

LARGE DEVIATIONS FOR THE ONE-DIMENSIONAL EDWARDS MODEL

March 6, 2002

Remco van der Hofstad^{1 2}
Frank den Hollander³
Wolfgang König⁴

Abstract: In this paper we prove a large deviation principle for the empirical drift of a one-dimensional Brownian motion with self-repulsion called the Edwards model. Our results extend earlier work in which a law of large numbers, respectively, a central limit theorem were derived. In the Edwards model a path of length T receives a penalty $e^{-\beta H_T}$, where H_T is the self-intersection local time of the path and $\beta \in (0, \infty)$ is a parameter called the strength of self-repulsion. We identify the rate function in the large deviation principle for the endpoint of the path as $\beta^{\frac{2}{3}} I(\beta^{-\frac{1}{3}} \cdot)$, with $I(\cdot)$ given in terms of the principal eigenvalues of a one-parameter family of Sturm-Liouville operators. We show that there exist numbers $0 < b^{**} < b^* < \infty$ such that: (1) I is linearly decreasing on $[0, b^{**}]$; (2) I is real-analytic and strictly convex on (b^{**}, ∞) ; (3) I is continuously differentiable at b^{**} ; (4) I has a unique zero at b^* . (The latter fact identifies b^* as the asymptotic drift of the endpoint.) The critical drift b^{**} is associated with a crossover in the optimal strategy of the path: for $b \geq b^{**}$ the path assumes local drift b during the full time T , while for $0 \leq b < b^{**}$ it assumes local drift b^{**} during time $\frac{b^{**}+b}{2b^{**}}T$ and local drift $-b^{**}$ during the remaining time $\frac{b^{**}-b}{2}T$. Thus, in the second regime the path makes an overshoot of size $\frac{b^{**}-b}{2}T$ in order to reduce its intersection local time.

2000 Mathematics Subject Classification. 60F05, 60F10, 60J55, 82D60.

Keywords and phrases. Self-repellent Brownian motion, intersection local time, Ray-Knight Theorems, large deviations, Airy function.

¹Department of Applied Mathematics, Delft University of Technology, Mekelweg 4, 2628 CD Delft, The Netherlands.

²Present address: Department of Mathematics and Computer Science, Eindhoven University of Technology, P.O. Box 513, 5600 MB Eindhoven, The Netherlands. rhofstad@win.tue.nl.

³EURANDOM, P.O. Box 513, 5600 MB Eindhoven, The Netherlands. denhollander@eurandom.tue.nl

⁴Institut für Mathematik, TU Berlin, Straße des 17. Juni 136, D-10623 Berlin, Germany. koenig@math.tu-berlin.de

1.1 The Edwards model

Let $B = (B_t)_{t \geq 0}$ be standard Brownian motion on \mathbb{R} starting at the origin ($B_0 = 0$). Let P be the Wiener measure and let E be expectation with respect to P . For $T > 0$ and $\beta \in (0, \infty)$, define a probability law \mathbb{Q}_T^β on paths of length T by setting

$$\frac{d\mathbb{Q}_T^\beta}{dP}[\cdot] = \frac{1}{Z_T^\beta} e^{-\beta H_T[\cdot]}, \quad Z_T^\beta = E(e^{-\beta H_T}), \quad (1.1)$$

where

$$H_T[(B_t)_{t \in [0, T]}] = \int_0^T du \int_0^T dv \delta(B_u - B_v) = \int_{\mathbb{R}} dx L(T, x)^2 \quad (1.2)$$

is the Brownian intersection local time up to time T . The first expression in (1.2) is formal only. In the second expression the Brownian local times $L(T, x)$, $x \in \mathbb{R}$, appear. The law \mathbb{Q}_T^β is called the T -polymer measure with strength of self-repellence β . The Brownian scaling property implies that

$$\mathbb{Q}_T^\beta((B_t)_{t \in [0, T]} \in \cdot) = \mathbb{Q}_{\beta^{\frac{2}{3}}T}^1\left((\beta^{-\frac{1}{3}}B_{\beta^{\frac{2}{3}}t})_{t \in [0, T]} \in \cdot\right). \quad (1.3)$$

It is known that under the law \mathbb{Q}_T^β the endpoint B_T satisfies the following central limit theorem:

Theorem 1.1 (Central limit theorem). *There are numbers $a^*, b^*, c^* \in (0, \infty)$ such that for any $\beta \in (0, \infty)$:*

- (i) *Under the law \mathbb{Q}_T^β , the distribution of the scaled endpoint $(|B_T| - b^* \beta^{\frac{1}{3}}T)/c^* \sqrt{T}$ converges weakly to the standard normal distribution.*
- (ii) $\lim_{T \rightarrow \infty} \frac{1}{T} \log Z_T^\beta = -a^* \beta^{\frac{2}{3}}$.

Theorem 1.1 is contained in [vdHdHK97, Theorem 2 and Proposition 1]. For the identification of a^*, b^*, c^* , see (2.4) below. Bounds on these numbers appeared in [vdH98, Theorem 3]. The numerical values are: $a^* \approx 2.19$, $b^* \approx 1.11$, $c^* \approx 0.63$. The law of large numbers corresponding to Theorem 1.1(i) was first obtained by Westwater [W84] (see also [vdHdH95, Section 0.6]).

The main object of interest in the present paper is the rate function I_β defined by ⁵

$$\begin{aligned} -I_\beta(b) &= \lim_{T \rightarrow \infty} \frac{1}{T} \log E\left(e^{-\beta H_T} \mathbb{1}_{\{|B_T| \approx bT\}}\right) \\ &= \lim_{T \rightarrow \infty} \frac{1}{T} \log \left\{ Z_T^\beta \mathbb{Q}_T^\beta(B_T \approx bT) \right\}, \quad b \in \mathbb{R}, \end{aligned} \quad (1.4)$$

where $B_T \approx bT$ is an abbreviation for $|B_T - bT| \leq \gamma_T$ for some $\gamma_T > 0$ such that $\gamma_T/T \rightarrow 0$ and $\gamma_T/\sqrt{T} \rightarrow \infty$ as $T \rightarrow \infty$. (We will see that the limit in (1.4) does not depend on the choice of γ_T .) It is clear from (1.3) that

$$\beta^{-\frac{2}{3}} I_\beta(\beta^{\frac{1}{3}}b) = I_1(b), \quad b \geq 0, \quad (1.5)$$

provided the limit in (1.4) exists for $\beta = 1$ and $b \geq 0$. Moreover,

$$I_\beta(b) = I_\beta(-b), \quad b \leq 0. \quad (1.6)$$

Therefore, we may restrict ourselves to $\beta = 1$ and $b \geq 0$. In the following we write $I = I_1$.

⁵In fact, I_β differs by a constant from what is usually called a rate function. This constant is $\lim_{T \rightarrow \infty} \frac{1}{T} \log Z_T^\beta = -a^* \beta^{\frac{2}{3}}$ (see Theorem 1.1(ii)). Hence, $I_\beta - a^* \beta^{\frac{2}{3}}$ is the true rate function.

1.2 Main results

Our first main result says that I exists and has the shape exhibited in Fig. 1.⁶

Theorem 1.2 (Large deviations). *Let $\beta = 1$.*

- (i) *For any $b \geq 0$, the limit $I(b)$ in (1.4) exists and is finite.*
- (ii) *I is continuous and convex on $[0, \infty)$, and continuously differentiable on $(0, \infty)$.*
- (iii) *There are numbers $a^{**} \in (a^*, \infty)$, $b^{**} \in (0, b^*)$ and $\rho(a^{**}) \in (0, \infty)$ such that $I(0) = a^{**}$, I is linearly decreasing on $[0, b^{**}]$ with slope $-\rho(a^{**})$, is real-analytic and strictly convex on (b^{**}, ∞) , and attains its unique minimum at b^* with $I(b^*) = a^*$ and $I''(b^*) = 1/c^{*2}$.*
- (iv) *$I(b) = \frac{1}{2}b^2 + \mathcal{O}(b^{-1})$ as $b \rightarrow \infty$.*

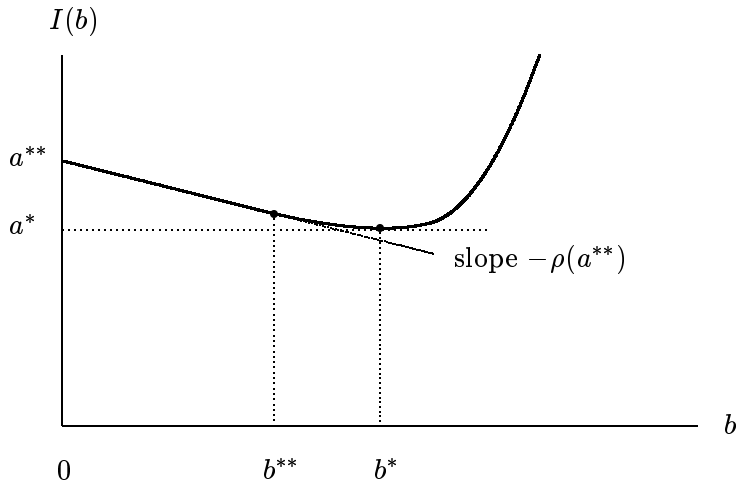


Fig. 1. Qualitative picture of $b \mapsto I(b)$.

The linear piece of the rate function has the following intuitive interpretation. If $b \geq b^{**}$, then the best strategy for the path to realize the large deviation event $\{B_T \approx bT\}$ is to assume local drift b during time T . In particular, the path makes no overshoot on scale T , and this leads to the real-analyticity and strict convexity of I on (b^{**}, ∞) . On the other hand, if $0 \leq b < b^{**}$, then this strategy is too expensive, since too small a drift leads to too large an intersection local time. Therefore the best strategy now is to assume local drift b^{**} during time $\frac{b^{**}+b}{2b^{**}}T$ and local drift $-b^{**}$ during the remaining time $\frac{b^{**}-b}{2b^{**}}T$. In particular, the path makes an overshoot on scale T , namely, $\frac{b^{**}-b}{2}T$, and this leads to the linearity of I on $[0, b^{**}]$. At the critical drift $b = b^*$, I is continuously differentiable.

For $b \rightarrow \infty$, $I(b)$ is determined by the Gaussian tail of B_T because the intersection local time H_T vanishes.

For the identification of $a^{**}, b^{**}, \rho(a^{**})$, see (2.5) below. The numerical values are: $a^{**} \approx 2.95$, $b^{**} \approx 0.85$, $\rho(a^{**}) \approx 0.78$. These estimates can be obtained with the help of the method in [vdH98].

There is an intimate connection between the rate function I and the two moment generating functions $\Lambda^+, \Lambda^- : \mathbb{R} \rightarrow \mathbb{R}$ given by

$$\Lambda^+(\mu) = \lim_{T \rightarrow \infty} \frac{1}{T} \log E(e^{-H_T} e^{\mu B_T} \mathbf{1}_{\{B_T \geq 0\}}), \quad (1.7)$$

⁶In [MS87, Corollary 2.6 and Remark 2.7] it was proved that $\lim_{T \rightarrow \infty} \frac{1}{T} \log E(e^{-H_T} | B_T = 0) = -a^{**}$, which essentially gives the existence of $I(0)$ with value a^{**} . Furthermore, the existence of $I(b^*)$ with value a^* follows from our earlier work [vdHdHK97, Proposition 1].

and the same formula for $\Lambda^-(\mu)$ with $\mathbb{1}_{\{B_T \geq 0\}}$ replaced by $\mathbb{1}_{\{B_T \leq 0\}}$. Obviously, $\Lambda^+(-\mu) = \Lambda^-(\mu)$ for any $\mu \in \mathbb{R}$, provided one limit exists.

Our second main result says that Λ^+ exists and has the shape exhibited in Fig. 2, and that its Legendre transform is equal to I on $[0, \infty)$.

Theorem 1.3 (Exponential moments). *Let $\beta = 1$.*

- (i) *For any $\mu \in \mathbb{R}$, the limit $\Lambda^+(\mu)$ in (1.7) exists and is finite.*
- (ii) *Λ^+ equals $-a^{**}$ on $(-\infty, -\rho(a^{**})]$, is real-analytic and strictly convex on $(-\rho(a^{**}), \infty)$, and satisfies $\lim_{\mu \downarrow -\rho(a^{**})} (\Lambda^+)'(\mu) = b^{**}$.*
- (iii) *$\Lambda^+(\mu) = \frac{1}{2}\mu^2 + \mathcal{O}(\mu^{-1})$ as $\mu \rightarrow \infty$.*
- (iv) *The restriction of I to $[0, \infty)$ is the Legendre transform of Λ^+ , i.e.,*

$$I(b) = \max_{\mu \in \mathbb{R}} [b\mu - \Lambda^+(\mu)], \quad b \geq 0. \quad (1.8)$$

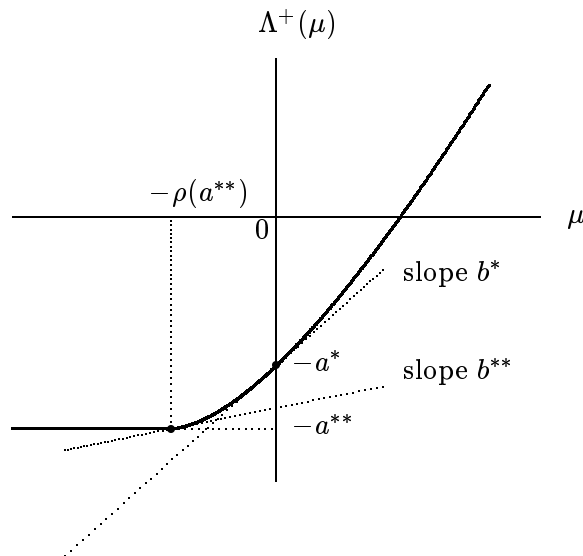


Fig. 2. Qualitative picture of $\mu \mapsto \Lambda^+(\mu)$.

