A localization test for observations

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Abstract

We consider randomly obtained observations from an infinite binary code (scenery). We provide a test that, with high probability, allows us from a finite string of observations determine the localization of the observed data. The main result of the paper is used to solve the following problem asked by Kesten: is it possible to reconstruct a 2-color scenery along the path of a recurrent random walk with jumps. In the subsequent paper the present result will be used to show the existence of the asked algorithm. Furthermore, the result of the paper directly implies that almost all independent sceneries can be distinguished when they are red along a random walk.

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1 Introduction and Result

1.1 Introduction

A (one dimensional) *scenery* ξ is a coloring of the integers $\mathbb{Z}$ with C_0 colors $\{1,\dots,C_0\}$. Two sceneries ξ, ξ' are called *equivalent*, $\xi \approx \xi'$, if one of them is obtained from the other by a translation or reflection. Let $(S(t))_{t\geq 0}$ be a recurrent random walk on the integers. Observing the scenery ξ along the path of this random walk, one sees the color $\xi(S(t))$ at time t. The *scenery reconstruction problem* is concerned with trying to retrieve the scenery ξ, given only the sequence of observations $\chi := (\xi(S(t)))_{t\geq 0}$. Quite obviously retrieving a scenery can only work up to equivalence. For an overview about scenery reconstruction we refer the reader to an excellent survey in [13].

In the present paper we consider the following problem: can a 2-color scenery be reconstructed, if it is observed along a random walk with jumps. Among others, this question was asked by H. Kesten in [13]. The main result of this paper, Theorem 5.3, is an important ingredient for solving the above-stated problem. Furthermore, it has another direct implication: it shows that we can distinguish 2 independent i.i.d. sequences observed along a path of a recurrent random walk with jumps.

Before explaining the main result, let us briefly mention a few people as well as some of their work in this area: Benjamini [1], Burdzy, Harris [5], Heicklen [6], [2], den Hollander [4], [3], Hoffman [6], Howard [9], [8], [7], Kalikow [10], Keane [5], [11], Kesten [12], [1], [14], Levin [16], Lindenstauss [17], Rudolph [6], Pemantle [16], Peres [16], Spitzer [14], Steif [4].

Furthermore, the following people have been working on scenery reconstruction: Loewe[18], Merkl [19], Rolles [23], Le Ny[15], Redig [15].

The research in scenery reconstruction was first motivated by the general problem of studying the properties of the color record χ. In particular, the research on scenery reconstruction started with the scenery distinguishing problem. The question was raised independently by Benjamini and by den Hollander and Keane. Later Kesten asked, whether one can recognize a single defect in a random scenery.

In order to provide an answer to Kesten's question, Matzinger in his Ph.D. thesis [20] proved a somewhat stronger result: typical sceneries can be reconstructed a.s. up to equivalence. The sceneries in Matzinger's setup are independent uniformly distributed random variables. He showed that almost every scenery can be almost surely reconstructed. In [13], Kesten noticed that Matzinger's proof in [20] heavily relies on the skip-free property of the random walk. He asked whether the result might still hold in the case of a random walk with jumps. Merkl, Matzinger and Loewe in [19] gave a positive answer to Kesten's question under a particular assumption: there are strictly more colors than possible single steps for the random walk.

The two color case, ($C_0 = 2$) is more difficult than the case investigated by Merkl, Matzinger and Loewe in [19]. Although several arguments in [19] do not use the fact that there are more than 2 colors, the central idea hopelessly fails in the 2-color case. To overcome the problem, the existence of a certain localization test becomes crucial. To provide such a test is the objective of the present paper. In a follow-up paper we will present the other ideas necessary for the 2-color scenery reconstruction with jumps.

1.2 Main assumptions

We define the main concepts of the paper: scenery, random walk and observations. Also, some notations will be introduced.

* **Scenery** $\xi = \{\xi(z)\}_{z\in\mathbb{Z}}$ is a family of i.i.d. Bernoulli random variables with parameter 1/2. We often use ψ for a non-random scenery, i.e. $\psi \in \{0,1\}^{\mathbb{Z}}$ is a value of random element ξ.

* In this paper, $S = \{S(t)\}_{t\in\mathbb{N}}$ is a recurrent random walk, that visits every integer z with positive probability. We assume S starts at origin, i.e. $S(0) = 0$. Another important assumption is that S has only a finite number of steps ("bounded jumps"). More precisely, we assume that the set $\{z : P(S(1) - S(0) = z) > 0\}$ is finite. Throughout this paper we denote

$$L := \max\{z : P(S(1) - S(0) = z) > 0\}.$$

Thus L stands for length of the maximum jump.
To simplify some proofs we also assume that S is symmetric (however, we do not believe that the symmetricity is necessary).
The random walk S and scenery ξ are independent.

* We denote by χ the **observations** :

$$\chi := \xi\left(S\left(0\right)\right), \xi\left(S\left(1\right)\right), \xi\left(S\left(2\right)\right), \ldots$$

and we interpret χ as a random function from $\mathbb{N}$ to $\{0,1\}$, so that $\chi(k) := \xi\left(S\left(k\right)\right)$ for all $k \in \mathbb{N}$.

* Let $f : D \to I$ be a map. For a subset $E \subset D$ we shall write $f|E$ for the restriction of f to the set E. Thus, when $[a,b] \in \mathbb{Z}$ is an integer interval and ξ is scenery (resp. ψ is a non-random scenery), then $\xi|[a,b]$ (resp. $\psi|[a,b]$) stands for the random vector $(\xi(a), \ldots, \xi(b))$ (resp. for vector $(\psi(a), \ldots, \psi(b))$). We also denote $\xi|[0,b] = \xi_0^b$. Similarly, we often denote $\chi|[0,b] = \chi_0^b$.

* Let $a = (a_1, \ldots, a_N)$, $b = (b_1, \ldots, b_{N+1})$ be two vectors with lengths N and $N+1$, respectively. We denote $a \preccurlyeq b$, if

$$a \in \{(b_1, \ldots, b_N), (b_2, \ldots b_{N_1})\}.$$

Thus, $a \preccurlyeq b$ if a can be obtained from b by "removing the first or the last element".

1.3 The theorem

The main result of the paper is the following theorem:

Theorem 1.1 *There exists constants $c > 0$ (not depending on n), $N < \infty$, $m(n) > n$, the maps*

$$g : \{0,1\}^{m+1} \mapsto \{0,1\}^{n^2+1}$$
$$\hat{g} : \{0,1\}^{m^2+1} \mapsto \{0,1\}^{n^2}$$

and the sequence of events $E^n_{\text{cell\_OK}} \in \sigma(\xi(z)|z \in [-cm, cm])$ such that:

1) $P(E^n_{\text{cell\_OK}}) \to 1$

2) *For all $n > N$ and $\psi_n \in E^n_{\text{cell\_OK}}$ we have:*

$$P\left(\hat{g}(\chi_0^{m^2}) \preccurlyeq g(\psi_0^m) \Big| S(m^2) = m, \xi = \psi_n\right) > 3/4.$$

3) *$g(\xi_0^m)$ is an i.i.d. binary vector where the components are Bernoulli with parameter $1/2$.*

Theorem says that there exists functions g and $\hat{g}$ as well as a set of typical sceneries, $E^n_{\text{cell\_OK}}$, such that 1) 2) and 3) hold. The function g, applied to the (piece of) underlying scenery, ξ_0^m, will be referred as g**-information vector**. It contains enough information about ξ_0^m : the vectors $g(\xi_0^m)$ and $g(\xi_{m+1}^{2m+2})$ are equal with probability $(1/2)^n$, only. The latter follows from the statement 3) of the theorem. On the other hand, the information $g(\xi_0^m)$ can be successfully estimated by $\hat{g}$ function: the statement 3) ensures that, conditioning on the event $\{S(m^2) = m\}$, the probability of $\hat{g}(\chi_0^{m^2}) \preccurlyeq g(\psi_0^m)$ is bigger than $1/2$, provided that ψ_0^m is typical. Thus, in this case $\hat{g}(\chi_0^{m^2})$ is a good estimator of $g(\psi_0^m)$. The construction of $\hat{g}$ will be referred as g**-information reconstruction algorithm**. The statement 1) of theorem concerns the set of typical sceneries, $E^n_{\text{cell\_OK}}$. These are the sceneries, for which the reconstruction algorithm successfully works. The statement 3) says that, by increasing n, one can make the set $E^n_{\text{cell\_OK}}$ as large as one wants.
As mentioned, the g-information and $\hat{g}$-estimators are the main issues for reconstructing the 2-color scenery observed along a recurrent random walk with jumps.

The paper is organized as follows. In order to explain the main ideas of the g-information reconstruction algorithm, in the following section we consider an simplified example. We hope that this example helps to understand the main structure of the algorithm.
In Chapter 2 we prove many auxiliary results needed for the following. In particular, the chapter 2 deals with the signal probabilities.
In Chapter 3 we consider the set of typical sceneries: $E^n_{\text{cell\_OK}}$. We prove that $P(E^n_{\text{cell\_OK}}) \to 1$.
Chapter 4 is devoted to the random walk - we investigate the typical behavior of the random walk (and observations) under the condition $\{S(m^2) = m\}$.
Finally, in Chapter 4 we formally define the functions g and $\hat{g}$ and we prove that the g-reconstruction algorithm works, i.e. we prove the main theorem.

1.4 3 colors example

1.4.1 Setup

The main goal of this paper is to define the **g- information reconstruction algorithm** as well as to prove that this works. Before starting it, in this section we present a simplified case.
Let ξ_0^m and $\chi_0^{m^2}$ denote the piece of scenery $\xi|[0,m]$ and the first m observations $\chi|[0,m]$, respectively. Recall that we want to construct two functions $g : \{0,1\}^{m+1} \to \{0,1\}^{n^2+1}$ and $\hat{g} : \{0,1\}^{m^2+1} \to \{0,1\}^{n^2}$ such that

1) with high probability

$$P\left(\hat{g}(\chi_0^{m^2}) \preccurlyeq g(\xi_0^m) \,\Big|\, S(m^2) = m\right).$$

2) $g(\xi_0^m)$ is an i.i.d. binary vector where the components are Bernoulli with parameter $\frac{1}{2}$.

In other words, **1)** states that, with high probability, we can reconstruct $g(\xi_0^m)$ from the observations, provided that random walk S goes in m^2 steps from 0 to m. (Remember that $\hat{g}(\chi_0^{m^2}) \preccurlyeq g(\xi_0^m)$ means that $\hat{g}(\chi_0^{m^2})$ and $g(\xi_0^m)$ are equal up to one bit.) Thus the function $\hat{g}$ represents a "reconstruction algorithm" which tries to reconstruct the information $g(\xi_0^m)$.

Since this is not yet the real case in which we are interested in this paper, during the present subsection we will not be very formal. For this subsection only, let us assume that the scenery ξ has three colors instead of two. This is our simplifying assumptions introduced for tutorial reasons. Thus, we assume that $\{\xi(z)\}$ satisfies all of the following three conditions:

a) $\{\xi(z) : z \in \mathbb{Z}\}$ are i.i.d. variables with state space $\{0,1,2\}$,

b) $\exp(n/\ln n) \leq 1/P(\xi(0) = 2) \leq \exp(n))$,

c) $P(\xi(0) = 0) = P(\xi(0) = 1)$.

We define $m = n^{2.5}(1/P(\xi(0) = 2))$. Because of **b)** this means

$$n^{2.5}\exp(n/\ln n) \leq m(n) \leq n^{2.5}\exp(n).$$

The thus defined scenery distribution is very similar to our usual scenery except that sometimes (quite rarely) there appear also 2's in this scenery.
We now introduce some necessary definitions.

Let $\bar{z}_i$ denote the i-th place in $[0,\infty)$ where we have a 2 in ξ. Thus $\bar{z}_1 := \min\{z \geq 0 | \xi(z) = 2\}$, $\bar{z}_{i+1} := \min\{z > \bar{z}_i | \xi(z) = 2\}$. We make the convention that $\bar{z}_0$ is the last location before zero where we have a 2 in ξ. For a negative integer $i < 0$, $\bar{z}_i$ designates the $i+1$-th point before 0 where we have a 2 in ξ. The random variables $\bar{z}_i$-s are called **signal carriers**. For each signal carrier, $\bar{z}_i$, we define the **frequency of ones** at $\bar{z}_i$. By this we mean the (conditional on ξ) probability to see 1 exactly after $e^{n^{0.1}}$

observations having been at $\bar{z}_i$. We denote that conditional probability by $h(\bar{z}_i)$ and will also write $h(i)$ for it. Formally:

$$h(i) := h(\bar{z}_i) := P\Big(\xi(S(e^{n^{0.1}}) + \bar{z}_i) = 1\Big|\xi\Big).$$

It is easy to see that the frequency of ones is equal to a weighted average of the scenery in a neighborhood of radius $Le^{n^{0.1}}$ of the point $\bar{z}_i$. That is $h(i)$ is equal to:

$$h(i) := \sum_{\substack{z\in[-Le^{n^{0.1}}, Le^{n^{0.1}}] \\ z\neq\bar{z}_i}} \xi(z) P\big(S(e^{n^{0.1}}) + \bar{z}_i = z\big) \tag{1.1}$$

(Of course this formula to hold assumes that there are no other two's in $[\bar{z}_i - Le^{n^{0.1}}, \bar{z}_i + Le^{n^{0.1}}]$ except the two at $\bar{z}_i$. This is very likely to hold, see event $E^n_{6\_2}$ below).
Let

$$g_i(\xi_0^m) := I_{[0,0.5)}(h(i)).$$

We now define some events that describe the typical behavior of ξ.

* Let $E^n_{6\_2}$ denote the event that in $[0,m]$ all the signal carriers are further apart than $\exp(n/(2\ln n))$ from each other as well as from the points 0 and m. By the definition of $P(\xi(i)=2)$, the event $P(E^n_{6\_2}) \to 1$ as $n\to\infty$.

* Let $E^n_{1\_2}$ be the event that in $[0,m]$ there are more than n^2+1 signal carrier points. Because of the definition of m, $P(E^n_{1\_2}) \to 1$ as $n\to\infty$.

When $E^n_{1\_2}$ and $E^n_{6\_2}$ both hold, we define $g(\xi_0^m)$ in the following way:

$$g\left(\xi_0^m\right) := \left(g_1\left(\xi_0^m\right), g_2\left(\xi_0^m\right), g_3\left(\xi_0^m\right), \dots, g_{n^2+1}\left(\xi_0^m\right)\right)$$

Conditional on $E^n_{1\_2} \cap E^n_{6\_2}$ we get that $g\left(\xi^m\right)$ is an i.i.d. random vector with the components being Bernoulli variables with parameter 1/2. Here the parameter 1/2 follows simply by symmetry of our definition [to be precise, $P(g_i\left(\xi_i^m\right) = 1) = 1/2 - P(h(i) = 1/2)$, but we disregard this small error term in this example] and the independence follows from the fact that the scenery is i.i.d. [indeed, $g_i(\xi_0^m)$ depends only on the scenery in a radius $Le^{n^{0.1}}$ of the point $\bar{z}_i$ and, due to E_{6_2}, the points $\bar{z}_i$ are further apart than $\exp(\frac{n}{2\ln n}) > L\exp(n^{0.1})$].
Hence, with almost no effort we get that when $E^n_{1\_2}$ and $E^n_{6\_2}$ both hold, then condition **2)** is satisfied. To be complete, we have to define the function g such that **2)** holds also outside $E^n_{1\_2}\cap E^n_{6\_2}$. We actually are not interested in g outside $E^n_{1\_2}\cap E^n_{6\_2}$ - it would be enough that we reconstruct g on $E^n_{1\_2}\cap E^n_{6\_2}$. Therefore, extend g in any possible way, so that $g\left(\xi_0^m\right)$ depends only on ξ_0^m and its component are i.i.d.

1.4.2 $\hat{g}$-algorithm

We show, how to construct a map $\hat{g}: \{0,1\}^{n^2} \mapsto \{0,1\}^n$ and an event $E_{OK} \in \sigma(\xi)$ such that $P(E_{OK})$ is close to 1 and for each scenery belonging to E_{OK} the probability

$$P\Big(\hat{g}(\chi_0^{m^2}) \preccurlyeq g(\xi_0^m) | S(m^2) = m\Big) \tag{1.2}$$

is also high. Note, when the scenery ξ is fixed, then the probability (1.2) depends on S.
The construction of $\hat{g}$ consists of several steps. The first step is the estimation of the frequency of one's $h(i)$. Note: due to E_{6_2} we have that in the region of our interest we can assume that all the signal carriers are further apart form each other than $\exp(n/(2\ln n))$. In this case we have that all the 2's observed in a time interval of length $e^{n^{0.3}}$ must come from the same signal carrier. We will thus take time intervals T of length $e^{n^{0.3}}$ to estimate the frequency of one's.
Let $T = [t_1, t_2]$ be a (non-random) time interval such that $t_2 - t_1 = e^{n^{0.3}}$. Assume that during time T the random walk is close to the signal carrier $\bar{z}_i$. Then every time we see a 2 during T this gives us a

stopping time which stops the random walk at $\bar{z}_i$. We can now use these stopping times to get a very precise estimate of $h(i)$. In order to obtain the independence (which makes proofs easier), we do not take all the 2's which we observe during T. Instead we take the 2's apart by at least $e^{n^{0.1}}$ from each other. To be more formal, let us now give a few definitions:

* Let $\nu_{t_1}(1)$ denote the first time $t > t_1$ that we observe a 2 in the observations χ after time t_1. Let $\nu_{t_1}(k+1)$ be the first time after time $\nu_{t_1}(k) + e^{n^{0.1}}$ that we observe a 2 in the observations χ. Thus $\nu_{t_1}(k+1)$ is equal to $\min\{t|\chi(t) = 2, t \geq \nu_{t_1}(k) + e^{n^{0.1}}\}$. We say that T is such that we can significantly estimate the frequency of one's for T, if there are more than $e^{n^{0.2}}$ stopping times $\nu_{t_1}(k)$ during T. In other words, we say that we can significantly estimate the frequency of one's for T, iff $\nu_{t_1}(e^{n^{0.2}}) \leq t_2 - e^{n^{0.1}}$.

* Let $\hat{X}_{t_1}(k)$ designate the Bernoulli variable which is equal to one iff $\chi(\nu_{t_1}(k) + e^{n^{0.1}}) = 1$. When $\nu_{t_1}(e^{n^{0.2}}) \leq t_2 - e^{n^{0.1}}$ we define $\hat{h}_T$ the estimated frequency of one's during T in the following obvious way:

$$\hat{h}_T := \frac{1}{e^{n^{0.2}}} \sum_{k=1}^{e^{n^{0.2}}} \hat{X}_{t_1}(k).$$

Suppose we can significantly estimate the frequency of one's for T. Assume $E^n_{6\_2} \cap E_{1\_2}$ hold. Then all the stopping times $\nu_{t_1}(e^{n^{0.2}})$ stop the random walk S at one signal carrier, say $\bar{z}_i$. Because of the strong Markov property of S we get then that, conditional on ξ, the variables $X_{t_1}(k)$ are i.i.d. with expectations h_i. Now use Höffding inequality to see

$$P(|\hat{h}_T - h(i)| > e^{-n^{0.2}/4}) \leq \exp(-(2e^{n^{0.2}/2})).$$

Hence, with high probability, $\hat{h}_T$ is a precise estimate for $h(i)$.
The obtained preciseness of $\hat{h}_T$ is of the great importance. Namely, it is of smaller order than the typical variation of $h(i)$. In other words, with high probability $|h(i) - h(j)|$ is of much bigger order than $\exp(-n^{0.2}/4)$, $i \neq j$. To see this, consider (1.1). Note that, for each z, $\mu_i(z) := P(S(e^{n^{0.1}}) + \bar{z}_i = z)$ is constant, and, conditional under the event that in the radius of $L\exp(n^{0.1})$ are no more 2's in the scenery than $\bar{z}_i$, we have that $\xi(\bar{z}_i + z)$ are iid Bernoulli variables with parameter $\frac{1}{2}$. Hence

$$Var[h(i)] \leq \sum_{[-Le^{n^{0.1}}, Le^{n^{0.1}}]} \frac{1}{4}\Big(\mu_{0.2}(z)\Big)^2.$$

Since our random walk is symmetric we get that $\sum_{z\in[-Le^{n^{0.1}}, Le^{n^{0.1}}]} \frac{1}{4}\left(\mu_{0.2}(z)\right)^2$ is equal to 1/4 times the probability that the random walk is back at the origin after $2e^{n^{0.1}}$ time. By the central local theorem that probability is of order $e^{-n^{0.1}/2}$. This is much bigger than the order of the precision of the estimation of the frequencies of one's, $e^{-n^{0.2}/4}$. Since $h(i)$ is approximately normal, it is possible to show that with high probability all frequencies $h(0), h(1), \ldots, h(n^2+1)$ are more than $\exp(-n^{0.11})$ apart from $\frac{1}{2}$. Moreover, by the similar argument it is possible to show: if $\{\bar{z}_i\}_{i\in I}$ is the set of signal carriers that S encounters during the time $[0, m^2]$, then for each pair $i, j \in I$, the frequencies of ones satisfy $|h(i) - h(j)| > \exp(-n^{0.11})$. Let $E^n_{3\_2}$ be the set on which both statements holds.

Define $E_{OK} := E^n_{1\_2} \cap E^n_{3\_2} \cap E^n_{6\_2}$. From now on we assume that E_{OK} hold and we describe the $\hat{g}$-construction algorithm in this case :

Phase I) Determine the intervals $T \subseteq [0, m^2]$ containing more than $e^{n^{0.2}}$ two's (in the observations.) Let T_j designate the j-th such interval. Recall that these are the intervals where we can significantly estimate the frequency of one's. Let K designate the total number of such time-intervals in $[0, m^2]$.
Let $\pi(j)$ designate the index of the signal carrier $\bar{z}_i$ the random walk visits during time T_j (due to $E^n_{6\_2}$, the visited signal carriers are further apart than $Le^{n^{0.2}}$ from each other and, hence, there is only one signal carrier that can get visited during time T_j. Thus the definition of $\pi(j)$ is correct.)

Phase II) Estimate the frequency of one's for each interval T_j, $j = 1,\dots,K$. Obtain thus, based on the observations $\chi_0^{m^2}$ only, the vector $(\hat{h}_{T_1},\dots,\hat{h}_{T_K}) = \Big(\hat{h}(\pi(1)),\hat{h}(\pi(2)),\dots,\hat{h}(\pi(K))\Big)$. Here, $\hat{h}(i)$ denotes the estimate of $h(i)$, obtained by time interval T_j, with $\pi(j) = i$.

The further construction of the $\hat{g}$-reconstruction algorithm bases on an important property of the mapping $\pi : \{1,\dots,K\} \to \mathbb{Z}$ - with high probability π is a skip free walk, i.e. $|\pi(j) - \pi(j+1)| \leq 1$. Hence, the random walk during time $[0, m^2]$ is unlikely to go from one signal carrier to another without signaling all those in-between. By signaling those in-between, we mean producing in the observations for each signal carrier $\bar{z}_i$ a time intervals of length $e^{n^{0.3}}$ for which one can significantly estimate the frequency of one's $h(i)$. In particular, the skip-freeness implies that $\pi(1) \in \{0,1\}$. The skip-freeness of π is proved in Theorem 5.2.
Let $\pi_* := \min\{\pi(j) : j = 1,\dots,K\}$. Now $\pi_* \leq 1$. Let $\pi^* := \max\{\pi(j) : j = 1,\dots,K\}$. If $S(m^2) = m$, then, by $E^n_{1\_2}$, $\pi^* > n^2$.

Phase III) Apply clustering to the vector $(\hat{h}_{T_1},\hat{h}_{T_2},\dots,\hat{h}_{T_K})$, i.e. define

$$C_i := \{\hat{h}_{T_j} : |\hat{h}_{T_j} - \hat{h}_{T_i}| \leq 2\exp(-n^{0.12})\},\ \hat{f}_i := \frac{1}{|C_i|}\sum_{j\in C_i}\hat{h}_{T_j}, \quad i = 1,\dots,K.$$

By $E^n_{3\_2}$, we have $5\exp(-n^{0.12}) < \exp(-n^{0.11}) < |h(i) - h(j)|$, if n is big enough. Hence, $\hat{h}_{T_j} \in C_i$ iff $\pi(i) = \pi(j)$. Thus, for each different i, j either $C_i = C_j$ or $C_i \cap C_j = \emptyset$. Hence, $\hat{f}_j$ is the average of all estimates of $h(\pi(j))$ and, therefore, $\hat{f}_j$ is a good estimate of $h(\pi(j))$. Obviously,

$$\hat{f}_i = \hat{f}_j \quad \text{iff} \quad \pi(i) = \pi(j). \tag{1.3}$$

Thus, we can denote $\hat{f}(\bar{z}_i) := \hat{f}_j$, if $\pi(j) = i$ and (1.3) implies $\hat{f}(\bar{z}_i)) \neq \hat{f}(\bar{z}_j)$, if $i \neq j$.
After phrase III we, therefore, end up with a sequence of estimators $\hat{f}(\bar{z}_{\pi(1)}),\dots,\hat{f}(\bar{z}_{\pi(K)})$ that correspond to the sequence of frequencies $h(\pi(1)),\dots,h(\pi(1))$. Or, equivalently, $j \mapsto \hat{f}_j$ is a path of a skip-free random walk π on the set of different reals $\{\hat{f}(\bar{z}_{\pi_*}),\dots,\hat{f}(\bar{z}_{\pi^*})\}$.
The problem is that the estimates, $\hat{f}(\bar{z}_{\pi(1)}),\dots,\hat{f}(\bar{z}_{\pi(K)})$ are in the wrong order, i.e. we are not aware of the values $\pi(j)$, $j = 1,\dots,K$. But having some information about the values $\pi(j)$ is necessary for estimating the frequencies $h(1),\dots,h(n^2+1)$. So the question is: How can get from the sequence $\hat{f}(\bar{z}(_{\pi(1)}),\dots,\hat{f}(\bar{z}_{\pi(K)})$ the elements $\hat{f}(\bar{z}_1),\dots,\hat{f}(\bar{z}_{n^2+1})$? Or, equivalently: after observing the path of π on $\{\hat{f}(\bar{z}_{\pi_*}),\dots,\hat{f}(\bar{z}_{\pi^*})\}$, how can we deduce $\hat{f}(\bar{z}_1),\dots,\hat{f}(\bar{z}_{n^2+1})$?

1.4.3 Real scenery reconstruction algorithm

We now present the so-called **real scenery reconstruction algorithm** - $\mathcal{A}^{\mathbb{R}}_n$. This algorithm is able to answer to the stated questions up to the (swift by) one element.
The algorithm works due to the particular properties of π and $\{\hat{f}(\bar{z}_{\pi_*}),\dots,\hat{f}(\bar{z}_{\pi^*})\}$. These properties are:

A1) $\pi(1) \in \{0,1\}$, i.e. the first estimated frequency of one's, $\hat{f}_1$ must be either an estimate of $h(1)$ or of $h(0)$. Unfortunately there is no way to find out which one of the two signal carriers $\bar{z}_0$ or $\bar{z}_1$ was visited first. This is why our algorithm can reconstruct the real scenery up to the first or last bit, only;

A2) $\pi(K) > n^2$. This is true, because we condition on $S(m^2) = m$ and we assume that there are at least $n^2 + 1$ 2-s in $[0, m]$ (event $E^n_{1\_2}$);

A3) π is skip-free;

A4) $\hat{f}(\bar{z}_i) \neq \hat{f}(\bar{z}_j)$ $\forall j \neq i, \quad i, j \in \{\pi_*,\dots,\pi^*\}$.

