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Abstract: Consider n i.i.d. random elements on $C[0,1]$. We show that under an appropriate strengthening of the domain of attraction condition natural estimators of the extreme-value index, which is now a continuous function, and the normalizing functions have a Gaussian process as limiting distribution. A key tool is the weak convergence of a weighted tail empirical process, which makes it possible to obtain the results uniformly on $[0,1]$.

Key words: Estimation, extreme value index, infinite dimensional extremes, weak convergence on C[0,1].
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1 Introduction

Recently considerable progress has been made in the interesting field of infinite dimensional extreme value theory. After the characterization of max-stable stochastic processes in $C[0,1]$ by Giné, Hahn and Vatan (1990), de Haan and Lin (2001, 2002) investigated the domain of attraction conditions and established weak consistency of estimators of the extreme value index, the centering and standardizing sequences, and the exponent measure. Statistics of infinite dimensional extremes will find various applications, e.g. in coast protection (flooding) and risk assessment in finance. Also from a mathematical point of view the research is challenging, because of the new features of $C[0,1]$-valued random elements, when compared to random variables or vectors, in particular the uniformity in $t \in [0,1]$ of the results asks for novel approaches.

It is the purpose of this paper to establish the weak convergence of estimators of the extreme value index, which is now an element of $C[0,1]$, and the normalizing sequences. In fact, we will show the asymptotic normality on $C[0,1]$ of the estimators under a suitable second order condition and present all the limiting processes involved in terms of one underlying Wiener process, which means that we have the simultaneous weak convergence of all the estimators. The results are on the one hand interesting in themselves, because the extreme value index measures the tail heaviness of the distribution of the data, and on the other hand the results are a major step forward in the estimation of probabilities of rare events in $C[0,1]$; see de Haan and Sinha (1999) for a study of this problem in the finite dimensional case.

In order to be more explicit let us now specify the setup and introduce notation. Let $\xi_1, \xi_2, \ldots$ be i.i.d. random elements on $C[0,1]$. Throughout assume that $P\{\inf_{t\in[0,1]} \xi_i(t) > 0\} = 1$ and that $F_t : I\!R \to [0,1]$, defined by $F_t(x) = P\{\xi_i(t) \le x\}$, is a continuous and strictly increasing function on its support. Define

$$U_t(s) = F_t^{\leftarrow}(1 - 1/s), s > 0, 0 \le t \le 1.$$

We assume that the domain of attraction condition holds, i.e.

$$\left\{ \left(\max_{i=1,\ldots,n} \xi_i(t) - b_t(n) \right) / a_t(n), \; t \in [0,1] \right\} \tag{1}$$

converges in distribution on $C[0,1]$ to a stochastic process, η say, with non-degenerate marginals, where $a_t(n) > 0$ and $b_t(n)$ are continuous (in t) normalizing functions, chosen in such a way that for each t

$$P\{\eta(t) \le x\} = \exp\left(-(1 + \gamma(t)x)^{-\frac{1}{\gamma(t)}} \right),$$

see de Haan and Lin (2001); we can and will take $b_t \equiv U_t$. Then $\gamma : [0,1] \to I\!R$, the extreme value index (function), is continuous. Define

$$\zeta_i(t) = \frac{1}{1 - F_t(\xi_i(t))},$$

$$\bar{\eta}(t) = (1 + \gamma(t)\eta(t))^{1/\gamma(t)},$$

and $\nu_s(E) = sP\{\zeta_i \in sE\}$, with $sE = \{sh : h \in E\}$. Clearly the $\zeta_i(t)$ are standard Pareto random variables, i.e. $P\{\zeta_i(t) \le x\} = 1 - 1/x$, $x \ge 1$. It follows from Theorem 2.8 in de Haan and Lin (2001), that there exist a homogeneous measure ν on $C[0,1]$, such that for any $g > 0$

$$P\{\bar{\eta} < g\} = \exp\{-\nu(\{f \in C[0,1], f \nless g\})\},$$

and

$$\nu_s \longrightarrow \nu, \text{ as } s \longrightarrow \infty,$$

weakly on $S_c := \{f \in C[0,1] : \sup_{t\in[0,1]} f(t) \ge c\}$, for any $c > 0$, and

$$\frac{U_t(nx) - U_t(n)}{a_t(n)} \to \frac{x^{\gamma(t)} - 1}{\gamma(t)}, \quad \text{as } n \longrightarrow \infty,$$

uniformly in $t \in [0,1]$ and locally uniformly in $x \in (0, \infty)$.

