# Periodic points of nonexpansive maps: a survey

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### Abstract

In this paper we survey the research on periodic points of non-expansive maps. Since the pioneering paper [AK87] by Akcoglu and Krengel in the nineteen-eighties remarkable progress has been made in this field. This paper brings together the main results and it discusses some of the open problems. At the same time we hope that it will be an invitation for others to become acquainted with the subject.

## 1 Introduction

Let $P$ be an $n \times n$ column stochastic matrix and let $f : \mathbb{R}^n \to \mathbb{R}^n$ be given by $f(x) = Px$ for $x \in \mathbb{R}^n$. It is then well-known that one can use the theory of Perron and Frobenius, concerning the eigenvalues of nonnegative matrices, to predict the asymptotic behaviour of the sequence of iterates $(f^k(x))_k$ for $x \in \mathbb{R}^n$. Indeed one can show (see [NV99, Section 9]) that there exists an integer $p \geq 1$ such that the sequence $(f^{kp}(x))_k$ converges for each $x \in \mathbb{R}^n$ to a periodic point of $f$, and moreover $p$ is the order of a permutation on $n$ letters. There are two properties of the map $f$ that cause this behaviour. To begin the map $f$ is *nonnegative*, that is to say it leaves the positive cone in $\mathbb{R}^n$ invariant. Secondly, the map $f$ is *nonexpansive* in the 1-norm, i.e.

$$\|f(x) - f(y)\|_1 \leq \|x - y\|_1 \quad \text{for all } x, y \in \mathbb{R}^n.$$

Here $\|z\|_1 = \sum_i |z_i|$ denotes the 1-*norm* for $z = (z_1, \ldots, z_n)$.

Surprisingly often the nonexpansiveness property is sufficient to give this type of asymptotic behaviour of the iterates. This is illustrated by the following remarkable theorem.

**Theorem 1.1.** *Let $\| \cdot \|$ be a norm on $\mathbb{R}^n$ for which the unit ball is a polyhedron and let $X \subset \mathbb{R}^n$ be closed. If $f : X \to X$ is nonexpansive with respect to this norm and there exists $\eta \in X$ such that $(\|f^k(\eta)\|)_k$ is bounded, then the following assertions are true.*

- *For each $x \in X$ there exists $p_x = p \geq 1$ such that $(f^{kp}(x))_k$ converges to a periodic point $\xi \in X$ of $f$ of minimal period $p$, that is $f^p(\xi) = \xi$ and $f^j(\xi) \neq \xi$ for $0 < j < p$.*

- *For each polyhedral norm there exists an integer $\rho(n)$, which only depends on the dimension, such that the minimal period of each periodic point of $f$ is at most $\rho(n)$.*

The main point of this theorem is that the nonexpansiveness property causes the limit behaviour of the iterates of certain nonlinear maps to be periodic. Important examples of polyhedral norms on $\mathbb{R}^n$ are the 1-norm and the *sup-norm*: $\|z\|_\infty = \max_i |z_i|$ for $z = (z_1, \ldots, z_n)$.

Theorem 1.1 raises a number of questions.

**Question 1.1.** *For which nonexpansive maps $f : X \to X$, with $X \subset \mathbb{R}^n$, is the asymptotic behaviour of the sequence of iterates $(f^k(x))_k$ periodic for each $x \in X$?*

**Question 1.2.** *Given a polyhedral norm and a domain $X \subset \mathbb{R}^n$, can one determine the finite set of integers $p \geq 1$ for which there exist a nonexpansive map $f : X \to X$ and a periodic point of $f$ of minimal period $p$?*

Of course, not every nonexpansive map exhibits this type of behaviour. One can think, for instance, of a rotation under an irrational angle in the plane. Such a map is nonexpansive with respect to the Euclidean norm. In connection with the second question Nussbaum [Nus90] made the following conjecture.

**Conjecture 1.1 (Nussbaum).** *The minimal period of each periodic point of a sup-norm nonexpansive map $f : X \to X$, with $X \subset \mathbb{R}^n$, is at most $2^n$.*

At present the conjecture is known to be true for the dimensions $n = 1, 2$, and 3 (see [LN92]).

The remainder of the paper has the following outline. In Section 2 we give a brief history of Theorem 1.1 and provide some motivation to study the iterative behaviour of nonexpansive maps. Section 3 is used to explain the main ideas behind the proof of Theorem 1.1. It moreover discusses some results concerning Question 1.2. In Section 4 we review the results on Question 1.2 in case the polyhedral norm is the 1-norm. Subsequently we discuss in Section 5 the connection between lattice homomorphisms and nonnegative nonexpansive maps. We conclude with Section 6 in which some numerical data is given.

# 2   Historical remarks and motivation

Pioneering research on the behaviour of nonlinear nonexpansive maps was done by Akcoglu and Krengel in the nineteen-eighties. In [AK87] they proved Theorem 1.1 in case the polyhedral norm is the 1-norm. Their results however did not provide an upper bound $\rho(n)$. An upper bound was obtained by Misiurewicz [Mis87]. In fact, he showed (in case of the 1-norm) that the minimal period of periodic points of 1-norm nonexpansive maps $f : X \rightarrow X$, with $X \subset \mathbb{R}^n$, is at most $n!2^m$, where $m = 2^n$. In his thesis [Wel87] Weller generalised the result of Akcoglu and Krengel to polyhedral norms on $\mathbb{R}^n$. Thereafter various upper bounds for $\rho(n)$ for different polyhedral norms were derived by: Blokhuis and Wilbrink [BW92], Lemmens, Nussbaum and Verduyn Lunel [LNV01], Lo [Lo89], Martus [Mar89], Nussbaum [Nus90], and Sine [Sin90].

Further investigations on the periodic points of 1-norm nonexpansive maps were made by Scheutzow. In [Sch88] and [Sch91] he showed that if $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$, with $f(0) = 0$, is a 1-norm nonexpansive map, then there exists an integer $p \geq 1$ such that the sequence $(f^{kp}(x))_k$ is convergent for each $x \in \mathbb{R}^n$, and moreover $p$ is a divisor of the least common multiple of the integers $1, 2, \ldots, 2n$. If in addition $f$ leaves the positive cone in $\mathbb{R}^n$ invariant, then $p$ divides the least common multiple of the integers $1, 2, \ldots, n$. Later his ideas were further developed by Nussbaum and Scheutzow in [Nus91b] and [NS98]. The results from these papers eventually allowed Nussbaum, Scheutzow, and Verduyn Lunel to give a complete characterization (in arithmetical and combinatorial constraints) of the possible minimal periods of periodic points of 1-norm nonexpansive maps $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ that leave the positive cone in $\mathbb{R}^n$ invariant and have zero as a fixed point (see [NSV98, Theorem 3.1]). Further improvements for the possible minimal periods of periodic points of general 1-norm nonexpansive maps $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ were obtained by Lemmens in [Lem01a] (see also [Lem01b]). We will give a detailed overview of these result in Section 4.

