How edge-reinforced random walk arises naturally

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Abstract

We give a characterization of a modified edge-reinforced random walk in terms of certain partially exchangeable sequences. In particular, we obtain a characterization of edge-reinforced random walk (introduced by Coppersmith and Diaconis) on a 2-edge-connected graph. Modifying the notion of partial exchangeability introduced by Diaconis and Freedman in [2], we characterize unique mixtures of reversible Markov chains under a recurrence assumption.

1 Introduction

In the 1920s the Cambridge philosopher W.E. Johnson gave the following characterization of Polya urns (see [7]): Let $X := (X_0, X_1, \dots)$ be an exchangeable sequence with values in a finite state space of cardinality $m \geq 3$. If the conditional probabilities $P(X_{n+1} = v | X_0, X_1, \dots, X_n)$ depend only on v and the number of times state v has been visited up to time n and if some natural technical conditions hold, then X has the same distribution as drawings from a Polya urn containing balls of m different colors. Johnson formulated his statement in terms of Dirichlet distributions rather than Polya urns, but it is well known that the two notions are equivalent (see e.g. [6], Section 2).

Diaconis [personal communication] conjectured that edge-reinforced random walk arises as naturally as Dirichlet distributions. In this article, we prove his conjecture in the sense that we generalize Johnson's statement for a modified edge-reinforced random walk.

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1.1 Result

Let $G = (V, E)$ be a locally finite connected graph with vertex set V and edge set E. We assume that G has no loops, i.e. each edge has two distinct endpoints. Parallel edges are allowed; thus two edges may have the same pair of endpoints. For an edge e we denote the set of its endpoints by $\bar{e}$. We call $\pi = (v_0, e_1, v_1, \ldots, e_n, v_n)$ an *admissible path* if $v_i \in V$ for $0 \leq i \leq n$, $e_i \in E$ and $\bar{e}_i = \{v_{i-1}, v_i\}$ for $1 \leq i \leq n$. We say that π has starting point v_0, endpoint v_n and length n. We denote by $k(v, \pi)$ the number of visits to vertex v and by $k(e, \pi)$ the number of traversals of edge e:

$$k(v, \pi) := |\{i \in \{0, 1, \ldots, n\} : v_i = v\}|, \tag{1.1}$$

$$k(e, \pi) := |\{i \in \{1, \ldots, n\} : e_i = e\}|; \tag{1.2}$$

here we write $|S|$ for the cardinality of a set S. We define a sequence $(X, Y) := (X_0, Y_1, X_1, Y_2, X_2, \ldots)$ of random variables to be a *nearest-neighbor random walk on G* if $Z_n := (X_0, Y_1, X_1, \ldots, Y_n, X_n)$ is an admissible path for all $n \geq 0$. We abbreviate $Z := (X, Y)$.

We denote by $\mathcal{P}$ the set of all transition matrices on $V \times E$. For a Markovian nearest-neighbor random walk on G with transition matrix p, we have $p(v, e, v', e') = 0$ if $\bar{e} \neq \{v, v'\}$ and $p(v, e, v', e')$ depends only on v' and e' if $\bar{e} = \{v, v'\}$. Therefore we write $p(v', e')$ instead of $p(v, e, v', e')$ in the following.

Definition 1.1 *We say that a nearest-neighbor random walk Z on G is a* unique mixture of Markov chains *if there exists a unique probability measure μ on $V \times \mathcal{P}$ such that for any admissible path $\pi = (u_0, e_1, u_1, \ldots, e_n, u_n)$*

$$P(Z_n = \pi) = \int_{V \times \mathcal{P}} \prod_{i=0}^{n-1} p(u_i, e_{i+1}) \mu(du_0, dp).$$

The measure μ is called the mixing measure. *If for μ-a.a. (u_0, p) the Markov chain with transition matrix p is reversible, then we say that the process is a* unique mixture of reversible Markov chains.

Diaconis and Freedman [2] call a nearest-neighbor random walk partially exchangeable if any two admissible paths with the same starting point and the same number of transition counts for all *directed* edges have the same probability. They prove that under a recurrence assumption their notion of partial exchangeability characterizes unique mixtures of Markov chains (Theorem (7), [2]). We introduce a more restrictive notion of partial exchangeability which characterizes unique mixtures of *reversible* Markov chains: We define two finite admissible paths π and π' to be *equivalent* if they have the same starting point and $k(e, \pi) = k(e, \pi')$ for all $e \in E$.

Definition 1.2 *We call a nearest-neighbor random walk Z* partially exchangeable *if $P(Z_n = \pi) = P(Z_n = \pi')$ for any equivalent paths π and π' of length n.*

Any process which is partially exchangeable in the sense of Definition 1.2 is partially exchangeable in the sense of Diaconis and Freedman. We prove:

Theorem 1.1 *Let Z be a nearest-neighbor random walk on a finite graph G satisfying*

$$P(X_n = X_0 \textit{ for infinitely many } n) = 1 \tag{1.3}$$

and for all $e \in E$ and all $u, v \in \bar{e}$

$$P\left(\textit{There exists } n \geq 0 \textit{ with } (X_n, Y_{n+1}, X_{n+1}) = (u, e, v)\right) = 1. \tag{1.4}$$

Then Z is a unique mixture of reversible Markov chains if and only if Z is partially exchangeable in the sense of Definition 1.2.

Assumption (1.4) says that every edge is traversed in both directions with probability 1.

A Markovian nearest-neighbor random walk on G with transition probabilities given by strictly positive weights on the edges is partially exchangeable in the sense of Definition 1.2 (transitions are made with probabilities proportional to the edge weights). We call such a Markov chain a *non-reinforced random walk.* A more interesting example of a nearest-neighbor random walk which is partially exchangeable in the sense of Definition 1.2 is *edge-reinforced random walk.* The process was introduced by Coppersmith and Diaconis in 1987 (see [1]) as follows: All edges are given strictly positive numbers as weights. In each step, the random walker traverses an incident edge with a probability proportional to its weight. Each time an edge is traversed, its weight is increased by 1.

Suppose G is 2-edge-connected, i.e. removing an edge does not make G disconnected. Let Z be a partially exchangeable nearest-neighbor random walk on G such that the conditional probabilities to traverse edge e in the next step depend only on the current location, the edge e, the local time accumulated at the present vertex, and the number of times e has been traversed in the past. If Z satisfies in addition some natural technical assumptions, then Z has the same distribution as a non-reinforced random walk or an edge-reinforced random walk. More precisely, we make the following assumptions on G and Z:

Assumption 1.1 *For all $v \in V$ degree$(v) \neq 2$.*

Assumption 1.2 *There exists $v_0 \in V$ with $P(X_0 = v_0) = 1$.*

Assumption 1.3 *For any admissible path π of length $n \geq 1$ starting at v_0 we have $P(Z_n = \pi) > 0$.*

Assumption 1.4 *Z is partially exchangeable in the sense of Definition 1.2.*

Assumption 1.5 *For all $v \in V$ and $e \in E$ there exists a function $f_{v,e}$ taking values in $[0,1]$ such that for all $n \geq 0$*

$$P(Y_{n+1} = e, X_{n+1} = v | Z_n) = f_{X_n,e}(k_n(X_n), k_n(e)).$$

For the last assumption, we need the following definition.

Definition 1.3 *We define the domain of definition $\mathrm{Def}(f_{v,e})$ of $f_{v,e}$ to be the set of all (k, k_e) such that there exists a path π from v_0 to v with $k(v, \pi) = k$ and $k(e, \pi) = k_e$. We set $\mathcal{D}_{v,e}(k) := \{k_e : (k, k_e) \in \mathrm{Def}(f_{v,e})\}$.*

Assumption 1.6 *For all $v \in V$ and $e \in E$ with $v \in \bar{e}$ there exist real-valued constants $b_{v,e}(2)$, $c_v(2)$ such that $f_{v,e}(2, j) = b_{v,e}(2) + c_v(2)j$ for all $j \in \mathcal{D}_{v,e}(2)$.*

We prove:

Theorem 1.2 *Suppose G is 2-edge-connected and satisfies Assumption 1.1. A nearest-neighbor random walk Z on G satisfies Assumptions 1.2-1.6 if and only if Z is an edge-reinforced random walk or a non-reinforced random walk starting at v_0, except that the conditional probability $P(Y_1 = e, X_1 = v | X_0 = v_0)$ may be different from the corresponding conditional probability for edge-reinforced/non-reinforced random walk.*

Without Assumption 1.1, Theorem 1.2 need not be true. If G is the graph consisting of two vertices which are connected by two parallel edges, then

Assumption 1.5 is vacuous (because Assumption 1.4 holds) and Theorem 1.2 does not hold (compare Zabell [7]).