Algorithm 1.2 *Let $\varkappa = (\varkappa_1,\varkappa_2,\dots,\varkappa_K)$ be the vector of real numbers such that the number of different reals in $\varkappa$ is at least $n^2 + 1$. The vector $\varkappa$ constitutes the input for $\mathcal{A}^{\mathbb{R}}_n$.*

Define $\mathcal{R}_1 := \varkappa_1$. From here on we proceed by induction on j : once $\mathcal{R}_j$ is defined, we define $\mathcal{R}_{j+1} : \varkappa_s$, with $s := 1 + \max\{j : \varkappa_j = \mathcal{R}_j\}$. Proceed until $j = n^2 + 1$ and put

$$\mathcal{A}_n^{\mathbb{R}}(\varkappa) := \big(\mathcal{R}_2, \mathcal{R}_3, \dots, \mathcal{R}_{n^2+1}\big).$$

The idea of the algorithm is very simple: take the first element $\varkappa_1$ of $\varkappa$ and consider all elements of the input vector $\varkappa$ that are equal to $\varkappa_1$ and find the one with the biggest index (the last $\varkappa_1$). Let j_1 be this index. Then take $\varkappa_{j_1+1}$ as the first output and look for the last $\varkappa_{j_1+1}$. Let the corresponding index be j_2 and take $\varkappa_{j_2+1}$ as the second output. Proceed so $n^2 + 1$ times.
Let us proof that the algorithm $\mathcal{A}_n^{\mathbb{R}}$ works. In our case the input vector is $\hat{f} := (\hat{f}_1, \dots, \hat{f}_K)$.

Proposition 1.1 *Let $\{\hat{f}(\bar{z}_{\pi_*}), \dots, \hat{f}(\bar{z}_{\pi^*})\}$ and π satisfy A1), A2), A3), A4). Then*

$$\mathcal{A}_n^{\mathbb{R}}(\hat{f}) \in \{(\hat{f}(\bar{z}_1), \dots, \hat{f}(\bar{z}_{n^2})), (\hat{f}(\bar{z}_2), \dots, \hat{f}(\bar{z}_{n^2+1}))\}, \quad \text{i.e.} \quad \mathcal{A}_n^{\mathbb{R}}(\hat{f}) \preccurlyeq (\hat{f}(\bar{z}_1), \dots, \hat{f}(\bar{z}_{n^2+1})).$$

Proof. By A1) we have that the first element of the input vector, $\hat{f}_1$, is either $\hat{f}(\bar{z}_1)$ or $\hat{f}(\bar{z}_0)$. Consider the first case. Thus $\mathcal{R}_1 = \hat{f}(\bar{z}_1)$. Proceed by induction: suppose that $\mathcal{R}_j = \hat{f}(\bar{z}_j)$, $j < n^2 + 1$. Let $i(j)$ be the index of the last $\hat{f}(\bar{z}_j)$ in vector $\hat{f}$. By A2), $i(j) < K$. Since π is skip-free and ends to the right of n^2, we have that after the last visits of $\hat{f}(\bar{z}_j)$, the next observation must be $\hat{f}(\bar{z}_{j+1})$. Hence, in this case, $(\mathcal{R}_1, \dots, \mathcal{R}_{n^2+1}) = (\hat{f}(\bar{z}_1), \dots, \hat{f}(\bar{z}_{n^2+1}))$ and $\mathcal{A}_n^{\mathbb{R}}(f) = (\hat{f}(\bar{z}_2), \dots, \hat{f}(\bar{z}_{n^2+1}))$.
Similarly, if the first element of the $\hat{f}$ is $\hat{f}(\bar{z}_0)$, then $(\mathcal{R}_1, \dots, \mathcal{R}_{n^2+1}) = (\hat{f}(\bar{z}_0), \dots, \hat{f}(\bar{z}_{n^2}))$ and $\mathcal{A}_n^{\mathbb{R}}(f) = (\hat{f}(\bar{z}_1), \dots, \hat{f}(\bar{z}_{n^2}))$. ■

Phase IV) Apply $\mathcal{A}_n^{\mathbb{R}}$ to $\hat{f}$. Denote the output $\mathcal{A}_n^{\mathbb{R}}(\hat{f})$ by $(f_1, \dots, f_{n^2})$. By Proposition1.1 we know

$$(f_1, \dots, f_n) \preccurlyeq (\hat{f}(\bar{z}_1), \dots, \hat{f}(\bar{z}_{n^2+1}). \tag{1.4}$$

Now recall that we are interested in reconstructing the $g_i(\xi_0^m) := I_{[0,5)}(h(i))$ rather than $\hat{h}(i)$. Thus, having estimates for $h(\bar{z}_i)$, namely $\hat{f}(\bar{z}_i)$, we use the obvious estimator for g_i: $I_{[0,0.5)}(f_i)$.

Phase V) Define the final output of $\hat{g}$

$$\hat{g}(\chi_0^{m^2}) := \Big(I_{[0.5,1]}(f_1), \dots I_{[0.5,1]}(f_{n^2})\Big).$$

Recall that because of $E_{3\text{-}2}^n$, with high probability all random variables $h(1), \dots, h(n^2 + 1)$ are more than $\exp(-n^{0.11})$ apart from $\frac{1}{2}$. Since $\exp(-n^{0.11})$ is much bigger than the preciseness of our estimate, with high probability we have $\hat{f}(\bar{z}_i) < 0.5$ iff $h(\bar{z}_i) < 0.5$. By (1.4) this means

$$\hat{g}(\chi_0^{m^2}) = \Big(I_{[0.5,1]}(f_1), \dots I_{[0.5,1]}(f_n^2)\Big) \preccurlyeq \Big(I_{[0.5,1]}(h(\bar{z}_1)), \dots I_{[0.5,1]}(h(z_{n^2+1}))\Big) = g(\xi_0^m).$$

Hence, when E_{OK} holds, then $\hat{g}$ is properly defined and the probability (1.2) is high. Since we are not interested in $\hat{g}$ when E_{OK} does not hold, we extend the definition of $\hat{g}$ arbitrary to E_{OK}^c.

2 Whole truth about signal probabilities

In the previous section we considered the case where the scenery has three colors $\{0,1,2\}$. The places of the 2's where called the signal carriers. The i-th such place was denoted by $\bar{z}_i$. In reality we have only two colors 0 and 1. Thus, we need to show that with 2 colors we also manage to define signal carriers $\bar{z}_i$ in such a way that the following holds:

a) Whenever the random walk passes close to a signal carrier, we can recognize that the random walk is close to a signal carrier point by looking at the observations (with high probability).

b) The probability to think that one recognizes a signal carrier when one is actually not close to one is so small that this type of mistake never happens up to time m^2.

c) When we pass a signal carrier we are able to estimate its frequency of one's with high enough precision (with high probability).

In the present section we define and investigate an important concept related to the signal carriers - Markov signal probability.

2.1 Definitions

In this subsection we will define the main notions of the section: delayed signal probability, strong signal probability and Markov signal probability. We also give a few equivalent characterizations of these concepts, and we try to explain their meaning. In the end of the subsection we give a formal definition of the frequency of ones.
At first, some definitions.

* Let $D \subset \mathbb{Z}$ and let $\zeta : D \longrightarrow \{0,1\}$. For example, ζ can be the original scenery, ξ or the observations, χ.
Let $T = [t_1, t_2] \subset D$ be an integer interval of length at least 3. Then we say that T is a **block** of ζ iff we have that

$$\zeta(t_1) = \zeta(t_2) \neq \zeta(t), \forall t \in]t_1, t_2[$$

We call $t_2 - t_1$ the length of the block T. The point t_1 is called the beginning of the block. For example, T is a block of ζ with length 4, if $\zeta | T = 01110$.

* Let $T = T(\chi) \subset \mathbb{N}$ be a time interval, possible depending from observations. For example, T can be a block of χ or $T = [t, t+n^2]$ can be a time interval of length n^2+1 such that $\chi(t) = \chi(t+1) = \cdots, \chi(t+n^2)$.
Let $I \subset \mathbb{Z}$ be an integral interval (a location set). We say that T **was generated** (by S) **on** I, iff $\forall t \in T, S(t) \in I$.

* We now define the delayed signal probability. To simplify the notations afterwards, define

$$M = M(n) := n^{1000} - n^2, \quad \tilde{M} := n^{1000} - 2n^2.$$

Fix an $z \in \mathbb{Z}$ and let S_z denote the random walk translated by z, i.e. for all $t \in \mathbb{N}$, $S_z(t) := S(t) + z$. We define the random variable δ_z^d in the following way:

$$\delta_z^d := P\left(\xi(S_z(M)) = \cdots = \xi\left(S_z\left(n^{1000}-1\right)\right) = \xi\left(S_z\left(n^{1000}\right)\right) \Big| \xi\right). \tag{2.1}$$

In other words, δ_z^d is the conditional probability (conditional on ξ) to observe only one color in time interval $[n^{1000} - n^2, n^2]$ if the random walk starts at z. We shall call δ_z^d **delayed signal probability at** z.
During time n^{1000} the random walk can not move more than Ln^{1000}. Thus, δ_z^d depends only on the scenery ξ in the interval $\left[z - Ln^{1000}, z + Ln^{1000}\right]$. Let, for each $z \in \mathbb{Z}$

$$I_z := [z - Ln^{1000}, z + Ln^{1000}]. \tag{2.2}$$

We have that δ_z^d is a random variable which is measurable with respect to $\sigma(\xi(s)|s \in I_z)$. Since the distribution of ξ is translation invariant, the distribution of δ_z^d does not depend on z.

* For some technical reason only, we need a stronger version of the delayed signal probability. Again, let $z \in \mathbb{Z}$. We define the **strong signal probability** at z, $\tilde{\delta}_z^d$, as follows

$$\tilde{\delta}_z^d := P\Big(\xi(S_z(M)) = \cdots = \xi(S_z(n^{1000})), \quad S_z(M+1), S_z(2), \ldots, S_z(n^{1000}) \in [z - L\tilde{M}, z + L\tilde{M}]\Big|\xi\Big).$$

Note that $\tilde{\delta}_z^d$ is measurable with respect to the sigma algebra $\sigma(\xi(s)|s \in [z - L\tilde{M}, z + L\tilde{M}])$.
Also note that, obviously, $\delta_z^c \geq \tilde{\delta}_z^c$. However, the difference is not too big. Indeed, by Höffding's inequality (see below), we have that, for some constant $d > 0$

$$\begin{aligned} \delta_z^d - \tilde{\delta}_z^d &= P\Big(\xi(S_z(M)) = \cdots = \xi(S_z(n^{1000})), \quad \exists s \in \{M, \ldots, n^{1000}\} : |z - S_z(s)| > L\tilde{M}\Big|\xi\Big) \\ &\leq P\Big(|S(M)| > L(\tilde{M} - n^2)\Big) \leq \exp(-dn^{999}). \end{aligned} \tag{2.3}$$

* We now define the Markov signal probability at z.

Let $z \in \mathbb{Z}$. Roughly speaking, the Markov signal probability at z, denoted by δ_z^M, is the conditional (on ξ) probability to have (at least) $n^2 + 1$ times the same color generated on I_z exactly $n^{1000} - n^2$ after we observe $n^2 + 1$ times the same color generated on I_z. In this formulation the part "after we observe a string of $n^2 + 1$ times the same color generated on I_z" needs to be clarified. The explanation is the following: every time there is in the observations n^2+1 times the same color generated on I_z, we introduce a stopping time $\nu_z(i)$. The position of the random walk at these stopping times defines a Markov chain with state space I_z. As we will prove later, this Markov chain $\{S(\nu_z(k))\}_{k\geq 1}$ converges very quickly to a stationary measure, say μ_z. So, by "M after we observe $n^2 + 1$ times the same color generated on I_z" we actually mean: "M after starting the random walk from an initial position distributed according to μ_z". Since the distribution of $S(\nu_z(i))$ converges quickly to μ_z, δ_z^M is close to the probability of observing $n^2 + 1$ times the same color generated on I_z exactly M after time $\nu_z(i)$. In other words, δ_z^M is close to the conditional (on ξ) probability of the event that we observe only one color in the time interval $[\nu_z(i) + n^{1000} - n^2, \nu_z(i) + n^{1000}]$ and that during that time interval the random walk S is in I_z. Thus (for k big enough) δ_z^M is close to:

$$P\Big(\chi(\nu_z(i) + M) = \cdots = \chi(\nu_z(i) + n^{1000}) \quad \text{and} \quad S(\nu_z(i) + M), \ldots, S(\nu_z(i) + n^{1000}) \in I_z|\xi\Big) \tag{2.4}$$

The ergodic theorem then implies that on the long run the proportion of stopping times $\nu_z(i)$ which are followed after M by the $n^2 + 1$ observations of the same color generated on I_z converges a.s. to δ_z^M. Actually, to make some subsequent proofs easier, we do not take a stopping time $\nu_z(i)$ after each $n^2 + 1$ observations of the same color generated on I_z. Rather we also ask that the stopping times be apart by at least $e^{n^{0.1}}$.
In order to prove how quickly we converge to the stationary measure, we also view the explained notions in terms of a regenerative process. The renewal times will be defined as the stopping times, denoted by $\vartheta_z(k)$, which stop the random walk at the point $z - 2Le^{n^{0.1}}$. To simplify some proofs, we also require that there is at least one stopping $\nu_z(i)$ between $\vartheta_z(k)$ and $\vartheta_z(k+1)$. Thus $\vartheta_z(0)$ denotes the first visit by the random walk S to the point $z - 2Le^{n^{0.1}}$. We define $\nu_z(1)$ to be the first time after $\vartheta_z(0)$ where there happens to be $n^2 + 1$ times the same color generated on I_z. Then, $\vartheta_z(1)$ is the first return of S to $z - 2Le^{n^{0.1}}$ after $\nu_z(1)$ and so on. Let us give the formal definitions of all introduced notions.

* Let $\vartheta_z(0)$ denote the first visit of S to the point $z - 2Le^{n^{0.1}}$. Thus

$$\vartheta_z(0) = \min\{\, t \geq 0 |\, S(t) = z - 2Le^{n^{0.1}}\}.$$

* Let $\nu_z(1)$ designate the first time after $\vartheta_z(0)$ where we observe n^2+1 zero's or one's in a row, generated

on I_z. More precisely:

$$\nu_z(1) := \min\left\{ t > \vartheta_z(0) \,\middle|\, \begin{array}{c} \chi(t) = \chi(t-1) = ... = \chi\left(t - n^2\right) \\ \text{and } S(t-n^2), S(t-n^2+1), \ldots, S(t) \in I_z \end{array} \right\}$$

Once $\nu_z(i)$ is well defined, define $\nu_z(i+1)$ in the following manner:

$$\nu_z(i+1) := \min\left\{ t > \nu_z(i) + e^{n^{0.1}} \,\middle|\, \begin{array}{c} \chi(t) = \chi(t-1) = ... = \chi\left(t - n^2\right) \\ \text{and } S(t-n^2), S(t-n^2+1), \ldots, S(t) \in I_z \end{array} \right\}$$

* Let $\vartheta_z(k)$ denote the consecutive visits of S to the point $z - 2Le^{n^{0.1}}$ provided that between two visits random walk S generates (at least once) $n+1$ consequtive 0-s or 1-s on I_z. More precisely,

$$\vartheta_z(k+1) := \min\{t > \vartheta_z(k) | S(t) = z - 2Le^{n^{0.1}}, \exists j : \vartheta_z(k) < \nu_z(j) < t\}, \quad k = 1, 2 \ldots.$$

Basically, the definition above says: if $\vartheta_z(k)$ is defined, we wait until we observe $n^2 + 1$ same colors generated on I_z. Since $S(\vartheta_z(k)) = z - 2Le^{n^{0.1}}$, then the first n^2+1 same colors generated on I_z can not happen earlier than $e^{n^{0.1}}$ times after $\vartheta_z(k)$. This means, the first n^2+1 same colors generated on I_z can not happen earlier than $e^{n^{0.1}}$ times after last stopping time ν_z, say $\nu_z(i)$ (this happens before $\vartheta_z(k)$). Thus, the first n^2+1 same colors generated on I_z is actually $\nu_z(i+1)$. Observing $\nu_z(i+1)$, we just wait for the next visit of S to the $z - 2Le^{n^{0.1}}$. This defines $\vartheta_z(k+1)$.

* Let $X_{z,i}$, $i = 1, 2, \ldots$ designate the Bernoulli variable which is equal to one if exactly after time M the stopping time $\nu_z(i)$ is followed by a sequence of n^2+1 one's or zero's generated on I_z. More precisely, $X_{z,i} = 1$ iff

$$\chi(\nu_z(i) + M) = \chi(\nu_z(i) + M + 1) = \cdots = \chi(\nu_z(i) + n^{1000})$$

and

$$S(j) \in I_z \quad \forall j = \nu_z(i) + M, \ldots, \nu_z(i) + n^{1000}$$

* Define $\kappa_z(0) := 0$. Let $\kappa_z(k)$ designate the number of stopping times $\nu_z(k)$ occurring during the time from $\vartheta_z(0)$ to $\vartheta_z(k)$. Thus $\kappa_z(k)$ is defined by the inequalities:

$$\nu_z(\kappa_z(k)) \le \vartheta_z(k) < \nu_z(\kappa_z(k) + 1).$$

For all k, $S(\vartheta_z(k)) = z - 2Ln^{1000}$. Hence, for all i, $\vartheta_z(k) \neq \nu_z(i)$ and the inequalities above are strict.

* Define the following variables:

$$\mathcal{X}_z(k) = \sum_{i=\kappa(k)+1}^{\kappa(k)} X_{z,i}, \quad \mathcal{Z}_z(k) {=} \kappa(k) - \kappa(k-1), \quad k = 1, 2, \ldots$$

Thus, $\mathcal{Z}_z(k)$ is the number of stopping times occurring during the time interval from time $\vartheta_z(k-1)$ to time $\vartheta_z(k)$. Note that $\mathcal{Z}_z(k) \ge 1$, $\forall k$. The random variable $\mathcal{X}_z(k)$ designates the number of such stopping times which, during the same time interval, have been followed exactly after time M by a sequence of n^2+1 0's or 1's generated on I_z. Note that conditional on ξ the variables $\mathcal{X}_z(1)$, $\mathcal{X}_z(2), \ldots$ are i.i.d. and the same holds for $\mathcal{Z}_z(1)$, $\mathcal{Z}_z(2), \ldots$.

* We define:

$$\delta_z^M := \frac{E\left[\mathcal{X}_z(1) \middle| \xi\right]}{E\left[\mathcal{Z}_z(1) \middle| \xi\right]} \tag{2.5}$$

We call δ_z^M **Markov signal probability** at z.

In the following we give some equivalent forms of (2.5).

Note that conditional on ξ, $X_{z,i}$ is a regenerative process with respect to the renewal $\kappa_z(k)$. Hence, conditioning on ξ, we have

$$\lim_{r\to\infty}\sum_{i=1}^{r}\frac{X_{z,i}}{r}=\lim_{k\to\infty}\sum_{i=1}^{\kappa(k)}\frac{X_{z,i}}{\kappa(k)}=\lim_{k\to\infty}\frac{\sum_{i=1}^{k}\mathcal{X}_z(i)}{\sum_{i=1}^{k}\mathcal{Z}_z(i)}=\frac{E\left[\mathcal{X}_{z,1}|\,\xi\right]}{E\left[\mathcal{Z}_{z,1}|\,\xi\right]}.\quad \text{a.s.} \tag{2.6}$$

We count (up to time r) all sequences of length n^2+1 of one's or zero's, generated on the interval I_z according to stopping times $\nu_z(i)$, $k=1,2,\ldots$. Among such sequences, the proportion of those sequences which are followed after exactly time M by another sequence of n^2+1 zero's or one's generated on the interval I_z converges a.s. to δ_z^M, as r goes to infinity.

On the other hand, the limit in (2.6) can be represented as follows. Fix ξ and z. Let $Y_i := S(\nu_z(i))$, $i=1,2,\ldots$ denote the Markov chain obtained by stopping the random walk S by $\nu_z(i)$. The state space of Y_i is I_z. Because of the nature of S, Y_i is finite, irreducible aperiodic and, therefore, ergodic Markov chain.

Let μ_z denote the stationary distribution of $\{Y_k\}$. In the present section the z is fixed, so we skip this from the notations and write μ. The measure μ is an discrete probability measure on I_z, so $\mu=(\mu(j))_{j\in I_z}$. For each state, $j\in I_z$ define the hitting times $\tau_j(l)$, $l=1,2,3,\ldots$. Formally,

$$\tau_j(1):=\min\{i\geq 1: Y_i=j\},\quad \tau_j(l):=\min\{i>\tau_j(l-1): Y_i=j\},\quad l=2,3\ldots.$$

Hence,

$$\frac{1}{r}\sum_{i=1}^{r}X_{z,i}=\sum_j\frac{N_j(r)}{r}\frac{1}{N_j(r)}\sum_{l=1}^{N_j(r)}X_{z,\tau_j(l)},$$

where $N_j(r):=\max\{l:\tau_j(l)\leq r\}$, $r=1,2,3,\ldots$. Since $\tau_j(l)$, $l=1,2,3,\ldots$ is a (delayed) renewal process with the corresponding renewal numbers $N_j(r)$ and with the expected renewal time $\frac{1}{\mu(j)}$ we get

$$\frac{N_j(r)}{r}\to\mu(j)\quad \text{a.s..}$$

On the other hand, $X_{z,i}$ is a regenerative process with respect to each $\tau_j(l)$, $l=1,2,3,\ldots$. Hence

$$\frac{1}{N_j(r)}\sum_{l=1}^{N_j(r)}X_{z,\tau_j(l)}\to E[X_{z,\tau_j(2)}],\quad \text{as}\quad r\to\infty\quad \text{a.s..}$$

Since $E[X_{z,\tau_j(2)}]=P(X_{z,\tau_j(2)}=1)$. The latter equals

$$P\Big(S_j(M),S_j(M+1),\cdots,S_j(n^{1000})\in I_z\quad \text{and}\quad \xi(S_j(M))=\xi(S_j(M+1))=\cdots=\xi(S_j(n^{1000})\Big).$$

This can be rewritten as

$$\sum_{l\in I_z}P(j,l)\delta_z(l),$$

where $P(j,l):=P(S(M)=j-l)$ and

$$\delta_z(l):=P\Bigg(S_l(0),S_l(1),\ldots,S_l(n^2)\in I_z\quad \text{and}\quad \xi(S_l(0))=\xi(S_l(1))=\ldots=\xi(S_l(n^2))\Bigg) \tag{2.7}$$

Hence

$$\delta_z^M=\sum_{j\in I_z}\mu(j)P\Big(S_j(M),S_j(M+1),\cdots,S_j(n^{1000})\in I_z,\quad \xi(S_j(M))=\cdots=\xi(S_j(n^{1000})\Big) \tag{2.8}$$

or

$$\delta_z^M = \sum_{j,l \in I_z} \mu(j) P(j,l) \delta_z(l). \tag{2.9}$$

Using the same notation, we have an equivalent form of delayed signal probability

$$\delta_z^d = \sum_{l=I_z} P(z,l) \delta_z(l). \tag{2.10}$$

The formula (2.9) can be interpreted as follows: let U be a random variable with distribution μ_z and let S be a random walk, independent of U. Let S_U denote the translation of S by U, i.e., for each t, $S_U(t) = U + S(t)$. Then (2.9) states

$$\delta_z^M = P\Big(\xi(S_U(M)) = \cdots = \xi(S_U(n^{1000})) \quad \text{and} \quad S_U(M), \cdots, S_U(n^{1000}) \in I_z|\xi\Big). \tag{2.11}$$

Thus, δ_z^M is a limit-version of (2.4).

* We now define the frequency of ones. To obtain the consistency with the Markov signal probability, we formally define the frequency of ones in terms of regenerative processes. However, we also derive the analogue of (2.11), which explains the meaning of the notion.
Let $U_{z,i} = \xi(S(\nu_z(i) + e^{n^{0.1}}))$ and define

$$\mathcal{U}_z(k) = \sum_{i=\kappa(k)+1}^{\kappa(k)} U_{z,i}.$$

Now, let

$$h(z) := \frac{E(\mathcal{U}_z(1)|\xi)}{E(\mathcal{Z}_z(1)|\xi)}.$$

The random variable $h(z)$ is $\sigma(\xi(i) : i \in [z - L(n^{1000} + e^{n^{0.1}}), z + L(n^{1000} + e^{n^{0.1}})]$- measurable; $h(z)$ is called as **frequency of ones** at z. As in (2.6), conditioning on ξ, we have

$$\lim_{r\to\infty} \sum_{i=1}^{r} \frac{\mathcal{U}_{z,i}}{r} = h(z) \quad \text{a.s..}$$

With the same argument as above, we get

$$\lim_{r\to\infty} \frac{1}{r} \sum_{i=1}^{r} U_{z,i} = \lim_{r\to\infty} \sum_j \frac{N_j(r)}{r} \frac{1}{N_j(r)} \sum_{l=1}^{N_j(r)} U_{z,\tau_j(l)} = \sum_j \mu(j) E(U_{z,\tau_j(2)}).$$

Now,

$$E(U_{z,\tau_j(2)}) = \sum_{i=j-Le^{n^{0.1}}}^{i=j+Le^{n^{0.1}}} \xi(i) P(S_j(i))$$

and, therefore

$$h(z) = \sum_{j=I_z} \mu(j) \sum_{i=j-Le^{n^{0.1}}}^{j+Le^{n^{0.1}}} \xi(i) P(S_j(i)) = \sum_{i=z-L(n^{1000}+e^{n^{0.1}})}^{z+L(n^{1000}+e^{n^{0.1}})} \xi(i) \sum_{j=I_z} \mu(j) P(S_j(e^{n^{0.1}}) = i). \tag{2.12}$$

Now, it is easy to see that in terms of U and S as in (2.11), i.e. U and S are independent, U has law μ_z, we have

$$h(z) = P(\xi(U + S(e^{n^{0.1}})) = 1) = E[\xi(U + S(e^{n^{0.1}}))|\xi], \tag{2.13}$$

2.2 Auxiliary results

In the present section we investigate the relations between δ_z^M and δ_z^d. Note that they only depend on the scenery ξ in the interval $[z - Ln^{1000}, z + Ln^{1000}]$. In other words,

$$\delta_z^M, \delta_z^d \in \sigma\Big(\xi(j)|j \in [z - Ln^{1000}, z + Ln^{1000}]\Big).$$

The distribution of both δ_z^M and δ_z^d does not depend on particular choice of z. Hence, w.l.o.g., in the following we consider the point $z = 0$, only.

Define $p_M := \max\{P(S(M) = z)|z \in \mathbb{Z}\}$.
We call a block **big**, if its length is bigger than $\frac{n}{\ln n}$.

Proposition 2.1 *For any $c_\delta \in [p_M, 2p_M]$ we have that the following holds:*

a $P(\delta_z^d \geq c_\delta) \leq \exp(-\alpha n/\ln n)$, where $\alpha := \ln(1.5)$

b $P\left(\delta_z^d \geq c_\delta\right) \geq (0.5)^n > \exp(-n)$

c If all blocks of $\xi|[z - Ln^{1000}, z + Ln^{1000}]$ are shorter than $n/\ln n + 1$, then $\delta_z^d < c_\delta$. Formally:

$$\left\{\delta_z^d \geq c_\delta\right\} \subseteq \left\{ \ [z - Ln^{1000}, z + Ln^{1000}] \text{ contains a big block of } \xi \ \right\}$$

d Conditional on $\left\{\delta_z^d \geq c_\delta\right\}$ it is likely that $[z - Ln^{1000}, z + Ln^{1000}]$ contains at most $0.5 \ln n$ big blocks of ξ. More precisely:

$$P\left(E_{\delta,z}^c \middle| \, \delta_z^d \geq c_\delta\right) \leq \left(2Ln^{1000}\right)^{0.5 \ln n} (0.5)^{-0.5n}$$

where

$$E_{\delta,z} := \left\{ \ [z - Ln^{1000}, z + Ln^{1000}] \text{ has less than } 0.5 \ln n \text{ big blocks of } \xi \ \right\}$$

In order to prove the Proposition 2.1, we use the following lemma. The proof of it can be found in [18]

Lemma 2.1 *There exists a constant $a > 0$ such that for each $t, r \in \mathbb{N}$, for each subset $I \subset \mathbb{Z}$, and for each $j \in I$ and for every mapping $\zeta : \mathbb{Z} \longrightarrow \{0, 1\}$ we have the following implication:*

if all blocks of ζ in I are shorter or equal to r, then

$$P\left(\begin{array}{c} \zeta\left(S_j\left(0\right)\right) = \zeta\left(S_j\left(1\right)\right) = \cdots = \zeta\left(S_j\left(t\right)\right) \\ \textit{and } S_j\left(0\right), S_j\left(1\right), ..., S_j\left(t\right) \in I \end{array} \right) \leq \exp\left(-\frac{at}{r^2}\right).$$

Proof that c holds: W.l.o.g. assume $z = 0$. Suppose that the length of all blocks of $\xi|[-Ln^{1000}, Ln^{1000}]$ is at most $n/\ln n$. Let $I := [-Ln^{1000}, Ln^{1000}]$. Denote $\delta(l) = \delta_0(l)$, where $\delta_0(l)$ is as in (2.7). If the all the blocks in I are not longer than $n/\ln n$ we get by Lemma2.1 that for all $j \in I$

$$\delta(j) \leq \exp\Big(-\frac{an^2}{(n/\ln n)^2}\Big) = n^{-a \ln n}.$$

By (2.10) we get that

$$\delta_0^d = \sum_{j=-Ln^{1000}}^{Ln^{1000}} P(0, j)\delta(j) \leq \sum_{j=-Ln^{1000}}^{Ln^{1000}} P(0, j)n^{-a \ln n} \leq n^{-a \ln n} \tag{2.14}$$

The expression on the right side of the last inequality is of smaller order than any negative polynomial order in n. By the local central limit theorem p_M is of order $n^{-\frac{M}{2}}$. Thus, for n big enough

$$\delta_0^d < p_M \leq c_\delta.$$

Proof that a holds: W.l.o.g. assume $z=0$. Define the event

$$E_z := \{\xi(z) = \xi(z+1) = \cdots = \xi(z + \frac{n}{\ln n})\}$$

Part c states that

$$\{\delta_0^d \geq c_\delta\} \subseteq \bigcup_{z \in [-Ln^{1000}, Ln^{1000}]} E_z.$$

Thus,

$$P\left(\delta_0^d \geq c_\delta\right) \leq \sum_{z=-Ln^{1000}}^{Ln^{1000}} P(E_z).$$

Now, clearly

$$P(E_z) = \exp\left(-\frac{\ln(2)n}{\ln n}\right)$$

So,

$$P(\delta_0^d \geq c_\delta) \leq 2Ln^{1000} \exp\left(-\frac{\ln(2)\, n}{\ln n}\right). \tag{2.15}$$

The dominating term in the product on the right side (2.15) is $\exp(-\ln(2)\, n/\ln n)$. Hence, for n big enough, the expression on the right side of (2.15) is smaller than $\exp(-\frac{\ln(1.5)n}{\ln n})$.