Throughout assume that $k = k(n)$ is a sequence of positive integers satisfying $k \to \infty$ and $k/n \to 0$, as $n \to \infty$. Fix $t \in [0,1]$. Let $\xi_{1,n}(t) \le \xi_{2,n}(t) \le \ldots \le \xi_{n,n}(t)$ be the order statistics of $\xi_i(t), i = 1, 2, \ldots, n$. We define the following statistical functions

$$M_n^{(r)}(t) = \frac{1}{k}\sum_{i=0}^{k-1}(\log \xi_{n-i,n}(t) - \log \xi_{n-k,n}(t))^r \quad r = 1, 2. \tag{2}$$

Set $\gamma^+(t) = \gamma(t) \vee 0$, $\gamma^-(t) = \gamma(t) \wedge 0$ and observe that $\gamma(t) = \gamma^+(t) + \gamma^-(t)$. Now we define estimators for $\gamma^+(t)$, $\gamma^-(t)$, $\gamma(t)$, $a_t(\frac{n}{k})$ and $b_t(\frac{n}{k})$ as in Dekkers, Einmahl and de Haan (1989):

$$\hat{\gamma}_n^+(t) = M_n^{(1)}(t) \text{ (Hill estimator)}; \tag{3}$$

$$\hat{\gamma}_n^-(t) = 1 - \frac{1}{2}\left(1 - \frac{(M_n^{(1)}(t))^2}{M_n^{(2)}(t)}\right)^{-1}; \tag{4}$$

$$\hat{\gamma}_n(t) = \hat{\gamma}_n^+(t) + \hat{\gamma}_n^-(t) \text{ (moment estimator)}; \tag{5}$$

$$\hat{U}_t(\frac{n}{k}) = \xi_{n-k,n}(t) \text{ (location estimator)}; \tag{6}$$

(7) $$\hat{a}_t(\frac{n}{k}) = \xi_{n-k,n}(t)\hat{\gamma}_n^+(t)(1-\hat{\gamma}_n^-(t)) \text{ (scale estimator).}$$

For fixed t these are well-known one-dimensional estimators (see e.g. de Haan and Rootzén (1993)).

The following weak consistency results have been shown in de Haan and Tao (2002).

Theorem 1.1 *As $k \to \infty$ and $\frac{k}{n} \to 0$, we have*

(8) $$\sup_{0\le t\le 1} \left|\hat{\gamma}_n^+(t) - \gamma^+(t)\right| \xrightarrow{P} 0,$$

(9) $$\sup_{0\le t\le 1} |\hat{\gamma}_n(t) - \gamma(t)| \xrightarrow{P} 0,$$

(10) $$\sup_{0\le t\le 1} \left|\frac{\hat{U}_t(\frac{n}{k}) - U_t(\frac{n}{k})}{a_t(\frac{n}{k})}\right| \xrightarrow{P} 0,$$

(11) $$\sup_{0\le t\le 1} \left|\frac{\hat{a}_t(\frac{n}{k})}{a_t(\frac{n}{k})} - 1\right| \xrightarrow{P} 0.$$

The main results of the paper are given in Section 2; the proofs are deferred to Section 3.

2 Main results

In this section we present our main result, dealing with the asymptotic normality of the estimators of which the weak consistency is shown in Theorem 1.1. In order to establish our main result we first present a result that is a key tool for its proof. This result deals with the weak convergence of a tail empirical process based on the $\zeta_i, i = 1, \ldots, n$. Write $C_{t,x} = \{h \in C[0,1] : h(t) \ge x\}$ and define

$$S_{n,t}(x) = \frac{1}{n}\sum_{i=1}^{n} 1_{\{\zeta_i \in C_{t,x}\}} = \frac{1}{n}\sum_{i=1}^{n} 1_{\{\zeta_i(t) \ge x\}}.$$

Denote the corresponding tail empirical process with

$$w_n(t,x) = \sqrt{k}\left(\frac{n}{k}S_{n,t}\left(x\frac{n}{k}\right) - \frac{1}{x}\right).$$

Let W be a zero-mean Gaussian process with $EW(B_1)W(B_2) = \nu(B_1 \cap B_2)$. Clearly for fixed $t \in [0,1]$, $\{W(C_{t,1/y}), y \le \frac{1}{c}\}$ is a standard Wiener process, since $\nu(C_{t,1/y_1} \cap C_{t,1/y_2}) = y_1 \wedge y_2$. For $\beta > 0$ and $c > 0$, set for any $(t,x),(s,y) \in [0,1] \times [c,\infty)$,

$$\begin{aligned} d((t,x),(s,y)) &= \sqrt{E\left(x^\beta W(C_{t,x}) - y^\beta W(C_{s,y})\right)^2} \\ &= \sqrt{\left(x^\beta - y^\beta\right)^2 \nu(C_{t,x} \cap C_{s,y}) + x^{2\beta}\nu(C_{t,x} \setminus C_{s,y}) + y^{2\beta}\nu(C_{s,y} \setminus C_{t,x})}. \end{aligned}$$

For convenient presentation and convenient application in the proofs of the main result, this result is presented in an approximation setting, with the random elements involved, defined on *one* probability space. So the random elements in this theorem are only equal in distribution to the original ones, but we do not add the usual tildes to the notation.