As a consequence of Theorem 1.3(ii), the maximum on the right-hand side of (1.8) is attained at some $\mu > -\rho(a^{**})$ if $b > b^{**}$ and at $\mu = -\rho(a^{**})$ if $0 \leq b \leq b^{**}$. Analogous assertions hold for Λ^- , in particular, the restriction of I to $(-\infty, 0]$ is the Legendre transform of Λ^- . Since $\Lambda^-(\mu) = \Lambda^+(-\mu)$, the moment generating function equals

$$\Lambda(\mu) = \lim_{T \rightarrow \infty} \frac{1}{T} \log E(e^{-H_T} e^{\mu B_T}) = \max\{\Lambda^+(\mu), \Lambda^-(\mu)\} = \Lambda^+(|\mu|), \quad \mu \in \mathbb{R}, \quad (1.9)$$

which is symmetric and strictly convex on \mathbb{R} , and non-differentiable at 0, with $\Lambda(0) = -a^*$ and $\lim_{\mu \downarrow 0} \Lambda'(\mu) = b^*$.

The outline of the present paper is as follows. In Section 2 we introduce some preparatory material that will be needed in the sequel. Two basic propositions are presented in Section 3: a representation for the probabilities of certain events under the Edwards measure, and an integrable majorant under which the dominated convergence theorem can be applied. In Section 4 we carry out the proofs of Theorems 1.2–1.3. Some more refined results about the Edwards model (which will be needed in a

forthcoming paper [vdHdHK02]) appear in Section 5. Finally, Section 6 contains a technical proof of a result used in Section 5.

2. PRELIMINARIES

In this section we provide some tools that are needed for the proofs of our main results in Section 1.2. These tools are taken from [vdHdH95], [vdHdHK97] and references cited therein. Section 2.1 introduces the Sturm-Liouville operators that determine the constants. Section 2.2 provides the ingredients that are needed for the formulation of the Ray-Knight Theorems (describing the joint distribution of the endpoint and the local times), and contains a mixing property. Section 2.3 gives a spectral decomposition of a function describing the “overshoots” of the path (i.e., the pieces outside the interval between the starting point and the endpoint) in terms of shifts of the Airy function, which plays an important role in various estimates.

2.1 Sturm-Liouville operators and definition of the constants

In [vdHdH95, Section 0.4] we introduced and analyzed a family of Sturm-Liouville operators $\mathcal{K}^a: L^2[0, \infty) \cap C^2[0, \infty) \rightarrow C[0, \infty)$, indexed by $a \in \mathbb{R}$, defined as

$$(\mathcal{K}^a x)(h) = 2hx''(h) + 2x'(h) + (ah - h^2)x(h), \quad h \geq 0. \quad (2.1)$$

The operator \mathcal{K}^a is symmetric and has a largest eigenvalue $\rho(a) \in \mathbb{R}$ with multiplicity one. The corresponding strictly positive (and L^2 -normalized) eigenfunction $x_a: [0, \infty) \rightarrow (0, \infty)$ is real-analytic and vanishes faster than exponential at infinity, more precisely,

$$\lim_{h \rightarrow \infty} h^{-\frac{3}{2}} \log x_a(h) = -\frac{\sqrt{2}}{3}. \quad (2.2)$$

The eigenvalue function $\rho: \mathbb{R} \rightarrow \mathbb{R}$ has the following properties:

- (a) ρ is real-analytic;
 - (b) ρ is strictly log-convex, strictly convex and strictly increasing;
 - (c) $\lim_{a \downarrow -\infty} \rho(a) = -\infty$, $\rho(0) < 0$, $\lim_{a \rightarrow \infty} \rho(a) = \infty$.
- (2.3)

In terms of this object, the numbers a^* , b^* , c^* appearing in Theorem 1.1 are defined as

$$\rho(a^*) = 0, \quad b^* = \frac{1}{\rho'(a^*)}, \quad c^{*2} = \frac{\rho''(a^*)}{\rho'(a^*)^3}, \quad (2.4)$$

while the numbers a^{**} , b^{**} appearing in Theorem 1.2 are defined as

$$a^{**} = 2^{\frac{1}{3}}(-a_0), \quad b^{**} = \frac{1}{\rho'(a^{**})}, \quad (2.5)$$

where a_0 (≈ -2.3381) is the largest zero of the Airy function:

$$\text{Ai is the unique solution of the Airy differential equation} \\ y''(h) = hy(h) \text{ that vanishes at infinity.} \quad (2.6)$$

From [vdHdHK97, Lemma 6] we know that $a^* < -a_0$. Therefore $a^{**} > a^*$, which in turn implies that $b^{**} < b^*$.

2.2 Squared Bessel processes, a Girsanov transformation, and a mixing property

The basic tools in our study of the Edwards model are the Ray-Knight Theorems, which give a description of the joint distribution of the local time process $(L(T, x))_{x \in \mathbb{R}}$ and the endpoint B_T . These will be summarized in Proposition 3.1 below. The key objects entering into this description are introduced here.

The first key ingredients are:

- (i) a squared two-dimensional Bessel process (BESQ^2), $X = (X_v)_{v \geq 0}$,
- (ii) a squared zero-dimensional Bessel process (BESQ^0), $X^* = (X_v^*)_{v \geq 0}$,

and their additive functionals

$$A(t) = \int_0^t X_v \, dv, \quad A^*(t) = \int_0^t X_v^* \, dv, \quad t \geq 0. \quad (2.7)$$

The respective generators of BESQ^2 and BESQ^0 are given by ⁷

$$Gf(h) = 2hf''(h) + 2f'(h), \quad G^*f(h) = 2hf''(h), \quad (2.8)$$

for sufficiently smooth functions $f: [0, \infty) \rightarrow \mathbb{R}$. For $h \geq 0$, we write \mathbb{P}_h and \mathbb{P}_h^* to denote the probability law of X and X^* given $X_0 = h$ and $X_0^* = h$, respectively. BESQ^2 takes values in $C^+ = C^+[0, \infty)$, the set of non-negative continuous functions on $[0, \infty)$. It has 0 as an entrance boundary, which is not visited in finite positive time with probability one. BESQ^0 takes values in $C_0^+ = C_0^+[0, \infty)$, the subset of those functions in C^+ that hit zero and afterwards stay at zero. It has 0 as an absorbing boundary, which is visited in finite time with probability one.

The second key ingredient is a certain Girsanov transformation, which turns BESQ^2 into a diffusion with strong recurrence properties. Namely, the process $(D_y^{(a)})_{y \geq 0}$ defined by

$$D_y^{(a)} = \frac{x_a(X_y)}{x_a(X_0)} \exp\left\{-\int_0^y [(X_v)^2 - aX_v + \rho(a)] \, dv\right\}, \quad y \geq 0, \quad (2.9)$$

is a martingale under \mathbb{P}_h for any $h \geq 0$ and hence serves as a density with respect to a new Markov process in the sense of a Girsanov transformation. More precisely, the transformed process, which we also denote by $X = (X_v)_{v \geq 0}$, has the transition density

$$\widehat{P}_y^a(h_1, h_2) \, dh_2 = \mathbb{E}_{h_1}(D_y^{(a)} \mathbb{1}_{\{X_y \in dh_2\}}), \quad y, h_1, h_2 \geq 0. \quad (2.10)$$

We write $\widehat{\mathbb{P}}_h^a$ to denote the probability law of the transformed process X given $X_0 = h$. This transformed process possesses the invariant distribution $x_a(h)^2 \, dh$, and so

$$\widehat{\mathbb{P}}^a = \int_0^\infty dh x_a(h)^2 \mathbb{P}_h^a \quad (2.11)$$

is its probability law in equilibrium. The transformed process is reversible under $\widehat{\mathbb{P}}^a$, since BESQ^2 is reversible with respect to the Lebesgue measure. Hence, $x_a(h_1)^2 \widehat{P}_y^a(h_1, h_2)$ is symmetric in $h_1, h_2 \geq 0$ for any $y \geq 0$.

The third key ingredient is the time-changed transformed process $Y = X \circ A^{-1} = (X_{A^{-1}(t)})_{t \geq 0}$. We write $\widetilde{\mathbb{P}}_h^a$ to denote the probability law of Y given $Y_0 = h$. This process possesses the invariant distribution $\frac{1}{\rho'(a)} h x_a(h)^2 \, dh$, and so

$$\widetilde{\mathbb{P}}^a = \frac{1}{\rho'(a)} \int_0^\infty dh h x_a(h)^2 \widetilde{\mathbb{P}}_h^a \quad (2.12)$$

is its probability law in equilibrium. Both transformed processes X and $Y = X \circ A^{-1}$ are ergodic.

The following mixing property will be used frequently in the sequel. By $\langle \cdot, \cdot \rangle$ we denote the inner product on $L^2 = L^2[0, \infty)$, and we write $\langle f, g \rangle_\circ = \int_0^\infty dh h f(h) g(h)$ for the inner product on L^2 weighted with the identity. The latter space will be denoted by $L^{2,\circ} = L^{2,\circ}[0, \infty)$.

⁷ BESQ^0 is sometimes called *Feller's diffusion*.

Proposition 2.1. Fix $a \in \mathbb{R}$ and fix measurable functions $f, g: [0, \infty) \rightarrow \mathbb{R}$ such that $f/\text{id}, g \in L^{2,\circ}$. For any family of measurable functions $f_s, g_s: [0, \infty) \rightarrow \mathbb{R}$, $s \geq 0$, such that $f_s/\text{id}, g_s \in L^{2,\circ}$, $s \geq 0$, and $f_s \rightarrow f, g_s \rightarrow g$ as $s \rightarrow \infty$ uniformly on compacts and in $L^{2,\circ}$, and for any family a_s , $s \geq 0$, such that $a_s \rightarrow a$ as $s \rightarrow \infty$,

$$\lim_{s \rightarrow \infty} \widehat{\mathbb{E}}^{a_s} \left(\frac{f_s(X_0)}{x_{a_s}(X_0)} \frac{g_s(Y_s)}{x_{a_s}(Y_s)} \right) = \langle f, x_a \rangle \frac{1}{\rho'(a)} \langle g, x_a \rangle_{\circ}. \quad (2.13)$$

This proposition is a slight extension of Proposition 3 in [vdHdHK97]; we omit the proof.

2.3 BESQ⁰, the Airy function, and a spectral decomposition

For $a < a^{**}$, introduce the function $y_a: [0, \infty) \rightarrow (0, \infty]$ defined by

$$y_a(h) = \mathbb{E}_h^* \left(e^{\int_0^\infty [aX_v^* - (X_v^*)^2] dv} \right). \quad (2.14)$$

(As a consequence of (2.17) and Proposition 2.2 below, the expectation on the right-hand side is infinite for $a > a^{**}$.) It is known (see [vdHdHK97], Lemma 5) that y_a is equal to a normalized scaled shift of the Airy function Ai:

$$y_a(h) = \frac{\text{Ai}(2^{-\frac{1}{3}}(h-a))}{\text{Ai}(-2^{-\frac{1}{3}}a)}, \quad h \geq 0, \quad (2.15)$$

It is well-known (see [E56, p. 43] and (6.2) below) that y_a vanishes faster than exponential at infinity:

$$\lim_{h \rightarrow \infty} h^{-\frac{3}{2}} \log y_a(h) = -\frac{\sqrt{2}}{3}. \quad (2.16)$$

An important role is played in the sequel by the function $w: [0, \infty)^2 \rightarrow [0, \infty)$ defined by

$$w(h, t) dt = \mathbb{E}_h^* \left(e^{-\int_0^\infty (X_v^*)^2 dv} \mathbb{1}_{\{A^*(\infty) \in dt\}} \right). \quad (2.17)$$

It is easily seen from (2.7) and (2.14) that $\int_0^\infty dt e^{at} w(h, t) = y_a(h)$ for $a < a^{**}$. We also have the following representation for $w(h, t)$ derived in [vdHdHK97, Lemma 7]:

$$w(h, t) = E_{\frac{h}{2}} \left(e^{-2 \int_0^t B_s ds} \mid T_0 = t \right) \varphi_h(t), \quad (2.18)$$

$$\varphi_h(t) = \frac{P_{\frac{h}{2}}(T_0 \in dt)}{dt} = (8\pi)^{-\frac{1}{2}} t^{-\frac{3}{2}} h e^{-\frac{h^2}{8t}}, \quad (2.19)$$

with $T_0 = \inf\{t > 0: B_t = 0\}$ the first time B hits zero. (We write P_h and E_h for probability and expectation with respect to standard Brownian motion B starting at $h \geq 0$, so that $P = P_0$, $E = E_0$.)

We will need the following expansion of the function w in terms of shifts of the Airy function:

Proposition 2.2.

(i) For any $\varepsilon > 0$,

$$w(h, t) = \sum_{k=0}^{\infty} e^{a^{(k)}(t-\varepsilon)} \langle w(\cdot, \varepsilon), \mathbf{e}_k(\cdot) \rangle \mathbf{e}_k(h), \quad h \geq 0, t \geq \varepsilon, \quad (2.20)$$

where

$$a^{(k)} = 2^{\frac{1}{3}} a_k, \quad \mathbf{e}_k(h) = c_k \text{Ai}(2^{-\frac{1}{3}}(h + a^{(k)})), \quad h \geq 0, \quad (2.21)$$

with a_k the k -th largest zero of Ai and with c_k chosen such that $\|\mathbf{e}_k\|_2 = 1$.

(ii) There exist constants $K_1, K_2, K_3 \in (0, \infty)$ such that

$$-a^{(k)} \sim K_1 k^{\frac{2}{3}}, \quad k \rightarrow \infty, \quad (2.22)$$

$$\int_0^\infty h e_k(h)^2 dh \leq K_2 k^{\frac{2}{3}} \quad \forall k, \quad (2.23)$$

$$\int_0^\infty \frac{1}{h} e_k(h)^2 dh \leq K_3 k^{\frac{1}{3}} \quad \forall k. \quad (2.24)$$

(Note that $a^{(0)} = -a^{**}$ by (2.5).)

Proof. (i) The proof comes in steps. We write c for a generic constant in $(0, \infty)$ whose value may change from appearance to appearance.