Let us now provide some motivation to study nonexpansive maps. One of the reasons to study nonexpansive maps is that they arise in applications. For instance, 1-norm nonexpansive maps can be used as models for diffusion processes on a finite state space. (see [AK87] and [Nus97]). A simple example of such a process is the following. Suppose there are $n$ containers $C_1, C_2, \ldots, C_n$ each having an infinite volume. Let $x_i$ denote the amount of sand in container $C_i$ and define $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ to be the distribution vector. With each container $C_i$, where $1 \leq i \leq n$, a sequence of buckets $(b_{ij})_{j \geq 1}$ is associated.

For each bucket $b_{ij}$ the volume is denoted by $a_{ij}$, and it is assumed that

$$\sum_{j=1}^{\infty} a_{ij} = \infty \quad \text{for } 1 \leq i \leq n.$$

We start the following procedure to pour sand from the containers into the buckets. For each container $C_i$ pour sand into bucket $b_{i1}$ until either $b_{i1}$ is full or $C_i$ is empty. If $b_{i1}$ is full, then pour the remaining sand in $C_i$ into $b_{i2}$ until either $b_{i2}$ is full or $C_i$ is empty. Continue in the same manner until $C_i$ is empty. If we let $M_{ik}(x)$ denote the amount of sand in bucket $b_{ik}$ after the procedure we find that

$$M_{ik}(x) = \min\{a_{ik}, \max\{x_i - \sum_{j=0}^{k-1} a_{ij}, 0\}\}, \quad \text{where } a_{i0} = 0.$$

Now let $\gamma : \{1, \ldots, n\} \times \mathbb{N} \to \{1, \ldots, n\}$ be a map. This map will serve as a rule to pour sand from the buckets back into the containers. For each bucket $b_{ik}$ pour sand in container $C_{\gamma(i,k)}$. The new distribution $y = (y_1, \ldots, y_n)$ of sand in the containers is given by

$$y_j = \sum_{\gamma(i,k)=j} M_{ik}(x) \quad \text{for } 1 \leq j \leq n.$$

More formally, we can define a map $f : \mathbb{K}^n \to \mathbb{K}^n$ by

$$f(x)_j = \sum_{\gamma(i,k)=j} M_{ik}(x) \quad \text{for } 1 \leq j \leq n \text{ and } x \in \mathbb{K}^n,$$

where $\mathbb{K}^n = \{x \in \mathbb{R}^n : x_i \geq 0 \text{ for } 1 \leq i \leq n\}$ is the *positive cone* in $\mathbb{R}^n$. The map $f$ is usually called a *sand-shift map* and it was introduced by Nussbaum in [Nus97].

To see that sand-shift maps are 1-norm nonexpansive one can use a result of Crandall and Tartar [CT80], which says: If $f : X \to X$, where $X = \mathbb{R}^n$ or $\mathbb{K}^n$, is *integral-preserving*, i.e. $\sum_i f(x)_i = \sum_i x_i$ for all $x \in X$, then $f$ is 1-norm nonexpansive if and only if $f$ is *order-preserving*, i.e. $f(x) \leq f(y)$ for all $x, y \in X$ with $x \leq y$. Here one should read the inequalities coordinate-wise.

An interesting class of sup-norm nonexpansive maps is provided by another observation of Crandall and Tartar [CT80]. If $f : \mathbb{R}^n \to \mathbb{R}^n$ is *additive homogeneous*, i.e. $f(x + h\mathbf{1}) = f(x) + h\mathbf{1}$ for all $h \in \mathbb{R}$ and $x \in \mathbb{R}^n$, then $f$ is sup-norm nonexpansive if and only if $f$ is order-preserving. Here $\mathbf{1}$ denotes the vector in $\mathbb{R}^n$ with all coordinates unity. Examples are so called *max-plus functions*, which are defined as follows. Let $A = (a_{ij})$ be a real $n \times n$ matrix and let $f : \mathbb{R}^n \to \mathbb{R}^n$ be given by

$$f(x)_i = \max_j\{a_{ij} + x_j\} \quad \text{for } 1 \leq i \leq n \text{ and } x \in \mathbb{R}^n.$$

Max-plus functions appear in various applications such as statistical mechanics (see [Nus91a]) and the analysis of discrete event systems (see [BCOQ92], [CG79], and [Gun98]).

Other, more general, examples of homogeneous and order-preserving maps are maps $g : \mathbb{R}^n \to \mathbb{R}^n$, where each component $g(x)_i$ consists of finitely many expressions of the form $x_j + c$, where $1 \leq j \leq n$ and $c \in \mathbb{R}$, which are joined by $\wedge$ or $\vee$ operations that are defined by $a \wedge b = \min\{a, b\}$ and $a \vee b = \max\{a, b\}$. An example is the map $g : \mathbb{R}^3 \to \mathbb{R}^3$ given by

$$g(x)_1 = (x_2 + 2) \vee (x_3 \wedge (x_1 - 3)),$$
$$g(x)_2 = (x_1 + 1) \wedge (x_2 + 5) \wedge (x_3 - 6),$$
$$g(x)_3 = (x_1 \vee (x_3 - 3)) \wedge ((x_2 - 2) \vee x_3) \quad \text{for } x \in \mathbb{R}^3.$$

These maps are often called *min-max functions*. From Theorem 1.1 it follows that the minimal period of periodic points of min-max functions $g : \mathbb{R}^n \to \mathbb{R}^n$ is bounded above by a number that only depends on the dimension $n$. It is believed that for these functions the optimal upper bound is $n$ choose $[n/2]$ (see [Gun98, page 25]), but at the present time no proof is known. A variety of other applications of nonexpansive maps can be found in: [Nus90], [Nus88], [Nus89], and [NS01]

Besides the applications there are theoretical reasons to be interested in nonexpansive maps. One such reason is that there exists a connection between the iterative behaviour of nonexpansive maps and the geometry of the underlying normed space (see [AGM89] and [DMR97]). Another reason is that Question 1.2, concerning the possible minimal periods of periodic points of nonexpansive maps, is related to nice problems in combinatorial geometry.

## 3    Limit sets of nonexpansive maps

To understand the iterative behaviour of a map $f : X \to X$ one has to study the structure of the $\omega$-*limit* sets:

$$\omega(x) = \{y \in X : y = \lim_{i \to \infty} f^{k_i}(x) \text{ for some integer sequence } k_i \to \infty\}.$$

Indeed to prove Theorem 1.1 it is sufficient to show the following assertion. If $X \subset \mathbb{R}^n$ is closed, $f : X \to X$ is nonexpansive with respect to a polyhedral norm $\| \cdot \|$, and $(\|f^k(\eta)\|)_k$ remains bounded for some $\eta \in X$, then there exists an integer $\rho(n)$ such that the cardinality of $\omega(x)$ is at most $\rho(n)$ for each $x \in X$.