A similar characterization for *directed-edge-reinforced random walk* on a *complete graph* has been obtained by Zabell [8]. In a directed-edge-reinforced random walk directed edges are reinforced; see [5] for the definition of the process. This model is easier to treat because there is independence between what happens at different vertices, and the assumption of a complete graph simplifies the proof considerably.

The remainder of the article is organized as follows: In Section 2, we describe a generalization of Theorem 1.2 for graphs which are not 2-edge-connected. Section 3 contains our results on mixtures of reversible Markov chains. In Section 4, we collect some graph-theoretical lemmas needed in our proofs. In Section 5 we prove Theorem 1.2 and its generalization from Section 2.

2 Result for a general graph

In this section, we state a generalization of Theorem 1.2. For $v \in V$ we denote by E_v the set of all edges incident to v:

$$E_v := \{e \in E : v \in \bar{e}\}. \tag{2.1}$$

For an admissible path π, we set

$$K(v, \pi) \quad := \quad \sum_{e \in E_v} k(e, \pi). \tag{2.2}$$

For a nearest-neighbor random walk Z, $v \in V$, and $e \in E$, we define

$$k_n(v) := k(v, Z_n), \quad k_n(e) := k(e, Z_n), \quad K_n(v) := K(v, Z_n).$$

Recall that a graph G' is called 2-*edge-connected* if removing an edge does not make G' disconnected. G' is 2-edge-connected if and only if for any two edges $e \neq e'$ in G' there exists a cycle containing both e and e'. A *bridge* is an edge whose deletion increases the number of connected components. There is no edge parallel to a bridge. A subgraph B of G is called a *block* of G if B is a bridge or a maximal 2-edge-connected subgraph of G. We denote the edge set of B by $E(B)$. The graph G decomposes into finitely many blocks $B_1, B_2, \ldots, B_m$ in the sense that the edge set E of G can be written as disjoint union of the $E(B_i)$'s. We write V_2 (E_2) for the set of all vertices (edges) contained in a 2-edge-connected block.

We define *modified edge-reinforced random walk* as follows:

Definition 2.1 *Let* $V \to \{0,1\}, v \mapsto d_v$ *be constant on any 2-edge-connected block of* G*, and let* $a_{v,e} > 0$*,* $v \in V$*,* $e \in E_v$*, with the property* $a_{v,e} = a_{u,e}$ *for all* $e \in E_2$*,* $u, v \in \bar{e}$*. We set* $a_v := \sum_{e \in E_v} a_{v,e}$*. We define* modified edge-reinforced random walk *with starting point* v_0 *to be a nearest-neighbor random walk* (X,Y) *on* G *with* $P(X_0 = v_0) = 1$ *and for all* $n \geq 0$

$$P(Y_{n+1} = e, X_{n+1} = v | Z_n) = \begin{cases} \dfrac{a_{X_n,e} + d_{X_n} \cdot k_n(e)}{a_v + d_{X_n} \cdot K_n(X_n)} & \text{if } \bar{e} = \{X_n, v\}, \\ 0 & \text{otherwise.} \end{cases}$$

In the definition of modified edge-reinforced random walk, we choose for each pair (v,e) with $e \in E_v$ a weight $a_{v,e} > 0$. If $\bar{e} = \{u,v\}$ and e is contained in a 2-edge-connected block, we require $a_{u,e} = a_{v,e}$. If $d_v = 0$, then the weights of (v,e) for $e \in E_v$ never change, whereas if $d_v = 1$ the weight of (v,e) increases by 1 after each traversal of e. Since $v \mapsto d_v$ is constant on any 2-edge-connected block B of G, the restriction of modified edge-reinforced random walk to B is either non-reinforced or edge-reinforced random walk. We prove:

Theorem 2.1 *Suppose* G *satisfies Assumption 1.1. A nearest-neighbor random walk* Z *on* G *satisfies Assumptions 1.2-1.6 if and only if the conditional probabilities*

$$P(Y_{n+1} = e, X_{n+1} = v | Z_n, k_n(X_n) \geq 2)$$

agree with the corresponding conditional probabilities for a modified edge-reinforced random walk on G *with starting point* v_0*.*

If G is the star-shaped graph with vertex set $V = \{v_0, v_1, \ldots, v_m\}$ and edges between v_0 and v_i for $1 \leq i \leq m$, then the weights of an edge-reinforced random walk observed at the times the random walker is at the central vertex v_0, obey the same dynamics as the number of balls in a Polya urn process where after each drawing the ball is returned with two additional balls of the same color. In this case, Assumption 1.5 is just Johnson's "sufficientness" postulate and we recover Johnson's result (compare [7], Corollary 2.2).

3 Mixtures of reversible Markov chains

In this section, we prove Theorem 1.1, and we derive conclusions for (modified) edge-reinforced random walk. We begin with a relation between $k(e,\pi)$ and $k(v,\pi)$.

Remark 3.1 *If π is an admissible path in G starting at v_0 and ending at v, then*

$$\sum_{e \in E_v} k(e,\pi) = 2k(v,\pi) - 1 - \delta_v(v_0); \tag{3.1}$$

here $\delta_v(v_0)$ denotes Kronecker's delta. In particular, $k(v,\pi)$ is determined by $k(e,\pi)$, $e \in E_v$, via equation (3.1).

We omit the elementary proof of Remark 3.1.

A *closed* path is a path with the same starting and endpoint. If $c := (u_0, e_1, \ldots, e_n, u_n)$ is a closed path and all e_i, $1 \le i \le n$, are distinct, then we call c a *cycle*.

Proof of Theorem 1.1. If Z is a reversible Markov chain, then its transition probabilities can be described by weights on the edges. Hence for a finite path π the probability $P(Z_n = \pi)$ depends only on $k(e,\pi)$, $e \in E$, and $k(v,\pi)$, $v \in V$. By Remark 3.1, $k(v,\pi)$ is uniquely determined by $k(e,\pi)$, $e \in E_v$. Hence Z is partially exchangeable in the sense of Definition 1.2, and the same is true if Z is a unique mixture of reversible Markov chains.

Conversely, suppose Z is partially exchangeable in the sense of Definition 1.2. Then Z is partially exchangeable in the sense of Diaconis and Freedman (see the comments before Definition 1.2). By Theorem (7) of [2], Z is a unique mixture of Markov chains. We denote the mixing measure by μ.

Suppose there exist $e \in E$ and $u \in \bar{e}$ such that $p(u,e) = 0$ on a set S of positive μ-measure. Using the definition of mixtures of Markov chains, we obtain $P(\text{There exists } n \ge 0 \text{ with } (X_n, Y_{n+1}, X_{n+1}) = (u,e,v)) \le 1 - \mu(S)$, which contradicts (1.4). Hence $p(u,e) > 0$ μ-a.s.. Thus for μ-a.a. $(v,p) \in V \times \mathcal{P}$, the Markov chain with transition matrix p is irreducible and since the state space is finite, recurrent.