1. Let \mathcal{K}^* be the second-order differential operator on $C_0^\infty = C_0^\infty[0, \infty)$, the set of smooth functions $x: [0, \infty) \rightarrow \mathbb{R}$ that vanish at zero, defined by

$$(\mathcal{K}^*x)(h) = \begin{cases} 2x''(h) - hx(h) & \text{if } h > 0, \\ 0 & \text{if } h = 0. \end{cases} \quad (2.25)$$

This operator is symmetric with respect to the L^2 -inner product on $L_0^2 = L^2 \cap C_0^\infty$. Furthermore, we can identify all the eigenvalues and eigenfunctions of \mathcal{K}^* in L_0^2 in terms of scaled shifts of the Airy function. Namely, a comparison of (2.6) and (2.25) shows that the k -th eigenspace is spanned by the eigenfunction $e_k: [0, \infty) \rightarrow \mathbb{R}$ given in (2.21) and the k -th eigenvalue is $a^{(k)}$, $k \in \mathbb{N}_0$.

2. We next show that \mathcal{K}^* has a compact inverse on L^2 . Therefore, this inverse has an orthonormal basis of eigenvectors in L^2 , and hence the same is true for \mathcal{K}^* itself. Consequently, $(e_k)_{k \in \mathbb{N}_0}$ is an orthonormal basis of L^2 . This fact will be needed later.

We begin by identifying the inverse of \mathcal{K}^* . To do so, we follow [G81]. Let

$$y_1(u) = \text{Bi}(2^{\frac{1}{3}}u) - \text{Bi}(0) \frac{\text{Ai}(2^{\frac{1}{3}}u)}{\text{Ai}(0)}, \quad y_2(u) = \text{Ai}(2^{\frac{1}{3}}u), \quad (2.26)$$

where Ai is the Airy function and Bi is another, linearly independent, solution to (2.6) (for the precise definitions of Ai and Bi , see [AS70, 10.4.1–10.4.3]). Hence, both y_1 and y_2 solve $\mathcal{K}^*y = 0$, y_1 satisfies the boundary condition at zero ($y_1(0) = 0$), while y_2 satisfies the boundary condition at infinity ($y_2 \in L^2$). Let $G: [0, \infty)^2 \rightarrow \mathbb{R}$ (Green function) be defined by

$$G(u, v) = Ky_1(u \wedge v)y_2(u \vee v) \quad \text{with} \quad K = -2y_1'(0)y_2(0). \quad (2.27)$$

Let Γ be the operator on L^2 defined by

$$(\Gamma y)(u) = \int_0^\infty G(u, v)y(v) dv. \quad (2.28)$$

According to [G81, Proposition 2.15], $x = \Gamma y$ is a weak solution of the equation $\mathcal{K}^*x = y$ with boundary condition $x(0) = 0$, for any $y \in L^2$. In fact, we can adapt the proof of [G81, Proposition 9.12] to see that Γ is the inverse of \mathcal{K}^* , since $\mathcal{K}^*x = 0$ does not have solutions in L^2 that satisfy the boundary condition $x(0) = 0$. Hence, we are done once we show that Γ is a compact operator.

3. By [G81, Theorem 8.54], it suffices to show that Γ is a Hilbert-Schmidt operator, i.e., G is square-integrable on $[0, \infty)^2$. In order to show this, we first note that (2.27) gives

$$\int_0^\infty du \int_0^\infty dv G^2(u, v) = 2K^2 \int_0^\infty du \int_0^u dv y_2(u)^2 y_1(v)^2. \quad (2.29)$$

Substitute (2.26) to see that, since $\text{Ai} \in L^2$, it suffices to show that

$$\int_0^\infty du \int_0^u dv \text{Ai}(u)^2 \text{Bi}(v)^2 < \infty. \quad (2.30)$$

Since Bi is locally bounded and $\text{Ai} \in L^2$, the latter amounts to

$$\int_1^\infty du \int_1^u dv \text{Ai}(u)^2 \text{Bi}(v)^2 < \infty. \quad (2.31)$$

We next use [AS70, 10.4.59 and 10.4.63], which shows that

$$\text{Ai}(u) \leq cu^{-\frac{1}{4}} e^{-\frac{2}{3}u^{\frac{3}{2}}}, \quad \text{Bi}(v) \leq cv^{-\frac{1}{4}} e^{\frac{2}{3}v^{\frac{3}{2}}}, \quad u, v \geq 1. \quad (2.32)$$

Hence

$$\int_1^\infty du \int_1^u dv \text{Ai}(u)^2 \text{Bi}(v)^2 \leq c^4 \int_1^\infty du u^{-\frac{1}{2}} \int_1^u dv v^{-\frac{1}{2}} e^{-\frac{4}{3}(u^{\frac{3}{2}} - v^{\frac{3}{2}})}. \quad (2.33)$$

Use partial integration to see that

$$\int_1^u dv v^{-\frac{1}{2}} e^{-\frac{4}{3}(u^{\frac{3}{2}} - v^{\frac{3}{2}})} = \frac{1}{2} \int_1^u dv v^{-1} \frac{d}{dv} \left(e^{-\frac{4}{3}(u^{\frac{3}{2}} - v^{\frac{3}{2}})} \right) \leq \frac{1}{2} \left[v^{-1} e^{-\frac{4}{3}(u^{\frac{3}{2}} - v^{\frac{3}{2}})} \right]_{v=1}^u \leq \frac{1}{2} u^{-1}, \quad u \geq 1. \quad (2.34)$$

Hence

$$\int_1^\infty du \int_1^u dv \text{Ai}(u)^2 \text{Bi}(v)^2 \leq \frac{1}{2} c^4 \int_1^\infty du u^{-\frac{3}{2}} < \infty. \quad (2.35)$$

This proves that Γ is a compact operator, so that $(e_k)_{k \in \mathbb{N}_0}$ is an orthonormal basis of L^2 .

4. To prove the expansion in (2.20), we now need the following:

Lemma 2.3. *For any $\varepsilon > 0$, the function w is a solution of the initial-boundary-value problem*

$$\begin{aligned} \partial_t w(h, t) &= \mathcal{K}^*(w(\cdot, t))(h), & h \geq 0, t > \varepsilon, \\ w(0, t) &\equiv 0, & t \geq \varepsilon, \end{aligned} \quad (2.36)$$

and the initial value $w(\cdot, \varepsilon)$ lies in C_0^∞ .

Proof. Use the Markov property at time $s > 0$ in (2.18) to see that, for any $h > 0$ and $t > s$,

$$w(h, t) = E_h \left(e^{-\int_0^s 2B_v dv} \mathbb{1}_{\{T_0 > s\}} w(2B_s, t - s) \right). \quad (2.37)$$

Now differentiate with respect to s at $s = 0$, to obtain

$$0 = -hw(h, t) + 2(\partial_h)^2 w(h, t) - \partial_t w(h, t) = \mathcal{K}^*(w(\cdot, t))(h) - \partial_t w(h, t). \quad (2.38)$$

This shows that the partial differential equation in (2.36) is satisfied on $(0, \infty)^2$. It is clear that it is also satisfied at the boundary where $h = 0$, since $w(0, t) = 0$ for all $t > 0$ (recall (2.17–2.18)). \square

5. From (2.18) it follows that $w(\cdot, \varepsilon) \in C_0^\infty$ for any $\varepsilon > 0$. A spectral decomposition in terms of the eigenvalues $(a^{(k)})_{k \in \mathbb{N}_0}$ and the eigenfunctions $(e_k)_{k \in \mathbb{N}_0}$ of \mathcal{K}^* shows that (2.36) has the solution given in (2.20).

(ii) In [AS70, 10.4.94, 10.4.96, 10.4.97, 10.4.105] the following asymptotics for the Airy function can be found. As $k \rightarrow \infty$,

$$-a_k \sim ck^{\frac{2}{3}}, \quad a_{k-1} - a_k \sim ck^{-\frac{1}{3}}, \quad \max_{[a_k, a_{k-1}]} |\text{Ai}| \sim ck^{-\frac{1}{6}}, \quad |\text{Ai}'(a_k)| \sim ck^{\frac{1}{6}}. \quad (2.39)$$

We will use these in combination with the observation that, by (2.6), Ai is convex (concave) between any two successive zeroes where it is negative (positive).

The first assertion in (2.39) is (2.22). To prove (2.23–2.24), we write the recursion

$$c_k^{-2} = \int_0^\infty \text{Ai}(2^{-\frac{1}{3}}(h + a^{(k)}))^2 dh = c_{k-1}^{-2} + 2^{\frac{1}{3}} \int_{a_k}^{a_{k-1}} \text{Ai}(h)^2 dh. \quad (2.40)$$

Using the second and third assertion in (2.39), we find that $\int_{a_k}^{a_{k-1}} \text{Ai}(h)^2 dh \asymp k^{-\frac{2}{3}}$ and hence that $c_k^{-2} \asymp k^{\frac{1}{3}}$. In a similar way, we find that

$$\int_0^\infty h \text{Ai}(2^{-\frac{1}{3}}(h + a^{(k)}))^2 dh \leq ck, \quad \int_0^\infty \frac{1}{h} \text{Ai}(2^{-\frac{1}{3}}(h + a^{(k)}))^2 dh \leq ck^{\frac{2}{3}}. \quad (2.41)$$

Combining (2.41) with (2.21) and $c_k^{-2} \asymp k^{\frac{1}{3}}$, we obtain (2.23–2.24). \square

3. TWO BASIC PROPOSITIONS

In this section we present the basic tools of our proofs. Section 3.1 introduces the Ray-Knight Theorems, which give a flexible representation for the probabilities of certain events under the Edwards measure. Section 3.2 exhibits an integrable majorant under which limits may be interchanged with integrals.

3.1 Ray-Knight representation

In this section we formulate the Ray-Knight Theorems that were already announced in Section 2.2. We do this in the compact form derived in [vdHdHK97, Section 1.2], which is best suited for the arguments in the sequel.

For any measurable set $G \subset C_0$, define $w_G: [0, \infty) \times [0, \infty) \rightarrow \mathbb{R}$ by

$$w_G(h, t) dt = \mathbb{E}_h^* \left(e^{-\int_0^\infty (X_v^*)^2 dv} \mathbb{1}_{\{X^* \in G\}} \mathbb{1}_{\{A^*(\infty) \in dt\}} \right). \quad (3.1)$$

It is clear that w_G is increasing in G . For $G = C_0$, w_{C_0} is identical to w defined in (2.17).

For $y \geq 0$, denote by $C^+[0, y]$ the set of non-negative continuous functions on $[0, y]$. Then the set $\mathcal{C}^+ = \bigcup_{y \geq 0} (\{y\} \times C^+[0, y])$ is the appropriate state space of the pair $(B_T, L(T, B_T - \cdot)|_{[0, B_T]})$ consisting of the endpoint $B_T (\geq 0)$ and the local time process between the endpoint B_T and the starting point 0.

Proposition 3.1 (Ray-Knight representation). *Fix $a \in \mathbb{R}$. Then, for any $T > 0$ and any measurable sets $G^+, G^- \subset C_0$ and $F \subset \mathcal{C}$,*

$$\begin{aligned} & e^{aT} E \left(e^{-HT} e^{-\rho(a)B_T} \mathbb{1}_{\{L(T, B_T + \cdot) \in G^+\}} \mathbb{1}_{\{(B_T, L(T, B_T - \cdot)|_{[0, B_T]}) \in F\}} \mathbb{1}_{\{L(T, - \cdot) \in G^-\}} \right) \\ &= \int_0^\infty dt_1 \int_0^\infty dt_2 \mathbb{1}_{\{t_1 + t_2 \leq T\}} e^{a(t_1 + t_2)} \\ & \quad \times \widehat{\mathbb{E}}^a \left(\mathbb{1}_{\{(A^{-1}(T - t_1 - t_2), X|_{[0, A^{-1}(T - t_1 - t_2)])} \in F\}} \frac{w_{G^+}(X_0, t_1)}{x_a(X_0)} \frac{w_{G^-}(Y_{T-t_1-t_2}, t_2)}{x_a(Y_{T-t_1-t_2})} \right). \end{aligned} \quad (3.2)$$

Proof. We briefly indicate how (3.2) comes about. Details can be found in [vdHdHK97, Section 1.2]. Recall the notation in Section 2.2. Fix $T > 0$. Then, according to the Ray-Knight Theorems, for any $t_1, t_2, h_1, h_2 \geq 0$ and $y > 0$, conditioned on the event

$$\{B_T = y\} \cap \{L(T, B_T) = h_1\} \cap \{L(T, 0) = h_2\} \cap \left\{ \int_{B_T}^\infty L(T, x) dx = t_1 \right\} \cap \left\{ \int_0^\infty L(T, -x) dx = t_2 \right\}, \quad (3.3)$$

the joint distribution of the processes

$$L(T, B_T + \cdot), \quad L(B_T - \cdot)|_{[0,y]}, \quad L(T, - \cdot), \quad (3.4)$$

on $C_0^+ \times C^+[0, y] \times C_0^+$ is equal to the joint distribution of the processes

$$X^{*,1}(\cdot), \quad X(\cdot)|_{[0,y]}, \quad X^{*,2}(\cdot), \quad (3.5)$$

under

$$\mathbb{P}_{h_1}^*(\cdot | A^*(\infty) = t_1) \otimes \mathbb{P}_{h_1}(\cdot | A(y) = T - t_1 - t_2, X_y = h_2) \otimes \mathbb{P}_{h_2}^*(\cdot | A^*(\infty) = t_2), \quad (3.6)$$

where X is BESQ² and $X^{*,1}$, $X^{*,2}$ are independent copies of BESQ⁰. In particular, the intersection local time in (1.2) has the representation

$$H_T \stackrel{\text{law}}{=} \int_0^\infty (X_v^{*,1})^2 dv + \int_0^y (X_v)^2 dv + \int_0^\infty (X_v^{*,2})^2 dv. \quad (3.7)$$

Use (2.10) for $y = A^{-1}(T - t_1 - t_2)$ and note that, on the event $\{A(T - t_1 - t_2) = y\} \cap \{X_0 = h_1, X_y = h_2\}$, (2.9) becomes

$$D_y^{(a)} = \frac{x_a(h_2)}{x_a(h_1)} \exp\left\{-\int_0^y (X_v)^2 dv\right\} e^{a(T-t_1-t_2)} e^{-\rho(a)y}, \quad (3.8)$$

which implies that

$$e^{aT} e^{-H_T} e^{-\rho(a)B_T} \stackrel{\text{law}}{=} \frac{x_a(h_1)}{x_a(h_2)} D_y^{(a)} e^{a(t_1+t_2)} e^{-\int_0^\infty (X_v^{*,1})^2 dv} e^{-\int_0^\infty (X_v^{*,2})^2 dv}. \quad (3.9)$$

Integrate the left-hand side with respect to P and the right-hand side with respect to the measure in (3.6), and absorb the term $D_y^{(a)}$ into the notation of the transformed diffusion. Integrate over $h_1, h_2 \geq 0$ and note that X_0 has the distribution $x_a(h_1)^2 dh_1$ under $\widehat{\mathbb{E}}^a$. Finally, use the notation in (3.1), to obtain (3.2). \square

3.2 Domination

In order to perform the limit $T \rightarrow \infty$ on the right-hand side of (3.2), we will need the dominated convergence theorem to interchange this limit with the integrals over t_1 and t_2 . The following proposition provides the required domination.