Proof that b holds: It is suffices to prove that

$$P(\delta_z^d \geq 2p_M) \geq (0.5)^n.$$

W.l.o.g assume $z=0$. Define $E := \{\xi(0) = \xi(1) = \cdots = \xi(n)\}$. We are going to show that

$$E \subseteq \{\delta_0^d \geq 2p_M\} \quad \text{and} \quad P(E) \geq \exp(-n).$$

Recall the definition of $\delta(j)$. If E holds, then for any $j \in [0,n]$ we have

$$\delta(j) \geq P\Big(S_j(t) \in [0,n], \forall t \in [0,n^2]\Big)$$

Now, because of the central limit theorem, there is a constant $b>0$ not depending on n, such that for all $j \in [n/3, 2n/3]$ we have:

$$P\Big(S_j(t) \in [0,n], \forall t \in [0,n^2]\Big) > b.$$

By the local central limit theorem, again, for all $j \in [n/3, 2n/3]$ we have, for n big enough, that

$$P(0,j) \geq \frac{p_M}{2}. \tag{2.16}$$

Using (2.10) and (2.16) we find that when E holds, then

$$\delta_0^d \geq \sum_{j=\frac{n}{3}}^{\frac{2n}{3}} bP(0,j) \geq \frac{bnp_M}{6} \tag{2.17}$$

For n big enough, obviously the right side of (2.17) is bigger than $2p_M$. This proves $E \subseteq \{\delta_0^d \geq 2p_M\}$. Furthermore, we have that $P(E) = 0.5^n$. The inequality $0.5^n > \exp(-n)$ finishes the proof.

Proof that d holds: W.l.o.g. assume $z = 0$. For a block T, the point $\inf T$ is called the beginning of the block. Let $t_1, t_2, \dots$ denote the beginnings of the consecutive big blocks in $[-Ln^{1000}, \infty)$. Define $t_0 := -Ln^{1000}$ and $g_i := t_i - t_{i-1}$, $i = 1, 2, \dots$. So, g_i measures the distances between consecutive big blocks. Clearly, g_i-s are iid. Note,

$$E^c_{\delta,0} \subset \Big\{ \sum_{i=1}^{0.5 \ln n} g_i \leq 2Ln^{1000} \Big\} \subset \cap_{i=1}^{0.5 \ln n} \Big\{ g_i < 2Ln^{1000} \Big\}.$$

Note

$$P(g_1 < 2Ln^{1000}) \leq \sum_{z=t_0}^{Ln^{1000}-1} P(\text{a big block begins at } z) \leq 2Ln^{1000}(0.5)^{\frac{n}{\ln n}}.$$

Hence,

$$P(E^c_{\delta,0}) \leq P(g_i \leq 2Ln^{1000})^{0.5 \ln n} = \big(2Ln^{1000}\big)^{0.5 \ln n}(0.5)^{0.5n}.$$

Combining this with b, we get

$$P(E^c_{\delta,0} | \delta_0^d > c_\delta) \leq \frac{P(E^c_{\delta,0})}{P(\delta_0^d > c_\delta)} \leq \big(2Ln^{1000}\big)^{0.5 \ln n}(0.5)^{-0.5n} \to 0.$$

Lemma 2.2

$$P\left(\delta_z^d \geq c_\delta\right)\left(2Ln^{1000}\right)^{-0.5 \ln n} \leq 2P\left(\delta_z^d \wedge \delta_z^M \geq c_\delta(1 - O(M^{-\frac{1}{2}}))\right).$$

2.3 Proof of Lemma 2.2

In the present subsection we prove Lemma 2.2. During the rest of the section we assume $z = 0$.
At first we define fences.

Fences

* An interval $[t, t + 4L - 1] \subset D$ is called a **fence** of ζ, if

$$\begin{gathered} 0 = \zeta(t) = \zeta(t+1) \cdots = \zeta(t+L-1) \neq \zeta(t+L) = \cdots = \zeta(t+2L-1) \neq \\ \zeta(t+2L) = \cdots = \zeta(t+3L-1) \neq \zeta(t+3L) = \cdots = \zeta(t+4L-1) \end{gathered}$$

The point $t + 2L$ is the **breakpoint** of the fence. So, T is a fence of ζ corresponding to the $L = 3$, iff $\zeta | T = 000111000111$.

Let $z_0 := -Ln^{1000}$ and let z_k, $k = 1, 2, \dots$ be defined inductively: z_k denotes the breakpoint of the first fence of scenery ξ in $[z_k + 4L, \infty)$. We call the points z_k the breakpoints of consecutive fences (of scenery ξ). Define $l_i := z_i - z_{i-1}$, $i = 1, 2, \dots$ and $N := \max\{k : z_{k-1} \leq Ln^{1000}\} < Ln^{1000}$. The random variables l_i measure the distances between the breakpoints of consecutive fences, they are iid. Let $l := Ln^{1000} - z_N$, $l \leq l_{N+1}$. The moment generating function of l_1, say $M(\lambda)$, does not depend on n and it is finite, if $\lambda > 0$ is small enough. Let $M := \exp(\lambda l_1) < \infty$ and choose $C > 1$ such that $\lambda C > 1$. Now define the event

$$E_b := \{l, l_i \leq Cn, \quad i = 1, 2, \dots, N\}$$

and apply the large deviation inequality to see $P(l_1 > Cn) = P(\lambda l_1 > \lambda Cn) < Me^{-\lambda Cn}$. Now,

$$P(E_b^c) \leq \sum_{i=1}^{Ln^{1000}} P(l_i > Cn) = Ln^{1000} P(l_1 > Cn) < Ln^{1000} Me^{-\lambda Cn}.$$

Applying b, we get

$$P(E_b^c|\delta_0^d \geq c_\delta) \leq \frac{P(E_b^c)}{P(\delta_0^d \geq c_\delta)} \leq Ln^{1000} M e^{(1-\lambda C)n} \to 0. \tag{2.18}$$

Mapping

Let $\mathcal{O}$ denote the all possible pieces of sceneries in $I := [-Ln^{1000}, Ln^{1000}]$, i.e. $\mathcal{O} := \{0,1\}^I$. The random variables δ_0^d, δ_0^M as well as the events $\{\delta_0^d > c_\delta\}$, $E_{\delta,0}$, E_b depend on the restriction of the scenery in I, only. Hence they can be defined on the probability space $(\mathcal{O}, 2^{\mathcal{O}}, P)$, where P stands for the normalized counting measure.
Define

$$\mathcal{C} := \{\delta_0^d > c_\delta\} \cap E_{\delta,0} \cap E_b \subset \mathcal{O}.$$

Hence $\mathcal{C}$ consists of all pieces of sceneries, η, with the following properties: $\delta_0^d(\eta)$ is bigger than c_δ, number or big blocks is less than $0.5 \ln n$ and the gaps between the breakpoints of the consecutive fences in I is at most Cn.

Let $\eta \in \mathcal{C}$ and let $z_0, z_1, \ldots, z_N$ be the breakpoints of consecutive fences (restricted to I) of η. Since $\eta \subset E_b$, we have $N \geq 2Ln^{999}$. Now partition the interval I as follows:

$$I = I_1 \cup I_2 \cup \cdots \cup I_N \cup I_{N+1}, \tag{2.19}$$

where $I_k := [z_{k-1}, z_k - 1]$, $k = 1, \ldots, N$, $I_{N+1} := [I_N, Ln^{1000}]$. Let $l(I_k) := z_k - k_{k-1}$ denote the length of I_k. We shall call the partition (2.19) the fence-partition corresponding to η. The fences guarantee that any block of η, that is longer than L is a proper subset of one interval I_k. Since $\eta \in \{\delta_0^d > c_\delta\} \cap E_{\delta,0}$, there is at least one and at most $0.5 \ln n$ big blocks. Let I_k^*, $k = 1, \ldots N^*$, $N^* \leq 0.5 \ln n$ denote the $k-th$ interval containing at least one big block. Similarly, let I_k^o, $k = 1, \ldots, N+1-N^*$ denote the $k-th$ interval with no big blocks. Clearly, most of the intervals I_k are without big blocks, in particular $\sum_k l(I_k^o) > Ln^{1000}$. Define

$$j^o := \min\{j : \sum_{k=1}^{j} l(I_k^o) > Ln^{1000}\}.$$

To summarize - to each $\eta \in \mathcal{C}$ corresponds an unique fence-partition, an unique labelling of the interval according to the blocks, and, therefore, unique j^o. We now define a mapping $B : \mathcal{C} \to \mathcal{O}$ as follows:

$$B(\eta) := (\eta|I_1^o, \eta|I_2^o, \ldots, \eta|I_{j^o}^o, \eta|I_1^*, \ldots, \eta|I_{N^*}^*, \eta|I_{j^o+1}^o, \ldots, \eta|I_{N+1-N^*}^o).$$

We also define the corresponding permutation

$$\Pi_\eta : I \to I, \quad \Pi_\eta(I) = (I_1^o, I_2^o, \ldots, I_{j^o}^o, I_1^*, \ldots, I_{N^*}^*, I_{j^o+1}^o, \ldots, I_{N+1-N^*}^o).$$

Thus, $B(\eta) = \eta \circ \Pi_\eta$.
Since all big blocks of η are contained in the intervals, I_k, the mapping B keeps all big blocks unchanged, it just removes them closer to the origin.
The mapping B is clearly not injective. However, $B(\eta_1) = B(\eta_2)$ implies that the fence-partitions corresponding to η_1 and η_2 consists of the same intervals, with possibly different order. Also the intervals with big blocks (marked with star) are the same, but possibly differently located. Moreover, the ordering of the similarly marked blocks corresponding to η_1 and η_2 are the same (i.e. if the 8-th interval, I_8, of the partition corresponding to η_1 is the 20-th interval, I_{20}, of the partition corresponding to η_2, then their marks are the same. If I_8 in its partition is the seventh interval with o ($I_8 = I_7^o$ in the partition corresponding to the η_1), then the same block in the second partition must be also the seventh interval with o ($I_{20} = I_7^o$ in the partition corresponding to η_2). Therefore, the partition corresponding to η_1 and η_2 differ on the location of the star-intervals, only. Since the number of intervals is smaller than $2Ln^{1000}$

and the number of star-intervals is at most $0.5\ln n$, the number of different partitions with the properties described above, is less than $(2Ln^{1000})^{0.5\ln n}$. This means

$$|B(\mathcal{C})|(2Ln^{1000})^{0.5\ln n} > |\mathcal{C}|. \tag{2.20}$$

Proof of Lemma 2.2: Because of the counting measure and (2.20) we get

$$\frac{P(B(\mathcal{C}))}{P(\mathcal{C})} = \frac{|B(\mathcal{C})|}{|\mathcal{C}|} > (2Ln^{1000})^{-0.5\ln n}.$$

By Propositions 2.2 and 2.3,

$$P(B(\mathcal{C})) \leq P\Big(\delta_0^d \wedge \delta_0^M \geq c_\delta(1 - O(M^{-\frac{1}{2}}))\Big).$$

By (2.18) and d, of Proposition 2.1, we get

$$\frac{P(\mathcal{C})}{P(\delta_0^d > c_\delta)} = P(E_{\delta,0} \cap E_b | \delta_0^d \geq c_\delta) > 0.5,$$

provided n is big enough. These relations yield

$$P\Big(\delta_0^d \wedge \delta_0^M \geq c_\delta(1 - O(M^{-\frac{1}{2}}))\Big) \geq (2Ln^{1000})^{-0.5\ln n} \cdot 0.5 \cdot P(\delta_0^d > c_\delta).$$

The lemma is proved.

Proposition 2.2 *For any $\varsigma \in B(\mathcal{C})$ we have*

$$\delta_0^d(\varsigma) \geq c_\delta[1 - O(M^{-\frac{1}{2}})].$$

Proof. Let $\varsigma \in B(\mathcal{C})$. Choose $\eta \in B^{-1}(\varsigma)$. Let $\{I_k\}$ be the fence-partition corresponding to η. Let $\delta_z^\eta(l)$, $\delta_z^\varsigma(l)$ denote the probabilities defined in (2.7), with ξ replaced by η and ς, respectively. As already noted, because of the fencing-structure, any sequence of consecutive one's or zero's can be generated on the one interval I_k, only. More precisely, if $l \in I_k$, then

$$\delta_0^\eta(l) = P\big(S_l(0), \dots, S_l(n^2) \in I_k\,,\ \eta(S_l(0)) = \dots = \eta(S_l(n^2))\big). \tag{2.21}$$

By the argument of the proof of c of Proposition 2.1, we get that each interval without big blocks, I_k^o, has the property: the probability of generating n^2+1 consecutive zeros or ones is smaller than $n^{-a\ln n}$. In other words $\delta_0^\eta(l) \leq n^{-a\ln n}, \quad \forall l \in I^o$, where $I^o := \cup_k I_k^o$. Denote $I^* := \cup_k I_k^*$. Now, by (2.10) and (2.21) we have

$$\begin{aligned}\delta_0^d(\eta) &= \sum_{l\in I} P(0,l)\delta_0^\eta(l) = \Big(\sum_{l\in I^o} + \sum_{l\in I^*}\Big) P(0,l)\delta_0^\eta(l)\\ &\leq \sum_{l\in I^o} P(0,l) n^{-a\ln n} + \sum_{l\in I^*} P(0,l)\delta_0^\eta(l)\\ &\leq n^{-a\ln n} + \sum_{l\in I^*} P(0,l)\delta_0^\eta(l) \leq n^{-a\ln n} + p_M \sum_{l\in I^*}\delta_0^\eta(l).\end{aligned}$$

Since $\eta \in \mathcal{C}$, $\delta_0^d(\eta) \geq c_\delta \geq p_M$, we have

$$\sum_{l\in I^*}\delta_0^\eta(l) \geq \frac{c_\delta - n^{-a\ln n}}{p_M} \geq 1 - \frac{n^{-a\ln n}}{p_M} = 1 - O\Big(\frac{\sqrt{M}}{n^{a\ln n}}\Big), \tag{2.22}$$

Clearly $O(\frac{\sqrt{M}}{n^{a\ln n}}) = o(n^{-\alpha})$, for all $\alpha \geq 0$.

Now consider $\varsigma = M(\eta)$. Let $J_1, J_2, \ldots J_{N+1}$ denote the new location of intervals I_i after applying mapping Π_η to I. Fix an $j \in I$ and let $j \in J_k$. The equation $\varsigma|J_k = \eta|I_k$ and (2.21) imply

$$\begin{aligned}\delta_0^\varsigma(j) &= P\big(S_j(0), \ldots, S_j(n^2) \in I, \varsigma(S_j(0)) = \cdots = \varsigma(S_j(n^2))\big)\\ &\geq P\big(S_j(0), \ldots, S_j(n^2) \in J_k, \varsigma(S_j(0)) = \cdots \varsigma(S_j(n^2))\big)\\ &= P\big(S_l(0), \ldots, S_l(n^2) \in I_k, \eta(S_l(0)) = \cdots = \eta(S_l(n^2))\big) = \delta_0^\eta(l),\end{aligned}$$

where $l = \Pi(j) \in I_k$. This means $\delta_0^\varsigma(j) \geq \delta_0^\eta(\Pi_\eta(j)), \quad \forall j \in I$. In particular,

$$\sum_{j \in J_k} \delta_0^\varsigma(j) \geq \sum_{l \in I_k} \delta_0^\eta(j) \tag{2.23}$$

If $I_1 = J_1$ and $I_{N+1} = J_{N+1}$, i.e. the first and last intervals do not contain big blocks, then, obviously, (2.23) is an equation.
Let $J^* = \Pi_\eta(I^*)$, i.e. J^* is the union of all intervals with big blocks in the new location. The length of I^* (and, therefore, that of J^*) is at most $0.5Cn\ln n$. Thus, J^* is at most $Cn + 0.5Cn\ln n$ from the origin. Let n be so big, that $Cn + 0.5Cn\ln n \leq n^2$. Then, $j \leq n^2$ for each for each $j \in J^*$. Denote

$$p_o = \min\{P(S(M) = i) : |i| \leq n^2\}.$$

Now from (2.22) and (2.23) we get

$$\begin{aligned}\delta_0^d(\varsigma) &= \sum_j P(0,j)\delta_0^\varsigma(l) \geq \sum_{j \in J^*} P(0,j)\delta_0^\varsigma(j) \geq \sum_{l \in I^*} P(0,j)\delta_0^\eta(l)\\ &\geq p_o \sum_{l \in I^*} \delta_0^\eta(l) \geq (c_\delta - n^{-a\ln n})\frac{p_o}{p_M} = c_\delta(1 - \frac{p_M - p_o}{p_M} - \frac{n^{-a\ln n}p_o}{c_\delta p_M})\\ &= c_\delta[1 - O(M^{-\frac{1}{2}})] - O(\frac{\sqrt{M}}{n^{a\ln n}}) = c_\delta[1 - O(M^{-\frac{1}{2}})].\end{aligned}$$

Proposition 2.3 *For any $\varsigma \in B(\mathcal{C})$ we have*

$$\delta_0^M(\varsigma) \geq c_\delta[1 - O(M^{-\frac{1}{2}})].$$

Proof. We use the notation and the results of the previous proof. By the representation (2.8) we have

$$\delta_0^M(\varsigma) = \sum_{i,j \in I} \mu(i)P(i,j)\delta_0^\varsigma(j) \geq \sum_{i,j \in J^*} \mu(i)P(i,j)\delta_0^\varsigma(j) \tag{2.24}$$

where $\mu = \{\mu(i)\}_{i \in I}$ is the stationary measure of $Y_k = S(\nu_0(k))$, $k = 1, 2, \ldots$.

Use LCLT to estimate

$$\begin{aligned}\min_{i,j \in J^*} P(j,i) &\geq \min\{P(i,j) : |i-j| \leq n^2\} \geq \frac{c}{\sqrt{M}}\exp\big(-\frac{dn^2}{M}\big) - O(M^{-1})\\ &= \frac{c}{\sqrt{M}}\Big(1 - O\big(\frac{n^2}{M}\big)\Big) - O(M^{-1}) = p_M\Big(1 - O\big(\frac{1}{\sqrt{M}}\big)\Big).\end{aligned} \tag{2.25}$$

with d, c being constants not depending on n.
Hence, because of (2.24), (2.22) and (2.25)

$$\begin{aligned}\delta_0^M(\varsigma) \geq& \mu(J^*)[p_M\big(1 - O(\frac{1}{\sqrt{M}})\big)]\frac{c_\delta - n^{-a\ln n}}{p_M}\\ =& \mu(J^*)\big(1 - O(\frac{1}{\sqrt{M}})\big)(c_\delta - n^{-a\ln n}) = \mu(J^*)\big(1 - O(\frac{1}{\sqrt{M}})\big)c_\delta.\end{aligned} \tag{2.26}$$

We now estimate $\mu(J^*)$. We shall show that

$$P(Y_{k+1} \in J^* | Y_k = j) \geq 1 - o(M^{-1}) \quad \forall j \in I.$$

Then $\mu(J^*) = \sum_j P(Y_{k+1} \in J^* | Y_k = j)\mu(j) \geq 1 - o(M^{-1})$ and, by (2.26)

$$\delta_0^M(\varsigma) \geq \mu(J^*) \geq (1 - o(M^{-1}))c_\delta[1 - O(M^{-\frac{1}{2}})] = c_\delta[1 - O(M^{-\frac{1}{2}})].$$

Estimation of $\mu(J^*)$

Fix an $j \in I$ and define ν as the first time after $e^{n^{0.1}}$ when $n^2 + 1$ consecutive 0-s or 1-s are generated on I. Formally,

$$\nu := \min \left\{ t \geq e^{n^{0.1}} \middle| \begin{array}{c} \chi(t) = \chi(t-1) = ... = \chi\left(t - n^2\right) \\ \text{and } S_j(i) \in I, \forall i = t - n^2, \ldots, t \end{array} \right\}$$

where $\chi = \varsigma \circ S_j$. Clearly

$$P(S_j(\nu) \in J^*) = P(Y_{k+1} \in J^* | Y_k = j).$$

Thus, it suffices to estimate $P(S_j(\nu) \in J^*)$.

At first note, by (2.22) and (2.23) we get $\sum_{j \in J^*} \delta_0^\eta(j) \to 1$. Since $|J^*| \leq n^2$ (and n is big enough), we deduce the existence of $j^* \in J^*$ such that

$$\delta_0^\eta(j^*) > \frac{1}{n^3}. \tag{2.27}$$

Then, note that because of the fences we have

$$\{S_j(\nu) \notin J^*\} = \{S_j(\nu - n^2), \ldots, S_j(\nu) \in I \backslash J^*, \ \chi(\nu - n^2) = \cdots = \chi(\nu)\}.$$

Now, let τ_k be the k-th visit after time $e^{n^{0.1}} - n^2$ to the interval I. Let τ_k^* be the k-th visit after time $e^{n^{0.1}} - n^2$ to the point j^*. Define the events

$$F_k := \{S_j(\tau_k - n^2), \ldots, S_j(\tau_k) \in I \backslash J^*, \ \chi(\tau_k - n^2) = \cdots = \chi(\tau_k)\}, \ k = 1, 2 \ldots$$

$$F_k' = \cup_{i=0}^{n^{2000}-1} \{S_j(\tau_k + i) = j^*\}, \quad k = 1, 2, \ldots$$

$$F_k^* = \{\chi(\tau_k^*) = \cdots = \chi(\tau_k^* + n^2)\}, \ k = 1, 2, \ldots$$

We consider the events

$$E_1 := \{\nu > \tau_{n^{2020}}\} \cup \{S_j(\nu) \in J^*\}, \ E_2 := \{\tau_{n^{10}}^* \leq \tau_{n^{2020}} - n^2\}, \ E_3 := \cup_{k=1}^{n^{10}} F_k^*$$

The event E_1 ensures that within first n^{2020} visits of S_j to I no consecutive 0's or 1's were generating on $I \backslash J^*$. The event E_2 ensures that before time $\tau_{n^{2020}} - n^2$ the random walk visits at least n^{10} times the point j^*. Finally, the event E_3 ensures that during these n^{10} visits of j^*, at least one of them is a beginning of n^2 consecutive 0's or 1's. If these events hold, then $\nu \leq \tau_{n^{2020}}$ and $S_j(\nu) \in J^*$. Thus

$$E_1 \cap E_2 \cap E_3 \subset \{S_j(\nu) \in J^*\}.$$

We now give the upper bounds to the probabilities $P(E_1), P(E_2), P(E_3)$.

1) Note, $E_1^c \subset \cup_{k=1}^{n^{2020}} F_k$, implying that $P(E_1^c) \leq \sum_{k=1}^{n^{2020}} P(F_k)$. For each k,

$$P(F_k) = \sum_{l \in I \backslash J^*} P[S_l(0), \ldots, S_l(n^2) \in I \backslash J^*, \varsigma(S_l(0)) = \cdots = \varsigma(S_l(n^2))] \times \\ \times P(S_j(\tau_k - n^2) = l).$$

There is no big blocks in $I\backslash J^*$, hence by the argument of c

$$P[S_l(0),\ldots,S_l(n^2)\in I\backslash J^*,\varsigma(S_l(0))=\cdots=\varsigma(S_l(n^2))]\leq n^{-a\ln n},$$

implying that

$$P(E_1^c)\leq n^{2020-a\ln n}.$$

2) To estimate $P(E_2)$ we use Höffding's inequality. Let By CLT there exists a constant $p>0$ not depending on n such that $P(F'_k)\geq p$. Also note that F'_k and F'_l are independent if $|k-l|\geq n^{2000}$. Hence, the set $\{F'_k\}$, $k=1,\ldots,n^{2020}$ contains a subset $\{F'_{k_i}\}$ $i=1,\ldots n^{20}$ consisting of independent events. Let $X_i:=I_{F'_{k_i}}$. Now, $\tau_{n^{2018}}+n^{2000}\leq\tau_{n^{2019}}\leq\tau_{n^{2020}}-n^2$, if n is big enough. This means

$$\Big\{\sum_{i=1}^{n^{18}}X_i\geq n^{10}\Big\}\subset E_2.$$

Now, when n is big enough, we have

$$\begin{aligned}P(E_2^c)&\leq P\Big(\sum_{i=1}^{n^{18}}X_i<n^{10}\Big)=P\Big(\sum_{i=1}^{n^{18}}(X_i-EX_i)<n^{10}-\sum_{i=1}^{n^{18}}EX_i\Big)\\&\leq P\Big(\sum_{i=1}^{n^{18}}(X_i-EX_i)<-(n^{18}p-n^{10})\Big)\leq P\Big(\sum_{i=1}^{n^{18}}(X_i-EX_i)<-n^{17}\Big)\leq\\&\leq\exp(-\frac{2n^{34}}{n^{18}})=\exp(-2n^{16}).\end{aligned}$$

3) Note F_l^*,F_k^* are independent, if $|k-l|>n^2$ Let $\{F_{k_i}^*\}$, $i=1,2,\ldots,n^7$ be a subset of $\{F_k^*\}$ consisting on independent events, only. By (2.27), $P(F_k^*)>\frac{1}{n^3}$, $\forall k$. Now

$$P(E_3^c)\leq P(\cap_{i=1}^{n^7}F_{k_i}^*)=\prod_{i=1}^{n^7}(1-P(F_{k_i}^*))\leq\Big(1-\frac{1}{n^3}\Big)^{n^7}. \tag{2.28}$$

The right side of (2.28) is smaller than $(0.5)^{n^4}$ if n is big enough.
Thus,

$$\begin{aligned}P(S_j(\nu)\in J^*)&\geq 1-[n^{2020-a\ln n}+\exp(-2n^{16})+(0.5)^{n^4}]\\&=1-O(n^{-2020+a\ln n})=1-o(M^{-1}).\end{aligned}$$

2.4 Corollaries

We now determine an important figure - the critical value c_r. Since we choose it within the interval $[p_M,2p_M]$, it has all properties stated in Proposition 2.1 and Lemma 2.2. However, we also have to ensure that with high probability the signal probabilities δ_z^d and δ_z^M are significantly away from c_r. By "significantly" we mean that the difference between these probabilities and c_r is bigger a polynomially small quantity in n. This polynomially small quantity will be denoted by Δ. Thus, c_r must be properly chosen and that will be done with the help of Corollary 2.2.
At first, some preliminary observations.