Theorem 2.1 *Suppose the conditions of the Introduction, in particular (1), hold and assume that there exist positive constants $d, K \in I\!R$, such that $P\{\zeta_1 \vee d \in E\} = 1$ with*

$$E = \left\{ h \in C[0,1] : h \ge 0,\ \frac{|h(t_1) - h(t_2)|}{h(t_1) \vee h(t_2)} \le K \left(\log \frac{1}{|t_1 - t_2|}\right)^{-3} \text{ for all } t_1, t_2 \in [0,1] \right\},$$

then, with a special construction, we have for any $0 \le \beta < \frac{1}{2}$ and $c > 0$

$$\sup_{0 \le t \le 1,\ x \ge c} x^\beta \left| w_n(t,x) - W(C_{t,x}) \right| \xrightarrow{P} 0, \text{ as } n \to \infty, \tag{12}$$

Define $Z(t,x) = x^\beta W(C_{t,x})$, then the process Z is bounded and uniformly d-continuous on $[0,1] \times [c,\infty)$.

Note that it is well-known that for *one* fixed t, the restriction $\beta < \frac{1}{2}$ is also needed for weak convergence of the (one-dimensional) tail empirical process. So our condition on β in the present infinite dimensional setting is the same as in dimension one. It is important to transform the ξ_i to processes with standard marginals, as we did by transforming to the ζ_i. Although the choice of standard Pareto marginals is convenient, it is also reasonable to transform to other marginal distributions, like the uniform-(0,1) distribution. Clearly, uniform-(0,1) marginals are obtained by taking $1/\zeta_i$. It is interesting to note and readily checked that the set E, defining the Lipschitz-type condition in Theorem 2.1, is invariant under this transformation.

We also need the following corollary which deals with certain quantiles and can be obtained by the usual 'inversion', from the tail empirical process theorem.

Corollary 2.2 *We have under the conditions of Theorem 2.1 for any $\alpha \in I\!R$*

$$\sup_{0 \le t \le 1} \left| \sqrt{k}\left(\left(\zeta_{n-k,n}(t)\frac{k}{n}\right)^\alpha - 1\right) - \alpha W(C_{t,1}) \right| \xrightarrow{P} 0, \text{ as } n \to \infty. \tag{13}$$

Finally we present the main result, which gives the asymptotic distributions of the estimators of γ^+, γ, a, and b in terms of the process W, figuring in Theorem 2.1.

Theorem 2.3 *Suppose the conditions of Theorem 2.1 are satisfied and the following second order condition holds:*
(14)
$$\left(\frac{\log U_t(sx) - \log U_t(s)}{a_t(s)/U_t(s)} - \frac{x^{\gamma^-(t)} - 1}{\gamma^-(t)}\right) / A_t(s) \to H_{\gamma^-(t),\rho(t)}(x), \quad \text{as } s \to \infty,$$

uniformly in $t \in [0,1]$ and locally uniformly in $x > 0$, with

$$H_{\gamma^-(t),\rho(t)}(x) = \int_1^x y^{\gamma^-(t)-1} \int_1^y u^{\rho(t)-1} du dy,$$

and $\rho \in C[0,1]$ with $\rho(t) \le 0$ for all $t \in [0,1]$. For any sequence $k = k(n)$ satisfying $k \to \infty$, $\frac{k}{n} \to 0$ and

$$\sqrt{k} \sup_{0\le t\le 1} \left|A_t\left(\frac{n}{k}\right)\right| \to 0, \tag{15}$$

$$\sqrt{k} \sup_{0\le t\le 1} \left|\frac{a_t\left(\frac{n}{k}\right)}{U_t\left(\frac{n}{k}\right)} - \gamma^+(t)\right| \to 0, \tag{16}$$

as $n \to \infty$, we have

$$\sup_{0\le t\le 1} \left|\sqrt{k}(\hat{\gamma}_n^+(t) - \gamma^+(t)) - \gamma^+(t)\mathcal{P}(t)\right| \xrightarrow{P} 0, \tag{17}$$

$$\sup_{0\le t\le 1} \left|\sqrt{k}(\hat{\gamma}_n(t) - \gamma(t)) - \Gamma(t)\right| \xrightarrow{P} 0, \tag{18}$$