Proposition 3.2 (Domination). *For any a_s , $s \geq 0$, in a compact subset of $(-\infty, a^{**})$, the map*

$$(t_1, t_2) \mapsto \sup_{s \geq 0} e^{a_s(t_1+t_2)} \widehat{\mathbb{E}}^{a_s} \left(\frac{w(X_0, t_1)}{x_{a_s}(X_0)} \frac{w(Y_s, t_2)}{x_{a_s}(Y_s)} \right) \quad (3.10)$$

is integrable over $(0, \infty)^2$.

Proof. Under the expectation in (3.10) we make a change of measure from the invariant distribution of X to the invariant distribution of Y , i.e., we replace $\widehat{\mathbb{E}}^{a_s}$ by $\widetilde{\mathbb{E}}^{a_s}$ and add a factor of $\rho'(a_s)/Y_0$. Fix $1 < p \leq q < \infty$ such that $\frac{1}{p} + \frac{1}{q} = 1$, apply Hölder's inequality and use the stationarity of Y under $\widetilde{\mathbb{P}}^{a_s}$. This gives, for any $t_1, t_2 > 0$, the bound

$$\widetilde{\mathbb{E}}^{a_s} \left(\frac{w(Y_0, t_1)}{Y_0 x_{a_s}(Y_0)} \frac{w(Y_s, t_2)}{x_{a_s}(Y_s)} \right) \leq W_p^{(1)}(t_1) W_q^{(2)}(t_2), \quad (3.11)$$

where the functions $W_p^{(1)}, W_q^{(2)}: (0, \infty) \rightarrow (0, \infty)$ are defined by

$$W_p^{(1)}(t) = \widetilde{\mathbb{E}}^{a_s} \left(\left(\frac{w(Y_0, t)}{Y_0 x_{a_s}(Y_0)} \right)^p \right)^{\frac{1}{p}}, \quad W_q^{(2)}(t) = \widetilde{\mathbb{E}}^{a_s} \left(\left(\frac{w(Y_0, t)}{x_{a_s}(Y_0)} \right)^q \right)^{\frac{1}{q}}. \quad (3.12)$$

Hence, it suffices to show that the maps

$$t \mapsto e^{a_s t} W_p^{(1)}(t), \quad t \mapsto e^{a_s t} W_q^{(2)}(t), \quad (3.13)$$

are integrable at zero and at infinity, uniformly in s , for a suitable choice of p and q . In the proof of Proposition 4 in [vdHdHK97] we showed that $W_p^{(1)}$ and $W_q^{(2)}$, with a_s replaced by a^* , are integrable at zero when $p < q$ with p, q sufficiently close to 2. An inspection of the proof shows that they are actually integrable at zero uniformly in s .

We will show that $t \mapsto e^{a_s t} W_2^{(1)}(t)$ and $t \mapsto e^{a_s t} W_2^{(2)}(t)$ are integrable at infinity uniformly in s . This will complete the proof because the left-hand side of (3.11) does not depend on p, q .

We use Proposition 2.2 with $\varepsilon = 1$ together with the representations (recall (2.12))

$$W_2^{(1)}(t) = \frac{1}{\sqrt{\rho'(a_s)}} \left(\int_0^\infty dh \frac{1}{h} w(h, t)^2 \right)^{\frac{1}{2}}, \quad W_2^{(2)}(t) = \frac{1}{\sqrt{\rho'(a_s)}} \left(\int_0^\infty dh h w(h, t)^2 \right)^{\frac{1}{2}}. \quad (3.14)$$

Using (2.20), the Cauchy-Schwarz inequality and the fact that $\|e_k\|_2 = 1$, we estimate

$$W_2^{(1)}(t) \leq \frac{1}{\sqrt{\rho'(a_s)}} \left(\|w(\cdot, 1)\|_2^2 \sum_{k_1, k_2=0}^\infty e^{(a^{(k_1)} + a^{(k_2)})(t-1)} \int_0^\infty \frac{1}{h} |e_{k_1}(h)| |e_{k_2}(h)| dh \right)^{\frac{1}{2}}, \quad t \geq 1. \quad (3.15)$$

Using the Cauchy-Schwarz inequality for the last integral, we obtain the bound

$$W_2^{(1)}(t) \leq \frac{\|w(\cdot, 1)\|_2}{\sqrt{\rho'(a_s)}} \sum_{k=0}^\infty e^{a^{(k)}(t-1)} \left(\int_0^\infty \frac{1}{h} e_k(h)^2 dh \right)^{\frac{1}{2}}, \quad t \geq 1. \quad (3.16)$$

In the same way, we find that

$$W_2^{(2)}(t) \leq \frac{\|w(\cdot, 1)\|_2}{\sqrt{\rho'(a_s)}} \sum_{k=0}^\infty e^{a^{(k)}(t-1)} \left(\int_0^\infty h e_k(h)^2 dh \right)^{\frac{1}{2}}, \quad t \geq 1. \quad (3.17)$$

Substitute (2.23–2.24) into (3.16–3.17) and use that $a^{(k)} \leq a^{(0)} = -a^{**}$, to estimate

$$W_2^{(1)}(t) \vee W_2^{(2)}(t) \leq c e^{-a^{**}(t-2)} \sum_{k=0}^\infty e^{a^{(k)}(t-1)} k^{\frac{1}{3}}, \quad t \geq 2. \quad (3.18)$$

By (2.22), the sum in the right-hand side converges. Since $a_s < a^{**}$, $s \geq 0$, is bounded away from a^{**} , it is now obvious that the maps $t \mapsto e^{a_s t} W_2^{(1)}(t)$ and $t \mapsto e^{a_s t} W_2^{(2)}(t)$ are integrable at infinity uniformly in s . \square

4. PROOF OF THEOREMS 1.2–1.3

In Sections 4.2–4.3 we give the proof of Theorems 1.2–1.3 with the help of Propositions 3.1–3.2. In Section 4.1 we derive a technical proposition that is needed along the way.

4.1 Growth rate of a restricted moment generating function

Abbreviate $B_{[0, T]} = \{B_t : t \in [0, T]\}$ for the range of the path up to time T . For $T > 0$ and $\delta, C \in (0, \infty]$, define events

$$\mathcal{E}(\delta; T) = \{B_{[0, T]} \subset [-\delta, B_T + \delta]\}, \quad (4.1)$$

$$\mathcal{E}^{\leq}(\delta, C; T) = \left\{ \max_{x \in [-\delta, \delta]} L(T, x) \leq C, \max_{x \in [B_T - \delta, B_T + \delta]} L(T, x) \leq C \right\}. \quad (4.2)$$

In words, on $\mathcal{E}(\delta; T)$ the path does not visit more than the δ -neighborhood of the interval between its starting point 0 and its endpoint B_T , while on $\mathcal{E}^{\leq}(\delta, C; T)$ its local times in the δ -neighborhoods of these two points are bounded by C . Note that both $\mathcal{E}(\infty; T)$ and $\mathcal{E}^{\leq}(\delta, \infty; T)$ are the full space.

Proposition 4.1. Fix $\mu > -\rho(a^{**})$. Then, for any $\delta, C \in (0, \infty]$ there exists a constant $K_1(\delta, C) \in (0, \infty)$ such that, for any $\mu_T \rightarrow \mu$ as $T \rightarrow \infty$,

$$e^{\rho^{-1}(-\mu_T)T} E\left(e^{-H_T} e^{\mu_T B_T} \mathbb{1}_{\mathcal{E}(\delta, T)} \mathbb{1}_{\mathcal{E}^{\leq}(\delta, C; T)} \mathbb{1}_{\{B_T \geq 0\}}\right) = K_1(\delta, C) + o(1). \quad (4.3)$$

Moreover, if $\mu = \mu_b$ solves $I(b) = \mu b - \Lambda^+(\mu)$, then the same is true when $\mathbb{1}_{\{B_T \geq 0\}}$ is replaced by $\mathbb{1}_{\{B_T \approx bT\}}$.

Proof. We may assume that $\mu_T > -\rho(a^{**})$ for all T . Fix $\delta, C \in (0, \infty]$ and choose a_T such that $\mu_T + \rho(a_T) = 0$, i.e., $a_T = \rho^{-1}(-\mu_T) < a^{**}$. Clearly, $\lim_{T \rightarrow \infty} a_T = \rho^{-1}(-\mu) < a^{**}$. Since, on $\mathcal{E}(\delta; T) \cap \{B_T \leq 2\delta\}$, we can estimate

$$H_T = 4\delta \int_{-\delta}^{3\delta} \frac{dx}{4\delta} L(T, x)^2 \geq 4\delta \left(\int_{-\delta}^{3\delta} \frac{dx}{4\delta} L(T, x) \right)^2 = \frac{T^2}{4\delta}, \quad (4.4)$$

we may insert the indicator of $\{B_T \geq 2\delta\}$ in the expectation on the left-hand side of (4.3), paying only a factor $1 + o(1)$ as $T \rightarrow \infty$.

1. Introduce the following subsets of C_0^+ , respectively, \mathcal{C}^+ (see below (3.1)):

$$G_{\delta, C}^{\leq} = \{g \in C_0^+ : g(\delta) = 0, \max g \leq C\}, \quad (4.5)$$

$$F_{\delta, C}^{\leq} = \left\{ (y, f) \in \mathcal{C}^+ : y \geq 2\delta, \max_{[0, \delta]} f \leq C, \max_{[y-\delta, y]} f \leq C \right\}. \quad (4.6)$$

Note that

$$\begin{aligned} & \mathcal{E}(\delta; T) \cap \mathcal{E}^{\leq}(\delta, C; T) \cap \{B_T \geq 2\delta\} \\ &= \{L(T, B_T + \cdot) \in G_{\delta, C}^{\leq}\} \cap \{L(T, -\cdot) \in G_{\delta, C}^{\leq}\} \cap \{(B_T, L(T, B_T - \cdot)|_{[0, B_T]}) \in F_{\delta, C}^{\leq}\}. \end{aligned} \quad (4.7)$$

Apply Proposition 3.1 for $a = a_T$ with $F = F_{\delta, C}^{\leq}$ and $G^+ = G^- = G_{\delta, C}^{\leq}$, to get

$$\begin{aligned} \text{l.h.s. of (4.3)} &= (1 + o(1)) \int_0^\infty dt_1 \int_0^\infty dt_2 \mathbb{1}_{\{t_1+t_2 \leq T\}} e^{a_T(t_1+t_2)} \\ &\times \widehat{\mathbb{E}}^{a_T} \left(\frac{w_{G_{\delta, C}^{\leq}}(X_0, t_1)}{x_{a_T}(X_0)} \mathbb{1}_{\{A^{-1}(T-t_1-t_2) \geq 2\delta\}} \mathbb{1}_{\{\max_{[0, \delta]} X \leq C\}} \right. \\ &\left. \times \mathbb{1}_{\{\max_{[A^{-1}(T-t_1-t_2)-\delta, A^{-1}(T-t_1-t_2)]} X \leq C\}} \frac{w_{G_{\delta, C}^{\leq}}(Y_{T-t_1-t_2}, t_2)}{x_{a_T}(Y_{T-t_1-t_2})} \right). \end{aligned} \quad (4.8)$$

2. In the case $C = \infty$, the last two indicators vanish and we can identify the limit of the integrand as $T \rightarrow \infty$ with the help of Lemma 2.1. Indeed, apply Lemma 2.1 for $f(\cdot) = w_{G_\delta}(\cdot, t_1)$ and $g(\cdot) = w_{G_\delta}(\cdot, t_2)$, where we put $G_\delta = G_{\delta, \infty}^{\leq} = \{g \in C_0^+ : g(\delta) = 0\}$. Then we obtain that the integrand converges to

$$e^{a(t_1+t_2)} \langle w_{G_\delta}(\cdot, t_1), x_a \rangle \frac{1}{\rho'(a)} \langle w_{G_\delta}(\cdot, t_2), x_a \rangle, \quad (4.9)$$

where we also use that $A^{-1}(\infty) = \infty$ because X never hits 0 (recall (2.7)). According to Proposition 3.2, we are allowed to interchange the limit $T \rightarrow \infty$ with the two integrals over t_1 and t_2 . This implies that (4.3) holds with $K_1(\delta, \infty)$ identified as

$$K_1(\delta, \infty) = \langle y_a^{(\delta)}, x_a \rangle \frac{1}{\rho'(a)} \langle y_a^{(\delta)}, x_a \rangle, \quad (4.10)$$

where $y_a^{(\delta)}(h)$ is defined as (recall (3.1))

$$y_a^{(\delta)}(h) = \int_0^\infty dt e^{at} w_{G_\delta}(h, t) = \mathbb{E}_h^* \left(e^{\int_0^\infty [aX_v^* - (X_v^*)^2] dv} \mathbb{1}_{\{X_\delta^* = 0\}} \right). \quad (4.11)$$

Trivially, $K_1(\delta, \infty) > 0$. Since $y_a^{(\delta)} \leq y_a$, it follows from (2.2) and (2.16) that $K_1(\delta, \infty) < \infty$.