As $X \subset \mathbb{R}^n$ is closed, the $\omega$-limits of $f$ are closed. Moreover, since $f$ has a bounded orbit the nonexpansiveness of $f$ implies that the $\omega$-limit sets of $f$ are bounded, and hence compact. Furthermore it is shown in [DS73] that for

each $x \in X$ the restriction of $f$ to $\omega(x)$ is an isometry that maps $\omega(x)$ onto itself. With these observations in mind Misiurewicz [Mis87] formulated the following property.

**Definition 3.1.** A set $S$ in $(\mathbb{R}^n, \|\cdot\|)$ has a *transitive and commutative family of isometries* if there exists a commutative family $\Gamma$ of isometries ( with respect to $\|\cdot\|$) of $S$ onto itself, such that for each $x, y \in S$ there exists $F_{x,y} \in \Gamma$ with $F_{x,y}(x) = y$.

He then showed the following proposition (see [Mis87, Lemma 1]).

**Proposition 3.1.** *Let $\|\cdot\|$ be a norm on $\mathbb{R}^n$ and let $X \subset \mathbb{R}^n$ be closed. If $f : X \to X$ is nonexpansive with respect to this norm and there exists $\eta \in X$ such that $(\|f^k(\eta)\|)_k$ remains bounded, then for each $x \in X$ the limit set $\omega(x)$ has a transitive and commutative family of isometries.*

Thus, in order to prove Theorem 1.1 it is sufficient to give for each polyhedral norm on $\mathbb{R}^n$ an upper bound for the cardinality of compact sets in $\mathbb{R}^n$ that have a transitive and commutative family of isometries.

In the next subsection we will see how an upper bound for the cardinality of such sets is derived in case the polyhedral norm is the sup-norm. By using the fact that for any polyhedral norm $\|\cdot\|$ on $\mathbb{R}^n$ the space $(\mathbb{R}^n, \|\cdot\|)$ can be isometrically embedded into $(\mathbb{R}^m, \|\cdot\|_\infty)$, where $m$ is sufficiently large, an upper bound can be derived for any polyhedral norm.

## 3.1   Upper bounds for the cardinality of limit sets

We begin by recalling several definitions. A sequence $x^1, x^2, \dots, x^m$ in $\mathbb{R}^n$ is called an *additive chain* (with respect to the sup-norm) if

$$\|x^1 - x^m\|_\infty = \sum_{i=1}^{m-1} \|x^i - x^{i+1}\|_\infty.$$

The *length* of a sequence is the number of distinct points in it.

For each $i = 1, \dots, n$ a partial ordering $\leq_i$ on $\mathbb{R}^n$ is defined by $x \leq_i y$ if $\|x - y\|_\infty = y_i - x_i$. A sequence $x^1, x^2, \dots, x^m$ is called an *i-chain* if

$$x^1 \leq_i x^2 \leq_i \dots \leq_i x^m \text{ or } x^m \leq_i x^{m-1} \leq_i \dots \leq_i x^1.$$

The set of all $i$ for which the sequence $x^1, x^2, \dots, x^m$ is an $i$-chain is denoted by $I(x^1, x^2, \dots, x^m)$.

By using the definition of the sup-norm one can verify that $x^1, x^2, \dots, x^m$ is an additive chain if and only if $I(x^1, x^2, \dots, x^m)$ is not empty. Furthermore, if

$x^{k_1}, \ldots, x^{k_r}$ is a subsequence of an additive chain $x^1, x^2, \ldots, x^m$ with $x^{k_1} = x^1$ and $x^{k_r} = x^m$, then it is easy to show that

$$I(x^{k_1}, \ldots, x^{k_r}) = I(x^1, \ldots, x^m).$$

For each $x, y \in \mathbb{R}^n$ we define the set

$$W(x, y) = \{z \in \mathbb{R}^n : x, y, z \text{ is an additive chain}\}.$$

Further let $W^o(x, y)$ denote the interior of $W(x, y)$ (with respect to the Euclidean norm). The definition of $W(x, y)$ is illustrated in Figure 1. It is not
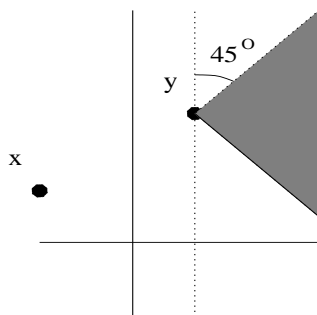


Figure 1: The set $W(x, y)$

difficult to show that

$$W^o(x, y) = \{z \in W(x, y) : I(x, y, z) = I(y, z) \text{ and } z \neq y\}.$$

By using these definitions the following result can be stated.

**Proposition 3.2.** *If $S$ is a compact subset of $\mathbb{R}^n$ and $S$ has a transitive and commutative family of sup-norm isometries, then $W^o(x, y) \cap S = \emptyset$ for all $x, y \in S$ with $x \neq y$.*

*Proof.* The argument goes by contradiction. So, suppose there exist $x, y, z \in S$ with $x \neq y$ and $z \in W^o(x, y)$. Since $x \neq y$ and $y \neq z$ we can find $\varepsilon > 0$ such that $\|x - y\|_\infty \geq \varepsilon$ and $\|y - z\|_\infty \geq \varepsilon$.

Define $\mathcal{F}$ to be the collection of additive chains in $S$ that start with the sequence $x, y, z$ and that are such that the distance between any two consecutive points in the sequence is at least $\varepsilon$. Since $S$ is a compact subset of $\mathbb{R}^n$ there exists an upper bound on the length of sequences in $\mathcal{F}$, say $r$.

Now let $x^1 = x, x^2 = y, x^3 = z, \ldots, x^r$ be a sequence of maximal length in $\mathcal{F}$. For integers $1 \leq k, l \leq r$ let $F_{k,l} : S \rightarrow S$ be a sup-norm isometry of the commutative family that maps $x^k$ to $x^l$, and put $x^{r+1} = F_{1,2}(x^r)$.

We claim that $x^2, x^3, \ldots, x^r, x^{r+1}$ is again an additive chain and the sup-norm distance between two consecutive elements in this sequence is at least $\varepsilon$. To prove this claim, we first show that the distance between consecutive elements is at least $\varepsilon$. It suffices to verify that $\|x^r - x^{r+1}\|_\infty \geq \varepsilon$. Since $x^r = F_{1,r}(x^1)$ we have that

$$
\begin{aligned}
\|x^r - x^{r+1}\|_\infty &= \|F_{1,r}(x^1) - F_{1,2}(F_{1,r}(x^1))\|_\infty \\
&= \|F_{1,r}(x^1) - F_{1,r}(F_{1,2}(x^1))\|_\infty \\
&= \|x^1 - x^2\|_\infty,
\end{aligned}
$$

so that

$$
\|x^r - x^{r+1}\|_\infty = \|x^1 - x^2\|_\infty, \tag{1}
$$

and this shows $\|x^r - x^{r+1}\|_\infty \geq \varepsilon$. From (1) it follows that

$$
\begin{aligned}
\|x^2 - x^{r+1}\|_\infty &= \|F_{1,2}(x^1) - F_{1,2}(x^r)\|_\infty \\
&= \|x^1 - x^r\|_\infty \\
&= \sum_{i=1}^{r-1} \|x^i - x^{i+1}\|_\infty \\
&= \|x^r - x^{r+1}\|_\infty + \sum_{i=2}^{r-1} \|x^i - x^{i+1}\|_\infty \\
&= \sum_{i=2}^{r} \|x^i - x^{i+1}\|_\infty,
\end{aligned}
$$

and hence $x^2, x^3, \ldots, x^{r+1}$ is an additive chain.