Proposition 2.4 *For any $j>2$, there exists an interval $[a,b]\subset[p_M,2p_M]$ of length $p_M/\big(n^{j+2}\big)$ such that*

$$P(\delta_0^d<b|\delta_0^d\geq a)\leq\frac{1}{n^j} \tag{2.29}$$

Proof. We do the proof by contradiction. Assume on the contrary that that there exists no interval $[a,b]\subset[p_M,2p_M]$ of length $l:=p_M/n^{j+2}$ such that (2.29) is satisfied. Let $a_i:=p_M+il$, $i=0,\ldots,n^{j+2}$. Since $[a_i,a_{i+1}]\subset[p_M,2p_M]$ is an interval of length l, by assumption

$$P(\delta_0^d\geq a_{i+1}|\delta_0^d\geq p_M+a_i)\leq\Big(1-\frac{1}{n^j}\Big),\quad i=1,\ldots,n^j-1.$$

Now, by b) of Proposition 2.1

$$e^{-n}<P(\delta^d\geq 2p_M)=\prod_{i=0}^{n^{j+2}-1}P(\delta_0^d\geq a_{i+1}|\delta_0^d\geq a_i)\leq\Big(1-\frac{1}{n^j}\Big)^{n^{j+2}}. \tag{2.30}$$

Since $(1-\frac{1}{n^j})^{n^j}<e^{-1}$, we have $(1-\frac{1}{n^j})^{n^{j+2}}<e^{-n^2}$. Thus, (2.30) implies $e^{-n}<e^{-n^2}$ - a contradiction. ■

Corollary 2.1 *Let $[x,y]\subset[p_M,2P_M]$ be an interval of length l. Then there exists an subinterval $[u,v]\subset[x,y]$ of length $\frac{l}{e^{2n}}$ such that*

$$P(\delta_0^d<v|\delta_0^d>u)\leq\frac{1}{e^n}. \tag{2.31}$$

Proof. The proof of the corollary follows the same argument that the proof of Proposition 2.4: (2.31) together with the statement b) of Proposition 2.1 yield the contradiction: $\exp(-n)<P(\delta_0^d\geq 2p_M)\leq P(\delta_0^d\geq v)\leq\Big[\big(1-\frac{1}{e^n}\big)^{e^n}\Big]^{e^n}<\exp(-e^n)$. ■

The next proposition proves the similar result for $\delta_0^M\wedge\delta_0^d$. Since we do not have the analogue of b) of Proposition 2.1, we use Lemma 2.2, instead.

Proposition 2.5 *Let $[a,b]\subset[p_M,2p_M]$ be such that $2p_M-b>p_M O(M^{-\frac{1}{2}})$. For any $i>2$ there exists an interval $[x,y]\subset[a,b]$ with length $(b-a)/n^{i+2}$ such that, for n big enough*

$$P(\delta_0^M<y|\delta_0^M\wedge\delta_0^d>x)\leq P(\delta_0^M\wedge\delta_0^d<y|\delta_0^M\wedge\delta_0^d>x)\leq\frac{1}{n^i}. \tag{2.32}$$

Proof. Suppose that such a (sub)interval does not exists. Then follow the argument of the previous proof to get

$$P\Big(\delta_0^M\wedge\delta_0^d\geq 2p_M(1-O(M^{-\frac{1}{2}}))\Big)\leq P\big(\delta_0^M\wedge\delta_0^d\geq b\big)\leq\Big(1-\frac{1}{n^i}\Big)^{n^{i+2}}<\exp(-n^2). \tag{2.33}$$

By Lemma 2.2 and b) of Proposition 2.1

$$P\Big(\delta_0^M\wedge\delta_0^d\geq 2p_M(1-O(M^{-\frac{1}{2}}))\Big)\geq 0.5(2Ln^{1000})^{-0.5\ln n}\exp(-n). \tag{2.34}$$

For n big enough, the right side of (2.34) is bigger than e^{-2n}. This contradicts (2.33). ■

The following corollary specifies c_r and Δ.

Corollary 2.2 *Let $\Delta:=\big(p_M/8\big)n^{-10054}$, $\tilde\Delta=\Delta e^{-2n}$. Then there exists $c_r\in[p_M+\Delta,2p_M-\Delta]$ such that, for n big enough, simultaneously,*

$$P\left(\delta_0^d\geq c_r-\Delta\right)\leq\exp((\ln n)^3)P\left(\delta_0^d\wedge\delta_0^M\geq c_r-\Delta\right); \tag{2.35}$$

$$P(\delta_0^M<c_r+\Delta|\delta_0^M\wedge\delta_0^d\geq c_r-\Delta)\leq n^{-10000} \tag{2.36}$$

and

$$P(\delta_0^d<c_r-\Delta+\tilde\Delta|\delta_0^d\geq c_r-\Delta)\leq\exp(-n). \tag{2.37}$$

Proof. By Proposition 2.4 there exists an interval $[a, b] \subset [p_M, 2p_M]$ of length p_M/n^{52} such that

$$\frac{P(\delta_0^d \geq b)}{P(\delta_0^d \geq a)} = P(\delta_0^d \geq b | \delta_0^d \geq a) > 1 - \frac{1}{n^{50}} > 0.5. \tag{2.38}$$

We now consider the interval $[a, \frac{a+b}{2}]$. Note that

$$2p_M - \frac{a+b}{2} \geq b - \frac{b+a}{2} = \frac{b-a}{2} = \frac{p_M}{2n^{52}} > p_M O(M^{-\frac{1}{2}}).$$

Now use Proposition 2.5 with $i = 10000$ to find a subset $[x, y] \in [a, \frac{a+b}{2}]$ with length $l := \frac{b-a}{2} n^{-10002} = \frac{p_M}{2} n^{-10054}$ such that (2.32) holds.
Let us now take $z = x + \frac{l}{4}$. By Corollary 2.1, there exists an subinterval $[u, u + \tilde{\Delta}] \in [x, z]$ with length $\frac{l}{4e^{2n}}$ such that

$$P(\delta_0^d < u + \tilde{\Delta} | \delta^d > u) \leq \exp(-n). \tag{2.39}$$

Now take $\Delta := \frac{l}{4} = \big(p_M/8\big) n^{-10054}$, $c_r := u + \Delta$. Since $[c_r - \Delta, c_r + \Delta] \subset [x, y]$, we have that

$$\begin{aligned}
&P(\delta_0^M < c_r + \Delta | \delta_0^M \wedge \delta_0^d > c_r - \Delta) \leq P(\delta_0^M \wedge \delta_0^d < c_r + \Delta | \delta_0^M \wedge \delta_0^d > c_r - \Delta) \leq \\
&P(\delta_0^M \wedge \delta_0^d < y | \delta_0^M \wedge \delta_0^d > c_r - \Delta) = \frac{P(\Delta - c_r < \delta_0^M \wedge \delta_0^d < y)}{P(\delta_0^M \wedge \delta_0^d > \Delta - c_r)} \leq \\
&\frac{P(y > \delta_0^M \wedge \delta_0^d > x) - P(x \leq \delta_0^M \wedge \delta_0^d \leq c_r - \Delta)}{P(\delta_0^M \wedge \delta_0^d > x) - P(x < \delta_0^M \wedge \delta_0^d \leq c_r - \Delta)} \leq \frac{P(y > \delta_0^M \wedge \delta_0^d > x)}{P(\delta_0^M \wedge \delta_0^d > x)} = \\
&P(\delta_0^M \wedge \delta_0^d < y | \delta_0^M \wedge \delta_0^d > x) \leq \frac{1}{n^{10000}}.
\end{aligned}$$

Hence, (2.36) holds.
Since $u = c_r - \Delta$, we also have that (2.37) holds.
It only remains to show that the chosen c_r also satisfies (2.35).
Clearly $\Delta > 2p_M O(M^{-\frac{1}{2}}) > c_r O(M^{-\frac{1}{2}})$. That implies

$$P\Big(\delta_0^d \wedge \delta_0^M \geq c_r(1 - O(M^{-\frac{1}{2}})\Big) \leq P(\delta_0^d \wedge \delta_0^M \geq c_r - \Delta).$$

Combine this with Lemma 2.2to get

$$P(\delta_0^d \geq c_r) 0.5 (2Ln^{1000})^{-0.5 \ln n} \leq P(\delta_0^d \wedge \delta_0^M \geq c_r - \Delta) \tag{2.40}$$

Since $[c_r - \Delta, c_r + \Delta] \subset [a, b]$ we have

$$P(\delta_0^d \geq a) \geq P(\delta_0^d \geq c_r - \Delta) \geq P(\delta_0^d \geq c_r) \geq P(\delta_0^d \geq b).$$

Now, by (2.38)

$$\frac{P(\delta_0^d \geq c_r)}{P(\delta_0^d \geq c_r - \Delta)} \geq \frac{P(\delta_0^d \geq b)}{P(\delta_0^d \geq a)} > 0.5.$$

The latter together with (2.40) implies

$$P(\delta_0^d \geq c_r - \Delta) \leq 0.25 (2Ln^{1000})^{0.5 \ln n} P(\delta_0^d \wedge \delta_0^M \geq c_r - \Delta) \tag{2.41}$$

Now, the relation

$$0.25 (2Ln^{1000})^{0.5 \ln n} \leq \exp((\ln n)^3)$$

together with (2.41) establishes (2.35). ■

3 Scenery-dependent events

In the present section we define and investigate the signal points and Markov signal points. We show that with big probability the location of the signal points follows certain clustering structure. This structure gives us the desired signal carriers in 2 colors case.

3.1 Signal points

We are now going to define the Markov signal points, strong signal points and signal points - these are the location points, where the corresponding signal probabilities are above the critical value c_r. The Markov signal points form the core of the signal carriers, the (strong) signal points will be used in our proofs. In an oversimplified way, we could say that the Markov signal points are places in the scenery ξ where the conditional probability to see in the observations some rare unusual pattern is above c_r. The unusual pattern is basically a string of n^2 same colors.
In the present subsection, with the help of the signal points, we define many other important notions, and we also investigate their properties.

In the following, Δ and c_r are as in Corollary 2.2. In particular, $\Delta = \frac{p_M}{8} n^{-10054}$.

* A (location) point $z \in \mathbb{Z}$ is called **signal point**, if $\delta_z^d > c_r - \Delta$.

* A (location) point $z \in \mathbb{Z}$ is called **strong signal point**, if $\tilde{\delta}_z^d > c_r - \Delta$.

* A (location) point $z \in \mathbb{Z}$ is called **Markov signal point**, if

$$\delta_z^d > c_r - \Delta \quad \text{and} \quad \delta_z^M > c_r - \Delta.$$

* We call a Markov signal point z **regular**, if $\delta_z^M > c_r + \Delta$.

* Let $\bar{z}_1$ be the first Markov signal point in $[0, \infty)$. Let $\bar{z}_k$ be defined inductively: $\bar{z}_k$ is the first Markov signal point in $[\bar{z}_{k-1} + 2Ln^{1000}, \infty)$. Let $\bar{z}_0$ be the first (smallest) Markov signal point in $(-\infty, 0]$. And let $\bar{z}_{-k}$ be defined inductively: $\bar{z}_{-k}$ is the first Markov signal point in $(-\infty, \bar{z}_{-(k-1)} - 2Ln^{1000}]$. Thus $\ldots, \bar{z}_{-2}, \bar{z}_{-1}, \bar{z}_0, \bar{z}_1, \bar{z}_2, \ldots$ is a sequence of ordered random variables which we shall call as **signal carriers points**.

* For given z, the set

$$\mathcal{N}_z := [z - L(n^{1000} + e^{n^{0.3}}), z - L(n^{1000}) \cup (z + Ln^{1000}, z + L(n^{1000} + e^{n^{0.3}})]$$

is called the **neighborhood** of z.

* We say that the neighborhood of z is **empty**, if $\mathcal{N}_z$ does not contain any block of ξ longer than $n^{0.35}$. Thus, $\{\mathcal{N}_z \text{ is empty }\} \subset \sigma(\xi_i, i \in \mathcal{N}_z)$.

* We say that z has **empty border**, if the set $I_z - [z - \tilde{M}, z + \tilde{M}]$ does not contain any block of ξ longer than $n^{0.35}$. Thus, $\{\mathcal{N}_z \text{ is empty }\} \subset \sigma(\xi_i, i \in I_z - [z - \tilde{M}, z + \tilde{M}])$.

* Let p, $\tilde{p}$ and p^d be the probability, that a fixed point is Markov signal point, strong signal point or signal point, respectively.
From (2.3), part a) of Proposition 2.1 and by (2.35) of Corollary 2.2 we know

$$p^d - \exp(-dn^{999}) < \tilde{p} \leq p^d; \tag{3.1}$$

$$p \leq p^d \leq \exp(-\frac{\alpha n}{\ln n}); \tag{3.2}$$

$$\frac{p^d}{p} \leq \exp((\ln n)^3). \tag{3.3}$$

* We now define a construction, which we are going to use later.
For each $j = 0, 1, 2, \ldots, 2Ln^{1000}$ partition the set $\mathbb{Z} \cap [-Ln^{1000} + j, \infty)$ into adjacent integer intervals of length $2Ln^{1000}$. Let $I_{k,j}$ denote the k-th interval of the partition who's first interval starts at $-Ln^{1000}+j$. Thus,

$$I_{1,j} = [j - Ln^{1000}, j + Ln^{1000}], \quad I_{2,j} = [j + Ln^{1000} + 1, j + 3Ln^{1000} + 1],$$

$$I_{3,j} = [j + 3Ln^{1000} + 2, j + 5Ln^{1000} + 2],$$

$$\ldots$$

$$I_{k,j} = [j + kLn^{1000} + k - 1, j + (k+2)Ln^{1000} + k - 1].$$

Let $z_{j,k}$ denote the midpoints of $I_{k,j}$. Hence

$$z_{j,1} = j, \quad z_{j,2} = j + 2Ln^{1000} + 1, \quad \ldots, \quad z_{j,k} = j + 2kLn^{1000} + (k-1).$$

For, each j, the intervals $I_{k,j}$, $k = 1, 2, \ldots$ are disjoint. Thus, the events

$$\{z_{k,j} \quad \text{is} \quad \text{a} \quad \text{Markov} \quad \text{signal} \quad \text{point}\}, \quad k = 1, 2, \ldots$$

are independent with the same probability p.

Let k' denote the integer valued random variable that shows the index of the first interval $I_{k,0}$ which has its midpoint being a Markov signal point. By such a counting we disregard the first interval. Thus, $k' > 1$ and, formally, k' is defined by the relations

$$\delta_{z_{2,0}} \wedge \delta^M_{z_{2,0}} \leq c_r - \Delta, \ \ldots \quad \delta^M_{z_{k'-1,0}} \wedge \delta^d_{z_{k'-1,0}} \leq c_r - \Delta, \ \delta^M_{z_{k',0}} \wedge \delta^d_{z_{k',0}} > c_r - \Delta$$

Clearly, $k' - 1$ is a geometrical random variable with parameter p and, hence, $Ek' = \frac{1}{p} + 1$.

* Let Z be the location of the first Markov signal point after $2Ln^{1000}$. Recall $\bar{z}_1$ is the location of the first Markov signal point after 0 Note, that for each $i \geq 0$, we have

$$P(\bar{z}_1 \leq i) < P(\cup_{j=0}^{i}\{i \text{ is a Markov signal point}\}) \leq pi \tag{3.4}$$

and

$$P(Z \leq i) \leq p(i - 2Ln^{1000}), \quad i \geq 2Ln^{1000}. \tag{3.5}$$

From (3.4) and (3.2) we get

$$P(\bar{z}_1 \leq 2Ln^{1000}) \leq p2Ln^{1000} \leq 2Ln^{1000} \exp(-\frac{\alpha n}{\ln n}) \to 0. \tag{3.6}$$

* We now estimate EZ. For this note: $Z \leq z_{k',0} = 2k'Ln^{1000} + k' - 1$ and

$$EZ \leq (\frac{1}{p} + 1)2Ln^{1000} + \frac{1}{p} \leq \frac{3}{p}Ln^{1000}. \tag{3.7}$$

From (3.3) we get

$$EZp^d \leq 3\frac{p^d}{p}Ln^{1000} \leq 3Ln^{1000} \exp((\ln n)^3).. \tag{3.8}$$

On the other hand by (3.5) we have, for each x, $EZ \geq xP(Z \geq x) \geq x(1 - px)$. Now, take $x = (2p)^{-1}$ and use (3.2) to get

$$EZ \geq \frac{1}{4p} \geq \frac{1}{4}\exp(\frac{\alpha n}{\ln n}). \tag{3.9}$$

* Take $m(n) = \ulcorner n^{2.5} EZ \urcorner$.

By (3.3) and b) of Proposition 2.1 we get

$$n^{2.5}EZ \leq \frac{3Ln^{1002.5}}{p^d}\exp((\ln n)^3) \leq 3Ln^{1002.5}\exp((\ln n)^3+n) < \exp(2n),$$

implying

$$\frac{1}{4}\exp(\frac{\alpha n}{\ln n}) \leq m < \exp(2n), \tag{3.10}$$

provided n is big enough.

* We now define the random variables we are going to use later

$$X_z := I_{\{\delta^d_z > c_r - \Delta, \quad \delta^M_i > c_r - \Delta\}}, \quad z = 0, 1, 2, \ldots.$$

Thus, X_z indicates, whether z is a Markov signal point or not. The random variables X_z are identically distributed with mean p.
We now estimate the number of Markov signal points in $[0, cm]$, where $c > 1$ is a fixed integer, not depending on n . For this define

$$E_0 := \Big\{\sum_{z=0}^{cm} X_z \leq n^{10000}\Big\}.$$

Thus, when E_0 holds, then the interval $[0, cm]$ contains at most n^{10000} Markov signal points.
To estimate $P(E_0)$ we use Markov inequality and (3.7)

$$\begin{aligned} P(E_0^c) &= P\Big(\sum_{i=0}^{cm} X_i > n^{10000}\Big) < \frac{(cm+1)p}{n^{10000}} \leq \frac{c(n^{2.5}EZ+1)p+1}{n^{10000}} \\ &< c3Ln^{1002.5-10000} + (c+1)n^{-10000} = o(1). \end{aligned}$$

* Finally, define $Z_0 < Z_1 < \cdots < Z_k < \cdots$ as follows:

$Z_0 := 0$, $Z_1 := Z$, and, then, let Z_{k+1} be the first Markov signal point that is greater than $2Ln^{1000} + Z_k$.

Note: the differences: Z, $Z_2 - Z_1$, $Z_3 - Z_2$, ..., $Z_{k+1} - Z_k, \ldots$ are iid. Also note:

$$\{\text{No Markov signal points in } [0, 2Ln^{1000}]\} = \{Z_i = \bar{z}_i \text{ for all } i\} := E^n_s. \tag{3.11}$$

From (3.6) we know that

$$P(E^n_s) \to 1. \tag{3.12}$$

3.2 Scenery-dependent events

We are now going describe the typical behavior of the signal points in the interval $[0, cm]$. Here $c > 1$ is a fixed integer, not depending on n. Among others we show that, with high probability, for all signal carrier points $\bar{z}_i$ in $[0, cm]$ the corresponding frequencies of ones, $h(\bar{z}_i)$, vary more than $e^{-n^{0.11}}$ (events $\bar{E}^n_3$ and $\bar{E}^n_4$ below). We also show that, with high probability, all signal points in $[0, cm]$ have empty neighborhood.
All the properties listed below depend on the scenery, ξ, only. Therefore we refer to them as the **scenery dependent events**.
We now define all scenery dependent events, $\bar{E}^n_1, \ldots, \bar{E}^n_9$ and prove the convergence of their probabilities. All the events will be defined on interval $[0, cm]$ where $c > 1$ is a fixed integer. Thus, if a point z is such that $\mathcal{N}_z \notin [0, cm]$, then by the neighborhood of z we mean $\mathcal{N}_z \cap [0, cm]$. This means $\bar{E}^n_i \in \sigma(\xi_z : z \in [0, cm])$. The exact value of c will be defined in the next chapter (in connection with the event $E^n_{2,S}$). During this chapter, c is assumed to be any fixed integer bigger than 1.

At first, we list the events of interest:

$\bar{E}_1^n := \{\bar{z}_{n^2+1} \leq m\}$;

$\bar{E}_2^n := \{$every signal point in $[0, cm]$ has an empty neighborhood$\}$;

$\bar{E}_3^n := \{$every pair $\bar{z}_1, \bar{z}'$ of signal carrier points in $[0, cm]$ satisfies : $|h(\bar{z}) - h(\bar{z}')| \geq e^{-n^{0.11}}$ if $\bar{z} \neq \bar{z}'\}$;

$\bar{E}_4^n := \{$every signal carrier point $\bar{z}$, in $[0, cm]$ satisfies : $|h(\bar{z}) - \frac{1}{2}| \geq e^{-n^{0.11}}\}$;

$\bar{E}_5^n := \{$every signal point $z \in [0, cm]$ satisfies $\delta_z^M \notin [c_r - \Delta, c_r + \Delta]\}$;

$\bar{E}_6^n := \{$ for all signal carrier points $\bar{z}_i$ in $[0, cm]$ we have $EZn^{11001} \geq |\bar{z}_i - \bar{z}_{i+1}| \geq EZn^{-11001}\}$;

$\bar{E}_7^n := \{$no signal carrier points in $[m - EZn^{-11001}, m + EZn^{-11001} \wedge cm] \cup [0, EZn^{-11001}]\}$;

$\bar{E}_8^n := \{$every strong signal point in $[0, cm]$ has empty border$\}$;

$\bar{E}_9^n := \{$every signal point in $[0, cm]$ is a strong signal point$\}$.

Proof that $P(\bar{E}_1^n) \to 1$

If $\bar{E}_1^n$ holds, then in $[0, m]$ we have more than n^2 signal carrier points .

Define random variables $Z_0 < Z_1 < \cdots < Z_k < \cdots$ as in (3.11). Let $E_{1a}^n := \{Z_{n^2+1} \leq m\}$. Since $E_s \cap E_{1a}^n \subset \bar{E}_1^n$, it suffices to show that $P(E_{1a}^n) \to 1$. To see this, we use Markov inequality:

$$P(E_{1a}^{nc}) = P(Z_{n^2+1} > m) \leq \frac{EZ_{n^2+1}}{m} \leq \frac{(n^2+1)}{n^{2.5}} \to 0.$$

Proof that $P(\bar{E}_2^n) \to 1$

$$\bar{E}_2^{nc} = \{\text{ there exists a signal point in } [0, cm] \text{ with non} - \text{empty neighborhood}\}.$$

Clearly,

$$\bar{E}_2^{nc} = \cup_{z=0}^{cm} E_2(z), \quad \text{where} \quad E_2(z) := \{z \text{ is a signal point and } \mathcal{N}_z \text{ is not empty}\}.$$

For each z, the events $\{\mathcal{N}_z$ is empty$\}$ and $\{\delta_z > c_r - \Delta\}$ are independent. Thus, for each z,

$$P(E_2(z)) = P(\delta_z > c_r - \Delta) P(\mathcal{N}_z \text{ is empty}) = p^d P(\mathcal{N}_z \text{ is not empty}).$$

We obviously have $P(\mathcal{N}_z$ is empty$) = P(\mathcal{N}_o$ is empty$)$ and

$$P(\mathcal{N}_o \text{ is not empty}) =$$
$$P(\mathcal{N}_o \text{ contains at least one block longer than } n^{0.3}) < 2L \exp(n^{0.3}) 2^{-n^{0.35}}.$$

Hence, from (3.8)

$$P(\bar{E}_2^{nc}) \leq cmp^d 2L \exp(n^{0.3}) (\frac{1}{2})^{n^{0.35}} \leq 6cn^{2.5} L^2 n^{1000} \exp((\ln n)^3 + n^{0.3}) 2^{-n^{0.35}}$$
$$= 6cL^2 n^{1002.5} \exp(n^{0.3} + (\ln n)^3) 2^{-n^{0.35}} \to 0,$$

if $n\to\infty$.

Proof that $P(\bar{E}_8^n)\to 1$

For each z, the events $\{\delta_z^d > c_r-\Delta\}$ and $\{z$ has empty border $\}$ are independent. Now use the same argument as in the previous proof.

Proof that $P(\bar{E}_5^n)\to 1$

Note

$$\bar{E}_5^{nc}=\{\text{there exists a non}-\text{regular Markov signal point } z\in[0,cm]\}.$$

As in the previous proof, write

$$\bar{E}_5^n=\cup_{z=0}^{cm}E_5(z),\quad \text{where}\quad E_5(z):=\{z \text{ is a non}-\text{regular Markov signal point}\}.$$

For each z,

$$\begin{aligned}P(E_5^c(z))&=P(\delta_z^M\wedge\delta_z^d>c_r-\Delta)P(\delta_z^M\le c_r+\Delta|\delta_z^M\wedge\delta_z^d>c_r-\Delta)\\&=pP(\delta_z^M\le c_r+\Delta|\delta_z^M\wedge\delta_z^d>c_r-\Delta).\end{aligned}$$

From (2.36) of Corollary 2.2 we have:

$$P(\delta_z^M\le c_r+\Delta|\delta_z^M\wedge\delta_z^d>c_r-\Delta)\le n^{-10^5}.$$

Thus, from (3.7) $P(\bar{E}_5^{nc})\le cmpn^{-10^5}\le c(n^{2.5}EZ+1)pn^{-10^5}=c3Ln^{1002.5-100000}+cpn^{-10^5}\to 0$, as $n\to\infty$.

Proof that $P(\bar{E}_9^n)\to 1$

We use the same argument as in the previous proof. Note

$$\bar{E}_9^{nc}=\{\text{there exists a signal point } z\in[0,cm] \text{ that is not strong signal point}\}.$$

As in the previous proof, write

$$\bar{E}_9^{nc}=\cup_{z=0}^{cm}E_9(z),\quad \text{where}\quad E_9(z):=\{z \text{ is a non}-\text{strong signal point}\}.$$

Recall (2.3): $\tilde{\delta}_z^d>\delta_z^d-\exp(-dn^{999})$. Since, for n big enough, $\exp(-dn^{999})<\tilde{\Delta}=\Delta\exp(-2n)$, we get

$$\tilde{\delta}_z^d>\delta_z^d-\tilde{\Delta}.$$

Now, for each z,

$$\begin{aligned}P(E_9(z))&=P(\delta_z^d>c_r-\Delta)P(\tilde{\delta}_z^d\le c_r-\Delta|\delta_z^d>c_r-\Delta)\\&=p^dP(\tilde{\delta}_z^d\le c_r-\Delta|\delta_z^d>c_r-\Delta)\le p^dP(\delta_z^d-\tilde{\Delta}\le c_r-\Delta|\delta_z^d>c_r-\Delta)\\&\le p^dP(\delta_z^d\le c_r-\Delta+\tilde{\Delta}|\delta_z^d>c_r-\Delta).\end{aligned}$$

By (2.37) of Corollary 2.2 we now have

$$P(E_9(z))\le p^d\exp(-n).$$

Hence, by (3.8)

$$P(\bar{E}_9^{nc})\le cmp^d\exp(-n)\le p^dc(EZn^{2.5}+1)\exp(-n)\le c3Ln^{1000}\exp{(\ln n)^3}\exp(-n)+o(1)=o(1).$$

Proof that $P(\bar{E}_6^n) \to 1$

Consider random variables $Z_0 < Z_1 < \cdots < Z_k < \cdots$ as in (3.11). Let $N = \max\{i : Z_i \leq cm\}$. Define

$$E_{6b}^n := \{Z_i - Z_{i-1} \leq EZn^{10001}, \quad i = 1, 2, \ldots, n^{1000}\} \tag{3.13}$$

$$\bar{E}_{6c}^n := \{Z_i - Z_{i-1} \geq EZn^{-11001}, \quad i = 1, 2, \ldots, n^{1000}\} \tag{3.14}$$

and note

$$E_s \cap E_{6b}^n \cap E_{6a}^n \cap \{N \leq n^{10000}\} \subset \bar{E}_6^n.$$

Since $E \subset \{N \leq n^{10000}\}$, we get $P(N \leq n^{10000}) \to 1$. We also know that $P(E_s) \to 1$. Thus, it suffices to show that $P(E_{6b}^{nc}), P(E_{6c}^{nc}) \to 0$ as $n \to \infty$. Now, by Markov inequality and (3.5) and (3.7):

$$P(E_{6b}^{nc}) = P(\exists 1 \leq i \leq n^{10000} \text{ such that : } Z_i - Z_{i-1} > EZn^{10001})$$

$$\leq \sum_{i=1}^{n^{10000}} P(Z_i - Z_{i-1} > EZn^{10001}) = n^{10000} P(Z > EZn^{10001}) \leq$$

$$n^{10000} \tfrac{EZ}{EZn^{10001}} = \tfrac{1}{n};$$

$$P(E_{6c}^{nc}) = P(\exists\ 1 \leq i \leq n^{10000} \text{ such that : } Z_i - Z_{i-1} < EZn^{-11001})$$

$$\leq \sum_{i=1}^{n^{10000}} P(Z_i - Z_{i-1} < EZn^{-11001}) \leq n^{10000} P(Z < EZn^{-11001}) <$$

$$pEZn^{-1001} \leq 3Ln^{1000-1001} = \tfrac{3L}{n}.$$

Proof that $P(\bar{E}_7^n) \to 1$

Consider the event

$$\{\text{no signal carrier points in } [0, EZn^{11001}]\}.$$

Every signal carrier point is a Markov signal point. Hence, for the proof suffices to show, that with high probability there is no Markov signal points in the interval $[0, EZn^{11001}]$.
Now, by (3.4) and (3.7)

$$P(\text{No Markov signal points in } [0, EZn^{11001}]) =$$

$$P(Z^o > EZn^{-11001}) \leq pEZn^{-11001} \leq 3Ln^{-11001+1000} = o(1).$$

Thus $P($No Markov signal points in $[0, EZn^{-11001}] \to 1$..
Now repeat the same argument for intervals $[m, m - EZn^{-11001}]$ and $[m, m + EZn^{-11001}]$.