Let $c = (u_0, e_1, \ldots, e_n, u_n)$ be a cycle. We set $\tau := \min\{i \ge 0 : X_i = u_0\}$; τ is the first hitting time of u_0. We denote by $Q_{v,p}$ the distribution of the Markov chain with transition matrix p which starts in v with probability 1. We write $Q_p(c)$ for the probability that the Markov chain with transition matrix p traverses the cycle c starting at a point in the cycle. We define $\theta(X_0, Y_1, X_1, Y_2, \ldots) := (X_1, Y_2, X_2, Y_3, \ldots)$; thus θ shifts the random walk Z by one step. We denote by θ^m the m^{th} iterate of θ. For $m, n \ge 0$, $(\theta^m Z)_n = (X_m, Y_{m+1}, X_{m+1}, \ldots, Y_{m+n}, X_{m+n})$ equals the random path of length n traversed by the random walker starting at time m. We calculate the probability that the process Z traverses c twice immediately after time

τ:

$$\begin{aligned}
q &:= P(\tau < \infty, (\theta^\tau Z)_n = c = (\theta^{\tau+n} Z)_n) \\
&= \int\limits_{V\times\mathcal{P}} Q_{v,p}(\tau < \infty, (\theta^\tau Z)_n = c = (\theta^{\tau+n} Z)_n)\mu(dv,dp) \\
&= \int\limits_{V\times\mathcal{P}} Q_{v,p}(\tau < \infty) Q_{u_0,p}(Z_n = c = (\theta^n Z)_n)\mu(dv,dp) \\
&= \int\limits_{V\times\mathcal{P}} Q_p(c)^2 \mu(dv,dp);
\end{aligned}$$

for the first equality we used that the process is a mixture of Markov chains, for the second equality we used the strong Markov property under $Q_{v,p}$, and for the last equality we used $Q_{v,p}(\tau < \infty) = 1$ which follows from recurrence of the Markov chain.

Using partial exchangeability, we see that the probability that the process Z traverses the reversed cycle $c^{\leftrightarrow} := (u_n, e_n, u_{n-1}, \ldots, e_1, u_0)$ twice immediately after time τ equals q, and the same argument as above yields

$$q = \int_{V\times\mathcal{P}} Q_p(c^{\leftrightarrow})^2 \mu(dv,dp).$$

Furthermore the probability to traverse first c and then $c^{\leftrightarrow}$ immediately after time τ also equals q:

$$q = \int_{V\times\mathcal{P}} Q_p(c) Q_p(c^{\leftrightarrow}) \mu(dv,dp).$$

Consequently,

$$\begin{aligned}
& \int_{V\times\mathcal{P}} \left[Q_p(c) - Q_p(c^{\leftrightarrow})\right]^2 \mu(dv,dp) \\
= & \int_{V\times\mathcal{P}} \left[Q_p(c)^2 - 2Q_p(c)Q_p(c^{\leftrightarrow}) + Q_p(c^{\leftrightarrow})^2\right] \mu(dv,dp) \\
= & \; q - 2q + q = 0,
\end{aligned}$$

and we conclude $Q_p(c) = Q_p(c^{\leftrightarrow})$ μ-a.s. It follows from Kolmogorov's cycle criterion (see e.g. [3], page 303), that for μ-a.a. (v,p) the Markov chain with transition matrix p is reversible. This completes the proof of Theorem 1.1. ■

Corollary 3.1 *Edge-reinforced random walk and modified edge-reinforced random walk on a finite graph G are unique mixtures of reversible Markov chains.*

Proof. The assumptions of Theorem 1.1 are satisfied for edge-reinforced random walk: By Lemma 2 in [4] the process is partially exchangeable; Proposition 1 in [4] implies (1.3) and (1.4). The claim for modified edge-reinforced random walk follows similarly. ■

For edge-reinforced random walk on a finite graph the mixing measure can be given explicitly. Let $\Delta := \{(x_e; e \in E) : x_e \geq 0, \sum_{e \in E} x_e = 1\}$, and let σ denote Lebesgue measure on Δ.

Theorem 3.1 *Let Z be edge-reinforced random walk on a finite graph G. There exists a function $\Phi : \Delta \to [0, \infty[$ such that for any admissible path $\pi = (v_0, e_1, v_1, \ldots, e_n, v_n)$ the following holds:*

$$P(Z_n = \pi) = \int_{\Delta} \prod_{i=1}^{n} \frac{x_{e_i}}{x_{v_{i-1}}} \Phi(x_e; e \in E) \, d\sigma(x_e; e \in E);$$

here $x_v := \sum_{e \in E_v} x_e$. The density Φ is given explicitly in Theorem 1 of [4], see also Diaconis [1]; Φ is strictly positive in the interior of Δ.

Proof. By Corollary 3.1, Z is a unique mixture of reversible Markov chains. Hence the mixing measure can be described as the image of a measure on Δ. Theorem 1 of [4] states that $\lim_{n\to\infty}(k_n(e)/n; e \in E)$ exists almost surely and has distribution $\Phi d\sigma$. The claim follows from Markov chain theory. ■

4 Some graph-theoretical lemmas

In this section, we collect some graph-theoretical results which will be needed in the proofs of Theorems 1.2 and 2.1. We assume in the whole section that degree$(v) \geq 3$ for all $v \in V$.

If a path π starts at v_0 and ends at v, then we say that π is a path from v_0 to v. For a vertex $v \in V$, we define $\Pi_{v_0,v}$ to be the set of all admissible paths from v_0 to v which visit v only in the last step:

$$\Pi_{v_0,v} := \{\pi \text{ admissible path from } v_0 \text{ to } v \text{ with } k(v, \pi) = 1\}. \tag{4.1}$$

Π_{v_0,v_0} contains only the trivial path $\pi = (v_0)$. If $\pi = (v_0, e_1, v_1, \ldots, e_n, v_n)$ is an admissible path, we say that π enters vertex v_n via e_n. We define subsets of E_v:

$$E_{v,\text{initial}} \;:=\; \{e \in E_v :\ \text{every } \pi \in \Pi_{v_0,v} \text{ enters } v \text{ via } e\}\,, \tag{4.2}$$

$$E_{v,\text{enter}} \;:=\; \{e \in E_v :\ \text{no } \pi \in \Pi_{v_0,v} \text{ enters } v \text{ via } e\}\,, \tag{4.3}$$

$$E_{v,\text{cycle}} \;:=\; \left\{\begin{matrix} e \in E_v : \text{there exist } e' \in E_v \setminus \{e\} \text{ and a cycle } c \\ \text{such that } e \text{ and } e' \text{ are both contained in } c \end{matrix}\right\}. \tag{4.4}$$

We have $E_{v_0,\text{initial}} = \emptyset$ and $E_{v_0,\text{enter}} = E_{v_0}$. If $E_{v,\text{initial}} \neq \emptyset$, then it contains precisely one edge which we denote by $e_{v,\text{initial}}$. Removing $e_{v,\text{initial}}$ from G makes the graph disconnected; thus e is a bridge. Hence $E_{v,\text{initial}} \cap E_{v,\text{cycle}} = \emptyset$. Note that $E_v = E_{v,\text{initial}} \cup E_{v,\text{enter}} \cup E_{v,\text{cycle}}$.

If π and π' are admissible paths with the property that the endpoint of π agrees with the starting point of π', then we write $\pi\pi'$ for the concatenation of π and π'. For a closed path π we denote by π^k the concatenation of k copies of π. By definition, π^0 equals the empty path. We set $\mathbb{N} := \{1, 2, \dots\}$ and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$.

The following lemma collects information about $\mathcal{D}_{v,e}(k)$; recall Definition 1.3.

Lemma 4.1 *1. If $k_1 \in \mathcal{D}_{v,e_1}(k)$, then there exist $e_2, e_3 \in E_v \setminus \{e_1\}$, $e_2 \neq e_3$, and $k_2, k_3 \in \{0,1\}$ such that for all $k_1', k_2', k_3' \in \mathbb{N}_0$ with $k_i' \equiv k_i$ mod 2 and $k_1' + k_2' + k_3' = 2k - 1 - \delta_v(v_0)$ there exists a path π from v_0 to v with $k(e_i, \pi) = k_i'$ for $i = 1, 2, 3$ and $k(e, \pi) = 0$ for all $e \in E_v \setminus \{e_1, e_2, e_3\}$.*

2. Let $v \in V$, $e \in E_v$ and $k \in \mathbb{N}$. We set

$$m_{v,e} := \begin{cases} 1 & \textit{if } e \in E_{v,\text{initial}}, \\ 0 & \textit{otherwise.} \end{cases} \qquad \Delta_{v,e} := \begin{cases} 1 & \textit{if } e \in E_{v,\text{cycle}}, \\ 2 & \textit{otherwise.} \end{cases}$$

There exists $M_{v,e}(k) \in \mathbb{N}_0$ such that

$$\mathcal{D}_{v,e}(k) \;=\; \{m_{v,e} + j \cdot \Delta_{v,e} : 0 \leq j \leq M_{v,e}(k)\}\,. \tag{4.5}$$

Proof. Let $e_1 \in E_v$ and $k \in \mathbb{N}$. We pick edges $e_2, e_3 \in E_v \setminus \{e_1\}$, $e_2 \neq e_3$, with the following constraints: If $v \neq v_0$ and $e_1 \notin E_{v,\text{initial}}$, then we choose $e_2 \in E_v \setminus (E_{v,\text{enter}} \cup \{e_1\})$. If $e_1 \in E_{v,\text{enter}} \cap E_{v,\text{cycle}}$, we choose $e_3 \in E_v \setminus \{e_1\}$ such that there exists a cycle containing both e_1 and e_3.