$$\sup_{0\le t\le 1} \left|\sqrt{k}\frac{\hat{U}_t(\frac{n}{k}) - U_t(\frac{n}{k})}{a_t(\frac{n}{k})} - \mathcal{U}(t)\right| \xrightarrow{P} 0, \tag{19}$$

$$\sup_{0\le t\le 1} \left|\sqrt{k}\left(\frac{\hat{a}_t(\frac{n}{k})}{a_t(\frac{n}{k})} - 1\right) - \mathcal{A}(t)\right| \xrightarrow{P} 0, \tag{20}$$

as $n \to \infty$, where $\mathcal{P}, \mathcal{Q}, \Gamma, \mathcal{U}$ and $\mathcal{A}$ are the following functions defined in terms of the process W:

$$\mathcal{P}(t) = \int_1^\infty W(C_{t,x}) \frac{dx}{x^{1-\gamma^-(t)}} - \frac{1}{1-\gamma^-(t)} W(C_{t,1}),$$

$$\mathcal{Q}(t) = 2\int_1^\infty W(C_{t,x}) \frac{x^{\gamma^-(t)}-1}{\gamma^-(t)} \frac{dx}{x^{1-\gamma^-(t)}} - 2((1-\gamma^-(t))(1-2\gamma^-(t)))^{-1} W(C_{t,1}),$$

$$\Gamma(t) = \left\{\gamma^+(t) - 2(1-\gamma^-(t))^2(1-2\gamma^-(t))\right\} \mathcal{P}(t) + \frac{1}{2}(1-\gamma^-(t))^2(1-2\gamma^-(t))^2 \mathcal{Q}(t),$$

$$\mathcal{U}(t) = W(C_{t,1}),$$

$$\mathcal{A}(t) = \gamma^+(t)(1-\gamma^-(t))W(C_{t,1}) + (3-4\gamma^-(t))(1-\gamma^-(t))\mathcal{P}(t) - \frac{1}{2}(1-\gamma^-(t))(1-2\gamma^-(t))^2 \mathcal{Q}(t),$$

$t \in [0,1]$.

Note that for the case $\inf_{t\in[0,1]} \gamma(t) > 0$ and $\sup_{t\in[0,1]} \rho(t) < 0$, it follows from the second order condition (14) that

$$\sup_{t\in[0,1]} \left| \frac{\rho(t)\left(\frac{a_t(s)}{U_t(s)} - \gamma^+(t)\right)}{A_t(s)} - 1 \right| \to 0, \text{ as } s \to \infty. \tag{21}$$

So in this case (16) is superfluous, since it follows from (21) and (15). Also note that, condition (15) can be replaced by the stronger, but easier condition: for some $\varepsilon > 0$

$$\sqrt{k}\left(\frac{n}{k}\right)^{\varepsilon+\sup_{t\in[0,1]}\rho(t)} \to 0.$$

For the case $\sup_{t\in[0,1]} \gamma(t) < 0$ and $\sup_{t\in[0,1]} \rho(t) < 0$, it follows from the second order condition (14) that conditions (15) and (16) can be replaced by the stronger condition: for some $\varepsilon > 0$

$$\sqrt{k}\left(\frac{n}{k}\right)^{\varepsilon+\sup_{t\in[0,1]}\rho(t)\vee\sup_{t\in[0,1]}\gamma(t)} \to 0.$$

When $\sup_{t\in[0,1]} \rho(t) = 0$ or $\gamma(t_1) = 0$ for some $t_1 \in [0,1]$ (this also implies $\rho(t_1) = 0$) we do not have a simple sufficient condition on the growth of k, but it is necessary that k grows slower than any power of n.

3 Proofs

Proof of Theorem 2.1 We only give a proof for the case $c = 1$; for general $c > 0$ the proof is similar. For any $\beta \in [0, \frac{1}{2})$, define

$$f_{t,x} = 1_{C_{t,x}} x^\beta,$$

$$\mathcal{F} = \{f_{t,x} : 0 \le t \le 1, x \ge 1\}.$$

Also define the random measures

$$\mathcal{Z}_{n,i} = \frac{1}{\sqrt{k}} \delta_{\zeta_i \frac{k}{n}};$$

$\mathcal{Z}_{n,i}$ is a random function on $\mathcal{F}$, with

$$\mathcal{Z}_{n,i}(f_{t,x}) = \frac{1}{\sqrt{k}} 1_{\{\zeta_i(t)\frac{k}{n} \ge x\}} x^\beta.$$

Then

$$x^\beta w_n(t,x) = \sum_{i=1}^n (\mathcal{Z}_{n,i}(f_{t,x}) - E\mathcal{Z}_{n,i}(f_{t,x}))$$

First we are going to prove the tightness of $\{\sum_{i=1}^n (\mathcal{Z}_{n,i}(f) - E\mathcal{Z}_{n,i}(f)), f \in \mathcal{F}\}$. We need the following version of Theorem 2.11.9 in van der Vaart and Wellner (1996).