3.3 Proof of $P(\bar{E}_3^n) \to 1$ and $P(\bar{E}_4^n) \to 1$

The proof relies on the rate of convergence in local central limit theorem (LCLT). In the next subsection we present some technical preliminaries related to the proof.

3.3.1 Some preliminaries

Let S be symmetric random walk with span 1. Denote $p_N(k) = P(S(N) = k)$. The random walk S has lattice $+\backslash - z, z \in Z$; its variance is σ^2.

Local CLT (Petrov, 75, Thm 6 p.197):

$$\sup_k \left|\sigma\sqrt{N}p_N(k) - \frac{1}{\sqrt{2\pi}}\exp\{-\frac{k^2}{2\sigma^2 N}\}\right| = O(\frac{1}{\sqrt{N}})$$

or

$$\sup_k \left|p_N(k) - \frac{1}{\sigma\sqrt{N}\sqrt{2\pi}}\exp\{-\frac{k^2}{2\sigma^2 N}\}\right| = O(\frac{1}{N}).$$

Denote

$$q_N(k) := \frac{1}{\sigma\sqrt{N}\sqrt{2\pi}}\exp\{-\frac{k^2}{2\sigma^2 N}\} \quad |k| \le LN.$$

Let $t_N := (\ln N)^b$, $b > 1$.
We estimate

$$\begin{aligned} |p_N^2(k) - q_N^2(k)| &\le (p_N(k) + q_N(k)) \sup_k |p_N(k) - q_N(k)| \\ &\le [2q_N(k) + O(\frac{1}{\sqrt{N}})]O(\frac{1}{N}) = O(\frac{1}{\sqrt{N}N}) \end{aligned}$$

and

$$\sum_{k>t_N+j}^{L\sqrt{N}} [p_N^2(k) - q_N^2(k)] \le (L\sqrt{N})O(\frac{1}{\sqrt{N}N}) = O(\frac{1}{N}), \quad j = -t_N, \cdots, t_N.$$

Estimate

$$\begin{aligned} &\frac{p_N^2(k)}{\sum_{k>t_N+j} p_N^2(k)} \le \frac{p_N^2(k)}{\sum_{k>t_N+j}^{L\sqrt{N}} p_N^2(k)} \le \frac{q_N^2(k) + O(\frac{1}{N})}{\sum_{k>t_N+j}^{L\sqrt{N}}[p_N^2(k) - q_N^2(k)] + \sum_{k>t_N+j}^{L\sqrt{N}} q_N^2(k)} \\ &\le \frac{O(\frac{1}{N})}{\sum_{k>t_N+j}^{L\sqrt{N}} q_N^2(k) - O(\frac{1}{N})}, \end{aligned}$$

for all k and $j = -t_N \dots, t_N$.
Now,

$$\sum_{k>t_N+j}^{L\sqrt{N}} q_N^2(k) = \frac{1}{2\sigma^2\pi N}\sum_{k>t_N+j}^{L\sqrt{N}} \exp(-\frac{k^2}{\sigma^2 N})$$

and

$$\sum_{k>t_N+j}^{L\sqrt{N}} \exp(-\frac{k^2}{\sigma^2 N}) \ge \sum_{k>2t_N}^{L\sqrt{N}} \exp(-\frac{k^2}{\sigma^2 N}) > \sum_{k>2t_N}^{L\sqrt{N}} \exp(-\frac{L^2}{\sigma^2}) = M(L\sqrt{N} - 2t_N).$$

Thus, for each $j = -t_N \dots, t_N$,

$$\sup_k \frac{p_N^2(k)}{\sum_{k>t_N+j} p_N^2(k)} \le \frac{O(\frac{1}{N})}{\frac{K}{N}(L\sqrt{N} - 2t_N) - O(\frac{1}{N})} = \frac{K_4}{K_1\sqrt{N} - K_2 t_N - K_3} = O(\frac{1}{\sqrt{N}}) \tag{3.15}$$

where K, K_1, K_2, K_3, K_4 are constants.

Let μ be a probability distribution on $\{-t_N, -t_N + 1, \dots, 0, \dots, t_N - 1, t_N\}$. Consider the convolutions

$$u_N(k) = \sum_{j=-t_N}^{t_N} p_N(k-j)\mu_j, \quad k = -(LN - t_N), \dots, LN + t_N. \tag{3.16}$$

If $p_N(k) \geq p_N(k+1)$ for all $k \geq 0$, then for each $k > t_N$, we have bounds

$$p_N(k+t_N) \leq u_N(k) \leq p_N(k-t_N). \tag{3.17}$$

In this case,

$$\sum_{k>t_N}^{t_N+LN} u_N(k) \geq \sum_{l>2t_N}^{N} p_N(l).$$

And from (3.15), taking $j = t_N$ we may deduce that

$$\sup_{t_N<k} \frac{u_N^2(k)}{\sum_{k>t_N} u_N^2(k)} \leq \sup_{0<k} \frac{p_N^2(k)}{\sum_{k>2t_N} p_N^2(k)} \leq O(\frac{1}{\sqrt{N}}). \tag{3.18}$$

Generally, choose an atom $\lambda := \mu_j > 0$. Then

$$u_N(k) \geq \lambda p_N(k+j), \quad u_N^2(k) \geq \lambda^2 p_N^2(k+j)$$

and

$$\sum_{k>t_N}^{t_N+LN} u_N^2(k) \geq \lambda^2 \sum_{k>t_N+j}^{N} p_N^2(k). \tag{3.19}$$

Since $\sup_{k>t_N} u_N^2(k) \leq \sup_{k>0} p_N^2(k)$, we get from (3.15)

$$\sup_{t_N \leq k} \frac{u_N^2(k)}{\sum_{k>t_N} u_N^2(k)} \leq \sup_k \frac{p_N^2(k)}{\lambda^2 \sum_{k>t_N+j} p_N^2(k)} = O(\frac{1}{N^{\frac{1}{4}}}). \tag{3.20}$$

In particular, from (3.20) follows

$$\frac{\sum u_N^3(k)}{\sum u_N^2(k)\sqrt{\sum u_N^2(k)}} \leq \max_k u_N(k) \frac{\sum u_N^2(k)}{\sum u_N^2(k)\sqrt{\sum u_N^2(k)}} \leq \max_k \frac{u_N(k)}{\sqrt{\sum u_N^2(k)}} \leq O\Big(\frac{1}{N^{\frac{1}{4}}}\Big). \tag{3.21}$$

Suppose that arrays $u_k := u_N(k)$ and $v_k := v_N(k)$, $t_N < k \leq LN + t_N$ both satisfy (3.21). Then

$$\frac{\sum(u_k^3 + v_k^3)}{\sum(u_k^2 + v_k^2)\sqrt{\sum(u_k^2 + v_k^2)}} \leq \max\{u_k, v_k\} \frac{\sum(u_k^2 + v_k^2)}{\sum(u_k^2 + v_k^2)\sqrt{\sum(u_k^2 + v_k^2)}} \tag{3.22}$$

$$\leq \max\{\max_k \frac{u_k}{\sqrt{\sum u_k^2}}, \max_k \frac{v_k}{\sqrt{\sum v_k^2}}\} = O(N^{-\frac{1}{4}}) \tag{3.23}$$

Let us make one more observation. Since $\exp(\frac{-9t_N^2}{2\sigma^2 N}) \to 1$, there exists a $c' > 0$ such that

$$\exp(\frac{-9t_N^2}{2\sigma^2 N}) > c'$$

for each N big enough. Thus, there exists a constant $c > 0$ such that

$$p_N(k) > \frac{c}{\sqrt{N}}, \quad \forall |k| \leq 3t_N$$

Take λ as previously. Then

$$u_N(k) \geq p(k+j)\lambda \geq \frac{c\lambda}{\sqrt{N}}.$$

Hence there exists $C > 0$: $u(l) \geq \frac{C}{\sqrt{N}}$ $\forall l$ such that $|l+j| \leq 3t_N$.
In particular

$$u_N(k) \geq \frac{C}{\sqrt{N}}, \quad -2t_N \geq k \leq 2t_N. \tag{3.24}$$

3.3.2 Proof that $P(\bar{E}_3^n) \to 1$

Define the random variables z_1, z_2, ... as follows: z_1 is the first Markov signal point in $[0,\infty)$, z_k is the first Markov signal point in $[z_{k-1}+e^{n^{0.3}},\infty)$. Note that a.s. there are infinitely many such points.
From the signal carrier part we know that, if each Markov signal point in $[0,cm]$ has empty neighborhood,i.e. $\bar{E}_2^n$ holds, then they form clusters which have the radius at most $2Ln^{1000}$ and lie at least $e^{n^{0.3}}$ apart from each other. In this case all signal carrier points in $[0,cm]$ coincide with z_i-s defined above. We define the event

$$E_{3a}^n := \Big\{ \text{for each} \quad i,j \leq n^{10000}, i \neq j \text{ we have } |h(z_i)-h(z_j)| \geq \exp(-n^{0.11}) \Big\}$$

and note

$$E_{3a}^n \cap \bar{E}_2^n \cap E \subset \bar{E}_3^n.$$

Since $P(\bar{E}_2^n \cap E) \to 1$, it suffices to show that $P(E_{3a}^n) \to 1$ as $n \to \infty$.

Consider z_i, z_j, $i \neq j$. For simplicity denote them as z and z' Let

$$\epsilon_n := \exp(-n^{0.11}).$$

Consider the event

$$E_n(i,j) := \{|h(z)-h(z')| \geq \epsilon_n\}.$$

For each $y \in Z$, define the random vector:

$$\xi_n(y) := \Big(\xi(y - Ln^{1000} - e^{n^{0.1}}),\, \xi(y - Ln^{1000} - e^{n^{0.1}} + 1) \ldots \xi(y + Ln^{1000}) \Big).$$

Now, let $\xi_n := \xi_n(z)$ and $\xi_n' := \xi_n(z')$. They are independent.

$$f_n := \sum_{k=z+Ln^{1000}+1}^{z+L(n^{1000}+e^{n^{0.1}})} u_n(k)\xi(k), \quad f_n' := \sum_{k=z'+Ln^{1000}+1}^{z'+L(n^{1000}+e^{n^{0.1}})} u_n'(k)\xi(k),$$

where

$$u_n(k) := \sum_{i=z-Ln^{1000}}^{z+Ln^{1000}} P(S_i(e^{n^{0.1}}) = k)\mu_i, \;\; u_n'(k) := \sum_{i=z'-Ln^{1000}}^{z'+Ln^{1000}} P(S_i(e^{n^{0.1}}) = k)\mu_i'$$

and μ_i, $i = z - Ln^{1000}, \cdots, z + Ln^{1000}$ and μ_i', $i = z' - Ln^{1000}, \cdots, z' + Ln^{1000}$ denote the atoms of the stationary measure corresponding to z and z', respectively.
Recall that by (2.13)

$$h(z) := \sum_{k=z-L(n^{1000}+e^{n^{0.1}})}^{z+L(n^{1000}+e^{n^{0.1}})} u_n(k)\xi(k), \quad f_n' := \sum_{k=z'-L(n^{1000}+e^{n^{0.1}})}^{z'+L(n^{1000}+e^{n^{0.1}})} u_n'(k)\xi(k).$$

Note that conditioning on ξ_n, the coefficients $u_n(k)$ become constants.
[More precisely, f_n has the same distribution as

$$\tilde{f}_n := \sum_{k>Ln^{1000}}^{L(n^{1000}+e^{n^{0.1}})} \tilde{u}_n(k)\xi(k),$$

with

$$\tilde{u}_n(k) := \sum_{j=-Ln^{1000}}^{Ln^{1000}} P(S_j(e^{n^{0.1}}) = k)\tilde{\mu}_j = \sum_{j=-Ln^{1000}}^{Ln^{1000}} P(S(e^{n^{0.1}}) = k-j)\tilde{\mu}_j,$$

with $\tilde{\mu} := \{\tilde{\mu}_j\} := \{\mu_{z+j}\}$, $-Ln^{1000} \leq j \leq Ln^{1000}$ being random probability measure independent of $\xi_{Ln^{1000}+1}, \dots \xi_{e^{n^{0.1}}}$. In this setup the conditioning on ξ_n means the conditioning on $\tilde{\mu}$.]
Hence

$$P\Big(\frac{f_n - Ef_n}{\sqrt{Df_n}} \leq x|\xi_n\Big) = P\Big(\frac{\sum_{k>Ln^{1000}}^{L(e^{n^{0.1}}+N^{1000})} u_n(k)(\xi_(k) - \frac{1}{2})}{\frac{1}{2}\sqrt{\sum_{k>Ln^{1000}}^{L(e^{n^{0.1}}+N^{1000})} u_n^2(k)}} \leq x|\xi_n\Big),$$

where $(u_n(k))$ are the fixed coefficients of type (3.16) (with $N = e^{n^{0.1}}$, $b = 10000$). Now Berry-Esseen inequality for independent random variables (see, e.g., Petrov, Thm 3, p.111) states

$$\sup_x \Big|P\Big(\frac{\sum u_n(k)(\xi(k) - \frac{1}{2})}{\frac{1}{2}\sqrt{\sum u_n^2(k)}} \leq x|\xi_n\Big) - \Phi(x)\Big| \leq A\frac{\sum u_n^3(k)}{\sum u_n^2(k)\sqrt{\sum u_n^2(k)}}, \tag{3.25}$$

with some constant A not depending on n and $u_n(k)$-s. By (3.21) (with $N = e^{n^{0.1}}$, $b = 10000$), the right side of (3.25) is bounded by $O(e^{\frac{-n^{0.1}}{4}})$. Here Φ stands for the standard normal distribution function.

By similar argument, conditioning on (ξ_n, ξ'_n) and using (3.22) instead of (3.21) we have

$$\sup_x \Big|P\big(\frac{f_n - f'_n - \mu_n}{\sigma_n} \leq x|\xi_n, \xi'_n\big) - \Phi(x)\Big| = O(e^{\frac{-n^{0.1}}{4}}), \tag{3.26}$$

with $\mu_n := E(f_n - f'_n)$, $\sigma_n := \sqrt{Df_n + Df'_n}$ (f_n and f'_n are independent.)
Denote now $g_n := h_n - f_n$, $g'_n := h'_n - f'_n$. The event $E_n(i,j)$ can now be rewritten

$$E_n^c(i,j) := \{f_n - f'_n \in g_n - g'_n + [-\epsilon_n, \epsilon_n]\}.$$

Given ξ_n and ξ'_n, the random variable $g_n - g'_n$ is a constant. By (3.26) we have

$$P(E_n^c(i,j)|\xi_n, \xi'_b) = P\Big(\tfrac{f'_n - f_n - \mu_n}{\sigma_n} \in \tfrac{g_n - g'_n + [-\epsilon_n, \epsilon_n] - \mu_n}{\sigma_n}|\xi_n, \xi'_n\Big) \leq$$

$$2\sup_x \Big|P\big(\tfrac{f'_n - f_n - \mu_n}{\sigma_n} \leq x|\xi_n, \xi'_n\big) - \Phi(x)\Big| + \sup\Big\{\Phi(a) - \Phi(b)\big|a - b = \tfrac{2\epsilon_n}{\sqrt{2\pi}\sigma_n}\Big\} \leq$$

$$O(e^{\frac{-n^{0.1}}{4}}) + \sqrt{\tfrac{2}{\pi}}\tfrac{\epsilon_n}{\sigma_n}.$$

Next, we estimate the standard deviation σ_n. For that note: because of (3.24) $u_n^2(z + Ln^{1000} + 1) \geq C^2 e^{-n^{0.1}}$, $u'^2_n(z' + Ln^{1000} + 1) \geq C^2 e^{-n^{0.1}}$ if n is big enough. Thus,

$$\sigma_n = \sqrt{Df_n + Df'_n} = \frac{1}{2}\sqrt{\sum u_N^2(k) + \sum u'^2_N(k)} > \frac{1}{2}\sqrt{2C^2 e^{n^{0.1}}} = \sqrt{2}Ce^{\frac{-n^{0.1}}{2}}.$$

Hence, for n big enough there exists a constant $C_2 < \infty$ such that

$$\sqrt{\frac{2}{\pi}}\frac{\epsilon_n}{\sigma_n} \leq \frac{1}{\sqrt{\pi}}\exp(-n^{0.11} + \frac{n^{0.1}}{2}) \leq C_2\exp(-n^{0.05}). \tag{3.27}$$

Thus, (3.27), (3.26) give

$$P(E^n(i,j)) \leq O(e^{\frac{-n^{0.11}}{4}}) + O(e^{-n^{0.05}}) = O(e^{-n^{0.05}}).$$

Now, by definition

$$E_{3a}^n = \cap_{i,j,i\neq j}^{n^{10000}} E^n(i,j)$$

and

$$P(E_{3a}^{nc}) \leq \sum_{i,j,i\neq j}^{n^{10000}} P(E^{nc}(i,j)) < n^{20000}O(e^{-n^{0.05}}) = o(1).$$

Outline of the proof that $P(\bar{E}_4^n)$ is close to one

Denote the Use (3.25) to get

$$\begin{aligned}
P(\bar{E}_4^{nc}|\xi_n) &= P(|f_n + g_n - 0.5| \leq \epsilon_n|\xi_n) = P(f_n + g_n \in [0.5 - \epsilon_n, 0.5 + \epsilon_n]|\xi_n) \\
&= P(f_n \in [(0.5 - g_n) - \epsilon_n, 0.5 - g_n + \epsilon_n]|\xi_n) \\
&= P\Big(\frac{f_n - Ef_n}{\sqrt{Df_n}} \in \Big[\frac{0.5 - Ef_n - g_n - \epsilon_n}{\sqrt{Df_n}}, \frac{0.5 - Ef_n - g_n + \epsilon_n}{\sqrt{Df_n}}\Big]|\xi_n\Big) \\
&\leq 2\sup_x P\Big(\frac{f_n - Ef_n}{\sqrt{Df_n}} \leq x|\xi_n\Big) + \sup\Big\{\Phi(a) - \Phi(b)\Big|a - b = \sqrt{\frac{2}{\pi}}\frac{\epsilon_n}{\sqrt{Df_n}}\Big\} \\
&\leq O(e^{\frac{-n^{0.1}}{4}}) + \sqrt{\frac{2}{\pi}}\frac{\epsilon_n}{\sqrt{Df_n}} = O(e^{-n^{0.05}}),
\end{aligned}$$

because $\sqrt{Df_n} > C\exp(-\frac{n^{0.1}}{2})$. The rest of the proof goes as previously.

* In the following we consider the scenery dependent events defined on $[-cm, cm]$. Do do that, we define the events $\tilde{E}_i^n$, $i = 1, \ldots, 9$, where $\tilde{E}_i^n$ is defined exactly as $\bar{E}_i^n$, with $[-cm, 0]$ instead of $[0, cm]$. Finally, we define the events

$$E_i^n := \tilde{E}_i^n \cap \bar{E}_i^n.$$

The results of the present section show that $\forall\ i = 1, \ldots, 9$,

$$P(E_i^n) \to 0, \quad n \to \infty.$$

3.4 What is the signal carrier?

Let us briefly summarize the main ideas of the previous sections.
Basically, a signal carrier is the place in the scenery, where the probability of generating at $n^2 + 1$ same colors is high. However, it is clear that such a place can not be too small. In the 3-color example the signal carrier depends on the one bit of the scenery, only. Now, in 2-color case it takes many more bits to make the scenery (locally) atypical. We saw in Proposition 2.1 that for z to be a signal point, it is necessary that the interval I_z has at least one big (longer than $n/\ln n$) block of ξ. Thus, a point z being a (Markov, strong) signal point, is actually the property of $\xi|I_z$ and it depends on at least $n/\ln n$ bits.
If z is a signal point, then the scenery ξ is atypical in the interval I_z: δ_z^d is high. Thus, signal points would be the candidates for the signal carriers, if, for each z, we could estimate δ_z^d. The latter would be easy, if we knew when the random walk visits z. Then just take all such visits and consider the proportion of those visits that were followed by $n^2 + 1$ same colors after M steps. Unfortunately, we do not know when the random walk S visits z. But we do know (we observe) when S generates blocks with length at least n^2. Thus we can take these observations (times) as the visits of (the neighborhood of) z and estimate the probability of generating $n^2 + 1$ same colors M step after previous $n^2 + 1$ same colors. This idea yields the Markov signal probability. The problem now is to localize the area where the random walk (during a given time period) can generate $n^2 + 1$ same colors in observations. If this area were too big, we could neither estimate the Markov signal probability nor to understand where we are. To localize the described area, we showed (event E_2^n) that signal points have empty neighborhood. In the next section we shall see that the probability to generate $n^2 + 1$ same colors on empty neighborhood is very small. This means, if S is close to a signal point z, then, with high probability, (and during a certain time period) all $n^2 + 1$ same colors in observations will be generated on I_z. The fact that all signal points have also empty borders (events E_8^n and E_9^n) makes the latter statement precise. Thus, a Markov signal point seems to be a reasonable signal carrier. But which one? Note, if z is a Markov signal point, i.e. I_z contains at least one big block, then, very likely the point $z + 1$ is a Markov signal point, too. In other words, Markov signal points come in clusters. However, when E_2^n holds, then each point in such a cluster has empty neighborhood. On the other hand, for z to be a Markov signal point, it is necessary

to have at least one big block of ξ in I_z. This means that the diameter of every cluster of Markov signal points is at most $2Ln^{1000}$. And the distances between the clusters are at least $L(e^{n^{0.3}} - n^{1000})$. Hence, in 2-color case one we should think of signal carriers as the clusters of Markov signal points (provided E_2^n holds, but this holds with high probability). However, to make some statements more formal, for each cluster we have one representant, namely the signal carrier point. Since the diameters of clusters is at most $2Ln^{1000}$, our definition of signal carrier points ensures that different signal carrier points belong to different cluster. If the cluster is located in $[0, \infty)$, then the signal carrier point is the most left (smallest) Markov signal point in the cluster; if the cluster is located in $(-\infty, 0)$, then signal carrier point is the right most (biggest) Markov signal point in the cluster. The event E_7^n ensures that there are no Markov signal points in the $2Ln^{1000}$-neighborhood of 0, so $\bar{z}_1$ and $\bar{z}_0$ belong to the different clusters, too. The exact choice of a signal carrier point is irrelevant. However, it is important to note that given a cluster, everything that makes this cluster a signal carrier cluster (namely, the big blocks of scenery) are inside the interval $I_{\bar{z}}$, where $\bar{z}$ is the signal carrier point corresponding to the cluster. In particular, all blocks in observations that are longer than n^2 will be generated on $I_{\bar{z}}$. This means that signal carrier points, $\bar{z}_i$ (or the corresponding intervals, $I_{\bar{z}_i}$ serve as the signal carriers as well. At least, if we are able to estimate $\delta^M_{\bar{z}_i}$ with great precision. This is the subject of the next section.

4 Events depending on random walk

In the previous section we saw: if all scenery dependent events hold, then the signal carrier points are good candidates for the signal carriers. In this case the signal is an untypically higher Markov signal probability. Hence, to observe this signal, we must be able to estimate the Markov signal probability. In the present section we define these estimators and in the next section we will see that they perform well, if the random walk, S, behaves typically. We describe the typical behavior of S in terms of several events depending on S. The main objective of the present section is to show that the (conditional) probability of such events tends to 1.

4.1 Some preliminaries

As argued in subsection 3.4, the main idea of the estimation of Markov signal probability is very simple - given a time interval T, consider all blocks in the observations $\chi|_T$ that are bigger than n^2. Among these observations calculate the proportions of such blocks that after exactly M step were followed by another such block. The time interval used by such estimation must be big enough to get precise estimate but, on the other hand, it must be in the correspondence with the size of (empty) neighborhood. Recall that the neighborhood $\mathcal{N}_z$ consisted of two intervals of length $Le^{n^{0.3}}$. Hence, the optimal size of the interval T is $e^{n^{0.3}}$.
We now define the necessary concepts related to the described estimate - stopping times (that stop when at least n^2+1 same colors were observed) and the Bernoulli variables that show the whether the stopping times were followed (after M step) by another n^2+1 same colors or not. For technical reasons after stopping the process, we wait at least $e^{n^{0.1}}$ steps until we look for the next block.

* Let $t>0$ and let $\hat{\nu}_t(1)$ be the smallest $s \geq t$ such that

$$\chi(t)=\chi(t-1)=\cdots=\chi(t-n^2). \tag{4.1}$$

We define the stopping times $\hat{\nu}_t(i)$, $i=2,3,\ldots$ inductively: $\hat{\nu}_t(i)$ is the smallest $t \geq \hat{\nu}_t(i-1)+e^{n^{0.1}}$ such that (4.1) holds.

* Let $X_{t,i}$ be the Bernoulli random variable that is one iff

$$\chi(\hat{\nu}_t(i)+M)=\chi(\hat{\nu}_t(i)+M+1)=...=\chi(\hat{\nu}_t(i)+M+n^2).$$

Let $T=T(t):=[t,t+e^{n^{0.3}}]$. Define

$$\hat{\delta}^M_T=\begin{cases}\frac{1}{e^{n^{0.2}}}\sum_{i=1}^{e^{n^{0.2}}}X_{t,i} & \text{if } \hat{\nu}_t(e^{n^{0.2}})<t+e^{n^{0.3}}-e^{n^{0.1}}\\ 0 & \text{otherwise.}\end{cases} \tag{4.2}$$

* We now define some analogues of $\hat{\nu}_t$ and X_t.
Let $z\in\mathbb{Z}$ and $t\in\mathbb{N}$.

Let $\nu_{z,t}(1)$ designate the first time after t where we observe n^2 zero's or one's in a row, generated on the interval I_z. More precisely:

$$\nu_{z,t}(1):=\min\left\{s>0\ \middle|\ \begin{array}{c}\chi(s)=\chi(s-1)=\cdots=\chi(s-n^2)\\ S(j)\in I_z,\ \forall j=s-n^2,\ldots,s\end{array}\right\}.$$

Once $\nu_{z,t}(i)$ is well defined, define $\nu_{z,t}(i+1)$ in the following manner:

$$\nu_{z,t}(i+1):=\min\left\{t\geq\nu_{z,t}(i)+e^{n^{0.1}}\ \middle|\ \begin{array}{c}\chi(s)=\chi(s-1)=...=\chi(s-n^2)\\ S(j)\in I_z,\ \forall j=s-n^2,\ldots,s\end{array}\right\}.$$

* Let $X_{z,t,i}$, $i = 1, 2, \ldots$ designate the Bernoulli variable which is equal to one if exactly after time M the stopping time $\nu_{z,t}(i)$ is followed by a sequence of $n^2 + 1$ one's or zero's generated on I_z. More precisely, $X_{z,t,i} = 1$ iff

$$\chi(\nu_{z,t}(i) + M) = \chi(\nu_{z,t}(i) + M + 1) = \cdots = \chi(\nu_{z,t}(i) + n^2) \quad \text{and}$$
$$S(\nu_{z,t}(i) + M), \ldots, S(\nu_{z,t}(i) + n^{1000}) \in I_z.$$

Define

$$\hat{\delta}^M_{z,t} := \frac{1}{e^{n^{0.2}}} \sum_{i=1}^{e^{n^{0.2}}} X_{z,t,i}.$$

As argued in subsection 2.1, $\{S(\nu_{z,t,i})\}$ is an ergodic Markov process with state space I_z and with the stationary measure I_z. Hence,

$$\frac{1}{j} \sum_{i=1}^{j} X_{z,t,i} \to \delta^M_z, \quad \text{a.s.}$$

Now we can apply some large deviation inequality to see that if $j \geq \exp(n^{0.2})$, then $\hat{\delta}^M_{z,t}$ gives a very precise estimate of δ^M_z.
The problem is that the random variables $X_{z,t,i}$ and, hence, the estimate $\hat{\delta}^M_{z,t}$ is *a priori* not observable. This is because we cannot observe whether $n^2 + 1$ same colors in observations were generated on I_z or not. Thus, we can not observe neither $\nu_{t,z}(i)$ nor $X_{t,z,i}$. However, the event $E^n_{3,S}$, stated below, ensures that with high probability $\hat{\delta}^M_{z,t}$ is the same as $\hat{\delta}^M_T$, provided that during the time interval T, the random walk S is close to z (the sense of closeness will specified later).