Suppose $0 \in \mathcal{D}_{v,e_1}(k)$. If $v = v_0$, we set $\pi := (v_0)$. Otherwise there exists $\pi \in \Pi_{v_0,v}$ which enters v via e_2. For $i = 1, 2, 3$, we write $\bar{e}_i = \{v, v_i\}$, and we set $\pi_i = (v, e_i, v_i, e_i, v)$, i.e. π_i crosses edge e_i back and forth starting from

v. For $j_1, j_2, j_3 \in \mathbb{N}_0$ with $2j_1 + 2j_2 + 2j_3 + 1 - \delta_v(v_0) = 2k - 1 - \delta_v(v_0)$, we define π' to be the concatenation of π with j_i copies of π_i: $\pi' := \pi\pi_1^{j_1}\pi_2^{j_2}\pi_3^{j_3}$. Using Remark 3.1, we see that π' is an admissible path with $k(v, \pi') = k$. Hence $(k, 2j_1) \in \mathrm{Def}(f_{v,e_1})$ for all $j_1 \geq 0$ up to a certain upper bound which can be obtained from (3.1). This proves the first part of the lemma in the case $k_1 \in \mathcal{D}_{v,e_1}(k)$ with k_1 even.

Suppose $1 \in \mathcal{D}_{v,e_1}(k)$. If $e_1 \in E_v \setminus E_{v,\mathrm{enter}}$, then we choose a path $\pi \in \Pi_{v_0,v}$ which enters v via e_1. If $e_1 \in E_{v,\mathrm{enter}}$, we choose $\tilde{\pi} \in \Pi_{v_0,v}$ which enters v via e_2. In the latter case, we have $e_1 \in E_{v,\mathrm{cycle}}$ because $1 \in \mathcal{D}_{v,e_1}(k)$. Hence we can extend $\tilde{\pi}$ to a path π by adding one traversal of a cycle which contains e_1 and e_3. A similar argument as above completes the proof of the first part of the lemma in the case $k_1 \in \mathcal{D}_{v,e_1}(k)$ with k_1 odd.

Clearly $0 \in \mathcal{D}_{v,e_1}(k)$ if $e_1 \notin E_{v,\mathrm{initial}}$, and $1 \in \mathcal{D}_{v,e_1}(k)$ if $e_1 \in E_{v,\mathrm{initial}} \cup E_{v,\mathrm{cycle}}$. The above argument implies that the right-hand side of (4.5) is contained in $\mathcal{D}_{v,e}(k)$. It remains to show the reversed inclusion. By Remark 3.1, each $j \in \mathcal{D}_{v,e}(k)$ satisfies $j \leq 2k - 1 - \delta_v(v_0)$. It is easy to see that $j \in \mathcal{D}_{v,e}(k)$ is odd if $e \in E_{v,\mathrm{initial}}$.

Claim: $j \in \mathcal{D}_{v,e}(k)$ is even if $e \in E_v \setminus (E_{v,\mathrm{cycle}} \cup E_{v,\mathrm{initial}})$.

Let $e \in E_v \setminus (E_{v,\mathrm{cycle}} \cup E_{v,\mathrm{initial}}) = E_{v,\mathrm{enter}} \setminus E_{v,\mathrm{cycle}}$, and let π be a path from v_0 to v. The path $\tilde{\pi}$ which is obtained by considering π only until the first time v is visited, belongs to $\Pi_{v_0,v}$ and enters v via an edge $e' \neq e$ because $e \in E_{v,\mathrm{enter}}$. In particular, $k(e, \tilde{\pi}) = 0$. The number of traversals of e cannot increase by 1 between two successive visits to v because $e \notin E_{v,\mathrm{cycle}}$. Hence $k(e, \pi)$ must be even. This finishes the proof of the second part of the lemma. ∎

5 Proofs of Theorems 1.2 and 2.1

Throughout this section, we assume that G satisfies Assumption 1.1 and Z is a nearest-neighbor random walk satisfying Assumptions 1.2-1.6. We will show that the conditional probabilities $P(Y_{n+1} = e, X_{n+1} = v | Z_n, k_n(X_n) \geq 2)$ agree with the corresponding conditional probabilities for modified edge-reinforced random walk. By Assumption 1.5, it suffices to show that the functions $f_{v,e}$ have the appropriate form. Lemma 5.2 below is the first step in this direction. Since the random walk is reflected at vertices of degree 1, we assume in the following degree$(v) \geq 3$ for all $v \in V$. We begin with a remark which collects some properties of the functions $f_{v,e}$.

Remark 5.1 *For $e \in E_v$, the function $f_{v,e}$ is strictly positive on its domain* $\mathrm{Def}(f_{v,e})$. *If k and k_e, $e \in E_v$, are such that there exists a path π from v_0*

to v with $k(v,\pi) = k$ and $k(e,\pi) = k_e$ for all $e \in E_v$, then

$$\sum_{e \in E_v} f_{v,e}(k, k_e) = 1. \tag{5.1}$$

Proof. Let $\pi = (v_0, e_1, \dots, e_n, v_n = v)$ be a path as in the statement of the remark. Then using Assumption 1.5, we obtain

$$1 = \sum_{e \in E_v} P(Y_{n+1} = e | Z_n = \pi) = \sum_{e \in E_v} f_{v,e}(k, k_e).$$

Combining Assumptions 1.5 and 1.3 we see that $f_{v,e}(k, k_e) > 0$. ∎

We will need the following elementary lemma:

Lemma 5.1 *Let $\alpha, \alpha', \alpha'', \beta, \beta', \gamma, \gamma', \delta$ be real numbers. If*

$$\alpha + \beta' + \gamma' + \delta = 1, \tag{5.2}$$
$$\alpha' + \beta + \gamma' + \delta = 1, \tag{5.3}$$
$$\alpha' + \beta' + \gamma + \delta = 1 \text{ and} \tag{5.4}$$
$$\alpha'' + \beta + \gamma + \delta = 1, \tag{5.5}$$

then $\alpha - \alpha' = \beta - \beta' = \gamma - \gamma'$ and $\alpha - \alpha' = \alpha' - \alpha''$. If only (5.2) and (5.3) hold, then $\alpha - \alpha' = \beta - \beta'$.

The following lemma states that $f_{v,e}$ is linear in the second argument.

Lemma 5.2 *For all $v \in V$, $e \in E_v$ and $k \geq 1$, there exist real-valued constants $b_{v,e}(k), c_v(k)$ such that for all $j \in \mathcal{D}_{v,e}(k)$ the following holds:*

$$f_{v,e}(k, j) = b_{v,e}(k) + c_v(k) j. \tag{5.6}$$

Here $b_{v,e}(k) > 0$ for $e \in E_v \setminus E_{v,\text{initial}}$ and $b_{v,e}(k) + c_v(k) > 0$ for $e \in E_{v,\text{initial}}$.