3. Next we return to (4.8) and consider the case $C \in (0, \infty)$. Note that the integrals over t_1 and t_2 can both be restricted to $[0, C\delta]$, since $w_{G_{\delta,C}^{\leq}}(h, t) = 0$ for $t > C\delta$ as is seen from (3.1) and (4.5).

Let us abbreviate $s = T - t_1 - t_2$. We first apply the Markov property for the process X at time δ and integrate over all values $z = A(\delta)$. Because of the appearance of the indicator of $\{\max_{[0,\delta]} X \leq C\}$, we may restrict to $z \in [0, C\delta]$ (recall (2.7)). We note that the additive functional of the process $(X_{\delta+t})_{t \geq 0}$ given that $A(\delta) = z$, denoted by $\tilde{A} = (\tilde{A}(t))_{t \geq 0}$, is given by $\tilde{A}(t) = A(t + \delta) - z$. Making the change of variables $s = \tilde{A}(t) + z$, we see that $A^{-1}(s) = \tilde{A}^{-1}(s - z) + \delta$ for any $s \geq 0$. Defining $f_{s,T}^{t_1}: (0, \infty)^2 \rightarrow [0, \infty)$ by

$$f_{s,T}^{t_1}(h, z) dh dz = x_{a_T}(h) \widehat{\mathbb{E}}^{a_T} \left(\frac{w_{G_{\delta,C}^{\leq}}(X_0, t_1)}{x_{a_T}(X_0)} \mathbb{1}_{\{A^{-1}(s) \geq 2\delta\}} \mathbb{1}_{\{\max_{[0,\delta]} X \leq C\}} \mathbb{1}_{\{X_\delta \in dh\}} \mathbb{1}_{\{A(\delta) \in dz\}} \right), \quad (4.12)$$

we thus obtain that the expectation under the integral in (4.8) can be written as

$$\begin{aligned} & \widehat{\mathbb{E}}^{a_T} \left(\frac{w_{G_{\delta,C}^{\leq}}(X_0, t_1)}{x_{a_T}(X_0)} \mathbb{1}_{\{A^{-1}(s) \geq 2\delta\}} \mathbb{1}_{\{\max_{[0,\delta]} X \leq C\}} \mathbb{1}_{\{\max_{[A^{-1}(s)-\delta, A^{-1}(s)]} X \leq C\}} \frac{w_{G_{\delta,C}^{\leq}}(Y_s, t_2)}{x_{a_T}(Y_s)} \right) \\ &= \int_0^{C\delta} dz \widehat{\mathbb{E}}^{a_T} \left(\frac{f_{s,T}^{t_1}(X_0, z)}{x_{a_T}(X_0)} \mathbb{1}_{\{\max_{[A^{-1}(s-z), A^{-1}(s-z)+\delta]} X \leq C\}} \frac{w_{G_{\delta,C}^{\leq}}(X_{A^{-1}(s-z)+\delta}, t_2)}{x_{a_T}(X_{A^{-1}(s-z)+\delta})} \right). \end{aligned} \quad (4.13)$$

(The tilde can be removed afterwards.) We next apply the Markov property for the process Y at time $s - z$ (respectively, the strong Markov property for the process X at time $A^{-1}(s - z)$), to write

$$\text{r.h.s. of (4.13)} = \int_0^{C\delta} dz \widehat{\mathbb{E}}^{a_T} \left(\frac{f_{s,T}^{t_1}(X_0, z)}{x_{a_T}(X_0)} \frac{g_T^{t_2}(Y_{s-z})}{x_{a_T}(Y_{s-z})} \right), \quad (4.14)$$

where $g_T^{t_2}$ is defined by

$$g_T^{t_2}(h) = x_{a_T}(h) \widehat{\mathbb{E}}_h^{a_T} \left(\mathbb{1}_{\{\max_{[0,\delta]} X \leq C\}} \frac{w_{G_{\delta,C}^{\leq}}(X_\delta, t_2)}{x_{a_T}(X_\delta)} \right). \quad (4.15)$$

4. We want to take the limit $s \rightarrow \infty$ in (4.14) (recall that $s = T - t_1 - t_2$) and use Proposition 2.1. Therefore we need dominated convergence. To establish this, we note that

$$\sup_{h \in [0, C]} \sup_{t \in [0, C\delta]} \sup_{T \geq 1} \frac{w(h, t)}{x_{a_T}(h)} = K < \infty \quad (4.16)$$

(see (2.17–2.19) and recall that x_a is bounded away from zero on $[0, C]$ and continuous in a). By (4.15–4.16), the last quotient in the right-hand side of (4.14) is bounded above by K . Substituting (4.12) into (4.14) and using that $w_{G_{\delta,C}^{\leq}} \leq w_{C_0^+} = w$, we therefore obtain

$$\begin{aligned} \text{integrand of r.h.s. of (4.14)} &\leq K \widehat{\mathbb{E}}^{a_T} \left(\frac{f_{s,T}^{t_1}(X_0, z)}{x_{a_T}(X_0)} \right) \\ &\leq K \frac{\widehat{\mathbb{E}}^{a_T} \left(\frac{w(X_0, t_1)}{x_{a_T}(X_0)} \mathbb{1}_{\{\max_{[0,\delta]} X \leq C\}} \mathbb{1}_{\{A(\delta) \in dz\}} \right)}{dz} \\ &\leq K^2 \frac{\widehat{\mathbb{P}}^{a_T}(A(\delta) \in dz)}{dz}. \end{aligned} \quad (4.17)$$

It is easy to see from (2.9) that the right-hand side of (4.17) is bounded uniformly in $T \geq 1$ and $z \in [0, C\delta]$. Therefore we have an integrable majorant for (4.14), which allows us to interchange the limit $s \rightarrow \infty$ with the integral over z .

5. In order to identify the limit as $s \rightarrow \infty$ of the integrand on the right-hand side of (4.14), we apply Lemma 2.1 to see that this integrand converges to $\langle f^{t_1}(\cdot, z), x_a(\cdot) \rangle \frac{1}{\rho'(a)} \langle g^{t_2}, x_a \rangle_\circ$, with f^{t_1} and g^{t_2} the pointwise limit of $f_{s,T}^{t_1}$ and $g_T^{t_2}$, respectively:

$$f^{t_1}(h, z) dh dz = x_a(h) \widehat{\mathbb{E}}^a \left(\frac{w_{G_{\delta,C}^{\leq}}(X_0, t_1)}{x_a(X_0)} \mathbb{1}_{\{\max_{[0,\delta]} X \leq C\}} \mathbb{1}_{\{X_\delta \in dh\}} \mathbb{1}_{\{A(\delta) \in dz\}} \right) \quad (4.18)$$

$$g^{t_2}(h) dh = x_a(h) \widehat{\mathbb{E}}_h^a \left(\mathbb{1}_{\{\max_{[0,\delta]} X \leq C\}} \frac{w_{G_{\delta,C}^{\leq}}(X_0, t_2)}{x_a(X_\delta)} \right). \quad (4.19)$$

Using this in (4.14) and interchanging the integral over z with the limit $s \rightarrow \infty$, we obtain that

$$\lim_{s \rightarrow \infty} (\text{l.h.s. of (4.13)}) = \langle f^{t_1}, x_a \rangle \frac{1}{\rho'(a)} \langle g^{t_2}, x_a \rangle_\circ \quad (4.20)$$

with $f^{t_1}(h) = \int_0^{C_\delta} dz f^{t_1}(h, z)$.

6. Finally, recall that $s = T - t_1 - t_2$ and that $e^{a(t_1+t_2)}$ times the left-hand side of (4.13) is equal to the integrand on the right-hand side of (4.8). According to Proposition 3.2, we are allowed to interchange the limit $T \rightarrow \infty$ with the two integrals over t_1 and t_2 . Hence we obtain that (4.3) holds with $K_1(\delta, C)$ identified as the integral over t_1, t_2 of the right-hand side of (4.20), which is a strictly positive finite number. This proves the statement with the indicator on $\mathbb{1}_{\{B_T \geq 0\}}$.

7. To prove the statement with $\mathbb{1}_{\{B_T \geq 0\}}$ replaced by $\mathbb{1}_{\{B_T \approx bT\}}$, we let $\mu = \mu_b$ solve $I(b) = \mu b - \Lambda^+(\mu)$. The statement follows when we show that for every $a \in \mathbb{R}$, we have that

$$e^{\rho^{-1}(-\mu)T} E \left(e^{a \frac{B_T - bT}{\sqrt{T}}} e^{-H_T} e^{\mu B_T} \mathbb{1}_{\mathcal{E}(\delta, T)} \mathbb{1}_{\mathcal{E} \leq (\delta, C; T)} \mathbb{1}_{\{B_T \geq 0\}} \right) = e^{\frac{a^2}{2} \sigma_b^2} K_1(\delta, C) + o(1) \quad (4.21)$$

for some $\sigma_b^2 \in (0, \infty)$. Indeed, (4.21) shows that $\mathbb{1}_{\{|B_T - bT| > \gamma_T, B_T \geq 0\}}$ is asymptotically negligible for any γ_T such that $\gamma_T / \sqrt{T} \rightarrow \infty$.

In order to prove (4.21), we rewrite the left-hand side as

$$e^{[\rho^{-1}(-\mu) - \rho^{-1}(-\mu_{a,T})]T - ab\sqrt{T}} e^{\rho^{-1}(-\mu_{a,T})T} E \left(e^{-H_T} e^{\mu_{a,T} B_T} \mathbb{1}_{\mathcal{E}(\delta, T)} \mathbb{1}_{\mathcal{E} \leq (\delta, C; T)} \mathbb{1}_{\{B_T \geq 0\}} \right), \quad (4.22)$$

where $\mu_{a,T} = \mu + \frac{a}{\sqrt{T}}$. Clearly, $\mu_{a,T} \rightarrow \mu$, so that the second factor converges to $K_1(\delta, C)$. We are therefore left to compute the exponential. We note that since $\mu = \mu_b$ solves $I(b) = \mu b - \Lambda^+(\mu)$, we have that $\rho'(-\mu_b) = 1/b$. Therefore,

$$\rho^{-1}(-\mu_{a,T}) = \rho^{-1}(-\mu) - \frac{a}{\sqrt{T}} \frac{1}{\rho'(-\mu)} + \frac{a^2}{2T} \frac{d^2}{d\mu^2} \rho^{-1}(-\mu) + o(T^{-1}). \quad (4.23)$$

Therefore,

$$e^{[\rho^{-1}(-\mu) - \rho^{-1}(-\mu_{a,T})]T - ab\sqrt{T}} = e^{\frac{a^2}{2} \frac{d^2}{d\mu^2} \rho^{-1}(-\mu)} (1 + o(1)), \quad (4.24)$$

which completes the proof with $\sigma_b^2 = -\frac{d^2}{d\mu^2} \rho^{-1}(-\mu_b)$. \square

4.2 Proof of Theorem 1.3(i–iii)

STEP 1. For any $\mu > -\rho(a^{**})$, the limit in (1.7) exists and equals $\Lambda^+(\mu) = -\rho^{-1}(-\mu)$. On $(-\rho(a^{**}), \infty)$, the function Λ^+ is real-analytic and strictly convex, and satisfies $\lim_{\mu \downarrow -\rho(a^{**})} (\Lambda^+)'(\mu) = b^{**}$.

Proof. Fix $\mu > -\rho(a^{**})$, apply Proposition 4.1 with $\delta = C = \infty$, and use the continuity of ρ , to obtain that the limit in the definition of $\Lambda^+(\mu)$ in (1.7) exists and equals $-\rho^{-1}(-\mu)$. This proves the first assertion. The remaining assertions follow from (2.3–2.5). \square

In the following step, we consider paths that never go below $-\delta$, have local times that are bounded by C in the δ -neighborhood of the starting point 0, and have the endpoint B_T close to 0. Recall that γ_T is a function that satisfies $\gamma_T/T \rightarrow 0$ and $\gamma_T/\sqrt{T} \rightarrow \infty$ as $T \rightarrow \infty$.

STEP 2. For any $\delta \in (0, \infty)$ and $C \in (0, \infty]$,

$$E\left(e^{-HT} \mathbb{1}_{\{B_T \in [0, \gamma_T]\}} \mathbb{1}_{\{\min_{[0, T]} B \geq -\delta\}} \mathbb{1}_{\{\max_{[-\delta, \delta]} L(T, \cdot) \leq C\}}\right) \geq e^{-a^{**}T + o(T)}, \quad T \rightarrow \infty. \quad (4.25)$$

Proof. Pick $a = a^{**}$ and apply Proposition 3.1 for

$$F = F_{\delta, C} = \{(y, f) \in \mathcal{C}^+ : y \leq \delta, \max_{[y-\delta, y]} f \leq C\}, \quad G^+ = C_0^+, \quad G^- = G_{\delta, C}^{\leq} \quad (4.26)$$

(recall (4.5)). Note that the event under the expectation on the left-hand side of (4.25) contains the event

$$\{L(T, B_T + \cdot) \in C_0^+\} \cap \{(B_T, L(T, B_T - \cdot)) \in F_{\delta, C}\} \cap \{L(T, -\cdot) \in G_{\delta, C}^{\leq}\}. \quad (4.27)$$

Also note that $e^{-\rho(a^{**})B_T} \leq 1$ when $B_T \geq 0$ because $\rho(a^{**}) > 0$. Therefore we find

$$\begin{aligned} \text{l.h.s. of (4.25)} &\geq \int_0^\infty dt_1 \int_0^\infty dt_2 \mathbb{1}_{\{t_1+t_2 \leq T\}} e^{-a^{**}s} \\ &\quad \times \widehat{\mathbb{E}}^{a^{**}} \left(\frac{w(X_0, t_1)}{x_{a^{**}}(X_0)} \mathbb{1}_{\{A^{-1}(s) \leq \delta\}} \mathbb{1}_{\{\max_{[A^{-1}(s)-\delta, A^{-1}(s)]} X \leq C\}} \frac{w_{G_{\delta, C}^{\leq}}(Y_s, t_2)}{x_{a^{**}}(Y_s)} \right), \end{aligned} \quad (4.28)$$

where we again abbreviate $s = T - t_1 - t_2$. Next we interchange the two integrals, restrict the t_2 -integral to $[0, \delta]$ and the t_1 -integral to $[T - t_2 - \delta, T - t_2]$, estimate $A^{-1}(s) \leq A^{-1}(\delta)$ for $s \leq \delta$, and integrate over $s = T - t_1 - t_2$, to get

$$\begin{aligned} &\text{l.h.s. of (4.25)} \\ &\geq \int_0^\delta dt_2 \int_0^\delta ds \widehat{\mathbb{E}}^{a^{**}} \left(\frac{w(X_0, T - t_2 - s)}{x_{a^{**}}(X_0)} \mathbb{1}_{\{A^{-1}(\delta) \leq \delta\}} \mathbb{1}_{\{\max_{[0, \delta]} X \leq C\}} \frac{w_{G_{\delta, C}^{\leq}}(Y_s, t_2)}{x_{a^{**}}(Y_s)} \right). \end{aligned} \quad (4.29)$$

Now we use Proposition 2.2(i) to estimate $w(X_0, T - s - t_2) \geq e^{-a^{**}T + o(T)}$, uniformly on the domain of integration. The remaining expectation on the right-hand side no longer depends on T and is strictly positive for any $\delta \in (0, \infty)$ and $C \in (0, \infty]$. \square

STEP 3. Λ^+ equals $-a^{**}$ on $(-\infty, -\rho(a^{**})]$.