* We now define the estimates for the frequency of ones. Again, we define a general, observable, estimate: $\hat{h}_t$ and its theoretical, *a priori* not-observable counterpart: $\hat{h}_{z,t}$.

Define

$$\hat{h}_t := \begin{cases} \frac{1}{e^{n^{0.2}}} \sum_{i=1}^{e^{n^{0.2}}} \chi(\nu_t(i) + e^{n^{0.1}}) & \text{if}, \hat{\nu}_t(e^{n^{0.2}}) < t + e^{n^{0.3}} - e^{n^{0.1}} \\ 0 & \text{otherwise.} \end{cases},$$

$$\hat{h}_{z,t} := \frac{1}{e^{n^{0.2}}} \sum_{i=1}^{e^{n^{0.2}}} \chi(\nu_{z,t}(i) + e^{n^{0.1}}).$$

* Finally, we define the stopping time that stop the walk, when a new signal carrier is visited.

Let $\ldots, \bar{z}_{-1}, \bar{z}_0, \bar{z}_1, \ldots$ denote the signal carrier-points in $\mathbb{R}$. Denote $I_i := I_{z_i}$ and let $\rho(k)$ denote the k-th visit of S to the one of the intervals I_i in the following manner: when an interval I_i is visited, then the next stop is on the different interval.
More precisely, let $\rho(0)$ be the first time $t \geq 0$ such that $S(t) \in \cup_i I_i$. Denote $I(\rho(k))$ the interval I_i visited by $\rho(k)$. Then define $\rho(k)$ inductively:

$$\rho(k+1) = \min\{t > \rho(k) | S(t) \in \cup_i I_i, \quad S(t) \notin I(\rho(k))\}.$$

4.2 Random walk-dependent events

In this section we define the events that characterize the typical behavior of the random walk S on the typical scenery on interval $[-cm, cm]$. The (piece of) scenery $\xi|[-cm, cm]$ is typical if it belongs to the

all scenery-dependent events E_i^n, $i = 1, \dots, 9$. Recall, that the events E_i^n are the same as $\bar{E}_i^n$ defined in Section 4.2 with $[0, cm]$ replaced by $[-cm, cm]$. Also recall that $c > 1$ is an arbitrary fixed constant not dependent on n, and $m = \ulcorner n^{2.5} EZ \urcorner$.
Hence, throughout the section we consider the sceneries belonging to the set

$$E_{\text{cell\_OK}} := \cap_{i=1}^{9} E_i^n. \tag{4.3}$$

Clearly, $E_{\text{cell\_OK}}$ depends on n. We know that $P(E_{\text{cell\_OK}}) \to 1$ if $n \to \infty$.

Let $\psi : \mathbb{Z} \to \{0, 1\}$ be a (non random) scenery. Let $P_\psi(\cdot)$ designate the measure obtained by conditioning on $\{\xi = \psi\}$ and as well as on $\{S(m^2) = m\}$. Thus,

$$P_\psi(\cdot) := P(\cdot | \xi = \psi, S(m^2) = m). \tag{4.4}$$

Let $P(\cdot|\psi)$ denote $P(\cdot|\xi = \psi)$.
We now list the events that describe the typical behavior of S. The objective of the section is to show: if n is big and $\psi_n :=: \psi \in E_{\text{cell\_OK}}$ then all listed events have big conditional probabilities P_ψ.
The events depending on random walk are:

$$E_{1,\text{S}}^n := \left\{ S(m^2) = m \right\};$$

$$E_{2,S}^n := \left\{ \forall t \in [0, m^2] \text{ we have that } S(t) \in [-cm, cm] \right\};$$

$$E_{3,S} := \{\forall t \in [0, m^2], \text{ it holds}: \ \hat{\delta}_T^M \le c_r, \text{ if } \delta_{S(s)}^d \le c_r - \Delta \ \forall s \in T(t)\};$$

$$E_{4,S} := \{\rho(n^{25000}) \ge m^2\};$$

$$E_{5,S}^n := \{\forall k \le n^{25000} \text{ we have: if } \rho(k) \le m^2 \text{ then } \hat{\nu}_{\rho(k)}(e^{n^{0.2}}) \le \rho(k) + e^{n^{0.3}} - e^{n^{0.1}}\};$$

$$E_{6,S}^n := \left\{ \begin{array}{c} \text{for any } t \in [0, m^2] \text{ satisfying } \chi(t) = \dots = \chi(t + n^2) \\ \text{there exists } s \in [t, t + n^2] \text{ such that } S(s) \\ \text{is contained in a block of } \xi \text{ bigger than } n^{0.35} \end{array} \right\};$$

$$E_{7,S}^n(z, t) = \left\{ \left| \hat{\delta}_{z,t}^M - \delta_z^M \right| < e^{-n^{0.12}} \right\}, \quad z \in \mathbb{Z}, \ t > 0;$$

$$E_{7,S}^n := \cap_{z=-cm}^{cm} \cap_{t=0}^{m^2} E_{13,S}^n(z, t);$$

$$E_{8,S}^n(z, t) = \left\{ \left| \hat{h}_{z,t} - h(z) \right| < e^{-n^{0.12}} \right\}, \quad z \in \mathbb{Z}, \ t > 0;$$

$$E_{8,S}^n := \cap_{z=-cm}^{cm} \cap_{t=0}^{m^2} E_{8,S}^n(z, t);$$

We now estimate the conditional probabilities of all listed events. In most cases prove statements like $P_\psi(E_{j,S}^n) \to 1$. This means: for arbitrary sequence $\psi_n \in E_{\text{cell\_OK}}^n$ we have

$$\lim_n P(E_{j,S}^n | S(m^2) = m, \xi = \psi_n) = 1.$$

4.3 Proofs

At first note that by LCLT, we have

$$P(E_{1,S}) = \frac{1}{m} + O(\frac{1}{m^2}).$$

Clearly, $E_{1,S}$ does not depend on ξ, i.e. $P(E_{1,S}|\psi) = P(E_{1,S})$. Using (3.10) we get

$$P(E_{1,S}|\psi) \geq \exp(-2n) - O(\exp(-4n)) \geq \exp(-3n). \tag{4.5}$$

From (4.5) follows that for any event E,

$$P_{\psi}(E) = \frac{P(E, S(m^2) = m|\psi)}{P(S(m^2) = m|\psi)} \leq \frac{P(E|\psi)}{\exp(-3n)}. \tag{4.6}$$

Proposition 4.1 *For each $\epsilon > 0$ there exists $c(\epsilon)$, independent of n, such that for each ψ, $P_{\psi}(E^n_{2,S}) \geq 1 - \epsilon$, provided n is big enough.*

Proof. At first note, that, for each n, the event $E^n_{2,S}$ is independent of the scenery ψ. Thus,

$$P_{\psi}(E^n_{2,S}) = P(E^n_{2,S}|S(m^2) = m).$$

Define

$$E^n_a(c) = \{\forall t \in [0, m^2] \text{ we have that } S(t) \leq cm\}$$

$$E^n_b(c) = \big\{\forall t \in [0, m^2] \text{ we have that } S(t) \geq -cm\}$$

Clearly,

$$E^n_{2,S} = E^n_a(c) \cap E^n_b(c).$$

We now find c, not depending on n such that $P_{\psi}(E^{nc}_a(c)), P_{\psi}(E^{nc}_b(c)) \leq \frac{\epsilon}{2}$.
Let us define the stopping time ϑ

$$\vartheta := \min\{t|S(t) > cm\}.$$

Let for all $j \in 1, \dots, L$

$$p_j := P\Big(S(m^2) = m,\, \vartheta \leq m^2 \text{ and } S(\vartheta) = cm + j\Big)$$

We have that

$$P\left(E^{nc}_a(c) \cap E^n_{1,S}\right) = \sum_{j=1}^{L} p_j$$

Our random walk, S, is symmetric. By the reflection principle, for all $j \in 1, \dots, L$ we have

$$p_j = P(S(m^2) = cm + j + (cm + j - m) = 2cm + 2j - m,\, \vartheta \leq m^2 \text{ and } S(\vartheta) = cm + j).$$

Thus $p_j \leq P\left(S\left(m^2\right) = 2cm - m + 2j\ \right)$ and

$$P\left(E^{nc}_a(c) \cap E^n_{1,S}\right) \leq \sum_{j=1}^{L} P\left(S(m^2) = m(2c-1) + 2j\right). \tag{4.7}$$

By LCLT, for big m, the right side of (4.7) can be made arbitrary small in comparison to $P\left(S\left(m^2\right) = m\right)$ by taking c big enough. In other words, there exists c , not depending on n such that

$$\frac{\sum_{j=1}^{L} P\left(S\left(m^2\right) = 2cm + m + 2j\ \right)}{P\left(S\left(m^2\right) = m\right)} \leq \frac{\epsilon}{2}.$$

This means

$$\frac{P\left(E^{nc}_a(c) \cap E^n_{1,S}\right)}{P\left(E^n_{1,S}\right)} = P_{\psi}\left(E^{nc}_a(c)\right) \leq \frac{\epsilon}{2}.$$

The similar argument gives $P_\psi\left(E_b^{nc}(c)\right) \leq \frac{\epsilon}{2}$. ■

* Note, that the choice of c does not depend on n. From now on, we fix c such that Proposition 4.1 holds with $\epsilon > \frac{1}{8}$. This particular c is used in the definition of all scenery-dependent events and, therefore, in the definition of $E_{\text{cell\_OK}}$ as well as in the definitions $E^n_{4,S}$, $E^n_{5,S}$.

* In the following we use often the following versions of Höffding's inequality:
Let $X_1, \dots, X_N$ be independent random variables with range in $[a, b]$. Denote their sum by S_N. Then

$$\begin{aligned} P(|S_N - ES_N| \geq \epsilon) &\leq 2\exp(-2\frac{\epsilon^2}{N(b-a)^2}) \leq \exp(-\frac{d'\epsilon^2}{N}); \\ P(\frac{1}{N}|S_N - ES_N| \geq \epsilon) &\leq 2\exp(-2\frac{\epsilon^2 N}{(b-a)^2}) \leq \exp(-d'\epsilon^2 N). \end{aligned} \tag{4.8}$$

For our random walk, this is

$$\begin{aligned} P(|S(N)| \geq \epsilon) &\leq 2\exp(-\frac{\epsilon^2}{4L^2 N}) \leq \exp(-\frac{d\epsilon^2}{N}) \\ P(|\frac{S(N)}{N}| \geq \epsilon) &\leq 2\exp(-\frac{\epsilon^2 N}{4L^2}) \leq \exp(-d\epsilon^2 N), \end{aligned} \tag{4.9}$$

for some $d', d > 0$.

We also use the following results: let $X_1, \dots, X_N$ be iid random variables with mean 0 and finite variance σ^2. Let $M_n^+ = \max_{i=1,\dots,N} S_i$, $M_n = \max_{i=1,\dots,N} |S_i|$. Then

$$\frac{M_N^+}{\sigma\sqrt{N}} \Rightarrow \sup_{0\leq t\leq 1} W_t, \quad \text{and} \quad \left(\frac{M_N}{\sigma\sqrt{N}}, \frac{S(N)}{\sigma\sqrt{N}}\right) \Rightarrow (\sup_{0\leq t\leq 1} |W_t|, W(1)), \tag{4.10}$$

where W_t is standard Brownian motion. It is well-known that $\forall x > 0$, $P(\sup_{0\leq t\leq 1} W_t \leq x) = 2\Phi(x) - 1$.

Proof that $\liminf_n P_\psi(E^n_{4,S}) \geq 1 - \frac{1}{8}$

For each n, fix an arbitrary $\psi_n \in E^n_{\text{cell\_OK}}$. Since $\psi_n \in E^n_{\text{cell\_OK}} \subset E^n_6$, we have that every pair $\bar{z}_i \neq \bar{z}_j$ of signal carrier points in $[-cm, cm]$ satisfies

$$|\bar{z}_j - \bar{z}_i| \geq EZn^{-11001}.$$

During this proof, let $\mu := EZ$.

Let

$$\begin{aligned} E_{a,4}(k) &:= \{|\rho(k+1) - \rho(k)| \geq (\mu)^2 n^{-25000}\} \\ E_{a,4} &:= \cap_{k=0}^{n^{25006}} E_{a,4}(k). \end{aligned}$$

Since $m \leq n^{2.5}\mu + 1$, we have $n^{25006} \times \mu^2 n^{-25000} = \mu^2 n^6 > m^2$ and, therefore,

$$E_{a,4} \cap E^n_{2,S} \subset E^n_{4,S}. \tag{4.11}$$

By Proposition 4.1, for n big enough, $P_\psi(E^n_{2,S}) \leq \frac{1}{8}$. Thus,

$$P_\psi(E^{nc}_{4,S}) \leq P_\psi(E^{nc}_{4,a}) + P_\psi(E^{nc}_{2,S}) \leq \frac{1}{8} + \sum_{k=0}^{n^{22506}} P_\psi(E^{nc}_{a,4}(k)). \tag{4.12}$$

We now bound $P_\psi(E^n_{a,4}(k))$.

Note that for each T_i, T_j, $i \neq j$, we have

$$\inf\{|t-s| : t \in T_i, j \in T_j\} \geq \mu n^{-11001} - 2Ln^{1000}. \tag{4.13}$$

By (3.9), $\mu^2 > n^{25000}$. This means, $\mu > n^{12500} \geq 2Ln^{12002}$. The latter implies, that

$$\mu n^{-11001} - 2Ln^{11001} \geq \mu n^{-11002}. \tag{4.14}$$

Denote $N(n) := \mu^2 n^{-25000}$. From (4.13) and (4.14):

$$\begin{aligned}\{|\rho(k+1)-\rho(k)| < N\} &\subset \{\sup_{l\leq N} |S(l)| > \mu n^{-11001} - 2Ln^{11001}\} \\ &\subset \{\sup_{l\leq N} |S(l)| > \mu n^{-11002}\}.\end{aligned}$$

Now use the following maximal inequality to estimate

$$P(\max_{l\leq N} |S(l)| > \mu n^{-11002}) \leq 3 \max_{l\leq N} P\Big(|S(l)| > \frac{\mu}{3} n^{-11002}\Big). \tag{4.15}$$

By Höffding's inequality, for each $l \leq N$

$$\begin{aligned}P\Big(|S(l)| \leq \frac{\mu}{3} n^{-11002}\Big) &\leq \exp(-\frac{d\mu^2 n^{-22004}}{9l}) \leq \exp\big(-\frac{d\mu^2 n^{-22004}}{9N}\big) \\ &\leq \exp\big(-\frac{dn^{2500-22004}}{9}\big) = \exp\big(-\frac{dn^{2996}}{9}\big).\end{aligned}$$

Hence,

$$P(E_{a,4}(k)) \leq \exp\big(-\frac{dn^{2996}}{9}\big), \; P(E_{a,4}) \leq n^{22506} \exp\big(-\frac{dn^{2996}}{9}\big).$$

By (4.6), we get

$$P_\psi(E^{nc}_{a,4}) \leq n^{22506} \exp\big(3n - \frac{dn^{2996}}{9}\big).$$

The right side of the last inequality tends to 0 if $n \to \infty$. Relation (4.11) now finish the proof.

Proof that $P_\psi(E^n_{3,S}) \to 1$

Let $t \geq 0$ be an integer and define the stopping times $\hat\nu^o_t(1), \hat\nu^o_t(2), \ldots$ as follows:
$\hat\nu^o_t(1)$ is the smallest time $s \geq t + e^{n^{0.1}}$ and

$$\chi(s-n^2) = \chi(s-n^2+1) = \cdots = \chi(s) \text{ and } \delta^d_{S(s)} \leq c_r - \Delta. \tag{4.16}$$

Once $\hat\nu^o_t(k)$ is well defined, define $\hat\nu^o_t(k+1)$ to be the smallest time $s \geq \hat\nu^o_t(k) + e^{n^{0.1}}$ such that (4.16) holds.

Let $X^o_{t,k}$ be the Bernoulli variable which is equal to one iff

$$\chi(\hat\nu^o_t(k)+M) = \chi(\hat\nu^o_t(k)+M+1) = \cdots = \chi(\hat\nu^o_t(k)+M+n^2).$$

Finally define

$$\hat\delta^M_{o,t} := \frac{1}{e^{n^{0.2}}} \sum_{k=1}^{e^{n^{0.2}}} X^o_{t,k}.$$

Let

$$E_{3,S}^n(t) := \left\{ \hat{\delta}_{o,t}^M < cr \right\}.$$

Clearly,

$$\bigcap_{t \in 0,\dots,m^2} E_{3,S}^n(t) \subseteq E_{3,S}^n, \quad \text{imlpying} \quad P(E_{3,S}^{nc}|\psi) \leq \sum_{t=0}^{m^2} P(E_{3,S}^{nc}(t)|\psi), \tag{4.17}$$

where ψ is an arbitrary fixed scenery.

Note, for any fixed scenery ψ, the random variables $X_{t,1}^o, X_{t,2}^o, \dots$ are clearly independent (but not necessarily identically distributed). However, for each i, $E(X_{t,i}^o|\psi) \leq c_r - \Delta$, implying that

$$c_r - \frac{1}{e^{n^{0.2}}} \sum_{i=1}^{e^{n^{0.2}}} E(X_{t,i}^o|\psi) \geq \Delta.$$

Recall $\Delta = \frac{p_M}{n^{10054}}$. We know that $\Delta \geq n^{-\beta}$, where β is an integer bigger than 11000. Thus, by (4.8)

$$\begin{aligned} P(E_{3,S}^{nc}(t)|\psi) &= P(\hat{\delta}_{o,t}^M \geq c_r|\psi) = P\Big(\frac{1}{e^{n^{0.2}}} \sum_{i=1}^{e^{n^{0.2}}} X_{t,i}^o \geq c_r|\psi \Big) \\ &\leq P\Big(\frac{1}{e^{n^{0.2}}} \sum_{i=1}^{e^{n^{0.2}}} (X_{t,i}^o - EX_{t,k}^o) \geq \Delta|\psi \Big) \leq \exp(-d'\Delta^2 e^{n^{0.2}}) \\ &\leq \exp\big(-(d' n^{-2\beta} e^{n^{0.2}})\big). \end{aligned}$$

Now, use (4.6), (4.17) and (3.10) to get

$$P_\psi(E_{3,S}^{nc}) \leq m^2 \exp(-d' n^{-2\beta} e^{n^{0.2}} + 3n) \leq \exp\big(7n - (d' n^{-2\beta} e^{n^{0.2}})\big) \to 0,$$

as $n \to \infty$.

Proof that $P_\psi(E_{6,S}^n) \to 1$

$$E_{6,S}^n(t) = \left\{ \begin{array}{c} \text{if } \chi(t) = \chi(t+1) = \cdots = \chi(t+n^2) \\ \text{then } \exists s \in [t, t+n^2] \text{ such that} \\ S(s) \text{ is contained in a block of } \xi \text{ longer than } n^{0.35} \end{array} \right\}.$$

We have that

$$E_{6,S}^n = \bigcap_{t \in [0,m^2]} E_{6,S}^n(t)$$

and thus

$$P_\psi(E_{6,S}^{nc}) \leq \sum_{t=0}^{m^2} P_\psi(E_{6,S}^{nc}(t)).$$

Note

$$E_{6,S}^{nc}(t) = \left\{ \begin{array}{c} \forall s \in [t, t+n^2] \text{ the random walk } S(s) \\ \text{is contained in a block of } \xi \text{ with length at most } n^{0.35} \\ \text{and } \chi(t) = \chi(t+1) = \cdots = \chi(t+n^2) \end{array} \right\}.$$

Now, fix a scenery ψ. Let $I = \mathbb{Z}/\cup B(\psi)$, where $B(\psi_n)$ is a block of ψ bigger than $n^{0.35}$ and the union is taken over all such blocks. Note $I = \cup_k I_k$, where I_k are disjoint intervals, at least $n^{0.35}$ far from each other. Thus, if $S(t) \in I_k$, then $S(t+s) \notin I_l$ for each $l \neq k$ and for each $s = 1, \dots, n^2$.

Hence

$$P(E^{nc}_{6,S}(t)|\psi) = \sum_{j\in I} P\Big(S(t),\ldots,S(t+n^2)\in I \text{ and } \chi(t)=\cdots=\chi(t+n^2)|S(t)=j\Big)P(S(t)=j)$$

$$\sum_k \sum_{j\in I_k} P\Big(S_j(0),\ldots,S(n^2)\in I_k \text{ and } \chi(t)=\cdots=\chi(t+n^2)\Big)P(S(t)=j).$$

By Lemma 2.1 there exists a constant $a>0$ not depending on n such that, for each j,

$$P\Big(S_j(0),\ldots,S(n^2)\in I_k \text{ and } \chi(t)=\cdots=\chi(t+n^2)\Big)\leq \exp\Big(-\frac{an^2}{n^{0.7}}\Big). \tag{4.18}$$

Then

$$P(E^{nc}_{6,S}(t)|\psi)\leq \exp(-an^{1.3})$$

Thus, by (4.6)

$$P_\psi(E^{nc}_{6,S}(t))\leq \exp\left(-an^{1.3}+3n\right)\to 0$$

and by (3.10)

$$m^2\exp(-an^{1.2}+3n)\leq e^{7n-an^{1.3}}\to 0.$$

Proof that $P_\psi(E^n_{7,S})\to 1$

Preliminaries

Recall the definitions of stopping times involved:

$\vartheta_z(k)$, $k=0,1,\ldots$ stands for consecutive visits of S to the point $z-2Le^{n^{0.1}}$, provided that between $\vartheta_z(k)$ and $\vartheta_z(k+1)$ at least once n^2+1 same colors have been generated on I_z;

$\nu_z(1)$ ($\nu_z(i)$, $i=2,3,\ldots$) is the first time after $\vartheta_z(0)$, (after $\nu_z(k-1)+e^{n^{0.1}}$) observing n^2+1 same colors generated on I_z.

In Section 2.1 the stopping times $\vartheta_z(k)$, $\nu_z(i)$ as well as random variables $X_{z,i}$ were used to define the random variables $\kappa_z(k)$, $\mathcal{X}_z(k)$ and $\mathcal{Z}_z(k)$. The latter were used to define δ^M_z.

We now fix an arbitrary time moment t and we define the counterparts of all above-mentioned stopping times and random variables starting from t.

In Section 4.1 we already defined the t counterpart of $\nu_z(i)$ and $X_{z,i}$, namely $\nu_{z,t}(i)$, and $X_{z,t,i}$, $i=1,2,\ldots$.

Recall that in the definition of $\nu_{z,t}(1)$, the starting point $\vartheta_z(0)$ was replaced by t, the induction for $\nu_{z,t}(i)$ is the same as the one for $\nu_z(i)$, $i=2,3,\ldots$.

The Bernoulli random variables $X_{z,t,i}$ were defined exactly as $X_{z,i}$ with sopping times $\nu_{z,t}(i)$ instead of $\nu_z(i)$.

We now define the t-counterpart of $\vartheta_z(k)$, $k=0,1,\ldots$.

* Let $\vartheta_{z,t}(0)=t$ and let

$$\vartheta_{z,t}(k):=\{\min s>\vartheta_{z,t}(k-1): S(s)=z-2Le^{n^{0.1}},\ \exists j: s>\nu_{z,t}(j)>\vartheta_{z,t}(k-1)\}.$$

We now use $\vartheta_{z,t}(k)$ to define the t-analogues of κ_z, $\mathcal{Z}_z$ and $\mathcal{X}_z$.

* More precisely, let $\kappa_{z,t}(0)=0$ and let $\kappa_{z,t}(k)$ be defined with the inequalities

$$\nu_{z,t}(\kappa_{z,t}(k))<\vartheta_{z,t}(k)<\nu_{z,t}(\kappa_{z,t}(k)+1).$$

Now, the definition of $\mathcal{Z}_{z,t}$ and $\mathcal{X}_{z,t}$ is straightforward:

$$\mathcal{X}_{z,t}(k)=\sum_{i=\kappa_{z,t}(k-1)+1}^{\kappa_{z,t}(k)} X_{z,t,i},\quad \mathcal{Z}_{z,t}(k)=\kappa_{z,t}(k)-\kappa_{z,t}(k-1),\quad k=1,2,\ldots$$

Note that, if ξ is fixed, then, for all $t>0$, the random variables $\mathcal{X}_{z,t}(1),\mathcal{X}_{z,t}(2),\ldots$ are independent and the random variables $\mathcal{X}_{z,t}(2),\mathcal{X}_{z,t}(3),\ldots$ are i.i.d. with the same distribution as $\mathcal{X}_z(k)$. The same holds for $\mathcal{Z}_{z,t}(1)$, $\mathcal{Z}_{z,t}(2),\ldots$. Also note, that $\mathcal{Z}_{z,t}(k)\geq 1$, $k=1,2,\ldots$.

Hence, for all $t>0$,

$$\delta_z^M=\delta_z^M(\xi)=\frac{E(\mathcal{X}_{z,t}(2)|\xi)}{E(\mathcal{Z}_{z,t}(2)|\xi)}=\lim_{k\to\infty}\frac{\sum_{i=1}^k\mathcal{X}_{z,t}(i)}{\sum_{i=1}^k\mathcal{Z}_{z,t}(i)}.$$

We are now going to show that for each ξ, t, z, the first $e^{n^{0.2}}$ observations of $X_{z,t,i}$ are enough to estimate $\delta_z^M(\xi)$ very precisely, i.e. $\hat{\delta}^M_{z,t}$ is close to δ_z^M.