Remark 5.2 *Assumption 1.6 is just the relation (5.6) for $k = 2$. If e.g. $v \neq v_0$, $E_v = \{e_1, e_2, e_3\}$ with $e_1 \in E_{v,\text{initial}}$, $e_2, e_3 \in E_{v,\text{cycle}}$, then (5.6) need not hold for $k = 2$ and $e = e_2$ without Assumption 1.6.*

Proof of Lemma 5.2. First we consider the case $k = 1$. Let $k_e \geq 0$, $e \in E_v$, such that $\sum_{e \in E_v} k_e = 2k - 1 - \delta_v(v_0) = 1 - \delta_v(v_0)$. If $v = v_0$, then $k_e = 0$, consequently $\mathcal{D}_{v,e}(1) = \{0\}$ and the claim is trivial. If $v \neq v_0$, then there exists precisely one edge $e_1 \in E_v$ such that $k_{e_1} = 1$ and $k_{e_2} = 0$ for all $e_2 \in E_v \setminus \{e_1\}$. Linearity of the function $j \mapsto f_{v,e_1}(k, j)$ is clear because

$\mathcal{D}_{v,e_1}(1)$ contains at most two elements, namely 0 and 1. Whenever $\mathcal{D}_{v,e_1}(1)$ contains only one element, (5.6) is automatically true and does not give any constraint on $c_v(1)$. If $\mathcal{D}_{v,e_1}(1) = \{0,1\}$, there exist $e_2 \in E_v \setminus \{e_1\}$ and paths π from v_0 to v with transition counts $k(e,\pi) = k_e$, $e \in E_v$, given by

$$k_{e_1} = 1,\ k_{e_2} = 0,\ k_{e_3} = 0 \text{ for all } e_3 \in E_v \setminus \{e_1, e_2\}, \text{ and}$$
$$k_{e_1} = 0,\ k_{e_2} = 1,\ k_{e_3} = 0 \text{ for all } e_3 \in E_v \setminus \{e_1, e_2\}.$$

We apply (5.1) with these values for k_e to obtain equations of the form (5.2) and (5.3) with $\alpha = f_{v,e_1}(1,1)$, $\alpha' = f_{v,e_1}(1,0)$, $\beta = f_{v,e_2}(1,1)$, $\beta' = f_{v,e_2}(1,0)$. Lemma 5.1 implies

$$f_{v,e_1}(1,1) - f_{v,e_1}(1,0) = f_{v,e_2}(1,1) - f_{v,e_2}(1,0).$$

Hence the increment does not depend on the edge.

For $k = 2$, (5.6) is true by Assumption 1.6.

Supposes $k \geq 3$. By Lemma 4.1, $\mathcal{D}_{v,e}(k)$ is a set of non-negative integers which can be ordered in such a way that any two successive elements differ by $\Delta_{v,e} \in \{1,2\}$. Since $k \geq 3$, there exists $j \in \mathcal{D}_{v,e}(k)$ with $j \geq m_{v,e} + 4$. First we show that the function $j \mapsto f_{v,e}(k,j)$ is linear on the sets $\mathcal{D}_{v,e}(k) \cap 2\mathbb{N}_0$ and $\mathcal{D}_{v,e}(k) \cap (1 + 2\mathbb{N}_0)$.

Claim 1: For all $e_1 \in E_v$ and $j \in \mathcal{D}_{v,e_1}(k)$ with $j \geq m_{v,e_1} + 4$

$$f_{v,e_1}(k,j) - f_{v,e_1}(k,j-2) = f_{v,e_1}(k,j-2) - f_{v,e_1}(k,j-4).$$

Let j be as in the assumption of Claim 1. By Lemma 4.1, there exist $e_2, e_3 \in E_v \setminus \{e_1\}$ with $e_2 \neq e_3$ and k_2, k_3 with $j + k_2 + k_3 = 2k - 1 - \delta_v(v_0)$ such that paths π with the following numbers of edge traversals for $e \in E_v$ are possible: $k(e,\pi) = 0$ for all $e \in E_v \setminus \{e_1, e_2, e_3\}$ and

$$\begin{array}{lll} k(e_1,\pi) = j, & k(e_2,\pi) = k_2, & k(e_3,\pi) = k_3, \\ k(e_1,\pi) = j-2, & k(e_2,\pi) = k_2 + 2, & k(e_3,\pi) = k_3, \\ k(e_1,\pi) = j-2, & k(e_2,\pi) = k_2, & k(e_3,\pi) = k_3 + 2, \text{ and} \\ k(e_1,\pi) = j-4, & k(e_2,\pi) = k_2 + 2, & k(e_3,\pi) = k_3 + 2. \end{array}$$

Applying equation (5.1) to these transition counts, we obtain equations of the form (5.2)-(5.5) with $\alpha = f_{v,e_1}(k,j)$, $\alpha' = f_{v,e_1}(k,j-2)$, $\alpha'' = f_{v,e_1}(k,j-4)$. Lemma 5.1 implies Claim 1.

Next, we show that the increment does not depend on the edge e.

Claim 2: For any $e_1, e_2 \in E_v$ we have

$$f_{v,e_1}(k, m_{v,e_1} + 2) - f_{v,e_1}(k, m_{v,e_1}) = f_{v,e_2}(k, m_{v,e_2} + 2) - f_{v,e_2}(k, m_{v,e_2}). \tag{5.7}$$

If $v \neq v_0$ we choose $e_1 \in E_v \setminus E_{v,\text{enter}}$; otherwise we choose $e_1 \in E_v$ arbitrarily. Let $e_2, e_3 \in E_v \setminus \{e_1\}$, $e_2 \neq e_3$, and let k_3 be such that $m_{v,e_1} + m_{v,e_2} + 2 + k_3 = 2k - 1 - \delta_v(v_0)$. There exist paths π with $k(e,\pi) = 0$ for all $e \in E_v \setminus \{e_1, e_2, e_3\}$ and

$$\begin{array}{lll} k(e_1,\pi) = m_{v,e_1} + 2, & k(e_2,\pi) = m_{v,e_2}, & k(e_3,\pi) = k_3, \\ k(e_1,\pi) = m_{v,e_1}, & k(e_2,\pi) = m_{v,e_2} + 2, & k(e_3,\pi) = k_3. \end{array}$$

Applying equation (5.1) to these transition counts and then using Lemma 5.1 yields (5.7). Since e_2 is arbitrary in $E_v \setminus \{e_1\}$, Claim 2 follows.

Finally we show that $j \mapsto f_{v,e}(k,j)$ is linear on $\mathcal{D}_{v,e}(k)$ for $e \in E_{v,\text{cycle}}$. Because of Claim 1 it suffices to prove **Claim 3:** If $e_1 \in E_{v,\text{cycle}}$, then

$$f_{v,e_1}(k,3) - f_{v,e_1}(k,1) = f_{v,e_1}(k,2) - f_{v,e_1}(k,0).$$

We can find edges $e_2, e_3 \in E_v \setminus \{e_1\}$, $e_2 \neq e_3$, and k_3 such that there exist paths π with $k(e,\pi) = 0$ for all $e \in E_v \setminus \{e_1, e_2, e_3\}$ and

$$\begin{array}{lll} k(e_1,\pi) = 3, & k(e_2,\pi) = m_{v,e_2}, & k(e_3,\pi) = k_3, \\ k(e_1,\pi) = 1, & k(e_2,\pi) = m_{v,e_2} + 2, & k(e_3,\pi) = k_3. \end{array}$$

Using (5.1) with these transition counts and Lemma 5.1 we obtain

$$f_{v,e_1}(k,3) - f_{v,e_1}(k,1) = f_{v,e_2}(k, m_{v,e_2} + 2) - f_{v,e_2}(k, m_{v,e_2}).$$

Since $m_{v,e_1} = 0$, the last equation together with Claim 2 implies Claim 3.

Let $k \geq 1$ and $e \in E_v \setminus E_{v,\text{initial}}$. There exists a path π from v_0 to v with $k(v,\pi) = k$ and $k(e,\pi) = 0$. Using Remark 5.1 we obtain $b_{v,e}(k) = f_{v,e}(k,0) > 0$. A similar argument shows $b_{v,e}(k) + c_v(k) > 0$ for $e \in E_{v,\text{initial}}$. ∎

Remark 5.3 *The set $\mathcal{D}_{v,e}(1)$ has cardinality 1 for all $e \in E_v$ if and only if $v = v_0$ or $E_{v,\text{initial}} \neq \emptyset$. In these cases, $c_v(1)$ can be chosen arbitrarily. For technical reasons, we choose in these cases $c_v(1) := c_v(3)$.*

Lemma 5.3 *If $c_v(k)c_v(k+1) = 0$ for some $k \geq 1$, then $c_v(k) = c_v(k+1) = 0$. In particular, we have either $c_v(k) = 0$ for all $k \geq 1$ or $c_v(k) \neq 0$ for all $k \geq 1$.*

