Instead, we consider $\mu = 0.5$, 1 and 2, giving rise to $\rho = \frac{1}{2}N$, $\rho = N$ and $\rho = 2N$, respectively. The next three remarks relate to these three different cases.

Remark 6.1. Interestingly, in case $\rho = \frac{\mu}{\lambda} << N$, the standard deviation, σ_{ROS}, of the sojourn time for ROS is almost identical to ES= ES_{ROS}. This can be observed by combining the relevant entries from Tables 1 and 2 or from the entry corresponding to $\mu = 0.5$ in Table 3. This suggests that for the considered parameter values, S_{ROS} is approximately exponentially distributed. Indeed, the following reasoning shows that S_{ROS} is approximately exponentially distributed when N is large and $\rho = \frac{\mu}{\lambda} = o(N)$. In this case, the number of customers, Y, at the repair facility is usually close to N. Hence S_{ROS} is the sum of a random number, L, of $\exp(\mu)$ distributed service times, and L is approximately geometrically distributed with parameter $1/N$ (the tagged customer has a chance $1/j$ to be the next one served, if there are $j-1$ other customers present). It is well known that the sum of a geometrically distributed number of independent, exponentially distributed stochastic variables is exponentially distributed. If

	$N=100$		$N=200$		$N=1000$	
μ	ES_{ROS}	σ_{ROS}	ES_{ROS}	σ_{ROS}	ES_{ROS}	σ_{ROS}
0.5	100	99	200	199	1000	999
1.0	8.2	10.4	11.5	15.2	25.2	34.7
2.0	0.97	1.10	0.99	1.13	1.00	1.15

Table 3: Mean and standard deviation of the sojourn times in the ROS model for number of stations N and service rate μ, with total traffic intensity $\Lambda = N\lambda = 1$

EY is reasonably close to N, the above reasoning still holds to a considerable extent: L still is geometrically distributed, but its parameter is no longer (approximately) $1/N$ but a random variable.

Remark 6.2. Let us briefly consider the other extreme case: $\frac{\mu}{N\lambda} >> 1$. It is easily seen, and well known, that now $\mathrm{P}(X = N) \approx 1$. The repair facility now behaves like an open $M/M/1$ queue with arrival rate $\Lambda = N\lambda$ and service rate μ. Hence $\mathrm{E}S = \mathrm{E}S_{ROS} \approx \frac{1}{\mu-\Lambda}$. In the standard $M/M/1$ queue with FCFS, the sojourn time is exponentially distributed, so $\sigma_{FCFS} = \mathrm{E}S$. In the $M/M/1$ queue with ROS, it follows from Cohen [4] p. 443 that the standard deviation of the sojourn time is inflated with a factor f as compared to the standard deviation of the standard $M/M/1$ queue with FCFS, where

$$f = \sqrt{1 + \frac{2(\Lambda/\mu)^2}{2 - \Lambda/\mu}}.$$

The entries in Table 3 with $\mu = 2$ are relevant to this case. For this case we find that $f = 1.15$, revealing a rather close agreement although $\frac{\mu}{N\lambda}$ only equals 2. Furthermore, note that $f \to 1$ for $\frac{\mu}{N\lambda} >> 1$, which again yields a coefficient of variation of S_{ROS} that approaches 1.

Remark 6.3. We finally consider the intermediate case $\rho = N - c\sqrt{N}$, $N \to \infty$. This case has already been studied by Vaulot [18] (see also Whitt [19]) for the Erlang loss model with N servers and offered traffic ρ, a model that is equivalent with the machine repair model. Vaulot [18] proved that

$$\sqrt{N}B_N(\rho) = \sqrt{N}B_N(N - c\sqrt{N}) \sim \frac{\phi(c)}{\Phi(c)}, \quad c \in \mathcal{R}, \quad N \to \infty, \tag{27}$$

with $\phi(c) = \frac{1}{\sqrt{2\pi}}\mathrm{e}^{-c^2/2}$ and $\Phi(c) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{c} \mathrm{e}^{-x^2/2}\mathrm{d}x$ the standard normal density and standard normal distribution, respectively. Substitution of (27) into (3) yields the approximations

$$\mathrm{E}X \approx N - \sqrt{\frac{2N}{\pi}} \tag{28}$$

and

$$\mathrm{E}Y \approx \sqrt{\frac{2N}{\pi}}, \tag{29}$$

and hence (use Little's formula $\mathrm{E}Y = \Lambda_R \mathrm{E}S$, with $\Lambda_R = \lambda \mathrm{E}X$ the input rate into the repair facility):

$$\mathrm{E}S \approx \frac{\sqrt{\frac{2N}{\pi}}}{\Lambda}. \tag{30}$$

The entries in Table 3 with $\mu = 1$ (hence $\rho = N$, so $c = 0$) are relevant for this case.

7 Conclusion

In this paper, we started from the assumption that the sojourn time in the repair facility of the machine repair model is related to the access delay experienced when using contention trees to transmit requests. To substantiate this intuition, we obtained expressions for the moments of this sojourn time and numerically compared these to the corresponding moments of the access delay obtained through simulation. These numerical experiments showed that the expected sojourn time in the repair stage shows a perfect match with the average access delay for both variants of the tree procedure. It was also shown that the variance of the sojourn time in the model with ROS service discipline gives a good approximation of the variance of the access delay when using free trees. Similarly, the variance of the sojourn time in the model with GROS service discipline gives a good approximation of the variance of the access delay when using blocked trees.

In the introduction it was pointed out that data transfer in cable networks consists of two stages and that the variance of the access delay is needed in understanding the average total delay in the two stages together. In the present paper we have concentrated on the first stage. To analyze the overall delay is a topic for further study.

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