Definition 3.1 *For any $\varepsilon > 0$, the bracketing number $N_{[\,]}(\varepsilon, \mathcal{F}, L_2^n)$ is the minimal number of sets N_ε in a partition $\mathcal{F} = \bigcup_{j=1}^{N_\varepsilon} \mathcal{F}_{\varepsilon j}$ of the index set into sets $\mathcal{F}_{\varepsilon j}$ such that, for every partitioning set $\mathcal{F}_{\varepsilon j}$*

$$\sum_{i=1}^n E^* \sup_{f,g \in \mathcal{F}_{\varepsilon j}} |\mathcal{Z}_{n,i}(f) - \mathcal{Z}_{n,i}(g)|^2 \le \varepsilon^2. \tag{22}$$

Theorem 3.2 *For each n, let $\mathcal{Z}_{n,1}, \mathcal{Z}_{n,2}, \ldots, \mathcal{Z}_{n,n}$ be independent stochastic processes with finite second moments indexed by a totally bounded semimetric space $(\mathcal{F}, d)$. Suppose*

$$\sum_{i=1}^n E^* \|\mathcal{Z}_{n,i}\|_{\mathcal{F}} 1_{\{\|\mathcal{Z}_{n,i}\|_{\mathcal{F}} > \lambda\}} \to 0, \; for\, every\, \lambda > 0, \tag{23}$$

where $\|\mathcal{Z}_{n,i}\|_{\mathcal{F}} = \sup_{f \in \mathcal{F}} |\mathcal{Z}_{n,i}(f)|$, and

$$\int_0^{\delta_n} \sqrt{\log N_{[\,]}(\varepsilon, \mathcal{F}, L_2^n)}\, d\varepsilon \to 0, \; for\, every\, \delta_n \downarrow 0. \tag{24}$$

Then the sequence $\sum_{i=1}^n (\mathcal{Z}_{n,i} - E\mathcal{Z}_{n,i})$ is asymptotically tight in $\ell^\infty(\mathcal{F})$ and converges weakly, provided the finite-dimensional distributions converge weakly.

We can define d on $\mathcal{F}$ by $d(f_{t,x}, f_{s,y}) = d((t,x),(s,y))$; see the first paragraph of Section 2. Since ν is a finite and hence tight measure on $\{x \in C[0,1] : \sup_{t\in[0,1]} x(t) \geq 1\}$, it is easy to check that our class of functions $\mathcal{F}$ is totally bounded under the metric d.

To prove (23), observe

$$\|\mathcal{Z}_{n,i}\|_{\mathcal{F}} = \frac{1}{\sqrt{k}} \sup_{0\leq t\leq 1} \left(\zeta_i(t)\frac{k}{n}\right)^{\beta}.$$

So

$$\begin{aligned}
&\sum_{i=1}^{n} E\|\mathcal{Z}_{n,i}\|_{\mathcal{F}} 1_{\{\|\mathcal{Z}_{n,i}\|_{\mathcal{F}}>\lambda\}} \\
&= \frac{n}{\sqrt{k}} E\left(\sup_{0\leq t\leq 1} \zeta_i(t)\frac{k}{n}\right)^{\beta} 1_{\{\sup_{0\leq t\leq 1}\zeta_i(t)\frac{k}{n} > (\sqrt{k}\lambda)^{1/\beta}\}} \\
&= \frac{n}{\sqrt{k}} \int_{(\sqrt{k}\lambda)^{1/\beta}}^{\infty} x^{\beta} dF_n(x) \\
&= -\frac{n}{\sqrt{k}} x^{\beta}(1-F_n(x))|_{(\sqrt{k}\lambda)^{1/\beta}}^{\infty} + \beta\frac{n}{\sqrt{k}} \int_{(\sqrt{k}\lambda)^{1/\beta}}^{\infty} x^{\beta-1}(1-F_n(x))dx, \qquad (25)
\end{aligned}$$

where $1 - F_n(x) = P\{\sup_{0\leq t\leq 1} \zeta_i(t)\frac{k}{n} \geq x\}$. Note that for $x > 0$

$$\begin{aligned}
&\frac{n}{k}(1-F_n(x)) = \frac{n}{k} P\{\sup_{0\leq t\leq 1} \zeta_i(t)\frac{k}{n} \geq x\} = \nu_{n/k}\{f \in C[0,1] : \sup_{0\leq t\leq 1} f(t) \geq x\} \\
&\to \nu\{f \in C[0,1] : \sup_{0\leq t\leq 1} f(t) \geq x\} = \frac{1}{x}\nu\{f \in C[0,1] : \sup_{0\leq t\leq 1} f(t) \geq 1\} =: \frac{C}{x},
\end{aligned}$$