From the claim it follows that $I(x^2, x^3, \ldots, x^{r+1})$ is nonempty. Now let $i \in I(x^2, x^3, \ldots, x^{r+1})$. As $z \in W^o(x, y)$ we know that

$$
I(x^2, x^3) = I(y, z) = I(x, y, z) = I(x^1, x^2, x^3).
$$

Combining this with $I(x^2, x^3, \ldots, x^{r+1}) \subset I(x^2, x^3)$ gives $i \in I(x^1, x^2, x^3)$. Therefore the extended sequence $x^1, x^2, \ldots, x^r, x^{r+1}$ is an $i$-chain and hence an additive chain in $\mathcal{F}$. This however, contradicts the fact that $r$ is maximal. $\square$

This result motivates the following definition. A set $S$ in $\mathbb{R}^n$ is called $\infty$-*separated* if $W^o(x, y) \cap S$ is empty for all $x, y \in S$ with $x \neq y$. By using this geometric property one can now easily obtain cardinality estimates.

**Proposition 3.3.** *If $S$ is an $\infty$-separated set in $\mathbb{R}^n$, then $|S| \leq (n + 1)^n$.*

*Proof.* Let $x^1, x^2, \ldots, x^m$ be an additive chain $S$ of length $m$. Then clearly

$$I(x^1, x^2) \supset I(x^1, x^2, x^3) \supset \ldots \supset I(x^1, x^2, \ldots, x^m). \qquad (2)$$

Observe that each inclusion in (2) is strict. Because if $I(x^1, x^2, \ldots, x^k) = I(x^1, x^2, \ldots, x^{k+1})$ for some $1 < k < m$, then

$$
\begin{aligned}
I(x^{k+1}, x^k, x^1) &= I(x^1, x^k, x^{k+1}) = I(x^1, x^2, \ldots, x^{k+1}) \\
&= I(x^1, x^2, \ldots, x^k) = I(x^1, x^k) = I(x^k, x^1).
\end{aligned}
$$

Hence $x^1 \in W^o(x^{k+1}, x^k)$, which contradicts the fact that $S$ is $\infty$-separated. Since $I(x^1, x^2) \subset \{1, 2, \ldots, n\}$ we conclude that the length of every additive chain in $S$ is at most $n + 1$.

Now for $x \in S$ and $1 \leq i \leq n$ let $h_i(x)$ be the length of the longest decreasing $i$-chain starting in $x$, and consider $h : S \to \{1, \ldots, n + 1\}^n$ given by $h(x) = (h_1(x), \ldots, h_n(x))$ for $x \in S$. Then for each $x, y \in S$ with $x \neq y$ we have that $\|x - y\|_\infty = |x_i - y_i| > 0$ for some $1 \leq i \leq n$, so that $h_i(x) \neq h_i(y)$. Therefore $h$ is injective and hence $|S| \leq (n + 1)^n$. $\qquad \square$

A combination of the Propositions 3.2 and 3.3 now shows that every compact set in $\mathbb{R}^n$ with a commutative family of sup-norm isometries has at most $(n + 1)^n$ elements. This together with the observations at the beginning of this section gives a proof of Theorem 1.1.

There exist several proofs of Theorem 1.1 in the literature. The proof presented here is based on ideas that can be found in [BW92], [LNV01], and [Mis87]. However, very different ideas were used by Martus in [Mar89]. Another nice proof of Theorem 1.1 is given by Nussbaum in [Nus90].

At this point one might wonder what the optimal upper bound for $\rho(n)$ in Theorem 1.1 is for a given polyhedral norm. This however, appears to be a very difficult combinatorial geometric question. Of course Nussbaum's conjecture says that in case of the sup-norm the optimal bound for $\rho_\infty(n)$ is $2^n$. But his conjecture is proved only for $n = 1, 2$, and 3 (see [LN92]). The best known general upper bound for $\rho_\infty(n)$ is $n!2^n$, which was obtained by Martus in [Mar89].

In case of the 1-norm even less is known. Misiurewicz [Mis87] proved for the 1-norm that $\rho_1(n) \leq n!2^m$, where $m = 2^n$. Improvements on this bound for $n = 2, 3, 4$ and 5 were given in [LNV01]. Indeed it is shown there that $\rho_1(n) \leq n^m$, where $m = 2^{n-1}$, for $n \geq 2$. But even in dimension 3 this last bound is expected to be far from sharp. It is generally believed that $\rho_1(n) = \mathcal{O}(c^n)$ for some $c \geq 2$.

The difficulty in case of the 1-norm is related to the fact that for a 1-norm nonexpansive map $f : X \to \mathbb{R}^n$, with $X \subset \mathbb{R}^n$, there may not exist a 1-norm nonexpansive map $F : \mathbb{R}^n \to \mathbb{R}^n$ that extends $f$ (see [Kir43] or [WW75]). For the sup-norm however there always exists, by the Aronszajn-Panitchpakdi Theorem [AP56], a sup-norm nonexpansive extension to the whole of $\mathbb{R}^n$.

## 3.2  Lower bounds for the maximum cardinality of limit sets

To get some feeling for the optimal upper bound for $\rho(n)$ one can try to find examples of periodic points which have a large minimal period. To generate such examples one can look for so called regular polygons.

**Definition 3.2.** A finite sequence of $p$ distinct points $x^0, x^1, \ldots, x^{p-1}$ in $(\mathbb{R}^n, \|\cdot\|)$ is called a *regular polygon of size $p$* or simply a *regular $p$-gon* if

$$\|x^{k+l} - x^k\| = \|x^l - x^0\| \quad \text{for all } k, l \geq 0,$$

where the indices are counted modulo $p$.

Of course, for every regular $p$-gon $x^0, x^1, \ldots, x^{p-1}$ in $(\mathbb{R}^n, \|\cdot\|)$ the map $f$ given by $f(x^i) = x^{i+1 \bmod p}$ for $0 \leq i < p$ is an isometry with respect to $\|\cdot\|$, and moreover $f$ has a periodic point of minimal period $p$. On the other hand, each periodic orbit of a nonexpansive map is a regular polygon. Therefore the optimal upper bound for the integer $\rho(n)$ in Theorem 1.1 for a given polyhedral norm is precisely the maximum size of a regular polygon in $\mathbb{R}^n$ under this norm.