Fix z,t,ψ and denote

$$\mathcal{Z}_k:=\mathcal{Z}_{z,t}(k),\quad \mathcal{X}_k:=\mathcal{X}_{z,t}(k),\quad X_i:=X_{k,t,i},\quad E\mathcal{X}=E(\mathcal{X}_2|\psi),\quad E\mathcal{Z}=E(\mathcal{Z}_2|\psi),\quad P(\cdot)=P(\cdot|\psi).$$

Thus

$$\delta_z^M=\delta_z^M(\psi)=\frac{E\mathcal{X}}{E\mathcal{Z}}.$$

Let $a=\ulcorner e^{3n^{0.1}}\urcorner$ and define

$$\mathcal{Z}_k^a=\mathcal{Z}_k\wedge a,\quad \mathcal{X}_k^a=\mathcal{X}_k\wedge a,\quad E\mathcal{X}^a:=E(\mathcal{X}_2^a|\psi),\quad E\mathcal{Z}^a:=E(\mathcal{Z}_2^a|\psi).$$

Finally, define

$$\Delta:=e^{-\frac{n^{0.2}}{4}}.$$

We consider the events

$$E_{7,a}=\Big\{\mathcal{Z}_k\leq a,\quad k=1,2,\ldots,e^{n^{0.2}}\Big\}$$
$$E_{7,b}=\Big\{\Big|\frac{\mathcal{X}_1^a+\cdots+\mathcal{X}_k^a}{k}-E\mathcal{X}^a\Big|\leq\frac{\Delta}{3},\quad \forall k\in[\frac{e^{n^{0.2}}}{a},e^{n^{0.2}}]\Big\}$$
$$E_{7,c}=\Big\{\Big|\frac{\mathcal{Z}_1^a+\cdots+\mathcal{Z}_k^a}{k}-E\mathcal{Z}^a\Big|\leq\frac{\Delta}{3},\quad \forall k\in[\frac{e^{n^{0.2}}}{a},e^{n^{0.2}}]\Big\}.$$

First step

At first we show that

$$E_{7,a}\cap E_{7,b}\cap E_{7,c}\subset E_{7S}^n(z,t). \tag{4.19}$$

Let $\bar{\imath}$ be (random) number defined by the inequalities

$$\mathcal{Z}_1+\cdots+\mathcal{Z}_{\bar{\imath}}\leq e^{n^{0.2}}<\mathcal{Z}_1+\cdots+\mathcal{Z}_{\bar{\imath}+1}. \tag{4.20}$$

Since $\mathcal{Z}_k\geq 1$, we have $\bar{\imath}\leq e^{n^{0.1}}$. Let $\bar{k}:=\mathcal{Z}_1+\cdots+\mathcal{Z}_{\bar{\imath}}$. Now,

$$\hat{\delta}^M_{z,t}=\frac{\sum_{i=1}^{e^{n^{0.2}}}X_i}{e^{n^{0.2}}}=\frac{\sum_{k=1}^{\bar{\imath}}\mathcal{X}_k+\sum_{i=\bar{k}+1}^{e^{n^{0.2}}}X_i}{\bar{k}+e^{n^{0.2}}-\bar{k}}=\frac{\frac{1}{\bar{\imath}}\sum_{k=1}^{\bar{\imath}}\mathcal{X}_k+\frac{1}{\bar{\imath}}\sum_{i=\bar{k}+1}^{e^{n^{0.2}}}X_i}{\frac{\bar{k}}{\bar{\imath}}+\frac{e^{n^{0.2}}-\bar{k}}{\bar{\imath}}}.$$

Denote

$$\Delta_I:=E(\mathcal{X}^a-\mathcal{X})+\frac{1}{\bar{\imath}}\sum_{i=1}^{\bar{\imath}}(\mathcal{X}_i-E\mathcal{X}^a)+\frac{1}{\bar{\imath}}\sum_{i=\bar{k}+1}^{e^{n^{0.2}}}X_i,$$

$$\Delta_{II}:=E(\mathcal{Z}^a-\mathcal{Z})+\frac{1}{\bar{\imath}}\sum_{i=1}^{\bar{\imath}}(\mathcal{Z}_i-E\mathcal{Z}^a)+\frac{1}{\bar{\imath}}\sum_{i=\bar{k}+1}^{e^{n^{0.2}}}Z_i.$$

Thus,

$$\hat{\delta}^M_{z,t} = \frac{E\mathcal{X} + \Delta_I}{E\mathcal{Z} + \Delta_{II}}.$$

Suppose now, that E_{7a} holds. Then, for each $i = 1, \ldots, e^{n^{0.2}}$, we have $\mathcal{Z}_i = \mathcal{Z}_i^a$, $\mathcal{X}_i = \mathcal{X}_i^a$. From (4.20) then follows that $e^{n^{0.2}} \leq \bar{\imath}a$, i.e.

$$e^{n^{0.2}} \geq \bar{\imath} \geq \frac{e^{n^{0.2}}}{a}. \tag{4.21}$$

When $\bar{\imath} = e^{n^{0.2}}$, then $e^{n^{0.2}} - \bar{k} = 0$, otherwise $e^{n^{0.2}} - \bar{k} \leq \mathcal{Z}_{i+1} \leq a$. Since $\sum_{i=\bar{\imath}+1}^{e^{n^{0.2}}} X_i \leq e^{n^{0.2}} - \bar{k}$, we get

$$\frac{1}{\bar{\imath}} \sum_{i=\bar{k}+1}^{e^{n^{0.2}}} X_i \leq \frac{e^{n^{0.2}} - \bar{k}}{\bar{\imath}} \leq \frac{a}{\bar{\imath}} \leq a^2 e^{-n^{0.2}} = \exp(6n^{0.1} - n^{0.2}) < \frac{\Delta}{6}, \tag{4.22}$$

provided n is big enough.
Hence, by (4.21) we have (recall that we assumed $E_{7,a}$)

$$\Big\{\Big|\frac{1}{\bar{\imath}}\sum_{k=1}^{\bar{\imath}}(\mathcal{X}_k - E\mathcal{X}^a)\Big| \leq \frac{\Delta}{3}\Big\} = \Big\{\Big|\frac{1}{\bar{\imath}}\sum_{k=1}^{\bar{\imath}}(\mathcal{X}_k^a - E\mathcal{X}^a)\Big| \leq \frac{\Delta}{3}\Big\} = \bigcup_{l=\frac{e^{n^{0.2}}}{a}}^{e^{n^{0.2}}} \Big\{\Big|\frac{1}{l}\sum_{k=1}^{l}(\mathcal{X}_k^a - E\mathcal{X}^a)\Big| \leq \frac{\Delta}{3}, \bar{\imath} = l\Big\}$$

$$\supset \Big\{\Big|\frac{1}{l}\sum_{k=1}^{l}(\mathcal{X}_k^a - E\mathcal{X}^a)\Big| \leq \frac{\Delta}{3}, l = \frac{e^{n^{0.2}}}{a}, \ldots, e^{n^{0.2}}\Big\} = E_{7,b}.$$

Similarly,

$$\Big\{\Big|\frac{1}{\bar{\imath}}\sum_{k=1}^{\bar{\imath}}(\mathcal{X}_k - E\mathcal{X}^a)\Big| \leq \frac{\Delta}{3}\Big\} \supset E_{7,c}.$$

Thus, by (4.22) on $E_{7a} \cap E_{7b} \cap E_{7c}$ we have

$$|\Delta_I| \leq |E\mathcal{X}^a - E\mathcal{X}| + 2\frac{\Delta}{3} = E(\mathcal{X} - \mathcal{X}^a) + 2\frac{\Delta}{3}$$
$$|\Delta_{II}| \leq |E\mathcal{Z}^a - E\mathcal{Z}| - 2\frac{\Delta}{3} = E(\mathcal{Z} - \mathcal{Z}^a) + 2\frac{\Delta}{3}.$$

Fix $k = 1, 2, \ldots$. Denote by $n_0, n_1, n_2, \ldots$ integers that satisfy $n_0 = 0$, $e^{2n^{0.1}} + 1 \geq n_i - n_{i-1} \geq e^{2n^{0.1}}$, $\forall i$. Let Y_j, $j = 0, 1, \ldots$ denote a Bernoulli random variable which is equal to 1 iff between the time $\nu(\vartheta(k) + 1 + n_j)$ and $\nu(\vartheta(k) + 1 + n_{j+1})$ random walk does not visit the point $z^* := z - 2Le^{n^{0.1}}$. The random variables Y_j are independent.
By definition, $\nu(i+1) - \nu(i) \geq e^{n^{0.1}}$. Hence, $\nu(\vartheta(k) + 1 + n_{j+1}) - \nu(\vartheta(k) + 1 + n_j) \geq e^{3n^{0.1}}$. At time $\nu(\vartheta(k) + 1)$, random walk is located on I_z and, therefore, no more than $3e^{n^{0.1}}$ from z^*. By (4.10), the probability to visit the point z^* within the time $e^{3n^{0.1}}$ starting from the $3e^{n^{0.1}}$-neighborhood of z^* goes to 1 if $n \to \infty$. Hence, $\sup_j P(Y_j = 1) \to 0$. Let n be so big, that $P(Y_j = 1) \leq e^{-1}$, for all j. This means, for each

$$P(\mathcal{Z}_k \geq te^{2n^{0.1}}) \leq P(Y_j = 1, j = 0, \ldots, \ulcorner t \urcorner - 1) \leq \exp(-\ulcorner t \urcorner) \leq \exp(-t), \quad k = 1, 2, \ldots \tag{4.23}$$

Now,

$$E(\mathcal{Z} - \mathcal{Z}^a) = \int_{\{\mathcal{Z} \geq a\}} \mathcal{Z} dP - aP(\mathcal{Z} \geq a) = aP(\mathcal{Z} \geq a) + \int_{(a,\infty)} P(\mathcal{Z} > x)dx - aP(\mathcal{Z} \geq a) = \int_{(a,\infty)} P(\mathcal{Z} > x)dx.$$

By (4.23)

$$\int_{(a,\infty)} P(\mathcal{Z} > x)dx \leq \int_a^\infty \exp(-xe^{-2n^{0.1}})dx \leq e^{2n^{0.1}} \exp(-ae^{-2n^{0.1}})) \leq e^{2n^{0.1}} \exp(-e^{n^{0.1}}).$$

Thus, for n big enough

$$E(\mathcal{Z} - \mathcal{Z}^a) \leq e^{2n^{0.1}} \exp(-e^{n^{0.1}}) \leq \frac{\Delta}{3}.$$

Since, $\mathcal{X} \leq \mathcal{Z}$, we get

$$E(\mathcal{X} - \mathcal{X}^a) = \int_{(a,\infty)} P(\mathcal{X} > x)dx \leq \int_{(a,\infty)} P(\mathcal{Z} > x)dx \leq \frac{\Delta}{3}.$$

Thus, on $E_{7a} \cap E_{7b} \cap E_{7c}$ we have

$$|\Delta_I|, |\Delta_{II}| \leq \Delta. \tag{4.24}$$

Now recall that we have

$$\hat{\delta}^M_{z,t} = \frac{E\mathcal{X} + \Delta_I}{E\mathcal{Z} + \Delta_{II}}.$$

Hence, by (4.24)

$$\frac{E\mathcal{X} - \Delta}{E\mathcal{Z} + \Delta} \leq \hat{\delta}^M_{z,t} \leq \frac{E\mathcal{X} + \Delta}{E\mathcal{Z} - \Delta}.$$

By Taylor's formula,

$$\frac{E\mathcal{X} - \Delta}{E\mathcal{Z} + \Delta} = \frac{E\mathcal{X}}{E\mathcal{Z}} - \Big(\frac{E\mathcal{X} + E\mathcal{Z}}{(E\mathcal{Z})^2}\Big)\Delta + o(\Delta).$$

Since $1 \leq E\mathcal{X} \leq E\mathcal{Z}$, the latter means (for Δ small enough)

$$\Big|\frac{E\mathcal{X} - \Delta}{E\mathcal{Z} + \Delta} - \frac{E\mathcal{X}}{E\mathcal{Z}}\Big| \leq \Big(\frac{E\mathcal{X} + E\mathcal{Z}}{(E\mathcal{Z})^2}\Big)\Delta + o(\Delta) \leq 2\Delta + o(\Delta) < 3\Delta.$$

Similarly

$$\Big|\frac{E\mathcal{X} + \Delta}{E\mathcal{Z} - \Delta} - \frac{E\mathcal{X}}{E\mathcal{Z}}\Big| < 3\Delta.$$

Now, $\delta^M_{z,t} = \frac{E\mathcal{X}}{E\mathcal{Z}}$ implying that

$$|\delta^M_z - \hat{\delta}^M_{z,t}| < 3\Delta < e^{-n^{0.12}}.$$

Thus, (4.19) holds.

Second step
We now show that $P(E^c_{7,a})$, $P(E^c_{7,b})$ and $P(E^c_{7,c})$ are $o(\exp(-n^{1000}))$.
By taking $t = e^{n^{0.1}}$ (4.23) yields

$$P(\mathcal{Z}_k > a) \leq \exp(-e^{n^{0.1}}), \quad k = 1, 2, \ldots.$$

Thus

$$P(E^c_{7,a}) \leq \exp(n^{0.2}) \exp(-e^{n^{0.1}}) = \exp(n^{0.2} - e^{n^{0.1}}) < \exp(-n^{1000}). \tag{4.25}$$

To estimate $P(E_{7,b})$ and $P(E_{7,c})$ we use Höffding's inequality. Fix $l \in [\frac{e^{n^{0.2}}}{a}, e^{n^{0.2}}]$. By (4.8) we have

$$P\Big(\Big|\frac{1}{l}\sum_{k=1}^{l}(\mathcal{X}^a_k - E\mathcal{X}^a_k)\Big| \geq \frac{\Delta}{6}\Big) \leq \exp\Big(-2l\big(\frac{\Delta}{a6}\big)^2\Big).$$

On the other hand, since $\mathcal{X}_k^a$, $k \geq 2$ are iid, we have

$$\Big|\frac{1}{l}\sum_{k=1}^{l} E\mathcal{X}_k^a - E\mathcal{X}^a\Big| = \frac{1}{l}|E\mathcal{X}^a - E\mathcal{X}_1^a| \leq \frac{2a}{l} \leq 2a^2 e^{-n^{0.2}} = 2\exp(6n^{0.1} - n^{0.2}) < \frac{\Delta}{6}.$$

Thus,

$$P\Big(\Big|\frac{1}{l}\sum_{k=1}^{l} \mathcal{X}_k^a - E\mathcal{X}^a\Big| \geq \frac{\Delta}{3}\Big) \leq P\Big(\Big|\frac{1}{l}\sum_{k=1}^{l}(\mathcal{X}_k^a - E\mathcal{X}^a)\Big| \geq \frac{\Delta}{6}\Big) \leq \exp\Big(-2l(\frac{\Delta}{a6})^2\Big) \leq \exp(-Ke^{n^{0.2}}\frac{\Delta^2}{a^3}),$$

where $K = \frac{2}{36}$. Now,

$$e^{n^{0.2}}\frac{\Delta^2}{a^3} = \exp(n^{0.2} - \frac{1}{2}n^{0.2} - 9n^{0.1}) = \exp(\frac{1}{2}n^{0.2} - 9n^{0.1}) > \exp(\frac{n^{0.2}}{4})$$

and

$$P\Big(\Big|\frac{1}{l}\sum_{k=1}^{l} \mathcal{X}_k^a - E\mathcal{X}^a\Big| \geq \frac{\Delta}{3}\Big) \leq \exp(-Ke^{\frac{n^{0.2}}{4}}).$$

Finally

$$P(E_{7,b}^c) \leq \sum_{l=\frac{e^{n^{0.2}}}{a}}^{e^{n^{0.2}}} P\Big(\Big|\frac{1}{l}\sum_{k=1}^{l} \mathcal{X}_k^a - E\mathcal{X}^a\Big| \geq \frac{\Delta}{3}\Big) < e^{n^{0.2}}\exp(-Ke^{\frac{n^{0.2}}{4}}) < \exp(-e^{n^{0.1}}) < \exp(-n^{1000}). \tag{4.26}$$

The same bound holds for $P(E_{7,c}^c)$.

Because of (4.19), (4.25) and (4.26) we now get

$$P(E_{7S}^{nc}(c,t)) \leq 3\exp(-n^{1000}). \tag{4.27}$$

The bound in (4.27) do not depend on chosen z, t and ψ. Note that on $[-cm, cm] \times [0, m^2]$ there are no more than $(cm)^3$ values of (z,t). Hence

$$P\left(E_{7,S}^{nc}\right) \leq \Sigma_{z\in[-cm,cm],t\in[0,m^2]} P\left(E_{7,S}^{nc}(z,t)\right).$$

From (4.27) it follows

$$P\left(E_{7,S}^{nc}\right) \leq (cm)^3 3\exp(-n^{1000}). \tag{4.28}$$

Recall, by (3.10) $(cm)^3 \leq c^3 e^{6n}$. Hence, the right side of (4.28) is less than $3c^3\exp(6n - n^{1000})$. This is of order $o(\exp(-3n))$. By (4.6) we, therefore, have

$$P_\psi(E_{7S}^{nc}) \to 0.$$

Outline of the proof that $P_\psi(E_{8,S}^n) \to 1$

Note that in the previous proof the exact nature of $X_{z,i}$, $\mathcal{X}_z(k)$ as well as $X_{z,t,i}$, $\mathcal{X}_{z,t}(k)$ were not used. Hence, the proof holds, if they were replaced by $U_{z,i}$, $\mathcal{U}_z(k)$, $\chi(\nu_{z,t}(i) + e^{n^{0.1}})$ and

$$\sum_{\kappa(k-1)+1}^{\kappa(k)} \chi(\nu_{z,t}(i) + e^{n^{0.1}}),$$

respectively. By (2.12) this proves that $P_\psi(E_{8,S}^n) \to 1$.

Proof that $P_\psi(E^n_{5,S}) \to 1$.

Fix $\psi_n \in E^n_{\text{cell\_OK}}$.

For each $k = 0,1,2,\ldots$, let $\tau_k(0) := \rho(k)$ and for each $j = 1,2,\ldots$, let $\tau_k(j)$ be the smallest time $t > \tau_k(j-1) + 2e^{n^{0.1}}$ for which $S(t) \in I(\rho(k))$.

Let $X_k(j)$ be the Bernoulli random variable which is equal to one iff during time $[\tau_k(j), \tau_k(j)+(n^{3000}+n^2)]$ we observe n^2+1 consecutive 0's or 1's. That is $X_k(j) = 1$ iff $\exists t \in [\tau_k(j), \tau_k(j)+n^{3000}]$ such that $\chi(t) = \chi(t+1) = \cdots = \chi(t+n^2)$.

Clearly, for each k, the random variables $X_k(j)$, $j = 0,1,2,\ldots$ are independent

At first we show that there exists a constant $a > 0$, not depending on n, such that for each k and j,

$$P(X_k(j) = 1) \geq n^{-a\ln n} = e^{-a\ln^2 n}. \tag{4.29}$$

Fix $k = 0,1,\ldots$ and let $I := I(\rho(k))$. Let $\bar{z}$ be the signal carrier point such that $I_{\bar{z}} = I$. Since $\bar{z}$ is a signal carrier point, then, by Corollary2.2 and c) of Proposition2.1, I contains at least one big block of ψ_n. Let $T = [a,b] \subset I$ be that block. Now, let $a < a^* < b^* < b$ be such that $a^* - a, b^* - a^*, b - b^* \geq \frac{|T|}{3} \geq \frac{\ln n}{3n}$. Denote $T^* = [a^*, b^*]$. Now,

$$P(X_k(j) = 1) \geq P(S(\tau_k(j) + n^{3000}) \in T^*) P(\chi(t) = \chi(t+1) = \cdots = \chi(t+n^2) | S(t) \in T^*).$$

Now, by LCLT

$$P(S(\tau_k(j) + n^{3000}) \in T^*) \geq \frac{1}{cn^{1500}} - O(\frac{1}{n^{3000}}) \geq n^{-1501},$$

provided that n is big enough.

Now, denote $N = (\frac{n}{\ln n})^2$ (w.l.o.g we assume that this is an integer) and estimate

$$\begin{aligned} &P(\chi(t) = \chi(t+1) = \cdots = \chi(t+n^2) | S(t) = j \in T^*) \geq P(S_j(i) \in T, \quad \forall i = 1,2,\ldots,n^2) \geq \\ &P\Big(\max_{i=1,\ldots,N} |S_j(i)| \leq \frac{|T|}{3}, S_j(N) \in T^*\Big)^{\ln^2 n} = P\Big(\max_{i=1,\ldots,N} \frac{|S_j(i)|}{\sqrt{N}} \leq \frac{1}{3}, \frac{S_j(N)}{\sqrt{N}} \in \frac{T^*}{\sqrt{N}}\Big)^{\ln^2 n}. \end{aligned} \tag{4.30}$$

Note: $|T^*| \geq \sqrt{N}$. By (4.10)

$$P\Big(\max_{i=1,\ldots,N} \frac{|S_j(i)|}{\sqrt{N}} \leq \frac{1}{3}, \frac{S_j(N)}{\sqrt{N}} \in \frac{T^*}{\sqrt{N}}\Big) \to P(\sup_{0\geq t\leq 1} |W_t| \leq \frac{1}{3\sigma}, W_1 \in I) > \gamma > 0.$$

Thus, for n big enough there exists $a < \infty$ such that the right side of (4.30) is bigger than $(\frac{1}{a})^{\ln^2 n} = n^{-c\ln n}$, with $c > 0$. Hence, (4.29) holds with $a = c+1$.

Define the following events:

$$E_a(k) = \left\{ \begin{array}{c} \text{if } \rho(k) \leq m^2 \\ \text{then during the time } [\rho(k), \rho(k) + e^{n^{0.3}} - e^{n^{0.1}}] \\ S \text{ visits } I(\rho(k)) \text{ more than } e^{n^{0.22}} \text{ times} \end{array} \right\} \quad k = 0,1,\ldots$$

and

$$E_a := \cap_{k=1}^{25000} E_a(k).$$

Also define

$$E_b(k) := \Big\{ \sum_{j=0}^{e^{n^{0.21}}} X_k(j) \geq e^{n^{0.2}} \Big\}, \quad E_b := \cap_{k=0}^{n^{25000}} E_b(k).$$

Now, clearly, on $E_a(k)$ we have $\tau_k(e^{n^{0.21}}) \leq \rho(k) - e^{n^{0.21}} - 2e^{n^{0.1}}$. Thus $E^n_{5,S}$ holds, if

$$\sum_{j=0}^{e^{n^{0.21}}} X_k(j) \geq e^{n^{0.2}}.$$

Hence

$$E_{5,S} \supset E_a \cap E_b \quad \text{and} \quad P_\psi(E^c_{5,S}) \leq P_\psi(E^c_a) + P_\psi(E^c_b).$$

We are now proving that $P_\psi(E^c_a) \to 0$ and $P_\psi(E^c_b) \to 0$.

Proof that $P_\psi(E^c_b) \to 0$
By (4.6) it is enough to show that

$$P(E^c_b|\psi_n) = o(e^{-3n}). \tag{4.31}$$

Note that for big n, $\exp(n^{0.2} - n^{0.21}) < EX_k(j)$, $\forall j$. Thus,

$$\exp(n^{0.2} - n^{0.21}) < \frac{1}{e^{n^{0.21}}} \exp(-n^{0.21}) \sum_{j=0}^{e^{n^{0.21}}} E(X_k(j)) =: \bar{m}.$$

By Höffding's inequality we obtain that for a constant $K > 0$

$$P(E^c_b(k)|\psi_n) = P\Big(\frac{1}{e^{n^{0.21}}} \sum_{j=0}^{e^{n^{0.21}}} X_k(j) < \exp(n^{0.2} - n^{0.21})\Big) \leq P\Big(\frac{1}{e^{n^{0.21}}} \sum_{j=0}^{e^{n^{0.21}}} X_k(j) < \frac{\bar{m}}{2}\Big) =$$

$$P\Big(\frac{1}{e^{n^{0.21}}} \sum_{j=0}^{e^{n^{0.21}}} \big(X_k(j) - EX_k(j)\big) < -\frac{\bar{m}}{2}\Big) \leq \exp(-K\bar{m}^2 e^{n^{0.21}}) \leq \exp(-Ke^{n^{0.21} - 2a\ln^2 n}).$$

Hence,

$$P(E^c_b|\psi_n) \leq n^{25000} \exp(-Ke^{n^{0.21} - 2a\ln^2 n}) = o(e^{-3n}).$$

Proof that $P_\psi(E^c_a) \to 0$
This proof is a little tricky because unlike the other proofs we have that $P(E_a|\psi_n)$ is much bigger than $P(S(m^2) = m)$.
Let $L = n^{100000}$ and consider the event

$$C = \Big\{S\left(m^2(1 - n^{-3L})\right) \in \big[m(1 - n^{-L}), m(1 + n^{-L})\big] = [m - \frac{m}{n^L}, m + \frac{m}{n^L}]\Big\}.$$

Here and in the rest of the proof we assume (w.l.o.g.) that all ratios and exponents are integers. Also define

$$E_c(k) = \left\{\rho(k) \notin [m^2(1 - n^{-3L}), m^2]\right\}, \quad k = 0, 1, \ldots, \quad E_c := \cup_{k=1}^{25000} E_c(k).$$

The event E_c means that no stopping time $\rho(k)$ occur in time-interval $[m^2(1 - n^{-3L}), m^2]$, the event $E_a \cap E_c$ satisfies

$$E_a \cap E_c = E^*_a := \cap_{k=1}^{25000} E^*_a(k),$$

where

$$E^*_a(k) = \left\{ \begin{array}{c} \text{if } \rho(k) \leq m^2(1 - n^{-L}) \\ \text{then during the time } [\rho(k), \rho(k) + e^{n^{0.3}} - e^{n^{0.1}}] \\ S \text{ visits } I(\rho(k)) \text{ more than } e^{n^{0.22}} \text{ times} \end{array} \right\}.$$

We now show that the probability of our interest, $P(E_a|E^n_{1,S}, \psi_n)$, can be very well approximated by the probability $P(E^*_a|C, \psi_n)$ and the latter goes to 0 if $n \to \infty$. We proceed in three steps.

1) At first note: since

$$C^c \cap E^n_{1,S} = \{S(m^2(1-n^{-3L})) \notin [m(1-n^L), m(1+n^L)], S(m^2) = m\},$$

we get, by Höffdigs inequality

$$\begin{aligned} P(C^c \cap E^n_{1,S}|\psi_n) =& P(C^c \cap E^n_{1,S}) = P(E^n_{1,S}|C^c)P(C^c) \leq P(E^n_{1,S}|C^c) \\ =& P\Big(\Big|S\Big(\frac{m^2}{n^{3L}}\Big)\Big| \geq \frac{m}{n^L}\Big) \leq \exp(-dn^L) = o(n^{-3n}). \end{aligned}$$

The latter implies

$$P_\psi(C^c) = o(1) \tag{4.32}$$

2) Secondly, use the relations

$$P(E^{*c}_a \cap E^n_{1,S} \cap C|\psi_n) \leq P(E^c_a \cap E^n_{1,S} \cap C|\psi_n) \leq P(E^{*c}_a \cap E^n_{1,S} \cap C|\psi_n) + P(E^c_c \cap E^n_{1,S}|\psi_n).$$

Since $\psi \in E^n_7$, it has no signal carrier points in $[m - EZn^{-11001}]$. Hence, $E^c_c \cap E^n_{1,S}$ can hold only, if during time interval $[m^2(1-n^{-3L}), m^2]$ the random walk covers the distance at least $EZn^{-11001} - Ln^{1000}$. Thus,

$$P(E^c_c \cap E^n_{1,S}|\psi_n) \leq P\Big(\max_{l=1,\dots,\frac{m^2}{n^{3L}}} |S(l)| \geq EZn^{-11001} - Ln^{1000}\Big) \leq P\Big(\max_{l=1,\dots,\frac{m^2}{n^{3L}}} |S(l)| \geq \frac{m}{n^{11003}} - Ln^{1000}\Big).$$

Now use the maximal inequality (4.15) together with Höffdings inequality to estimate

$$\begin{aligned} P\Big(\max_{l=1,\dots,\frac{m^2}{n^{3L}}} |S(l)| \geq \frac{m}{n^{11003}} - Ln^{1000}\Big) \leq& \max_{l=1,\dots,\frac{m^2}{n^{3L}}} 3P\Big(|S(l)| \\ &\geq \frac{1}{3}\frac{m}{n^{12000}}\Big) \leq 3\exp(-dn^{3L-12000}) = o(e^{-3n}). \end{aligned}$$

This implies

$$\frac{P(E^c_a \cap C \cap E^n_{1,S}|\psi_n) - P(E^{*c}_a \cap C \cap E^n_{1,S}, |\psi_n)}{P(E^n_{1,S}|\psi_n)} = P_\psi(E^c_a \cap C) - P_\psi(E^{*c}_a \cap C) = o(1) \tag{4.33}$$

3) Finally, note that

$$P(E^{*c}_a \cap E^n_{1,S} \cap C|\psi_n) = P(E^{*c}_a \cap C|\psi_n)P(E^n_{1,S}|E^{*c}_a \cap C, \psi_n) = P(E^{*c}_a \cap C|\psi_n)P(E^n_{1,S}|C, \psi_n).$$

On the other hand,

$$P(E^n_{1,S}|\psi_n) \geq P(E^n_{1,S} \cap C|\psi_n) = P(E^n_{1,S}|C, \psi_n)P(C|\psi_n).$$

Hence,

$$P_\psi(E^{*c}_a \cap C) = \frac{P(E^{*c}_a \cap E^n_{1,S} \cap C|\psi_n)}{P(E^n_{1,S}|\psi_n)} \leq \frac{P(E^{*c}_a \cap C|\psi_n)P(E^n_{1,S}|C, \psi_n)}{P(E^n_{1,S}|C, \psi)P(C|\psi_n)} = P(E^{*c}_a|C, \psi_n). \tag{4.34}$$

By CLT, $P(C|\psi_n) = P(S\left(m^2(1-n^{-3L}\right) \in [m - \frac{m}{n^L}, m + \frac{m}{n^L}])$ is of order $\frac{1}{n^K}$ for some big $K > 0$. We now estimate the probability $P(E^{*c}_a|\psi_n)$.
To do that fix k and let $T_1, T_2, \dots$ denote the waiting times of S between visits of the point $S(\rho(k))$ (when

we start at the time $\rho(k)$). Although $ET_i = \infty$, it is known that $ET_i^{\frac{1}{3}} =: K' < \infty$ (see, e.g. [19]). The quantity K', obviously, does not depend on n. Thus, by Markov's inequality we have

$$\begin{aligned} P(E_a^{*c}) &\leq P\Big(\sum_{i=1}^{e^{n^{0.22}}} T_i > e^{n^{0.3}} - e^{n^{0.1}}\Big) = P\left(\Big(\sum_{i=1}^{e^{n^{0.22}}} T_i\Big)^{\frac{1}{3}} > \big(e^{n^{0.3}} - e^{n^{0.1}}\big)^{\frac{1}{3}}\right) \\ &\leq P\Big(\sum_{i=1}^{e^{n^{0.22}}} T_i^{\frac{1}{3}} > \big(e^{n^{0.3}} - e^{n^{0.1}}\big)^{\frac{1}{3}}\Big) \leq \frac{e^{n^{0.22}} K'}{\big(e^{n^{0.3}} - e^{n^{0.1}}\big)^{\frac{1}{3}}} \leq e^{-n^{0.25}}. \end{aligned}$$

Thus, $P(E_{a*}^c) \leq n^{25000} e^{-n^{0.25}} = o(n^{-K})$ implying that

$$P(E_{a*}^c | C, \psi) \leq \frac{P(E_{a*}^c | \psi_n)}{P(C|\psi_n)} = o(1). \tag{4.35}$$

To complete the proof, use (4.32), (4.33), (4.35), (4.35) to get

$$\begin{aligned} P_\psi(E_a^c) &\leq P_\psi(E_a^c \cap C) + P_\psi(C^c) = P_\psi(E_a^{*c} \cap C) + P_\psi(E_a^c \cap C) - P(E_a^{*c} \cap C) + o(1) \\ &\leq P(E_a^{*c} | C, \psi_n) + o(1) = o(1). \end{aligned}$$

5 Combinatorics of g and ĝ

In this section we show: if all scenery dependent events and random walk dependent events hold, then our estimates $\hat{\delta}^M_T$ and $\hat{h}_t$ are precise. This means, we can observe our signals and, just like in our 3-color example, we can well estimate the g-function.
At first we give the definition of g-function in 2-colors case.