with $C \geq 1$. Since $\frac{n}{k}(1-F_n)$ is non-increasing and since the function $x \mapsto \frac{C}{x}$ is continuous and tends to 0 if $x \to \infty$, we have that this convergence is uniform on $[b,\infty)$, for any $b > 0$. So for n large enough, we have for $x \geq b$ that

$$\frac{1}{x} \leq \frac{n}{k}(1-F_n(x)) \leq \frac{2C}{x}. \qquad (26)$$

Hence the right hand side of (25) is bounded from above by

$$\begin{aligned}
&2C\sqrt{k}(\sqrt{k}\lambda)^{\frac{\beta-1}{\beta}} + 2\beta C\sqrt{k}\int_{(\sqrt{k}\lambda)^{1/\beta}}^{\infty} x^{\beta-2}dx \\
&= 2C\tfrac{1}{1-\beta}\lambda^{\frac{\beta-1}{\beta}} k^{\frac{2\beta-1}{2\beta}} \to 0.
\end{aligned}$$

That is (23).

Next we will prove (24). For any (small) $\varepsilon > 0$, let $a = \varepsilon^{3/(2\beta-1)}$, $\delta = \exp\{-\varepsilon^{-1}\}$ and $\theta = 1/(1-K\varepsilon^3)$. Define

$$\mathcal{F}(a) = \{f_{t,x} \in \mathcal{F}, x > a\},$$

$$\mathcal{F}(l,j) = \{f_{t,x} \in \mathcal{F}, l\delta \le t < (l+1)\delta, \theta^j \le x \le \theta^{j+1}\},$$

then we have the 'partition' $\mathcal{F} = \mathcal{F}(a) \cup \bigcup_{l=0}^{\lceil \frac{1}{\delta} \rceil} \bigcup_{j=0}^{\lfloor \frac{\log a}{\log \theta} \rfloor} \mathcal{F}(l,j)$. First we check (22) for $\mathcal{F}(a)$:

$$\begin{aligned}
& \sum_{i=1}^{n} E \sup_{f,g \in \mathcal{F}(a)} |\mathcal{Z}_{n,i}(f) - \mathcal{Z}_{n,i}(g)|^2 \\
= \; & nE \sup_{f,g \in \mathcal{F}(a)} (\mathcal{Z}_{n,i}(f) - \mathcal{Z}_{n,i}(g))^2 \\
\le \; & 4nE \sup_{f \in \mathcal{F}(a)} \mathcal{Z}_{n,i}^2(f) \\
\le \; & \frac{4n}{k} E \left(\sup_{0 \le t \le 1} \zeta_i(t) \frac{n}{k} \right)^{2\beta} 1_{\{\sup_{0 \le t \le 1} \zeta_i(t) \frac{k}{n} \ge a\}} \\
= \; & \frac{4n}{k} \int_a^\infty x^{2\beta} dF_n(x) \\
\le \; & \frac{8C}{1-2\beta} a^{2\beta - 1} \\
= \; & \frac{8C}{1-2\beta} \varepsilon^3 \le \varepsilon^2,
\end{aligned}$$

where the last but one inequality follows from integration by parts and (26). Now we consider (22) for the $\mathcal{F}(l,j)$. First note that

$$\sup_{f \in \mathcal{F}(l,j)} \mathcal{Z}_{n,i}(f) \le \frac{1}{\sqrt{k}} 1_{\{\sup_{l\delta \le t < (l+1)\delta} \zeta_i(t) \frac{k}{n} \ge \theta^j\}} \theta^{(j+1)\beta}.$$

From the Lipschitz-type condition in the theorem it follows that $\sup_{l\delta \le t < (l+1)\delta} \zeta_i(t) \frac{k}{n} \ge \theta^j$ implies that almost surely

$$\sup_{l\delta \le t < (l+1)\delta} \zeta_i(t) - \zeta_i(l\delta) \le K\varepsilon^3 \sup_{l\delta \le t < (l+1)\delta} \zeta_i(t),$$

and hence $\zeta_i(l\delta)\frac{k}{n} \ge \theta^{j-1}$, so

$$\sup_{f \in \mathcal{F}(l,j)} \mathcal{Z}_{n,i}(f) \le \frac{1}{\sqrt{k}} 1_{\{\zeta_i(l\delta) \frac{k}{n} \ge \theta^{j-1}\}} \theta^{(j+1)\beta}.$$