Proof. Let $k \geq 2$, and let $e_1, e_2 \in E_v \setminus E_{v,\text{initial}}$, $e_1 \neq e_2$. We assume e_i has endpoints v and v_i for $i = 1,2$. Let π be any path from v_0 to v with $k(v,\pi) = k$. We abbreviate $k_i := k(e_i,\pi)$, $k_{v_i} := k(v_i,\pi)$. Let $\tilde{\pi}_i := (v, e_i, v_i, e_i, v)$ be the path which traverses e_i back and forth starting at v. We define $\pi_1 := \pi\tilde{\pi}_1\tilde{\pi}_2$, $\pi_2 := \pi\tilde{\pi}_2\tilde{\pi}_1$. By partial exchangeability (Assumption 1.4), π_1 and π_2 have the same probability. Using Assumption 1.5 we can write the probability of π_i as a product of values of $f_{u,e}$, $u \in V$, $e \in E$. The factors corresponding to the transitions in π agree for π_1 and π_2. Since $f_{u,e}$ is strictly positive on its domain by Remark 5.1, all these factors cancel and we obtain

$$\begin{aligned} & f_{v,e_1}(k,k_1) f_{v_1,e_1}(k_{v_1}+1,k_1+1) f_{v,e_2}(k+1,k_2) f_{v_2,e_2}(k_{v_2}+1,k_2+1) \\ = \; & f_{v,e_2}(k,k_2) f_{v_2,e_2}(k_{v_2}+1,k_2+1) f_{v,e_1}(k+1,k_1) f_{v_1,e_1}(k_{v_1}+1,k_1+1). \end{aligned}$$

The last equality implies

$$f_{v,e_1}(k,k_1) f_{v,e_2}(k+1,k_2) = f_{v,e_2}(k,k_2) f_{v,e_1}(k+1,k_1). \tag{5.8}$$

Suppose $c_v(k) = 0$. Using Lemma 5.2, we can rewrite (5.8) as follows:

$$\begin{aligned} & b_{v,e_1}(k)\left[b_{v,e_2}(k+1) + c_v(k+1)k_2\right] \\ = \; & b_{v,e_2}(k)\left[b_{v,e_1}(k+1) + c_v(k+1)k_1\right]. \end{aligned} \tag{5.9}$$

We apply the last equation first with $k_1 = 0, k_2 = 2$ and then with $k_1 = 2, k_2 = 0$ (recall $e_1, e_2 \in E_v \setminus E_{v,\text{initial}}$ and $k \geq 2$; hence there exist paths π from v_0 to v with $k(e_i,\pi) = k_i$ for $i = 1,2$.). This yields

$$b_{v,e_1}(k)\left[b_{v,e_2}(k+1) + 2c_v(k+1)\right] = b_{v,e_2}(k) b_{v,e_1}(k+1), \tag{5.10}$$

$$b_{v,e_1}(k) b_{v,e_2}(k+1) = b_{v,e_2}(k)\left[b_{v,e_1}(k+1) + 2c_v(k+1)\right]. \tag{5.11}$$

Subtracting (5.11) from (5.10), we obtain

$$2b_{v,e_1}(k)c_v(k+1) = -2b_{v,e_2}(k)c_v(k+1). \tag{5.12}$$

Since $e_1, e_2 \in E_v \setminus E_{v,\text{initial}}$, we have $b_{v,e_1}(k), b_{v,e_2}(k) > 0$ by Lemma 5.2. If $c_v(k+1) \neq 0$, then left and right-hand side of (5.12) have different signs. Hence $c_v(k+1) = 0$. If we assume $c_v(k+1) = 0$, then we obtain (5.9) with k and $k+1$ interchanged and the same argument shows $c_v(k) = 0$.

It remains to consider the case $k = 1$. If $v = v_0$ or $E_{v,\text{initial}} \neq \emptyset$, we chose $c_v(1) := c_v(3)$ (compare Remark 5.3). In these cases $c_v(1)c_v(2) = c_v(3)c_v(2)$, and the claim follows from what we proved above. In the remaining cases, $v \neq v_0$ and $E_{v,\text{initial}} = \emptyset$. Then there exist $e_1, e_2 \in E_v$, $e_1 \neq e_2$, such that $\mathcal{D}_{v,e_i} = \{0,1\}$ for $i = 1,2$, and we can apply (5.9) with $k_1 = 0, k_2 = 1$ and $k_1 = 1, k_2 = 0$. The desired result follows. ∎

Lemma 5.4 *Suppose* $c_v(k) \neq 0$. *We set* $K_v := 2k - 1 - \delta_v(v_0)$, $a_{v,e}(k) := b_{v,e}(k)/c_v(k)$ *for* $e \in E_v$, *and* $a_v(k) := \sum_{e \in E_v} a_{v,e}(k)$. *If* $(k, k_e) \in \mathrm{Def}(f_{v,e})$, *then*

$$f_{v,e}(k, k_e) = \frac{a_{v,e}(k) + k_e}{a_v(k) + K_v}.$$

Proof. We abbreviate $b_v(k) := \sum_{e' \in E_v} b_{v,e'}(k)$. Let $(k, k_e) \in \mathrm{Def}(f_{v,e})$. There exists a path π from v_0 to v with $k(v, \pi) = k$ and $k(e, \pi) = k_e$. We set $k_{e'} := k(e', \pi)$ for all $e' \in E \setminus \{e\}$. By Remark 3.1, we have $\sum_{e' \in E_v} k_{e'} = K_v$. Using (5.1) and Lemma 5.2 we obtain

$$1 = \sum_{e' \in E_v} f_{v,e'}(k, k_{e'}) = \sum_{e' \in E_v} b_{v,e'}(k) + c_v(k) \sum_{e' \in E_v} k_{e'} = b_v(k) + c_v(k) K_v.$$

Dividing the last equality by $c_v(k)$ yields

$$\frac{1}{c_v(k)} = \frac{b_v(k)}{c_v(k)} + K_v = a_v(k) + K_v. \tag{5.13}$$

Another application of Lemma 5.2 yields

$$f_{v,e}(k, k_e) = b_{v,e}(k) + c_v(k) k_e = c_v(k) \left[a_{v,e}(k) + k_e \right] = \frac{a_{v,e}(k) + k_e}{a_v(k) + K_v};$$

for the last equation we used (5.13). ∎

Lemma 5.5 *If* $c_v(1) \neq 0$, *then there exist constants* $a_{v,e} > 0$ *for* $e \in E_v \setminus E_{v,\mathrm{initial}}$ *and* $a_{v,e} > -1$ *for* $e \in E_{v,\mathrm{initial}}$, *such that for all* $k \geq 2$ *and* $k_e \in \mathcal{D}_{v,e}(k)$ *the following holds:*

$$f_{v,e}(k, k_e) = \frac{a_{v,e} + k_e}{a_v + K_v}$$

with $a_v := \sum_{e' \in E_u} a_{v,e'}$ *and* $K_v := 2k - 1 - \delta_v(v_0)$.

Proof. Let $k \geq 2$. Suppose $c_v(1) \neq 0$. By Lemma 5.4 it suffices to show $a_{v,e}(k) = a_{v,e}(2)$. We choose $e_1, e_2 \in E_v$, $e_1 \neq e_2$, with the constraint $e_1 \in E_v \setminus E_{v,\mathrm{enter}}$ if $v \neq v_0$. Let π be any path from v_0 to v with $k(v, \pi) = k$. We set $k_i := k(e_i, \pi)$. Applying Lemma 5.4 to the factors in (5.8) we obtain

$$\frac{a_{v,e_1}(k) + k_1}{a_v(k) + K_v} \cdot \frac{a_{v,e_2}(k+1) + k_2}{a_v(k+1) + K_v + 2} = \frac{a_{v,e_2}(k) + k_2}{a_v(k) + K_v} \cdot \frac{a_{v,e_1}(k+1) + k_1}{a_v(k+1) + K_v + 2};$$

recall $K_v := 2k - 1 - \delta_v(v_0)$. The denominators are equal, hence the enumerators are equal:

$$[a_{v,e_1}(k) + k_1]\,[a_{v,e_2}(k+1) + k_2] = [a_{v,e_2}(k) + k_2]\,[a_{v,e_1}(k+1) + k_1]\,. \quad (5.14)$$

We apply (5.14) with $k_1 = 1 - \delta_v(v_0)$, $k_2 = 0$ and $k_1 = 1 - \delta_v(v_0)$, $k_2 = 2$ to obtain

$$\begin{aligned} [a_{v,e_1}(k) + k_1]\, a_{v,e_2}(k+1) &= a_{v,e_2}(k)\,[a_{v,e_1}(k+1) + k_1]\,, \quad (5.15)\\ [a_{v,e_1}(k) + k_1]\,[a_{v,e_2}(k+1) + 2] &= [a_{v,e_2}(k) + 2]\,[a_{v,e_1}(k+1) + k_1]\,. \end{aligned}$$

Subtracting both equations yields

$$-2\,[a_{v,e_1}(k) + k_1] = -2\,[a_{v,e_1}(k+1) + k_1]\,,$$

which implies $a_{v,e_1}(k) = a_{v,e_1}(k+1)$. From (5.15) we conclude $a_{v,e_2}(k) = a_{v,e_2}(k+1)$.