5.1 Definition of g

In this subsection we give the formal definition of function

$$g : \{0,1\}^{m+1} \mapsto \{0,1\}^{n^2+1}.$$

The function g depends on n. When n is fixed, we choose $m = \ulcorner n^{2.5} EZ \urcorner$, where the random variable Z is the location of the first Markov signal point after $2Ln^{1000}$ in ξ. We now consider the signal carrier points $\bar{z}_1, \bar{z}_2, \ldots,$ in $[0, m]$. Define the following subset of $\{0,1\}^{m+1}$:

$$E^* := \{\psi \in \{0,1\}^{m+1} : \bar{z}_1(\psi) \geq L(e^{n^{0.1}} + n^{1000}), \bar{z}_{n^2+1} \leq m - L(e^{n^{0.1}} + n^{1000})\}.$$

Here, $\bar{z}_i(\psi) = \infty$, if the piece of scenery ψ has less than i signal carrier points.
Clearly $E^n_{\text{cell\_OK}} \subset E^*$. If $\psi \in E^*$, then for each $\bar{z}_i(\psi)$ we define the vector of the frequency of ones $h(i)$, $i = 1, \ldots, n^2+1$. Recall from (2.13) that

$$h(i) = h(\bar{z}_i(\psi)) = P(\psi(U + S(e^{n^{0.1}})) = 1),$$

where U is a random variable with distribution $\mu(\bar{z}_i)$.
Now, if $\psi \in E^*$, let

$$g_i(\psi) = \begin{cases} 1 & \text{, if } h(i) > 0.5 \\ 0 & \text{, if } h(i) < 0.5 \\ \bar{z}_i(\psi) & \text{otherwise.} \end{cases} \tag{5.1}$$

When $\psi \notin E^*$, define

$$g_i(\psi) = \psi(i), \quad , i = 1, 2, \ldots, n^2+1. \tag{5.2}$$

Definition 5.1 *$g(\psi) = (g_1(\psi), \ldots, g_{n^2+1}(\psi))$, where $g_i(\psi)$ is (5.1), if $\psi \in E^*$ and $g_i(\psi)$ is (5.2), if $\psi \notin E^*$.*

The Definition 5.1 ensures that $g(\psi)$ depends on ξ^m_0, only, and $(g_1(\xi), \ldots, g_{n^2+1}(\xi))$ is i.i.d. random vector, with the components being Bernoulli random variables with parameter $\frac{1}{2}$.

5.2 Definition of $\hat{g}$

We are now going to formalize the construction of the $\hat{g}$-function. The function $\hat{g} : \{0,1\}^{m^2+1} \mapsto \{0,1\}^{n^2}$ aims to estimate the (non-observable) function g. The argument of $\hat{g}$ is the vector of observations $\chi^{m^2}_0 := (\chi(0), \ldots, \chi(m^2))$, and the estimate is given up to first or last bit. In other words, $\hat{g}$ aims to achieve $\hat{g}(\chi^{m^2}) \preccurlyeq g(\xi|[0,m])$.
The algorithm for computation $\hat{g}$ has 5 phases and it differs from the $\hat{g}$-reconstruction algorithm for 3-colors case by the first step, only. The rest of the construction is the same.

1. For all $T = [t, t + e^{n0.3}] \subset [0, m^2]$ compute the estimate of Markov signal probability $\hat{\delta}^M_T$. Select all intervals $T_1 = [t_1, t_1 + e^{n^{0.3}}], T_2 = [t_2, t_2 + e^{n^{0.3}}], \ldots, T_K = [t_K, t_K + e^{n^{0.3}}]$, $t_1 < t_2 < \cdots < t_K$, where the estimated Markov signal probability were higher than c_r. Here K stands for the number of such intervals.

2. For all selected intervals estimate the frequency of ones. Thus, we obtain the estimates $\hat{h}_{T_1}, \ldots, \hat{h}_{T_K}$, $i = 1, \ldots, K$.

3. Define clusters

$$C_i := \{\hat{h}_{T_j} : |\hat{h}_{T_j} - \hat{h}_{T_i}| \leq 2\exp(-n^{0.12})\}, \ \hat{f}_i := \frac{1}{|C_i|} \sum_{j \in C_i} \hat{h}_{T_j}, \quad i = 1, \ldots, K.$$

4. Apply real scenery construction algorithm $\mathcal{A}_n^{\mathbb{R}}$ (see subsection 1.3) to the vector $(\hat{f}_1, \ldots, \hat{f}_K)$. Denote the output, $\mathcal{A}_n^{\mathbb{R}}(\hat{f}_1, \ldots, \hat{f}_K)$, by

$$(f_1, \ldots, f_{n^2}). \tag{5.3}$$

If the number of different reals in $(\hat{f}_1, \ldots, \hat{f}_K)$ is less than n^2 (e.g. $K \leq n^2$), then complete the vector (5.3) arbitrarily.

5. Define the final output of $\hat{g}$ as follows

$$\hat{g}(\chi^{m^2}) := (I_{[0.5,1]}(f_1), \ldots, I_{[0.5,1]}(f_{n^2})).$$

5.3 Main result

We are now going to prove the main result - when all previously stated events hold, then $\hat{g}$ algorithm **works**, i.e. $\hat{g}(\chi_0^{m^2}) \preccurlyeq g(\xi_0^m)$.
Recall $E^n_{\text{cell\_OK}} = \cap_{i=1}^9 E_i^n$. Similarly define the union of random walk dependent events $E_S^n := \cap_{i=1}^8 E_{i,S}^n$. Finally, let $E_{\text{g}-\text{works}}$ be the event that $\hat{g}$ works, i.e.

$$E_{\text{g}-\text{works}} := \Big\{\hat{g}(\chi_0^{m^2}) \preccurlyeq g(\xi_0^m)\Big\}. \tag{5.4}$$

At first we show that the step 1 in the definition of $\hat{g}$ works properly, i.e. an time interval T is selected (i.e. $\hat{\delta}_T^M > c_r$) only if during the time T the random walk is close to an unique signal carrier point $\bar{z}$. The closeness is defined in the following sense: we say that during time period T, the random walk S is close to z, if there exists $s \in T$ such that $S(s) \in I_z$.

Proposition 5.1 *Suppose $E^n_{\text{cell\_OK}} \cap E_S^n$ holds. Let $T = [t, t + e^{n^{0.3}}] \subset [0, m]$. If during T the random walk is close to a signal point z, and $\hat{\nu}_t(e^{n^{0.2}}) \leq t + e^{n^{0.3}} - e^{n^{0.1}}$, then $\hat{\delta}_T^M = \hat{\delta}_{z,t}^M$ and $\hat{h}_T = \hat{h}_{z,t}$.*

Proof. Since ξ and S are independent, we fix $\xi = \psi \in E^n_{\text{cell\_OK}}$ and show that the claim of the proposition holds.
Let S be close to the signal point z. By $E_2^n \cap E_8^n \cap E_9^n$, the point z has empty neighborhood and empty borders. Hence, in the area

$$([z - L(n^{1000} + e^{n^{0.3}}), z + L(n^{1000} + e^{n^{0.3}})] - [z - L\tilde{M}, z + L\tilde{M}]) \cap [-cm, cm]$$

there are no blocks that are bigger than $n^{0.35}$. Recall that $\tilde{M} = n^{1000} - 2n^2$. Since $2n^{0.35} < n^{0.4} < n^2$, this means: all blocks with length at least $n^{0.4}$ must lay inside the interval $[z - L(n^{1000} - n^2), z + L(n^{1000} - n^2)]$. In particular, this implies - if, during the time T the random walk S visits a block bigger than $n^{0.4}$, then during the n^2 step before and after that visit, it must stay in the interval I_z. Formally: if $\exists s \in T$: $S(s) \in B$, then

$$S(s - n^2), S(s - n^2 + 1), \ldots, S(s + n^2 - 1), S(s + n^2) \in I_z. \tag{5.5}$$

Here B stands for a block of ψ with length at least $n^{0.4}$.
We now take advantage of the event $E_{6,S}^n$: the random walk cannot generate $n^2 + 1$ same colors, if it does not visit block bigger than $n^{0.4}$. By (5.5) this means that all $n^2 + 1$ same colors must be generated

on I_z. Hence, inside the time interval T the stopping times $\hat{\nu}_t(i)$ are equal to the stopping times $\nu_{z,t}(i)$. Similarly, $X_{t,i} = X_{z,t,i}$, provided $\hat{\nu}_t(i) + n^{1000} \leq t + e^{n^{0.3}}$.
Now by assumption, there are at least $e^{n^{0.2}}$ stopping times $\hat{\nu}_t(i)$ in $[t, t+e^{n^{0.3}} - e^{n^{0.1}}]$ These stopping times are then equal to $\nu_{z,t}(i)$. Similarly, $X_{t,i} = X_{z,t,i}$, $i = 1, \ldots, e^{n^{0.2}}$. The latter means that the observable estimates $\hat{\delta}^M_T$ and $\hat{h}_T$ equals the non-observable estimates $\hat{\delta}^M_{z,t}$ and $\hat{h}_{z,t}$, respectively. ■

Corollary 5.1 *Suppose $E^n_{\text{cell\_OK}} \cap E^n_S$ holds. Let $T = [t, t + e^{n^{0.3}}] \subset [0, m]$. If during T the random walk is close to a signal point z, then $\hat{\delta}^M_T > 0$ implies that $\hat{h}_T = \hat{h}_{z,t}$ and $\hat{\delta}^M_T = \hat{\delta}^M_{z,t}$.*

Proof. By definition, $\hat{\delta}^M_T > 0$ if in the time interval $[t, t + e^{n^{0.3}} - e^{n^{0.1}}]$ there are at least $e^{n^{0.2}}$ stopping times $\hat{\nu}_t(i)$. Now Proposition 5.1 applies. ■

Lemma 5.1 *Suppose $E^n_{\text{cell\_OK}} \cap E^n_S$ holds. Let $T = [t, t+e^{n^{0.3}}] \subset [0, m]$ be such that $\hat{\delta}^M_T > c_r$. Then there exists an unique signal carrier point $\bar{z} \in [-cm, cm]$ such that S is close to $\bar{z}$ during T and $\hat{\delta}^M_T = \hat{\delta}^M_{\bar{z},t}$.*

Proof. Fix $\xi = \psi \in E^n_{\text{cell\_OK}}$. At first note, since E^n_2 holds, then all signal points in $[-cm, cm]$ have empty neighborhood. Together with d) of Proposition 2.1 it means – all signal points in $[-cm, cm]$ are in clusters with diameter less than $2Ln^{1000}$. The distance between the any two clusters, i.e. the distance between closest signal points in these clusters, is bigger than $e^{n^{0.3}}$. Moreover, by $E^n_8 \cap E^n_9$, all signal points have empty borders.
If $E^n_{2,S}$ holds, then during time $[0, m^2]$, our random walk stays in $[-cm, cm]$. Together with the clustering structure of the signal points, this means: if during the time interval $T \subset [0, m^2]$ of length $e^{n^{0.3}}$ the random walk S is close to some signal points, then they all belong to the same cluster. Hence, during T, S can be close to at most one signal carrier point (recall, every cluster has one representant, the signal carrier point). We have to show that if $\hat{\delta}^M_T > c_r$, then there exists at least one signal carrier point $\bar{z}$ such that, (during T) S is close to $\bar{z}$.
During T, the random walk S has 3 options :

- S is not close to any signal point
- S is close to the signal points that are not Markov signal points
- S is close to a Markov signal point.

If S is not close to any signal point, then by $E^n_{3,S}$, $\hat{\delta}^M_T \leq c_r$. This excludes the first possibility. Hence, $\hat{\delta}^M_T > c_r$ cannot happen, if during T, S is not close to any signal point.
Suppose now that there exists a signal point z such that (during T) S is close to z. By assumption we have $\hat{\delta}^M_T > c_r > 0$. By Corollary 5.1 we have that $\hat{\delta}^M_T = \hat{\delta}^M_{z,t}$. Now we reap benefit from the events E^n_5 and $E^n_{7,S}$. The event E^n_5 ensures that z is regular, i.e. $|\delta^M_z - c_r| \geq \Delta > e^{-n^{0.12}}$ (recall, Δ is polynomially small). On the other hand, the event $E^n_{7,S}$ ensures $|\hat{\delta}^M_T - \delta^M_z| = |\hat{\delta}^M_{z,t} - \delta^M_z| \leq \exp(-n^{0.12})$. Thus on $E^n_5 \cap E^n_{7,S}$ we have

$$\hat{\delta}^M_T > c_r \quad \text{iff} \quad \delta^M_z > c_r - \Delta. \tag{5.6}$$

Suppose we have the second possibility – S is close to some signal points, but not close to any Markov signal points. Then z is not a Markov signal point. Hence, (5.6) ensures that $\hat{\delta}^M_T \leq c_r$. This contradicts our assumption that $\hat{\delta}^M_T > c_r$. Hence, z must be a Markov signal point and our third option holds.
Thus $\hat{\delta}^M_T > c_r$ implies that during T, the random walk S is close to a Markov signal point. By clustering structure we know that S is close to a cluster of signal points with at least one Markov signal points. In subsection 3.4 we argued that such a cluster serves as the signal carrier. However, to complete the proof we must show that, during T, S is also close to the corresponding signal carrier point, say $\bar{z}$.
The points $\bar{z}$ and z belong to the same cluster, i.e. $|\bar{z} - z| < 2Ln^{1000}$. Consider the interval

$$J_z := [z - L(\exp(n^{0.3}), z + L(\exp(n^{0.3})] \cap [-cm, cm].$$

This is the region, where random walk S stays during the time T. We know that the intervals I_z and $I_{\bar{z}}$ both have empty neighborhood and empty borders. Thus in all blocks of $\psi|J_z$ that are longer than $n^{0.4}$ must lie in $I_z \cap I_{\bar{z}}$ (by c of Proposition 2.1, in $I_z \cap I_{\bar{z}}$ there is at least one big block of ψ). Now argue as in the proof of Proposition 5.1: because of $E^n_{6,S}$, to generate n^2+1 consecutive 0's or 1's, S must visit a block with length at least $n^{0.4}$. To have $\hat{\delta}^M_T > 0$, during T, S must have at least $e^{n^{0.2}}$ such visits. All those blocks are in $I_z \cap I_{\bar{z}} \subset I_{\bar{z}}$. Thus, when $\hat{\delta}^M_T > 0$, then during T, S visits $\bar{z}$ at least $e^{n^{0.2}}$ times. This means that during T, S is close to $\bar{z}$. By Corollary 5.1 we now get $\hat{\delta}^M_T = \hat{\delta}^M_{z,t}$. ■

Theorem 5.2 *If $E^n_{\text{cell\_OK}}$ and E^n_S both hold, then, for n big enough, $\hat{g}$ works. In other words,*

$$E^n_{\text{cell\_OK}} \cap E_S \subset E_{\text{g}-\text{works}}. \tag{5.7}$$

Proof. Suppose $E^n_{\text{cell\_OK}} \cap E^n_S$ hold. Fix $\xi = \psi \in E^n_{\text{cell\_OK}}$ and let

$$g(\psi) = (g_1(\psi), \dots, g_{n^2+1}(\psi))$$

We have to show: if E^n_S holds, then given the observations $\chi_0^{m^2}$, the function

$$\hat{g}(\chi_0^{m^2}) := (I_{[0.5,1]}(f_1), \dots, I_{[0.5,1]}(f_{n^2}))$$

is equal to $\hat{g}(\psi)$ up to the first or last bit.
Let $\chi_0^{m^2}$ be the observations. Apply the $\hat{g}$-construction algorithm.
1) At the first step we pick the intervals $T_1 = [t_1, t_1 + e^{n^{0.1}}], \dots, [t_K, t_K + e^{n^{0.1}}]$ such that for each j, $\hat{\delta}^M_T > c_r$, $j = 1, \dots, K$. By Lemma 5.1 we know that each interval T_j corresponds to exactly one signal carrier point, say $\bar{z}_{\pi(j)}$.

Let us investigate the mapping $\pi : \{1, \dots, K\} \mapsto \mathbb{Z}$, where $\pi(j)$ is the index of the signal carrier corresponding to the interval T_j. We now show that π posses the properties A1), A2), A3) that are familiar from the subsection 1.3.3.

A1) $\pi(1) \in \{0, 1\}$

A2) $\pi(K) \geq n^2 + 1$

A3) π is skip-free, i.e. $\forall j,\ |\pi(j) \pm \pi(j)| \leq 1$.

All these properties hold because of $E_{4,S} \cap E_{5,S}$. Indeed, during the time interval $[0, m^2]$ the random walk starts at 0 and, according to the event $E^n_{1,S}$, ends at m. Let $\bar{z}_1 \dots, \bar{z}_u$ denote all signal carrier points of ψ in $[0, m]$. By E^n_1, $u > n^2$. The maximal jump of S is L and, therefore, on its way, S visits all intervals $I_{\bar{z}_1}, \dots I_{\bar{z}_u}$. Recall that the stopping times $\rho(k)$ denote the first visits of the new interval (the first visit of the next interval, not necessarily new for the past). By $E^n_{4,S} \cap E^n_{5,S}$, for each k such that $\rho(k) < m^2$ we have: there is at least $e^{n^{0.2}}$ stopping times $\hat{\nu}_{\rho(k)}(i)$ in $T := [\rho(k), \rho(k) + e^{n^{0.3}} - e^{n^{0.1}}]$. Let $\bar{z}$ be the signal carrier point such that $S(\rho(k)) \in I_{\bar{z}}$. Thus the assumptions of Proposition 5.1 hold and $\hat{\delta}^M_T = \hat{\delta}^M_{\bar{z},t}$. Moreover, by (5.6) we have that $\hat{\delta}^M_T > c_r$, i.e. the interval T will be selected in the first step of $\hat{g}$ reconstruction.
To summarize: the random walk starts at 0, by convention the first signal carrier point in $[0, \infty)$ is $\bar{z}_1$, the biggest signal carrier point in $(-\infty, 0]$ is $\bar{z}_0$. From Lemma 5.1 we know - during T_1, S must be close to a signal carrier point. On the other hand $[\rho(0), \rho(0) + e^{n^{0.3}}]$ is the first time interval, during which S is close to a signal carrier point. We know that this interval will be selected. Hence $\pi(1) \in \{0, 1\}$.
On its way S visits all signal carrier interval $I_{\bar{z}_1}, \dots I_{\bar{z}_u}$. Right after the first visit in a new signal carrier, $\rho(k)$, the random walk produces an interval $T = [\rho(k), \rho(k) + e^{n^{0.3}}]$ that will be selected. Together with Lemma 5.1 the latter yields that π is skip-free.
Recall that $\bar{z}_u$ is the last signal carrier point in $[0, m]$. Thus, the last signal carrier interval S visits during $[0, m^2]$ is $\bar{z}_u$ or $\bar{z}_{u+1}$. By E^n_7 we know that $\bar{z}_u$ lays in $[0, m - Le^{n^{0.3}}]$ is at least $\bar{z}_u$. Hence, if $S(\rho(k)) \in I_{\bar{z}_u}$,

then $[\rho(k), \rho(k) + e^{n^{0.3}}]$ will be selected. We now get that last selected interval corresponds to the signal carrier that is at least $\bar{z}_{n^2+1}$. Thus $\pi(K) \geq n^2 + 1$.
Let $\pi_* := \min\{\pi(j) : j = 1, \ldots, K\}$, $\pi^* := \max\{\pi(j) : j = 1, \ldots, K\}$. We just saw that $\pi_* \leq 1$, $\pi^* \geq n^2+1$ and π is a skip-free random walk on $\{\pi_*, \pi_* + 1, \ldots, \pi^*\}$.
The rest of the algorithm was already argued in section 1.3. However, in the following we give a bit more formal explanation.

2) At the second step we calculate $\hat{h}_{T_1}, \ldots, \hat{h}_{T_K}$. By Lemma 5.1 we know that, for each $j = 1, \ldots, K$

$$\hat{h}_{T_j} = \hat{h}_{\bar{z}_{\pi(j)}, t_j}.$$

3) Since $E^n_{8,S}$ holds, we know that, for each $j = 1, \ldots, K$,

$$|\hat{h}_{T_j} - h(\bar{z}_{\pi(j)})| = |\hat{h}_{\bar{z}_{\pi(j)}, t_j} - h(\bar{z}_{\pi(j)})| < \exp(-n^{0.12}).$$

This means: if $\pi(i) = \pi(j)$ then $|\hat{h}_{T_i} - \hat{h}_{T_j}| \leq 2\exp(-n^{0.12})$.
On the other hand, by E^n_3 we know that $\pi(i) \neq \pi(j)$ implies

$$|h(\bar{z}_{\pi(j)}) - h(\bar{z}_{\pi(i)})| \geq \exp(-n^{0.11}). \tag{5.8}$$

We assume n to be big enough to satisfy $\exp(-n^{0.12}) < 5\exp(-n^{0.11})$. Hence $\pi(i) \neq \pi(j)$ implies that $|\hat{h}_{T_i} - \hat{h}_{T_j}| > 2\exp(-n^{0.12})$. Thus, if $E^n_{8,S} \cap E^n_3$, then for each $i, j = 1, \ldots, k$ we have

$$\hat{h}_j \in C_i \quad \text{iff} \quad \pi(i) = \pi(j). \tag{5.9}$$

Hence the clusters C_i and C_j are either the same or disjoint; $C_i = C_j$ iff $\pi(j) = \pi(i)$. The same, obviously, holds for the averages:

$$\hat{f}_j = \hat{f}_i \quad \text{iff} \quad \pi(i) = \pi(j).$$

Let for each $i = \{\pi_*, \pi_* + 1, \ldots, \pi^*\}$, $\hat{f}(\bar{z}_i) = \hat{f}_j$, if $\pi(j) = i$. Hence, $\hat{f}(\bar{z}_i)$ is the estimate of $h(\bar{z}_i)$ and

$$\hat{f}_j = \hat{f}(\bar{z}_{\pi(j)}), \quad j = 1, \ldots, K.$$

Hence, $j \mapsto \hat{f}_j$ can be considered as the observations of the skip-free random walk π on the different reals $\{\hat{f}(\bar{z}_{\pi_*}), \hat{f}(\bar{z}_{\pi_*+1}), \ldots \hat{f}(\bar{z}_{\pi^*})\}$.

4) The real scenery construction algorithm $\mathcal{A}^{\mathbb{R}}_n$ is now able to reproduce the numbers $\hat{f}(z_1), \ldots, \hat{f}(z_{n^2+1})$ up to the first or last number. Thus

$$(f_1, \ldots, f_{n^2}) = \mathcal{A}^{\mathbb{R}}(\hat{f}_1, \ldots, \hat{f}_K) \preccurlyeq (\hat{f}(\bar{z}_1), \ldots, \hat{f}(\bar{z}_{n^2+1})).$$

5) By E^n_4, we have that $|h(\bar{z}_i) - 0.5| \leq \exp(-n^{0.11})$. From (5.8) and (5.9), it follows

$$|\hat{f}_i - h(\bar{z}_{\pi(i)})| \leq \exp(-n^{0.12}).$$

The latter implies

$$\hat{f}(\bar{z}_i) \geq 0.5 \quad \text{iff} \quad h(\bar{z}_i) \geq 0.5.$$

Hence , for each $i = 1, \ldots, n^2 + 1$ we have that $I_{[0.5,1]}(\hat{f}(\bar{z}_i)) = I_{[0.5,1]}(h(\bar{z}_i))$. Thus

$$\hat{g}(\chi_0^{m^2}) = \Big(I_{[0.5,1]}(f_1), \ldots I_{[0.5,1]}(f^2_n)\Big) \preccurlyeq \Big(I_{[0.5,1]}(h(\bar{z}_1)), \ldots I_{[0.5,1]}(h(z_{n^2+1}))\Big) = g(\psi).$$

■

We now state the main result of the paper.

Theorem 5.3 *There exists constants $c > 0$ (not depending on n), $N < \infty$, $m(n) > n$, the maps*

$$g : \{0,1\}^{m+1} \mapsto \{0,1\}^{n^2+1}$$
$$\hat{g} : \{0,1\}^{m^2+1} \mapsto \{0,1\}^{n^2}$$

and the sequence of events $E^n_{\text{cell\_OK}} \in \sigma(\xi(z)|z \in [-cm, cm])$ such that:

1) $P(E^n_{\text{cell\_OK}}) \to 1$

2) *For all $n > N$ and $\psi_n \in E^n_{\text{cell\_OK}}$ we have:*

$$P\left(\hat{g}(\chi_0^{m^2}) \preccurlyeq g(\psi_0^m) \middle| S(m^2) = m, \xi = \psi_n\right) > 3/4.$$

3) *$g(\xi_0^m)$ is an i.i.d. binary vector where the components are Bernoulli with parameter $1/2$.*

Proof. Fix $c > 0$ such that Proposition 4.1 holds for $\epsilon = \frac{1}{8}$. Use this particular c to define all scenery dependent events as well as all random walk-dependent vents.
The intersection of all scenery-dependent events is $E^n_{\text{cell\_OK}}$. In Section 3.2 we proved that $P(E^n_{\text{cell\_OK}}) \to 1$. Hence **1)** holds.
Now consider the event E^n_S. Use Theorem 5.2 to find the integer $N_1 < \infty$ such that for each $n > N_1$, (5.4) hold. Then, for each $n > N_1$, $\psi_n \in E^n_{\text{cell\_OK}}$ we have

$$P(g(\chi_0^{m^2}) \preccurlyeq g(\xi_0^m)|S(m^2) = m, \xi = \psi_n) \geq P(E^n_S|S(m^2) = m, \xi = \psi_n) = P_\psi(E^n_S).$$

In the Section 4.3 we proved that $\liminf_n P_\psi(E^n_S) \geq 1 - \frac{1}{8}$. Let N_2 be so big that $P_\psi(E_n) > \frac{3}{4}$ $\forall n > N_1$. Take $N := N_1 \vee N_2$. With such N, **2**) holds.
Finally, the statement **3**) follows from the definition of g in Section 5.1. ■

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