Similarly it can be shown that

$$\inf_{f \in \mathcal{F}(l,j)} \mathcal{Z}_{n,i}(f) \ge \frac{1}{\sqrt{k}} 1_{\{\zeta_i(l\delta) \frac{k}{n} \ge \theta^{j+2}\}} \theta^{j\beta}.$$

This yields

$$
\begin{aligned}
&\sum_{i=1}^{n} E^* \sup_{f,g\in\mathcal{F}(l,j)} |\mathcal{Z}_{n,i}(f) - \mathcal{Z}_{n,i}(g)|^2 \\
\leq\ & nE^* \left(\sup_{f\in\mathcal{F}(l,j)} \mathcal{Z}_{n,i}(f) - \inf_{f\in\mathcal{F}(l,j)} \mathcal{Z}_{n,i}(f) \right)^2 \\
\leq\ & \frac{n}{k} E \left(1_{\{\zeta_i(l\delta)\frac{k}{n}\geq\theta^{j-1}\}} \theta^{(j+1)\beta} - 1_{\{\zeta_i(l\delta)\frac{k}{n}\geq\theta^{j+2}\}} \theta^{j\beta} \right)^2 \\
\leq\ & \frac{n}{k} E \left((\theta^{(j+1)\beta} - \theta^{j\beta}) 1_{\{\zeta_i(l\delta)\frac{k}{n}\geq\theta^{j-1}\}} + \theta^{j\beta} 1_{\{\theta^{j+2}>\zeta_i(l\delta)\frac{k}{n}\geq\theta^{j-1}\}} \theta^{j\beta} \right)^2 \\
\leq\ & 2\theta^{2(j+1)\beta}(1-\theta^{-\beta})^2 \frac{1}{\theta^{j-1}} + 2\theta^{2j\beta} \left(\frac{1}{\theta^{j-1}} - \frac{1}{\theta^{j+2}} \right) \\
\leq\ & 2\theta^{j+1}(1-\theta^{-\frac{1}{2}})^2 \frac{1}{\theta^{j-1}} + 2\theta^{j} \left(\frac{1}{\theta^{j-1}} - \frac{1}{\theta^{j+2}} \right) \leq 3(K\varepsilon^3 + 3K\varepsilon^3) \leq \varepsilon^2.
\end{aligned}
$$

It is easy to see that the number of elements of the partition is bounded by $\exp(2/\varepsilon)$, which leads to (24). Hence by Theorem 3.2 we proved the asymptotic tightness condition.

It remains to prove that the finite-dimensional distributions of $\sum_{i=1}^{n}(\mathcal{Z}_{n,i} - E\mathcal{Z}_{n,i})$ converge weakly. This follows from the fact that multivariate weak convergence follows from weak convergence of linear combinations of the components and the (univariate) Lindeberg-Feller central limit theorem. It is easily seen that the Lindeberg condition is fulfilled for the linear combinations, since the $f_{t,x}$ are made up of indicators and hence bounded.

The fact that Z is bounded and uniformly d-continuous follows from the general theory of weak convergence and properties of Gaussian processes; see Section 1.5 in van der Vaart and Wellner (1996). □

Proof of Corollary 2.2 Write $V_{n,t} = \zeta_{n-k,n}(t)\frac{k}{n}$. We first show the result for $\alpha = -1$, i.e.

$$
\sup_{0\leq t\leq 1} \left| \sqrt{k} \left(\frac{1}{V_{n,t}} - 1 \right) + W(C_{t,1}) \right| \xrightarrow{P} 0. \tag{27}
$$

Clearly

$$
\sup_{0\leq t\leq 1} \left| \sqrt{k} \left(\frac{1}{V_{n,t}} - 1 \right) + w_n \left(t, \frac{1}{V_{n,t}} \right) \right| \xrightarrow{P} 0,
$$

so (12), with $\beta = 0$, yields

$$
\sup_{0\leq t\leq 1} \left| \sqrt{k} \left(\frac{1}{V_{n,t}} - 1 \right) + W \left(C_{t,\frac{1}{V_{n,t}}} \right) \right| \xrightarrow{P} 0.
$$

Now by the boundedness and uniform d-continuity of W, we obtain (27). Finally, write

$$\sqrt{k}(V_{n,t}^{\alpha}-1)=\sqrt{k}(V_{n,t}^{-1}-1)\frac{V_{n,t}^{\alpha}-1}{V_{n,t}^{-1}-1}.$$

Since by (27)

$$\sup_{0\le t\le 1}\left|\frac{V_{n,t}^{\alpha}-1}{V_{n,t}^{-1}-1}+\alpha\right|\xrightarrow{P}0,$$

we obtain, again using (27), (13). □

Proof of Theorem 2.3 First from (14) we can prove: for any $\varepsilon>0$ there exist $s_\varepsilon>0$ such that if $s>s_\varepsilon$ and $x\ge 1$ we have for all $0\le t\le 1$