An example of a regular $2^n$-gon in $\mathbb{R}^n$ under the sup-norm is formed by the set of vertices of the $n$-dimensional cube $\{x \in \mathbb{R}^n : x_i = \pm 1 \text{ for } 1 \leq i \leq n\}$, as the distance between any two distinct points in this set is 2. This example shows that $\rho_\infty(n)$ is at least $2^n$ for each $n \geq 1$, and hence the upper bound suggested in Nussbaum's conjecture is sharp.

Regular polygons with an exponential size under the 1-norm are harder to obtain. However, several constructions for such regular polygons were given by Lemmens, Nussbaum, and Verduyn Lunel in [LNV01]. These constructions yield the following result.

**Theorem 3.1.** *For each $n \geq 3$ there exists a regular polygon of size $3 \cdot 2^{n-1}$ in $\mathbb{R}^n$ under the 1-norm.*

An example of a regular 12-gon in dimension 3 under the 1-norm is given by the sequence $x^0, -x^0, x^1, -x^1, \ldots, x^5, -x^5$, where

$$\begin{array}{llll}
x^0 & = & (0, 1, 2), & x^1 & = & (0, 2, 1), \\
x^2 & = & (1, 2, 0), & x^3 & = & (2, 1, 0), \\
x^4 & = & (2, 0, 1), & x^5 & = & (1, 0, 2).
\end{array}$$

Theorem 3.1 shows that $\rho_1(n) \geq 3 \cdot 2^{n-1}$ for each $n \geq 3$. At the present time no regular $p$-gons in $\mathbb{R}^n$, with $p > 3 \cdot 2^{n-1}$ are known under the 1-norm. Furthermore for $n = 3$ it can be shown that there exist regular polygons under the 1-norm of size 1, 2, 3, 4, 5, 6, 7, 8, and 12 (see [Lem01a]), but it

is an open problem to decide whether there exist regular polygons of size 9, 10, and 11 in this space. As an aside we like to mention that the maximum size of a "trivial" regular polygon in $\mathbb{R}^n$ under the 1-norm, that is a regular polygon in which any two distinct points are at the same 1-norm distance, is not known. It is generally believed that the maximum size of such regular polygons is $2n$, but at present this is only proved for $n = 1$, 2, 3, and 4 (see [BCL98], [GK83], and [KLS00]).

# 4   Periods of 1-norm nonexpansive maps

In the previous section we have seen that not much is known for the possible minimal periods of periodic points of 1-norm nonexpansive maps $f : X \to X$ when $X$ can be an arbitrary subset of $\mathbb{R}^n$. On the other hand, if $X$ is the whole of $\mathbb{R}^n$ or $X$ is the positive cone in $\mathbb{R}^n$, then there are many detailed results. In this section we will give an overview of these results. We begin by discussing nonnegative 1-norm nonexpansive maps.

## 4.1   Nonnegative 1-norm nonexpansive maps

Motivated by the models for diffusion processes on a finite state space one has studied in [Nus91b], [NS98], [NSV98], [NV99], and [Sch88] the set $P^*(n)$ which consists of integers $p \geq 1$ for which there exist a 1-norm nonexpansive map $f : \mathbb{K}^n \to \mathbb{K}^n$, with $f(0) = 0$, and a periodic point of $f$ of minimal period $p$. Surprisingly the set $P^*(n)$ admits a complete characterization in terms of arithmetical and combinatorial constraints. Indeed it is shown in [NS98] and [NSV98] together that $P^*(n)$ is precisely the set of possible periods of an admissible array on $n$ symbols. Here an admissible array is defined as follows.

**Definition 4.1.** Let $(L, <)$ be a finite totally ordered set and let $\Sigma$ be a set with $n$ elements. For each $i \in L$ let $\vartheta_i : \mathbb{Z} \to \Sigma$ be a map. The sequence $\vartheta = (\vartheta_i : \mathbb{Z} \to \Sigma \mid i \in L)$ is called an *admissible array on $n$ symbols* if the maps $\vartheta_i$ satisfy the following properties:

(i)   For each $i \in L$ there exists an integer $p_i$ with $1 \leq p_i \leq n$ such that the map $\vartheta_i : \mathbb{Z} \to \Sigma$ is periodic with period $p_i$ and moreover $\vartheta_i(s) \neq \vartheta_i(t)$ for each $1 \leq s < t \leq p_i$.

(ii)   If $m_1 < m_2 < \ldots < m_{r+1}$ is an increasing sequence of distinct points in $L$ and

$$\vartheta_{m_i}(s_i) = \vartheta_{m_{i+1}}(t_i) \quad \text{for } 1 \leq i \leq r,$$

then

$$\sum_{i=1}^{r}(t_i - s_i) \not\equiv 0 \bmod \rho, \text{ where } \rho = \gcd(\{p_{m_i} : 1 \le i \le r+1\}).$$

Here $\gcd(S)$ denotes the greatest common divisor of the elements of the set $S$. The *period* of an admissible array is said to be the least common multiple of $p_i$, where $i \in L$. Thus, if one defines for each $n \in \mathbb{N}$ the set

$$Q(n) = \{p \in \mathbb{N} : p \text{ is the period of an admissible array on } n \text{ symbols}\}, \quad (3)$$

then the characterization of the set $P^*(n)$ reads as follows.

**Theorem 4.1 ([NSV98], Theorem 3.1).** $P^*(n) = Q(n)$ *for each* $n \in \mathbb{N}$.

The main idea behind the proof of the inclusion $P^*(n) \subset Q(n)$ is to relate to each periodic point of a 1-norm nonexpansive map $f : \mathbb{K}^n \to \mathbb{K}^n$, with $f(0) = 0$, a so called lower semi-lattice homomorphism. Via this lower semi-lattice homomorphism an admissible array can be constructed such that its period corresponds to the minimal period of the original periodic point. Let us explain this procedure in more detail.

### 4.1.1  Lower semi-lattice homomorphisms

We begin by collecting several definitions. On $\mathbb{R}^n$ a partial ordering $\le$ is defined by $x \le y$ if $x_i \le y_i$ for each $1 \le i \le n$. In particular, we write $x < y$ if $x \le y$ and $x \ne y$. Of course, $x \le y$ if and only if $y - x \in \mathbb{K}^n$. Further for $x, y \in \mathbb{R}^n$ we let $x \wedge y$ be the vector in $\mathbb{R}^n$ given by $(x \wedge y)_i = \min\{x_i, y_i\}$ for $1 \le i \le n$. Similarly, $x \vee y$ denotes the vector with coordinates $(x \vee y)_i = \max\{x_i, y_i\}$ for $1 \le i \le n$.