Proof. For $\mu \leq -\rho(a^{**})$, define $\Lambda_-^+(\mu)$ and $\Lambda_+^+(\mu)$ as in (1.7) with \lim replaced by \liminf and \limsup , respectively. Since Λ_+^+ is obviously non-decreasing, we have $\Lambda_+^+(\mu) \leq \Lambda^+(-\rho(a^{**}) + \varepsilon)$ for $\mu \leq -\rho(a^{**})$ and any $\varepsilon > 0$. Using Step 1 and the continuity of ρ , we see that $\lim_{\varepsilon \downarrow 0} \Lambda^+(-\rho(a^{**}) + \varepsilon) = -\rho^{-1}(\rho(a^{**})) = -a^{**}$, which shows that $\Lambda_+^+(\mu) \leq -a^{**}$. In order to get the reversed inequality for $\Lambda_-^+(\mu)$, bound

$$E\left(e^{-HT} e^{\mu B_T} \mathbb{1}_{\{B_T \geq 0\}}\right) \geq E\left(e^{-HT} e^{\mu B_T} \mathbb{1}_{\{B_T \in [0, \gamma_T]\}}\right) \geq e^{\mu \gamma_T} E\left(e^{-HT} \mathbb{1}_{\{B_T \in [0, \gamma_T]\}}\right), \quad (4.30)$$

take logs, divide by T , let $T \rightarrow \infty$ and use Step 2, to obtain that $\Lambda_-^+(\mu) \geq -a^{**}$. Since $\Lambda_-^+ \leq \Lambda_+^+$, this implies the assertion. \square

STEP 4. $\Lambda^+(\mu) = \frac{1}{2}\mu^2 + \mathcal{O}(\mu^{-1})$ as $\mu \rightarrow \infty$.

Proof. According to Step 1, we have $\Lambda^+(\mu) = -\rho^{-1}(-\mu)$ for $\mu > -\rho(a^{**})$. Hence, in order to obtain the asymptotics for $\Lambda^+(\mu)$ as $\mu \rightarrow \infty$, we need to obtain the asymptotics for $\rho(a)$ as $a \rightarrow -\infty$. In the following we consider $a < 0$.

We use Rayleigh's Principle (see [G81, Proposition 10.10]) to write (recall (2.1))

$$\begin{aligned} \rho(a) &= \sup_{x \in L^2 \cap C^2: \|x\|_2=1} \langle \mathcal{K}^a x, x \rangle \\ &= \sup_{x \in L^2 \cap C^2: \|x\|_2=1} \int_0^\infty [-2hx'(h)^2 + (ah - h^2)x(h)^2] dh. \end{aligned} \quad (4.31)$$

Substituting $x(h) = (-a)^{\frac{1}{4}}y((-a)^{\frac{1}{2}}h)$, we get

$$\rho(a) = (-a)^{\frac{1}{2}} \sup_{y \in L^2 \cap C^2: \|y\|_2=1} \int_0^\infty [-2hy'(h)^2 - (h + h^2(-a)^{-\frac{3}{2}})y(h)^2] dh. \quad (4.32)$$

Hence, we have the upper bound $\rho(a) \leq V(-a)^{\frac{1}{2}}$ with

$$V = \sup_{y \in L^2 \cap C^2: \|y\|_2=1} \int_0^\infty [-2hy'(h)^2 - hy(h)^2] dh. \quad (4.33)$$

By completing the square under the integral and partially integrating the cross term, we easily see that $y^*(h) = \frac{1}{\sqrt{2}}e^{-h/\sqrt{2}}$ is the maximizer of (4.33) and $V = -\sqrt{2}$. Substituting y^* into (4.32), we can also bound $\rho(a)$ from below:

$$\rho(a) \geq -\sqrt{2}(-a)^{\frac{1}{2}} - (-a)^{-1} \int_0^\infty h^2 y^*(h)^2 dh. \quad (4.34)$$

Therefore,

$$\rho(a) = -\sqrt{2}(-a)^{\frac{1}{2}} + \mathcal{O}(|a|^{-1}), \quad a \rightarrow -\infty. \quad (4.35)$$

Consequently,

$$\Lambda^+(\mu) = -\rho^{-1}(-\mu) = \frac{1}{2}\mu^2 + \mathcal{O}(\mu^{-1}), \quad \mu \rightarrow \infty. \quad (4.36)$$

□

Steps 1, 3 and 4 complete the proof of Theorem 1.3(i–iii).

4.3 Proof of Theorem 1.2 and 1.3(iv)

For $b \in \mathbb{R}$, define $I_-(b)$ and $I_+(b)$ as in (1.4) with \lim replaced by \limsup and \liminf , respectively.

STEP 5. For any $b > b^{**}$, the limit in (1.4) exists and (1.8) holds.

Proof. Fix $b > b^{**}$.

1. To derive ' \geq ' in (1.8) for I_- instead of I , bound, for any $\mu \in \mathbb{R}$,

$$\begin{aligned} E(e^{-H_T} \mathbb{1}_{\{|B_T - bT| \leq \gamma_T\}}) &\leq e^{-\mu bT + |\mu| \gamma_T} E(e^{-H_T} e^{\mu B_T} \mathbb{1}_{\{|B_T - bT| \leq \gamma_T\}}) \\ &\leq e^{-\mu bT + |\mu| \gamma_T} E(e^{-H_T} e^{\mu B_T} \mathbb{1}_{\{B_T \geq 0\}}), \end{aligned} \quad (4.37)$$

where the last inequality holds for any T sufficiently large because $\gamma_T/T \rightarrow 0$ as $T \rightarrow \infty$. Take logs, divide by T , let $T \rightarrow \infty$, use (1.7) and minimize over $\mu \in \mathbb{R}$, to obtain

$$-I_-(b) \leq \min_{\mu \in \mathbb{R}} [-\mu b + \Lambda^+(\mu)]. \quad (4.38)$$

This shows that ' \geq ' holds in (1.8) for I replaced by I_- .

2. To derive ‘ \leq ’ in (1.8), bound, for any $\mu \in \mathbb{R}$,

$$\begin{aligned} & E\left(e^{-H_T} \mathbb{1}_{\{|B_T - bT| \leq \gamma_T\}}\right) \\ & \geq E\left(e^{-H_T} \mathbb{1}_{\mathcal{E}(\delta, T)} \mathbb{1}_{\{|B_T - bT| \leq \gamma_T\}} \mathbb{1}_{\{B_T \geq 0\}}\right) \\ & \geq e^{-\mu b T - |\mu| \gamma_T} P^{\mu, \delta, T}(|B_T - bT| \leq \gamma_T) E\left(e^{-H_T} e^{\mu B_T} \mathbb{1}_{\mathcal{E}(\delta, T)} \mathbb{1}_{\{B_T \geq 0\}}\right), \end{aligned} \quad (4.39)$$

where $P^{\mu, \delta, T}$ denotes the probability law whose density with respect to P is proportional to $e^{-H_T} e^{\mu B_T} \mathbb{1}_{\mathcal{E}(\delta, T)} \mathbb{1}_{\{B_T \geq 0\}}$.

3. Let μ_b be the maximizer of the map $\mu \mapsto \mu b - \Lambda^+(\mu)$. (Note that, by Step 1, the maximizer is unique and is characterized by $(\Lambda^+)'(\mu_b) = b$.) Next we argue that

$$\lim_{T \rightarrow \infty} P^{\mu_b, \delta, T}(|B_T - bT| \leq \gamma_T) = 1. \quad (4.40)$$

Indeed, pick $\varepsilon_T = \gamma_T / cT > 0$ (with $c > 0$ to be specified later) and estimate

$$\mathbb{1}_{\{B_T \geq bT + \gamma_T\}} \leq e^{\varepsilon_T [B_T - bT - \gamma_T]}. \quad (4.41)$$

This implies, with the help of Step 1 and Proposition 4.1 with $\mu_T = \mu + \varepsilon_T$, $C = \infty$, that

$$P^{\mu_b, \delta, T}(B_T \geq bT + \gamma_T) \leq e^{-\varepsilon_T [bT + \gamma_T]} e^{[\Lambda^+(\mu_b + \varepsilon_T) - \Lambda^+(\mu_b)]T} (1 + o(1)), \quad T \rightarrow \infty. \quad (4.42)$$

A Taylor expansion of Λ^+ around μ_b , in combination with the observation that $(\Lambda^+)'(\mu_b) = b$ and $c = (\Lambda^+)''(\mu_b) > 0$, yields that the right-hand side of (4.42) is equal to

$$e^{\frac{\varepsilon_T^2}{2} T [1 + \mathcal{O}(\varepsilon_T)] - \varepsilon_T \gamma_T} = e^{-\frac{\gamma_T^2}{2cT} [1 + \mathcal{O}(\frac{\gamma_T}{T})]}, \quad T \rightarrow \infty. \quad (4.43)$$

The right-hand side vanishes as $T \rightarrow \infty$ because $\gamma_T / T \rightarrow 0$ and $\gamma_T / \sqrt{T} \rightarrow \infty$. This shows that $\lim_{T \rightarrow \infty} P^{\mu_b, \delta, T}(B_T \geq bT + \gamma_T) = 0$. Analogously, replacing ε_T by $-\varepsilon_T$, we can prove that $\lim_{T \rightarrow \infty} P^{\mu_b, \delta, T}(B_T \leq bT - \gamma_T) = 0$. Hence, (4.40) holds.

4. Use (4.40) in (4.39) for $\mu = \mu_b$, take logs, divide by T , let $T \rightarrow \infty$, and use Step 1 and Proposition 4.1, to obtain

$$-I_+(b) \geq -\mu_b b + \Lambda^+(\mu_b) = -\max_{\mu \in \mathbb{R}} [\mu b - \Lambda^+(\mu)]. \quad (4.44)$$

This shows that ‘ \leq ’ holds in (1.8) for I replaced by I_+ . Combine (4.38) and (4.44) to obtain that $I_- = I = I_+$ and that (1.8) holds on (b^{**}, ∞) . \square

STEP 6. For any $b \geq 0$, $I_-(b) \geq -b\rho(a^{**}) + a^{**}$.

Proof. Estimate

$$\mathbb{1}_{\{|B_T - bT| \leq \gamma_T\}} \leq \mathbb{1}_{\{B_T \leq bT + \gamma_T\}} \leq e^{-\rho(a^{**})[B_T - bT - \gamma_T]}, \quad (4.45)$$

to obtain, for T sufficiently large,

$$\begin{aligned} E\left(e^{-H_T} \mathbb{1}_{\{|B_T - bT| \leq \gamma_T\}}\right) & \leq 2E\left(e^{-H_T} \mathbb{1}_{\{|B_T - bT| \leq \gamma_T\}} \mathbb{1}_{\{B_T \geq 0\}}\right) \\ & \leq 2e^{b\rho(a^{**})T + \gamma_T \rho(a^{**})} E\left(e^{-H_T} e^{-\rho(a^{**})B_T} \mathbb{1}_{\{B_T \geq 0\}}\right). \end{aligned} \quad (4.46)$$

According to the definition of Λ^+ in (1.7), the expectation in the right-hand side is equal to $e^{\Lambda^+(-\rho(a^{**}))T + o(T)}$. We therefore obtain that $I(b) \geq -b\rho(a^{**}) - \Lambda^+(-\rho(a^{**}))$. Now Step 3 concludes the proof. \square

STEP 7. For any $0 \leq b \leq b^{**}$, $I_+(b) \leq -b\rho(a^{**}) + a^{**}$.

Proof. Fix $0 \leq b \leq b^{**}$, pick $b' > b^{**}$ and put $\alpha = b/b' \in [0, 1)$. We split the path $(B_s)_{s \in [0, T]}$ into two pieces: $s \in [0, \alpha T]$ and $s \in [\alpha T, T]$. First we bound from below by inserting several indicators:

$$\begin{aligned} & E\left(e^{-H_T} \mathbb{1}_{\{|B_T - bT| \leq \gamma T\}}\right) \\ & \geq E\left(e^{-H_T} \mathbb{1}_{\{|B_{\alpha T} - b'\alpha T| \leq \gamma T/2\}} \mathbb{1}_{\{\max_{[0, \alpha T]} B \leq B_{\alpha T} + \delta\}} \mathbb{1}_{\{\max_{[B_{\alpha T} - \delta, B_{\alpha T} + \delta]} L(\alpha T, \cdot) \leq C\}}\right) \\ & \quad \times \mathbb{1}_{\{|\tilde{B}_{(1-\alpha)T}| \leq \gamma T/2\}} \mathbb{1}_{\{\min_{[0, (1-\alpha)T]} \tilde{B} \geq -\delta\}} \mathbb{1}_{\{\max_{[B_{\alpha T} - \delta, B_{\alpha T} + \delta]} \tilde{L}((1-\alpha)T, \cdot) \leq C\}}. \end{aligned} \quad (4.47)$$

Here, $(\tilde{B}_s)_{s \in [0, (1-\alpha)T]}$ is the Brownian motion with $\tilde{B}_s = B_{\alpha T + s} - B_{\alpha T}$, and $\tilde{L}((1-\alpha)T, x) = L(T, x) - L(\alpha T, x)$, $x \in \mathbb{R}$, are its local times.