Let $e \in E_v \setminus E_{v,\text{initial}}$. The equation $a_{v,e}(k+1) = a_{v,e}(k)$ is equivalent to $\frac{b_{v,e}(k+1)}{c_v(k+1)} = \frac{b_{v,e}(k)}{c_v(k)}$. Since $b_{v,e}(k), b_{v,e}(k+1) > 0$ by Lemma 5.2, $c_v(k)$ and $c_v(k+1)$ have the same sign. In particular, $c_v(k)$ has the same sign as $c_v(2)$ for all $k \geq 2$. Suppose $c_v(2) < 0$. Then $c_v(k) < 0$ for all $k \geq 2$ and by (5.13),

$$0 > \frac{1}{c_v(k)} = a_v(k) + K_v = a_v(2) + 2k - 1 - \delta_v(v_0).$$

Since the right-hand side diverges to infinity for $k \to \infty$, we obtained a contradiction and we conclude $c_v(2) > 0$. Hence $a_{v,e}(2) = \frac{b_{v,e}(2)}{c_v(2)} > 0$ for all $e \in E_v \setminus E_{v,\text{initial}}$. By Lemma 5.2, $a_{v,e}(2) + 1 = \frac{b_{v,e}(2) + c_v(2)}{c_v(2)} > 0$ for $e \in E_{v,\text{initial}}$, hence $a_{v,e}(2) > -1$ in this case. ■

Remark 5.4 *For any $e_1, e_2 \in E_{v_0}$, we have $\frac{a_{v_0,e_1}(1)}{a_{v_0,e_1}(2)} = \frac{a_{v_0,e_2}(1)}{a_{v_0,e_2}(2)}$. If G is 2-edge-connected, then the claim of Lemma 5.5 is true for $k = 1$ if $v \neq v_0$.*

Proof. If $v = v_0$, we can apply (5.14) with $k_1 = k_2 = 0$ to obtain the desired result.

Suppose G is 2-edge-connected and $v \neq v_0$, and let $e_1, e_2 \in E_v$, $e_1 \neq e_2$. Then there exist paths $\pi \in \Pi_{v_0,v}$ with $k(e_1,\pi) = k(e_2,\pi) = 0$ and $k(e_1,\pi) = 0$, $k(e_2,\pi) = 1$. Hence we can apply (5.14) for these values of k_1, k_2 to obtain $a_{v,e_1}(1) = a_{v,e_1}(2)$. ■

Lemma 5.6 *If $c_v(1) = 0$, then $f_{v,e}(k, k_e) = b_{v,e}(1)$ for all $e \in E_v$ and $(k, k_e) \in \mathrm{Def}(f_{v,e})$.*

Proof. Suppose $c_v(1) = 0$. We know from Lemmas 5.2 and 5.3 that $f_{v,e}(k, k_e) = b_{v,e}(k)$ for all $(k, k_e) \in \mathrm{Def}(f_{v,e})$. It remains to show $b_{v,e}(k) = b_{v,e}(1)$ for all $k \geq 1$.

Let $e_1, e_2 \in E_v$, $e_1 \neq e_2$. We write $\bar{e}_i = \{v, v_i\}$ for the set of endpoints of e_i, and we set $\tilde{\pi}_i := (v, e_i, v_i, e_i, v)$, $i = 1, 2$. Let $\pi \in \Pi_{v_0,v}$, and let $k \geq 0$. We extend π adding in two different ways $k+1$ copies of $\tilde{\pi}_1$ and one copy of $\tilde{\pi}_2$: We define $\pi_1 := \pi\tilde{\pi}_1^{k+1}\tilde{\pi}_2$, $\pi_2 := \pi\tilde{\pi}_1^k\tilde{\pi}_2\tilde{\pi}_1$. By partial exchangeability (Assumption 1.4), π_1 and π_2 have the same probability. Using Assumption 1.5 we can write the probabilities of π_1 and π_2 as products of values of $f_{u,e}$, $u \in V$, $e \in E$. The factors arising from the transitions in $\pi\tilde{\pi}_1^k$ are the same for π_1 and π_2. By Remark 5.1, all these factors are strictly positive. Furthermore the contributions for the traversals of e_i starting from v_i, $i = 1, 2$, are the same for π_1 and π_2 by Assumption 1.5. After all these cancellations have been done, only the factors corresponding to the traversals of e_i starting from v, $i = 1, 2$ remain, and we obtain

$$b_{v,e_1}(k+1)b_{v,e_2}(k+2) = b_{v,e_2}(k+1)b_{v,e_1}(k+2),$$

which implies

$$\frac{b_{v,e_2}(k+2)}{b_{v,e_2}(k+1)} = \frac{b_{v,e_1}(k+2)}{b_{v,e_1}(k+1)}. \tag{5.16}$$

Since $e_2 \neq e_1$ was arbitrary in E_v, we conclude for $e \in E_v$

$$b_{v,e}(k) = b_{v,e}(1)\prod_{j=1}^{k-1}\frac{b_{v,e}(j+1)}{b_{v,e}(j)} = b_{v,e}(1)\prod_{j=1}^{k-1}\frac{b_{v,e_1}(j+1)}{b_{v,e_1}(j)}; \tag{5.17}$$

here the empty product is defined to be 1. Let $k_e := k(e, \pi\tilde{\pi}_1^k)$ for $e \in E$ and $k_v := k(v, \pi\tilde{\pi}_1^k)$. Then $k_v = k+1$. Combining Remark 5.1, Lemma 5.2 and (5.17), we obtain

$$1 = \sum_{e\in E_v} f_{v,e}(k_v, k_e) = \sum_{e\in E_v} b_{v,e}(k+1) = \sum_{e\in E_v} b_{v,e}(1)\prod_{j=1}^{k}\frac{b_{v,e_1}(j+1)}{b_{v,e_1}(j)}$$

for all $k \geq 0$, and we conclude $\frac{b_{v,e_1}(k+1)}{b_{v,e_1}(k)} = 1$ for all $k \geq 1$. Consequently, $b_{v,e_1}(k) = b_{v,e_1}(1)$ for all $k \geq 1$. Since $e_1 \in E_v$ is arbitrary, the claim follows. ∎

Lemma 5.7 *1. If $c_v(1) = 0$, then $c_{v'}(1) = 0$ for all vertices v' contained in the same 2-edge-connected block as v.*

2. *If $c_v(1) \neq 0$ for all v in a 2-edge-connected block B, then $a_{u,e} = a_{v,e}$ for all edges e in B and $u, v \in \bar{e}$.*

Proof. Let v be contained in a 2-edge-connected block, and suppose $c_v(1) = 0$. There exists a cycle $c = (u_0, e_1, u_1, \ldots, e_n, u_n)$ in G with $u_0 = v$. We set

$$\begin{aligned} \mathcal{U}_0 &:= \left\{ j : 0 \leq j \leq n-1, c_{u_j}(1) = 0 \right\}, \\ \mathcal{U}_1 &:= \left\{ j : 0 \leq j \leq n-1, c_{u_j}(1) \neq 0 \right\}. \end{aligned}$$