A set $V \subset \mathbb{R}^n$ is called a *lower semi-lattice* if $x \wedge y \in V$ for all $x, y \in V$. If in addition, $x \vee y \in V$ for all $x, y \in V$, then $V$ is called a *lattice*. If $S$ is a subset of $\mathbb{R}^n$, then $V_S$ denotes the smallest (in the sense of inclusion) lower semi-lattice that contains $S$. The set $V_S$ is called the *lower semi-lattice generated by $S$*. A map $g : V \to V$, where $V$ is a lower semi-lattice is said to be a *lower semi-lattice homomorphism* if $g(x) \wedge g(y) = g(x \wedge y)$ for all $x, y \in V$. In a similar way *lattice homomorphisms* can be defined. The relation of these notions with nonnegative 1-norm nonexpansive maps is given by the following observation of Scheutzow [Sch88].

**Theorem 4.2.** *Let* $f : \mathbb{K}^n \to \mathbb{K}^n$ *be a 1-nonexpansive map, with* $f(0) = 0$, *and let* $\xi \in \mathbb{K}^n$ *be a periodic point of $f$ of minimal period $p$. If $V \subset \mathbb{K}^n$ is the lower semi-lattice generated by* $\{f^j(\xi) : 0 \le j < p\}$, *then the restriction of $f$ to $V$ is a lower semi-lattice homomorphism that maps $V$ onto itself.*

This results motivates a further study of periodic points of lower semi-lattice homomorphisms. To do so several more notions have to be introduced. If $A$ is a subset of a lower semi-lattice $V$ in $\mathbb{R}^n$ and there exists $\beta \in V$ such that $\alpha \le \beta$ for each $\alpha \in A$, we say that $A$ is *bounded above* in $V$, and $\beta$ is called an *upper bound of A* in $V$. By replacing $\le$ with $\ge$ *lower bounds* can be defined in the same manner. If $A$ is bounded above in $V$, then there exists a unique $\alpha \in V$ upper bound of $A$ in $V$ such that $\gamma < \alpha$ implies $\gamma$ is not an upper bound of $A$ in $V$, and $\alpha$ is called the *supremum* of $A$ in $V$, which will be denoted by $\sup_V(A)$. Analogously the *infimum* of $A$ in $V$, denoted $\inf_V(A)$, is said to be the unique lower bound $\alpha \in V$ of $A$ such that no $\beta > \alpha$ is a lower bound of $A$ in $V$.

For each $x$ in a finite lower semi-lattive $V$ the *height* of $x$, denoted $h_V(x)$, is defined by

$$h_V(x) = \sup\{k \ge 0 : \text{ there exist } y^0, \dots, y^k \in V \text{ such that}$$
$$y^k = x \text{ and } y^j < y^{j+1} \text{ for } 0 \le j < k\}. \quad (4)$$

If no $y \in V$ exists with $y < x$, then we say that $h_V(x) = 0$.

For every $x \in V$ put $S_x = \{y \in V : y < x\}$. An element $x \in V$ is called *irreducible* in $V$ if either $S_x$ is empty or

$$x > \sup_V(S_x). \quad (5)$$

If $x \in V$ is irreducible in $V$ and $S_x$ is nonempty, then $I_V(x)$ is said to be

$$I_V(x) = \{i : x_i > z_i\}, \quad \text{where } z = \sup_V(S_x). \quad (6)$$

In case $S_x$ is empty, that is $x = \inf_V(V)$, then $I_V(x) = \{1, 2, \dots, n\}$.

Using these notions we can now state the following result of Scheutzow [Sch88]. A proof of this version of the lemma can be found in [NS98].

**Lemma 4.1.** *Let $j \in \mathbb{Z}$, let $V$ be a finite lower semi-lattice in $\mathbb{R}^n$, and let $f : V \to V$ be a lower semi-lattice homomorphism of $V$ onto itself. If $y \in V$ and $f^j(y) \ne y$, then $y$ and $f^j(y)$ are incomparable, and $h_V(y) = h_V(f^j(y))$, where $h_V(\cdot)$ is the height function given in (4). If $y$ is irreducible in $V$, then $f^j(y)$ is irreducible in $V$. If $\eta \in V$ and $\zeta \in V$ are not comparable, and $\eta$ and $\zeta$ are irreducible in $V$, then*

$$I_V(\eta) \cap I_V(\zeta) = \emptyset. \quad (7)$$

*If $y \in V$ is irreducible in $V$ and $y$ is a periodic point of $f$ of minimal period $p$, then $1 \le p \le n$.*

The following technical definition forms the base from which the admissible arrays will be constructed.

**Definition 4.2.** Let $W$ be a lower semi-lattice in $\mathbb{R}^n$, let $g : W \to W$ be a lower semi-lattice homomorphism, and let $\xi \in W$ be a periodic point of $g$ of minimal period $p$. Let $V$ denote the lower semi-lattice generated by $\{g^j(\xi) : j \geq 0\}$ and let $f$ be the restriction of $g$ to $V$. A finite sequence $(y^i)_{i=1}^m \subset V$ is called a *complete sequence for* $\xi$, if it satisfies:

 (i)  For $1 \leq i \leq m$ we have $y^i \leq \xi$.

 (ii)  For $1 \leq i \leq m$ the element $y^i$ is irreducible in $V$.

 (iii)  If $p_i$ is the minimal period of $y^i$ under $f$, then $p$ is the least common multiple of $\{p_i : 1 \leq i \leq m\}$.

 (iv)  For $1 \leq i < m$ we have $h_V(y^i) \leq h_V(y^{i+1})$, where $h_V(\cdot)$ is the height function given by equation (4).

 (v)  For $1 \leq i < j \leq m$, the sets $\{f^k(y^i) : k \geq 0\}$ and $\{f^k(y^j) : k \geq 0\}$ are disjoint.

 (vi)  For $1 \leq i < j \leq m$, the elements $y^i$ and $y^j$ are not comparable.

The next result says that every periodic point of a lower semi-lattice homomorphism has a complete sequence (see [NS98, Proposition 1.1]).

**Proposition 4.1.** *If $W$ is a lower semi-lattice in $\mathbb{R}^n$, $g : W \to W$ is a lower semi-lattice homomorphism, and $\xi \in W$ is a periodic point of $g$, then there exists a complete sequence for $\xi$.*

Using the complete sequences one can now construct the admissible arrays.

### 4.1.2   Admissible arrays

Let $W$ be a lower semi-lattice in $\mathbb{R}^n$ and let $g : W \to W$ be a lower semi-lattice homomorphism. Suppose that $\xi \in W$ is a periodic point of $g$ of minimal period $p$. Let $V$ denote the lower semi-lattice generated by $\{g^j(\xi) : j \geq 0\}$ and let $f$ be the restriction of $g$ to $V$. Now by Proposition 4.1 there exists a complete sequence $(y^i)_{i=1}^m \subset V$ for $\xi$. Moreover it follows from property (ii) in Definition 4.2 and Lemma 4.1 that $f^j(y^i)$ is irreducible in $V$ for $1 \leq i \leq m$ and $j \in \mathbb{Z}$, so that the set $I_V(f^j(y^i))$ (as defined in (6)) is nonempty for $1 \leq i \leq m$ and $j \in \mathbb{Z}$. Let $p_i$ denote the minimal period of $y^i$ under $f$. Select for $1 \leq i \leq m$ and $0 \leq j < p_i$ an integer $a_{ij} \in I_V(f^j(y^i))$, and define for $1 \leq i \leq m$ and general $j \in \mathbb{Z}$ the integer $a_{ij}$ by

$$a_{ij} = a_{ik}, \quad \text{where } 0 \leq k < p_i \text{ and } j \equiv k \mod p_i.$$

The semi-infinite matrix $(a_{ij})$, where $1 \leq i \leq m$ and $j \in \mathbb{Z}$, is called an *array of* $\xi$. Now there exists the following connection with the admissible arrays on $n$ symbols (see [NS98, Propostion 1.2]).