On the event under the expectation in the right-hand side, we may estimate

$$H_T = H_{\alpha T} + \tilde{H}_{(1-\alpha)T} + 2 \int_{B_{\alpha T} - \delta}^{B_{\alpha T} + \delta} L(\alpha T, x) \tilde{L}((1-\alpha)T, x) dx \leq H_{\alpha T} + \tilde{H}_{(1-\alpha)T} + 4\delta C^2, \quad (4.48)$$

where $\tilde{H}_{(1-\alpha)T}$ denotes the intersection local time for the second piece. Using the Markov property at time αT , we therefore obtain the estimate

$$\begin{aligned} & E\left(e^{-H_T} \mathbb{1}_{\{|B_T - bT| \leq \gamma T\}}\right) \\ & \geq e^{-4\delta C^2} E\left(e^{-H_{\alpha T}} \mathbb{1}_{\{|B_{\alpha T} - b'\alpha T| \leq \gamma T/2\}} \mathbb{1}_{\{\max_{[0, \alpha T]} B \leq B_{\alpha T} + \delta\}} \mathbb{1}_{\{\max_{[B_{\alpha T} - \delta, B_{\alpha T} + \delta]} L(\alpha T, \cdot) \leq C\}}\right) \\ & \quad \times E\left(e^{-H_{(1-\alpha)T}} \mathbb{1}_{\{|B_{(1-\alpha)T}| \leq \gamma T/2\}} \mathbb{1}_{\{\min_{[0, (1-\alpha)T]} B \geq -\delta\}} \mathbb{1}_{\{\max_{[-\delta, \delta]} L((1-\alpha)T, \cdot) \leq C\}}\right). \end{aligned} \quad (4.49)$$

(The tilde can be removed afterwards.) Now use Proposition 4.1 (in combination with an argument like in parts 2-3 of the proof of Step 5) for the first term (with T replaced by αT) and use Step 2 for the second term (with T replaced by $(1-\alpha)T$), to conclude that

$$I(b) \leq \alpha I(b') + (1-\alpha)a^{**} = \frac{b}{b'}(I(b') - a^{**}) + a^{**}. \quad (4.50)$$

Let $b' \downarrow b^{**}$, use the continuity of I in b^{**} , and note that $I(b^{**}) - a^{**} = -b^{**}\rho(a^{**})$ by Step 5, to conclude the proof. \square

STEP 8. *Theorems 1.2 and 1.3(iv) hold.*

Proof. Steps 1 and 5 allow us to identify I on (b^{**}, ∞) as $I(b) = -b\rho(a_b) + a_b$, where a_b solves $\rho'(a_b) = 1/b$ (the maximum in (1.8) is attained at $\mu = -\rho(a_b)$). From this and (2.3–2.5) it follows that

$$I'(b) = -\rho(a_b), \quad I''(b) = -\rho'(a_b) \frac{d}{db} a_b = \frac{[\rho'(a_b)]^3}{\rho''(a_b)} > 0, \quad b > b^{**}. \quad (4.51)$$

In particular, I is real-analytic and strictly convex on (b^{**}, ∞) . Since $a_{b^{**}} = a^{**}$, it in turn follows that

$$\min_{b \geq 0} I(b) = \min_{b > b^{**}} I(b) = I(b^*) = a^*, \quad (4.52)$$

where a^* solves $\rho(a^*) = 0$ (the minimum is attained at $b^* = 1/\rho'(a^*)$). This, together with Steps 5–7, proves Theorem 1.2(i–iii).

Step 5 shows that (1.8) holds on (b^{**}, ∞) . To show that it also holds on $[0, b^{**}]$, use Step 3 to get

$$-b\rho(a^{**}) + a^{**} = \max_{\mu \in \mathbb{R}} [b\mu - \Lambda^+(\mu)], \quad 0 \leq b \leq b^{**}, \quad (4.53)$$

since the maximum is attained at $\mu = -\rho(a^{**})$. Recall from Steps 6–7 that the left-hand side is equal to $I(b)$. Thus we have proved Theorem 1.3(iv). \square

Finally, Theorem 1.2(iv) is an immediate consequence of Theorem 1.3(iii–iv). \square

5. ADDENDUM 1: AN EXTENSION OF PROPOSITION 4.1

At this point we have completed the proof of the main results in Section 1. In Sections 5–6 we derive an extension of Proposition 4.1 that will be needed in a forthcoming paper [vdHdHK02]. In that paper we show that several one-dimensional polymers models in discrete space and time, such as the weakly self-avoiding walk, converge to the Edwards model, after appropriate scaling, in the limit of vanishing self-repulsion. The proof is based on a coarse-graining argument, for which we need Proposition 5.1 below.

Recall the events in (4.1–4.2). For $\delta \in (0, \infty)$, $\alpha \in [0, \infty)$, define the event

$$\mathcal{E}^{\geq}(\delta, \alpha; T) = \left\{ \max_{x \in [B_T - \delta, B_T + \delta]} L(T, x) \geq \alpha \delta^{-\frac{1}{2}} \right\}. \quad (5.1)$$

Note that $\mathcal{E}^{\geq}(\delta, 0; T)$ is the full space.

Proposition 5.1. *Fix $\mu > -\rho(a^{**})$. Then:*

(i) *For any $\delta \in (0, \infty)$ and $\alpha \in [0, \infty)$ there exists a $K_2(\delta, \alpha) \in (0, \infty)$ such that*

$$e^{\rho^{-1}(-\mu)T} E \left(e^{-HT} e^{\mu B_T} \mathbb{1}_{\mathcal{E}(\delta, T)} \mathbb{1}_{\mathcal{E}^{\geq}(\delta, \alpha; T)} \mathbb{1}_{\{B_T \geq 0\}} \right) = K_2(\delta, \alpha) + o(1), \quad T \rightarrow \infty. \quad (5.2)$$

(ii) *For any $\alpha \in (0, \infty)$,*

$$\lim_{\delta \downarrow 0} \frac{K_2(\delta, \alpha)}{K_1(\delta, \infty)} = 0, \quad (5.3)$$

where $K_1(\delta, \infty)$ is the constant in Proposition 4.1 (recall (4.10)).

Proof. (i) As in the proof of Proposition 4.1, we may insert the indicator on $\{B_T \geq 2\delta\}$ in the expectation on the left-hand side of (5.2) and add a factor of $1 + o(1)$.

Introduce the following measurable subsets of C_0^+ , respectively, \mathcal{C}^+ :

$$G_{\delta, \alpha}^{\geq} = \{g \in C_0^+ : g(\delta) = 0, \max g \geq \alpha \delta^{-\frac{1}{2}}\}, \quad (5.4)$$

$$F_{\delta, \alpha}^{\geq} = \left\{ (y, f) \in \mathcal{C}^+ : y \geq 2\delta, \max_{[0, \delta]} f \geq \alpha \delta^{-\frac{1}{2}} \right\}. \quad (5.5)$$

Note from (4.1) and (5.1) that

$$\begin{aligned} \mathcal{E}(\delta; T) \cap \mathcal{E}^{\geq}(\delta, \alpha; T) \cap \{B_T \geq 2\delta\} &= \{L(T, -\cdot) \in G_{\delta}\} \cap \\ &\left(\{L(T, B_T + \cdot) \in G_{\delta, \alpha}^{\geq}\} \cup \left[\{(B_T, L(T, B_T - \cdot)|_{[0, B_T]}) \in F_{\delta, \alpha}^{\geq}\} \cap \{L(T, B_T + \cdot) \in G_{\delta}\} \right] \right) \end{aligned} \quad (5.6)$$

with $G_{\delta} = \{g \in C_0^+ : g(\delta) = 0\}$.

Pick $a \in \mathbb{R}$ such that $\mu + \rho(a) = 0$, i.e., $a = \rho^{-1}(-\mu) < a^{**}$. Apply Proposition 3.1 twice for $G^- = G_{\delta}$ and the two choices: (1) $F = F_{\delta, \alpha}^{\geq}$, $G^+ = G_{\delta}$; (2) $F = \mathcal{C}^+$, $G^+ = G_{\delta, \alpha}^{\geq}$. Sum the two

resulting equations, to obtain

$$\begin{aligned} \text{l.h.s. of (5.2)} &= (1 + o(1)) \int_0^\infty dt_1 \int_0^\infty dt_2 \mathbb{1}_{\{t_1+t_2 \leq T\}} e^{a(t_1+t_2)} \\ &\times \widehat{\mathbb{E}}^a \left(\left[\frac{w_{G_\delta}(X_0, t_1)}{x_a(X_0)} \mathbb{1}_{\{A^{-1}(T-t_1-t_2) \geq 2\delta\}} \mathbb{1}_{\{\max_{[0,\delta]} X \geq \alpha\delta^{-\frac{1}{2}}\}} \right. \right. \\ &\quad \left. \left. + \frac{w_{G_{\delta,\alpha}^\geq}(X_0, t_1)}{x_a(X_0)} \right] \frac{w_{G_\delta}(Y_{T-t_1-t_2}, t_2)}{x_a(Y_{T-t_1-t_2})} \right). \end{aligned} \quad (5.7)$$

In the same way as in the proof of Proposition 4.1, we obtain that (recall (4.10–4.11))

$$\lim_{T \rightarrow \infty} (\text{r.h.s. of (5.7)}) = K_2(\delta, \alpha) \quad (5.8)$$

with

$$K_2(\delta, \alpha) = \left[\widehat{\mathbb{E}}_h^a \left(\frac{y_a^{(\delta)}(X_0)}{x_a(X_0)} \mathbb{1}_{\{\max_{[0,\delta]} X \geq \alpha\delta^{-\frac{1}{2}}\}} \right) + \langle x_a, y_a^{(\delta,\alpha)} \rangle \right] \frac{1}{\rho'(a)} \langle x_a, y_a^{(\delta)} \rangle_\circ, \quad (5.9)$$

where $y_a^{(\delta)}$ is defined in (4.11) and $y_a^{(\delta,\alpha)}$ is defined as (recall (3.1))

$$y_a^{(\delta,\alpha)}(h) = \int_0^\infty dt e^{at} w_{G_{\delta,\alpha}^\geq}(h, t) = \mathbb{E}_h^* \left(e^{\int_0^\infty [aX_v^* - (X_v^*)^2] dv} \mathbb{1}_{\{X_\delta^* = 0\}} \mathbb{1}_{\{\max_{[0,\delta]} X^* \geq \alpha\delta^{-\frac{1}{2}}\}} \right). \quad (5.10)$$

The right-hand side of (5.9) is a strictly positive finite number.

(ii) Fix $\alpha \in (0, \infty)$. From (4.10) and (5.8) we see that $K_2(\delta, \alpha)/K_1(\delta, \infty) = K^{(1)}(\delta, \alpha) + K^{(2)}(\delta, \alpha)$ with

$$K^{(1)}(\delta, \alpha) = \frac{\int_0^\infty dh x_a(h) y_a^{(\delta)}(h) \widehat{\mathbb{P}}_h^a(\max_{[0,\delta]} X \geq \alpha\delta^{-\frac{1}{2}})}{\langle x_a, y_a^{(\delta)} \rangle}, \quad K^{(2)}(\delta, \alpha) = \frac{\langle x_a, y_a^{(\delta,\alpha)} \rangle}{\langle x_a, y_a^{(\delta)} \rangle}. \quad (5.11)$$

To prove (5.3), we need the following lemma.

Lemma 5.2. *Fix $a < a^{**}$ and $\alpha \in (0, \infty)$. Then:*

(i) *There exists $d = d(\alpha) > 0$ such that, for any $R > 0$ and any $\delta > 0$ sufficiently small,*

$$\sup_{h \in [0, R]} \widehat{\mathbb{P}}_h^a \left(\max_{[0, \delta]} X \geq \alpha\delta^{-\frac{1}{2}} \right) \leq c e^{-d\delta^{-\frac{1}{4}}} e^{c\sqrt{R}}, \quad (5.12)$$

$$\sup_{h \in [0, R]} \frac{y_a^{(\delta,\alpha)}(h)}{y_a(h)} \leq c e^{-d\delta^{-\frac{1}{4}}} e^{c\sqrt{R}}. \quad (5.13)$$

(ii) *For any $\delta > 0$ sufficiently small,*

$$\inf_{h \in [0, \delta]} y_a^{(\delta)}(h) \geq c. \quad (5.14)$$

Proof. The proof is deferred to Section 6. □

We use Lemma 5.2 to show that

$$\lim_{\delta \downarrow 0} K^{(1)}(\delta, \alpha) = \lim_{\delta \downarrow 0} K^{(2)}(\delta, \alpha) = 0, \quad (5.15)$$

which yields (5.3).

First note that, with the help of (5.14), the common denominator in (5.11) may be estimated from below by

$$\langle x_a, y_a^{(\delta)} \rangle \geq \int_0^\delta dh x_a(h) y_a^{(\delta)}(h) \geq c \int_0^\delta dh x_a(h) \geq c\delta, \quad (5.16)$$

where we use that x_a is bounded away from zero on $[0, \delta]$.

In order to estimate the numerator of $K^{(1)}(\delta, \alpha)$ from above, we split the integral in the numerator into two parts: $h \leq R$ and $h > R$. In the integral over $h \leq R$, estimate $y_a^{(\delta)} \leq y_a$ and use (5.12), to get the upper bound $ce^{-d\delta^{-\frac{1}{4}}}e^{c\sqrt{R}}$. In the integral over $h > R$, estimate $y_a^{(\delta)} \leq y_a$, estimate the probability against one and use (2.2) and (2.16), to get the upper bound $ce^{-cR^{\frac{3}{2}}}$. Pick R such that $c\sqrt{R} = \frac{d}{2}\delta^{-\frac{1}{4}}$, to obtain that the numerator of $K^{(1)}(\delta, \alpha)$ is at most $ce^{-\frac{d}{2}\delta^{-\frac{1}{4}}}$.