Note that $0 \in \mathcal{U}_0$. Let π be a path from v_0 to v with $k(v,\pi) \geq 2$. We set $k_e := k(e,\pi)$ for $e \in E$, $k_{v'} := k(v',\pi)$ and $K_{v'} := 2k_{v'} - 1 - \delta_{v'}(v_0)$ for $v' \in V$. We extend π adding one traversal of c or of the reversed cycle $c^{\leftrightarrow}$: We set $\pi_1 := \pi c$, $\pi_2 := \pi c^{\leftrightarrow}$. By partial exchangeability (Assumption 1.4), π_1 and π_2 have the same probability. Using Assumption 1.5, we obtain

$$\begin{aligned} & f_{u_0,e_1}(k_v, k_{e_1}) \prod_{j=2}^{n} f_{u_{j-1},e_j}(k_{u_{j-1}} + 1, k_{e_j}) \\ = \quad & f_{u_0,e_n}(k_v, k_{e_n}) \prod_{j=1}^{n-1} f_{u_j,e_j}(k_{u_j} + 1, k_{e_j}). \end{aligned}$$

Lemmas 5.5 and 5.6 imply

$$\begin{aligned} & b_{u_0,e_1}(1) \prod_{j \in \mathcal{U}_0 \setminus \{0\}} b_{u_j,e_{j+1}}(1) \prod_{j \in \mathcal{U}_1} \frac{a_{u_j,e_{j+1}} + k_{e_{j+1}}}{a_{u_j + K_{u_j} + 2}} \\ = \quad & b_{u_0,e_n}(1) \prod_{j \in \mathcal{U}_0 \setminus \{0\}} b_{u_j,e_j}(1) \prod_{j \in \mathcal{U}_1} \frac{a_{u_j,e_j} + k_{e_j}}{a_{u_j + K_{u_j} + 2}}. \end{aligned}$$

Since the denominators on both sides agree, the same is true for the enumerators:

$$\begin{aligned} & b_{u_0,e_1}(1) \prod_{j \in \mathcal{U}_0 \setminus \{0\}} b_{u_j,e_{j+1}}(1) \prod_{j \in \mathcal{U}_1} \left[a_{u_j,e_{j+1}} + k_{e_{j+1}} \right] \\ = \quad & b_{u_0,e_n}(1) \prod_{j \in \mathcal{U}_0 \setminus \{0\}} b_{u_j,e_j}(1) \prod_{j \in \mathcal{U}_1} \left[a_{u_j,e_j} + k_{e_j} \right]. \end{aligned} \tag{5.18}$$

For $m \geq 0$ we extend π adding m traversals of c or $c^{\leftrightarrow}$: We set $\pi_{1,m} := \pi c^m$, $\pi_{2,m} := \pi (c^{\leftrightarrow})^m$. An analogous argument as above shows that

$$(\alpha_1)^m \prod_{l=0}^{m-1} \varphi_1(l) \quad = \quad (\alpha_2)^m \prod_{l=0}^{m-1} \varphi_2(l) \tag{5.19}$$

with

$$\begin{aligned}
\alpha_1 &:= b_{u_0,e_1}(1) \prod_{j\in\mathcal{U}_0\setminus\{0\}} b_{u_j,e_{j+1}}(1) \\
\alpha_2 &:= b_{u_0,e_n}(1) \prod_{j\in\mathcal{U}_0\setminus\{0\}} b_{u_j,e_j}(1) \\
\varphi_1(l) &:= \prod_{j\in\mathcal{U}_1} \left[a_{u_j,e_{j+1}} + k_{e_{j+1}} + l\right] \\
\varphi_2(l) &:= \prod_{j\in\mathcal{U}_1} \left[a_{u_j,e_j} + k_{e_j} + l\right].
\end{aligned}$$

It follows from (5.19) that

$$\frac{\alpha_1}{\alpha_2} = \exp\left(\frac{1}{m}\sum_{l=0}^{m-1} \ln\left(\frac{\varphi_2(l)}{\varphi_1(l)}\right)\right); \tag{5.20}$$

note that $\alpha_1, \alpha_2 > 0$ and $\varphi_1, \varphi_2 > 0$ on $\mathbb{N}_0$ by Remark 5.1. Since φ_1 and φ_2 are polynomials with leading coefficient 1, $\lim_{l\to\infty} \ln(\varphi_2(l)/\varphi_1(l)) = 0$, and the same is true for the Cesaro mean: $\lim_{m\to\infty} m^{-1}\sum_{l=0}^{m-1} \ln(\varphi_2(l)/\varphi_1(l)) = 0$. Taking the limit as $m \to \infty$ in (5.20) yields $\alpha_1/\alpha_2 = 1$; thus $\alpha_1 = \alpha_2$. Since (5.19) is valid for all $m \geq 1$, we conclude $\varphi_1(m) = \varphi_2(m)$ for all $m \geq 0$. Since φ_1 and φ_2 are polynomials of degree $\leq n$, φ_1 and φ_2 must be identical. In particular, the zeros of φ_1 and φ_2 agree:

$$\left\{-a_{u_j,e_{j+1}} - k_{e_{j+1}}; j \in \mathcal{U}_1\right\} = \left\{-a_{u_j,e_j} - k_{e_j}; j \in \mathcal{U}_1\right\}. \tag{5.21}$$

Suppose $\mathcal{U}_1 \neq \emptyset$. By assumption, $\mathcal{U}_0 \neq \emptyset$. Hence there exists $j_0 \in \mathcal{U}_1 \setminus \{n-1\}$ such that $j_0 + 1 \notin \mathcal{U}_1$. Recall that in the above argument, π can be any path from v_0 to u_0 with $k(u_0, \pi) \geq 2$. We can choose π in such a way that

$$a_{u_{j_0},e_{j_0+1}} + k(e_{j_0+1}, \pi) > \max\left\{a_{u_j,e_j} + k(e_j, \pi); j \in \mathcal{U}_1\right\}.$$

This contradicts (5.21), and we conclude $\mathcal{U}_1 = \emptyset$. Since for any two edges in a 2-edge-connected block there exists a cycle containing both edges, the first part of the lemma follows.

Suppose $c_v(1) \neq 0$ for all vertices v contained in a 2-edge-connected block B. Let $c = (u_0, e_1, u_1, e_2, \ldots, e_n, u_n)$ be a cycle in B. A similar argument as above shows that

$$\prod_{l=0}^{m-1} \tilde{\varphi}_1(l) = \prod_{l=0}^{m-1} \tilde{\varphi}_2(l) \tag{5.22}$$

with $\tilde{\varphi}_1(l) := \prod_{j=1}^n \left[a_{u_{j-1},e_j} + k_{e_j} + l\right]$, $\tilde{\varphi}_2(l) := \prod_{j=1}^n \left[a_{u_j,e_j} + k_{e_j} + l\right]$. Again, $\tilde{\varphi}_i > 0$ on $\mathbb{N}_0$, $i = 1, 2$, and it follows from (5.22) that $\tilde{\varphi}_1(l) = \tilde{\varphi}_2(l)$ for all $l \geq 0$. Consequently $\tilde{\varphi}_1$ and $\tilde{\varphi}_2$ are identical, in particular they have the same zeros. Since $k_e = k(e, \pi)$, $e \in E$, and π was an arbitrary path from v_0 to u_0 with $k(u_0, \pi) \geq 2$, we conclude that $a_{u_{j-1},e_j} = a_{u_j,e_j}$ for $1 \leq j \leq n$. Since for any two edges in a 2-edge-connected block, there exists a cycle in B containing both edges, the second part of the lemma follows. ■

Proof of Theorem 2.1. It is not hard to see that a modified edge-reinforced random walk starting at v_0 satisfies Assumptions 1.2 - 1.6; in order to show partial exchangeability, one uses that edge-reinforced random walk is partially exchangeable by Lemma 2 of [4].

Let Z be a nearest-neighbor random walk satisfying Assumptions 1.2 - 1.6. Using Lemmas 5.5-5.7 together with Assumption 1.5 we conclude that the conditional probabilities $P(Y_{n+1} = e, X_{n+1} = v | Z_n, k_n(X_n) \geq 2)$ agree with the corresponding conditional probabilities for modified edge-reinforced random walk. We remark that this statement is trivial if degree$(v) = 1$. This completes the proof of the theorem. ■

Proof of Theorem 1.2. A modified edge-reinforced random walk on a 2-edge-connected graph is either a non-reinforced or an edge-reinforced random walk. Hence Theorem 1.2 follows from Theorem 2.1 and Remark 5.4. ■

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