**Proposition 4.2.** *Let $W$ be a lower semi-lattice in $\mathbb{R}^n$, $g : W \to W$ be a lower semi-lattice homomorphism, and $\xi \in W$ be a periodic point of $g$ of minimal period $p$. Let $(a_{ij})$, where $1 \leq i \leq m$ and $j \in \mathbb{Z}$, be an array of $\xi$. Further, let $L = \{1, \ldots, m\}$ be equipped with the usual ordering and let $\Sigma = \{1, 2, \ldots, n\}$. If $\vartheta = (\vartheta_i : \mathbb{Z} \to \Sigma \mid i \in L)$ is defined by*

$$\vartheta_i(j) = a_{ij} \quad \text{for } i \in L \text{ and } j \in \mathbb{Z},$$

*then $\vartheta$ is an admissible array on $n$ symbols with period $p$.*

A combination of Theorem 4.2 and Propositions 4.1 and 4.2 yields the inclusion $P^*(n) \subset Q(n)$. The other inclusion $Q(n) \subset P^*(n)$ is shown in [NSV98]. In fact, the following stronger result is proved there.

**Theorem 4.3 ([NSV98], Theorem 3.1).** *For each $p \in Q(n)$ there exist a sand-shift map $f : \mathbb{K}^n \to \mathbb{K}^n$ and a periodic point of $f$ of minimal period $p$.*

Although the set $Q(n)$ is described in terms of arithmetical and combinatorial constraints it is difficult to compute it. Despite this difficulty the set $Q(n)$ has been determined up to dimension 50 in [NV99] with the aid of a computer. From these computations it follows that the set $Q(n)$ has a highly irregular structure, and therefore we do not think that there exists a simple description of $Q(n)$. In Table 1 in Section 6 we only show the largest elements of $Q(n)$ for $1 \leq n \leq 20$. To conlude we like to mention that there are several overviews of the results discussed in this subsection, see for instance [Nus97], [Nus92], and [VL00].

## 4.2    1-Norm nonexpansive maps on the whole space

Another interesting set of periods that has been studied is the set $R(n)$, which consists of integers $p \geq 1$ for which there exist a 1-norm nonexpansive map $f : \mathbb{R}^n \to \mathbb{R}^n$ and a periodic point of $f$ of minimal period $p$. It turns out that the results for nonnegative 1-norm nonexpansive maps say something about the set $R(n)$. More precisely, it is shown in [Sch91] and [Nus97] that $R(n) \subset P^*(2n)$ for all $n \geq 1$, so that by Theorem 4.1 the inclusion $R(n) \subset Q(2n)$ holds for each $n \geq 1$. This upper bound however, is not optimal. A sharper bound was obtained by Lemmens in [Lem01a] (see also [Lem01b]). We will discuss this upper bound in this subsection.

To obtain a sharper bound one uses an observation, which follows from the proof of the inclusion $R(n) \subset P^*(2n)$: *For each $p \in R(n)$ there exist a 1-norm nonexpansive map $f : \mathbb{K}^{2n} \to \mathbb{K}^{2n}$, with $f(0) = 0$, and a periodic point $\xi$ of $f$ of minimal period $p$, such that $f^j(\xi) \in \mathbb{E}^{2n}$ for each $j \geq 0$, where $\mathbb{E}^{2n} = \{(x, y) \in \mathbb{K}^n \times \mathbb{K}^n : x \wedge y = 0\}$.* As $\mathbb{E}^{2n}$ is a lower semi-lattice in $\mathbb{K}^{2n}$ this observation and the results from the previous subsection

suggest that one should study the arrays of periodic points of lower semi-lattice homomorphisms $g : W \to W$, where $W \subset \mathbb{E}^{2n}$.

It turns out that one can derive two additional properties for such arrays. These properties motivate the notion of a strongly admissible array on $2n$ symbols, which we will give now. To exhibit the definition of a strongly admissible array a final piece of notation is needed. If $a \in \{1, 2, \ldots, 2n\}$, then we write $a^+ = a + n$ if $1 \leq a \leq n$, and $a^+ = a - n$ if $n + 1 \leq a \leq 2n$.

**Definition 4.3.** Suppose that $(L, <)$ is a finite totally ordered set and let $\Sigma = \{1, 2, \ldots, 2n\}$. Assume that $\vartheta = (\vartheta_i : \mathbb{Z} \to \Sigma \mid i \in L)$ is an admissible array on $2n$ symbols, and let $p_i$ denote the period of $\vartheta_i$, for $i \in L$. We call $\vartheta$ a *strongly admissible array on $2n$ symbols* if the maps $\vartheta_i$ satisfy:

(i) If $m_1, m_2$ are distinct elements of $L$ and $\vartheta_{m_1}(s) = \vartheta_{m_2}(t)^+$, then

$$s - t \not\equiv 0 \mod \pi, \quad \text{where } \pi = \gcd(p_{m_1}, p_{m_2}).$$

(ii) If $m_1 < m_2 < \ldots < m_{r+1}$ is an increasing sequence of distinct elements in $L$ such that

$$\vartheta_{m_i}(s_i) = \vartheta_{m_{i+1}}(t_i) \quad \text{for } 1 \leq i \leq r,$$

and if $\vartheta_{m_1}(u) = \vartheta_{m_{r+1}}(v)^+$ for some $u, v \in \mathbb{Z}$, then

$$\sum_{i=1}^{r}(t_i - s_i) \not\equiv (v - u) \mod \rho, \quad \text{where } \rho = \gcd(\{p_{m_i} : 1 \leq i \leq r + 1\}).$$

The connection between the strongly admissible arrays and the arrays of periodic points of lower semi-lattice homomorphisms $g : W \to W$, where $W \subset \mathbb{E}^{2n}$, is given in the following proposition.

**Proposition 4.3.** *Let $W$ be a lower semi-lattice in $\mathbb{E}^{2n}$, $g : W \to W$ be a lower semi-lattice homomorphism, and $\xi \in W$ be a periodic point of $g$ of minimal period $p$. Let $(a_{ij})$, where $1 \leq i \leq m$ and $j \in \mathbb{Z}$, be an array of $\xi$. Further let $L = \{1, \ldots, m\}$ be equipped with the usual ordering and let $\Sigma = \{1, \ldots, 2n\}$. If $\vartheta = (\vartheta_i : \mathbb{Z} \to \Sigma \mid i \in L)$ is defined by*

$$\vartheta_i(j) = a_{ij} \quad \text{for } i \in L \text{ and } j \in \mathbb{Z},$$

*then $\vartheta$ is a strongly admissible array on $2n$ symbols with period $p$.*

Now if one defines for each $n \geq 1$ the set

$$T(n) = \{p \in \mathbb{N} : p \text{ is the period of a strongly admissible array}$$
$$\text{on } 2n \text{ symbols}\}, \quad (8)$$

then by using Proposition 4.3 the following theorem can be obtained.