In the same way we show, with the help of (5.13), that the numerator of $K^{(2)}(\delta, \alpha)$ in (5.11) is at most $ce^{-\frac{d}{2}\delta^{-\frac{1}{4}}}$. Now combine the two estimates with (5.16) to obtain (5.15). \square

6. ADDENDUM 2: PROOF OF LEMMA 5.2

We will need the following asymptotics for x_a and y_a , which are refinements of (2.2) and (2.16), respectively.

STEP 1. For $a < a^{**}$,

$$\lim_{h \rightarrow \infty} \frac{1}{\sqrt{h}} \log [e^{\frac{\sqrt{2}}{3}h^{\frac{3}{2}}} x_a(h)] = \lim_{h \rightarrow \infty} \frac{1}{\sqrt{h}} \log [e^{\frac{\sqrt{2}}{3}h^{\frac{3}{2}}} y_a(h)] = \frac{a}{\sqrt{2}}. \quad (6.1)$$

Proof. The statement for y_a is well-known, and follows from (2.15) together with the asymptotics of the Airy function given by (see [E56, p. 43])

$$\text{Ai}(h) = \frac{1}{2\pi h^{\frac{1}{4}}} e^{-\frac{2}{3}h^{\frac{3}{2}}} [1 + o(1)], \quad h \rightarrow \infty. \quad (6.2)$$

To prove the statement for x_a , use [CL55, Theorem 2.1, pp. 143–144]. To this end, define

$$\zeta_1(h) = x_a(h^2), \quad \zeta_2(h) = h^{-2}\zeta_1'(h). \quad (6.3)$$

Then the eigenvalue equation $\mathcal{K}^a x_a = \rho(a)x_a$ (recall (2.1)) can be written as (see also [CL55, equation (5.3)])

$$\zeta'(h) = h^2 B(h)\zeta(h), \quad (6.4)$$

with

$$\zeta(h) = \begin{pmatrix} \zeta_1(h) \\ \zeta_2(h) \end{pmatrix}, \quad B(h) = \begin{pmatrix} 0 & 1 \\ 2 - \frac{2a}{h^2} + \frac{2\rho(a)}{h^4} & -\frac{3}{h^3} \end{pmatrix}. \quad (6.5)$$

Note that $B(h) = \sum_{n=0}^{\infty} h^{-n} B^{(n)}$ ($B^{(0)} \neq 0$) is a convergent power series in h^{-1} , with $B^{(0)}$ having eigenvalues $\lambda_{1,2} = \pm\sqrt{2}$. Therefore (6.4) has formal solutions of the form

$$Z(h) = P(h)h^R e^{Q(h)}, \quad (6.6)$$

where the columns of the matrix Z are the two linearly independent solutions to (6.4), $P(h) = \sum_{n=0}^{\infty} h^{-n} P^{(n)}$ ($\det(P^{(0)}) \neq 0$) is a formal power series in h^{-1} , R is a complex diagonal matrix, and $Q(h) = \frac{1}{3}h^3 Q^{(0)} + \frac{1}{2}h^2 Q^{(1)} + hQ^{(2)}$ is a matrix polynomial with $Q^{(0)}, Q^{(1)}, Q^{(2)}$ diagonal. In our case,

$$Q^{(0)} = \text{diag}\{-\sqrt{2}, +\sqrt{2}\}, \quad Q^{(1)} = 0, \quad Q^{(2)} = \text{diag}\left\{\frac{a}{\sqrt{2}}, -\frac{a}{\sqrt{2}}\right\}. \quad (6.7)$$

From the proof of [CL55, Theorem 2.1] it follows that $P(h), R, Q(h)$ can be chosen to be real, because $B(h), \lambda_1, \lambda_2$ are real. On [CL55, p. 151] there is the further remark that for every formal solution there exists an actual solution with the same asymptotics.

We need the solution that is in $L^2[0, \infty)$. By construction, we compute, for $R = \text{diag}\{r_1, r_2\}$ (with r_1, r_2 some functions of a),

$$h^R e^{\frac{1}{3}h^3 Q^{(0)} + \frac{1}{2}h^2 Q^{(1)} + hQ^{(2)}} = \begin{pmatrix} h^{r_1} e^{-\frac{\sqrt{2}}{3}h^3 + \frac{a}{\sqrt{2}}h} & 0 \\ 0 & h^{r_2} e^{\frac{\sqrt{2}}{3}h^3 - \frac{a}{\sqrt{2}}h} \end{pmatrix}. \quad (6.8)$$

Therefore the solution in $L^2[0, \infty)$ must be

$$\zeta(h) = h^{r_1} e^{-\frac{\sqrt{2}}{3}h^3 + \frac{a}{\sqrt{2}}h} \sum_{n=0}^{\infty} h^{-n} \begin{pmatrix} P_{11}^{(n)} \\ P_{21}^{(n)} \end{pmatrix}, \quad (6.9)$$

where $P_{ij}^{(n)}$ denotes the element in the i -th row and the j -th column of the matrix $P^{(n)}$. Now return to (6.3) to read off the claim. \square

Pick a' such that $a < a' < a^{**}$ and define (recall (2.9))

$$M_t = \frac{D_t^{(a')}}{D_t^{(a)}} = \frac{x_a(X_0) x_{a'}(X_t)}{x_{a'}(X_0) x_a(X_t)} e^{-t[\rho(a') - \rho(a)]} e^{(a' - a) \int_0^t X_v dv}, \quad t \geq 0. \quad (6.10)$$

STEP 2. For any $h \geq 0$, $(M_t)_{t \geq 0}$ is a martingale under $\widehat{\mathbb{P}}_h^a$.

Proof. Fix $0 \leq s < t$. If ϕ_s denotes the time-shift by $s \geq 0$ (i.e., $(\phi_s \circ f)((X_t)_{t \geq 0}) = f((X_{s+t})_{t \geq 0})$ for any bounded and measurable function f), then it is clear that $M_t = M_s(\phi_s \circ M_{t-s})$. Hence, using the Markov property at time s , we see that, for any $h \geq 0$,

$$\widehat{\mathbb{E}}_h^a(M_t | M_s) = M_s \widehat{\mathbb{E}}_h^a(\phi_s \circ M_{t-s} | M_s) = M_s \widehat{\mathbb{E}}_h^a(\widehat{\mathbb{E}}_{X_s}(M_{t-s})). \quad (6.11)$$

Now use that, for any $x \geq 0$, according to the construction of the transformed process in (2.9–2.10),

$$\widehat{\mathbb{E}}_x^a(M_{t-s}) = \mathbb{E}_x(D_{t-s}^{(a)} M_{t-s}) = \mathbb{E}_x(D_{t-s}^{(a')}) = 1. \quad (6.12)$$

\square

STEP 3. Proof of (5.12).

Proof. Use Step 2, Doob's martingale inequality and (6.12), to obtain

$$\widehat{\mathbb{P}}_h^a\left(\max_{[0, \delta]} M \geq K\right) \leq \frac{1}{K} \max_{t \in [0, \delta]} \widehat{\mathbb{E}}_h^a(M_t) = \frac{1}{K}, \quad h \geq 0, K > 0. \quad (6.13)$$

Next note that by Step 1, for any $R > 0$,

$$\inf_{[0, R]} \frac{x_a}{x_{a'}} \geq e^{-c\sqrt{R}}. \quad (6.14)$$

Substitute this into (6.10), to get

$$M_t \geq c \frac{x_{a'}(X_t)}{x_a(X_t)} e^{-c\sqrt{R}} \widehat{\mathbb{P}}_h^a\text{-a.s.}, \quad 0 \leq t \leq 1, \quad 0 \leq h \leq R. \quad (6.15)$$

Pick $g_a: [0, \infty) \rightarrow (0, \infty)$ to be the largest increasing function not exceeding $x_{a'}/x_a$ anywhere on $[0, \infty)$. Then, by (6.15), $M_t \geq c g_a(X_t) e^{-c\sqrt{R}} \widehat{\mathbb{P}}_h^a\text{-a.s.}, \quad 0 \leq t \leq 1, \quad 0 \leq h \leq R$. Now use (6.13) to

estimate, for $0 \leq h \leq R$,

$$\begin{aligned} \widehat{\mathbb{P}}_h^a \left(\max_{[0, \delta]} X \geq \alpha \delta^{-\frac{1}{2}} \right) &= \widehat{\mathbb{P}}_h^a \left(\max_{t \in [0, \delta]} c g_a(X_t) \geq c g_a(\alpha \delta^{-\frac{1}{2}}) \right) \\ &\leq \widehat{\mathbb{P}}_h^a \left(\max_{t \in [0, \delta]} M_t \geq c g_a(\alpha \delta^{-\frac{1}{2}}) e^{-c\sqrt{R}} \right) \\ &\leq \frac{1}{c g_a(\alpha \delta^{-\frac{1}{2}})} e^{c\sqrt{R}}. \end{aligned} \quad (6.16)$$

By Step 1, it is possible to pick g_a such that

$$g_a(h) \geq e^{-c\sqrt{h}}, \quad h \rightarrow \infty. \quad (6.17)$$

This implies the bound in (5.12) with $d(\alpha) = \sqrt{\alpha}$. \square

STEP 4. *Proof of (5.13).*

Proof. Fix $a < a^{**}$. Define

$$D_t^{(a,*)} = \frac{y_a(X_t^*)}{y_a(X_0^*)} e^{\int_0^t [aX_v^* - (X_v^*)^2] dv}, \quad t \geq 0. \quad (6.18)$$

Then it is easy to check (see [RY94, Section VIII.3]) that $(D_t^{(a,*)})_{t \geq 0}$ is a martingale under \mathbb{P}_h^* for any $h \geq 0$ (y_a is a strictly positive solution to the differential equation $2y_a''(h) = (h - a)y_a(h)$ on $[0, \infty)$; recall (2.6) and (2.15)). Hence, analogously to (2.10), we may construct a transformed process via a Girsanov transformation by taking $D_t^{(a,*)}$ formally as a density with respect to BESQ^0 . Denote by $\widehat{\mathbb{P}}_h^{a,*}$ and $\widehat{\mathbb{E}}_h^{a,*}$ probability and expectation with respect to this transformed process starting at $h \geq 0$.

Recall that $y_a(0) = 1$. We have the following representation for the function $y_a^{(\delta, \alpha)}$ (recall (5.10)):

$$y_a^{(\delta, \alpha)}(h) = y_a(h) \widehat{\mathbb{P}}_h^{a,*} (X_\delta^* = 0, \max X^* \geq \alpha \delta^{-\frac{1}{2}}). \quad (6.19)$$

The proof of (5.13) is now analogous to Steps 2–3. Indeed, use (6.19), drop the restriction $X_\delta^* = 0$, and proceed analogously. Step 1 provides the necessary asymptotic bounds for y_a and y_a' , provided that $a < a' < a^{**}$. \square

STEP 5. *Proof of (5.14).*

Proof. We return to the right-hand side of (4.11) and obtain a lower bound by inserting the indicator of the event $\{\max X^* \leq 2\delta\}$. On this event, we may estimate the exponential from below by c . Hence, for $0 \leq h \leq \delta$,

$$y_a^{(\delta)}(h) \geq c \mathbb{P}_h^* (\max X^* \leq 2\delta, X_\delta^* = 0) = c [\mathbb{P}_h^* (X_\delta^* = 0) - \mathbb{P}_h^* (\max X^* > 2\delta, X_\delta^* = 0)]. \quad (6.20)$$

Using the Markov property at the first time the BESQ^0 hits 2δ , we see that the latter probability is at most $\mathbb{P}_{2\delta}^* (X_\delta^* = 0)$. Since the first probability is decreasing in h , we therefore have the bound

$$y_a^{(\delta)}(h) \geq c (\mathbb{P}_\delta^* (X_\delta^* = 0) - \mathbb{P}_{2\delta}^* (X_\delta^* = 0)). \quad (6.21)$$

Now use that $\mathbb{P}_h^* (X_\delta^* = 0) = e^{-h/2\delta}$ for any $h, \delta \geq 0$ (see [RY94, Corollary XI(1.4)]), to complete the proof. \square

REFERENCES

- [AS70] M. ABRAMOWITZ and I. STEGUN, Handbook of Mathematical Functions (9th edition), Dover Publications Inc., New York, 1970.

- [CL55] E.A. CODDINGTON and N. LEVINSON, *Theory of Ordinary Differential Equations*, McGraw-Hill, New York, 1955.
- [E56] A. ERDÉLYI, *Asymptotic Expansions*, Dover, Toronto, 1956.
- [G81] D.H. GRIFFEL, *Applied Functional Analysis*, Wiley, New York, 1981.
- [vdH98] R. VAN DER HOFSTAD, *On the constants in the central limit theorem for the one-dimensional Edwards model*, J. Stat. Phys. 90, 1295–1306, 1998.
- [vdHdH95] R. VAN DER HOFSTAD and F. DEN HOLLANDER, *Scaling for a random polymer*, Comm. Math. Phys. 169, 397–440, 1995.
- [vdHdHK97] R. VAN DER HOFSTAD, F. DEN HOLLANDER and W. KÖNIG, *Central limit theorem for the Edwards model*, Ann. Probab. 25, 573–597, 1997.
- [vdHdHK02] R. VAN DER HOFSTAD, F. DEN HOLLANDER and W. KÖNIG, *Weak interaction limits for one-dimensional random polymers*, preprint 2002.
- [MS87] P. MARCH and A.-S. SZNITMAN, *Some connections between excursion theory and the discrete Schrödinger equation with random potentials*, Probab. Th. Rel. Fields 75, 11–53, 1987.
- [RY94] D. REVUZ and M. YOR, *Continuous Martingales and Brownian Motion* (2nd edition), Springer, Berlin, 1994.
- [W84] J. WESTWATER, *On Edwards model for polymer chains*, in: Trends and Developments in the Eighties (S. Albeverio and P. Blanchard, eds.), Bielefeld Encounters in Math. Phys. 4/5, World Scientific, Singapore, 1984.