**Theorem 4.4 ([Lem01b], Theorem 2.1).** *$R(n) \subset T(n)$ for each $n \in \mathbb{N}$.*

The set $T(n)$ is computed for $1 \leq n \leq 10$ in [Lem01a]. A list of the largest elements of $T(n)$ for $1 \leq n \leq 10$ is given in Table 2 in Section 6. It turns out that $T(n)$ is much smaller than $Q(2n)$ and moreover that $R(n) = T(n)$ for $n = 1, 2, 3, 4, 6, 7,$ and 10. However, it is unknown whether the sets $R(n)$ and $T(n)$ are equal for all $n \in \mathbb{N}$. To determine $R(n)$ up to $n = 10$ it only remains to be decided whether or not $18 \in R(5)$, $90 \in R(8)$, and $126 \in R(9)$ (see [Lem01a]).

# 5    Lattice homomorphisms and nonexpansive maps

We saw in the previous section that there exists a strong connection between lower semi-lattice homomorphisms and nonnegative 1-norm nonexpansive maps. It has been observed by Nussbaum in [Nus94] that this connection can be extended to other nonnegative nonexpansive maps. In particular to maps that are nonexpansive in a strictly monotone norm. A norm $\| \cdot \|$ on $\mathbb{R}^n$ is called *strictly monotone* if $\|x\| < \|y\|$ for each $0 \leq x < y$. More precisely Nussbaum's result can be stated as follows.

**Theorem 5.1 ([Nus94]).** *If $f : \mathbb{K}^n \to \mathbb{K}^n$, with $f(0) = 0$, is nonexpansive in a strictly monotone norm and $f$ is order-preserving, then the following assertions are true:*

- *For each $x \in \mathbb{K}^n$ there exists an integer $p_x = p \geq 1$ such that $(f^{kp}(x))_k$ converges to a periodic point of $f$ of minimal period $p$.*

- *If $\xi \in \mathbb{K}^n$ is a periodic point of $f$ of minimal period $p$, and $V$ denotes the lattice generated by $\{f^j(\xi) : 0 \leq j < p\}$, then the restriction of $f$ to $V$ is a lattice homomorphism that maps $V$ onto itself.*

- *If $\xi \in \mathbb{K}^n$ is periodic point of $f$ of minimal period $p$, then $p \in Q(n)$, where $Q(n)$ is given in (3).*

It is known that in this theorem the assumption that $f$ is order-preserving is necessary. Indeed one can find a map $f : \mathbb{K}^n \to \mathbb{K}^n$, with $f(0) = 0$, which is nonexpansive in the Euclidean norm, but not order-preserving, for which the first assertion in Theorem 5.1 does not hold (see [Nus92, pp. 224]).

Further we like to remark that Theorem 5.1 can not be applied in case the map $f : \mathbb{K}^n \to \mathbb{K}^n$ is nonexpansive in the sup-norm, since this norm is not strictly monotone. In fact, there seems to be no relation between sup-norm nonexpansive maps and lattice homomorphisms. Therefore we feel that

different ideas are needed to obtain a good upper bound for the minimal period of periodic points of sup-norm nonexpansive maps $f : \mathbb{K}^n \to \mathbb{K}^n$, with $f(0) = 0$.

Further Theorem 5.1 raises the following problem. Decide for a given strictly monotone norm whether the set $Q(n)$ is the optimal upper bound in the third assertion in Theorem 5.1. At present there exist no results in this direction, except for the 1-norm.

Another remark we like to make is that the second statement in Theorem 5.1 is suggested by the structure of the fixed point set of these nonexpansive maps. More precisely, one can show that if $f : \mathbb{K}^n \to \mathbb{K}^n$, with $f(0) = 0$, is nonexpansive in a strictly monotone norm and $f$ is order-preserving, then $\{x \in \mathbb{K}^n : f(x) = x\}$ is a lattice (see [NSV98, Proposition 2.1]). This statement is well-known and easy to see if in addition $f$ is assumed to be a linear map (compare [AAB93] or [Ran01]). Indeed, if $z \in \{x : f(x) = x\}$, then as $f$ is order-preserving $z = f(z) \leq f(z \vee 0)$. Since $0 = f(0) \leq f(z \vee 0)$ it follows that $z \vee 0 \leq f(z \vee 0)$. On the other hand, as $f$ is nonexpansive $\|f(z \vee 0)\| \leq \|z \vee 0\|$. Therefore by using the fact that $\|\cdot\|$ is strictly monotone we obtain $f(z \vee 0) = z \vee 0$. If we now apply the linearity of $f$, then it follows from the expressions $y \vee z = (y - z) \vee 0 + z$ and $y \wedge z = -((-y) \vee (-z))$ that $\{x : f(x) = x\}$ is a lattice.

To conclude we like to mention that it is interesting to understand under which additional assumptions a map $f : \mathbb{R}^n \to \mathbb{R}^n$, with $f(0) = 0$, which is nonexpansive in a strictly monotone norm, satisfies the first assertion in Theorem 5.1. It is remarkable that even for linear maps there are no results concerning this problem.

# 6 Numerical data

In the first table we list the largest element of $Q(n)$ for $1 \leq n \leq 20$. This table is taken from [NV99]. The second table contains the largest element of $T(n)$ for $1 \leq n \leq 10$ and is taken from [Lem01a]. We like to emphasize that more detailed lists can be found in [NV99] and [Lem01a].

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Table 1: The largest element of $Q(n)$ for $1 \leq n \leq 20$.

| $n$ | largest element $Q(n)$ | $n$ | largest element $Q(n)$ |
|---|---|---|---|
| 1 | 1 | 11 | 60 |
| 2 | 2 | 12 | 120 |
| 3 | 3 | 13 | 120 |
| 4 | 4 | 14 | 168 |
| 5 | 6 | 15 | 180 |
| 6 | 12 | 16 | 336 |
| 7 | 12 | 17 | 420 |
| 8 | 24 | 18 | 420 |
| 9 | 24 | 19 | 840 |
| 10 | 60 | 20 | 1680 |

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Table 2: The largest element of $T(n)$ for $1 \leq n \leq 10$.

| $n$ | largest element $T(n)$ |
|---|---|
| 1 | 2 |
| 2 | 4 |
| 3 | 6 |
| 4 | 12 |
| 5 | 20 |
| 6 | 30 |
| 7 | 60 |
| 8 | 90 |
| 9 | 140 |
| 10 | 210 |

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