(28)

$$\left|\left(\frac{\log U_t(sx)-\log U_t(s)}{a_t(s)/U_t(s)}-\frac{x^{\gamma^-(t)}-1}{\gamma^-(t)}\right)/A_t(s)-H_{\gamma^-(t),\rho(t)}(x)\right|\le\varepsilon(1+x^{\gamma^-(t)+\varepsilon});$$

the proof follows along the lines of that for the one-dimensional situation in de Haan and Stadtmüller (1996). Inequality (28) implies

$$\left|\frac{\log U_t(sx)-\log U_t(s)}{a_t(s)/U_t(s)}-\frac{x^{\gamma^-(t)}-1}{\gamma^-(t)}\right|\le|A_t(s)|(C_\varepsilon+x^\varepsilon). \tag{29}$$

where $C_\varepsilon\in(0,\infty)$ is a constant. Note that

$$M_n^{(1)}(t)=\frac{1}{k}\sum_{i=1}^{k-1}\log U_t(\zeta_{n-i,n}(t))-\log U_t(\zeta_{n-k,n}(t)).$$

Hence we have for sufficiently large n

$$\frac{1}{k}\sum_{i=0}^{k-1}\frac{\left(\frac{\zeta_{n-i,n}(t)}{\zeta_{n-k,n}(t)}\right)^{\gamma^-(t)}-1}{\gamma^-(t)}-|A_t(\zeta_{n-k,n}(t))|\left(C_\varepsilon+\frac{1}{k}\sum_{i=0}^{k-1}\left(\frac{\zeta_{n-i,n}(t)}{\zeta_{n-k,n}(t)}\right)^{\varepsilon}\right)$$

$$\le\quad\frac{M_n^{(1)}(t)}{a_t(\zeta_{n-k,n}(t))/U_t(\zeta_{n-k,n}(t))}$$

$$\le\quad\frac{1}{k}\sum_{i=0}^{k-1}\frac{\left(\frac{\zeta_{n-i,n}(t)}{\zeta_{n-k,n}(t)}\right)^{\gamma^-(t)}-1}{\gamma^-(t)}+|A_t(\zeta_{n-k,n}(t))|\left(C_\varepsilon+\frac{1}{k}\sum_{i=0}^{k-1}\left(\frac{\zeta_{n-i,n}(t)}{\zeta_{n-k,n}(t)}\right)^{\varepsilon}\right). \tag{30}$$

As before, write $V_{n,t} = \zeta_{n-k,n}(t)\frac{k}{n}$. Next

$$\sqrt{k}\left(\frac{1}{k}\sum_{i=0}^{k-1}\frac{\left(\frac{\zeta_{n-i,n}(t)}{\zeta_{n-k,n}(t)}\right)^{\gamma^-(t)}-1}{\gamma^-(t)} - \frac{1}{1-\gamma^-(t)}\right)$$

$$= \sqrt{k}\left(\int_{V_{n,t}}^{\infty}\frac{\left(\frac{x}{V_{n,t}}\right)^{\gamma^-(t)}-1}{\gamma^-(t)}d(-\frac{n}{k}S_{n,t}(x\frac{n}{k})) - \frac{1}{1-\gamma^-(t)}\right)$$

$$= \sqrt{k}\left(V_{n,t}^{-\gamma^-(t)}\int_{V_{n,t}}^{\infty}\frac{n}{k}S_{n,t}(x\frac{n}{k})x^{\gamma^-(t)-1}dx - \int_1^{\infty}x^{\gamma^-(t)-2}dx\right)$$

$$= V_{n,t}^{-\gamma^-(t)}\int_{V_{n,t}}^{\infty}w_n(t,x)x^{\gamma^-(t)-1}dx$$

$$+\sqrt{k}(V_{n,t}^{-\gamma^-(t)}-1)\int_{V_{n,t}}^{\infty}x^{\gamma^-(t)-2}dx + \sqrt{k}\int_{V_{n,t}}^{1}x^{\gamma^-(t)-2}dx$$

So

$$\sqrt{k}\left(\frac{1}{k}\sum_{i=0}^{k-1}\frac{\left(\frac{\zeta_{n-i,n}(t)}{\zeta_{n-k,n}(t)}\right)^{\gamma^-(t)}-1}{\gamma^-(t)} - \frac{1}{1-\gamma^-(t)}\right) - \mathcal{P}(t)$$

$$= V_{n,t}^{-\gamma^-(t)}\int_{V_{n,t}}^{\infty}(w_n(t,x) - W(C_{t,x}))x^{\gamma^-(t)-1}dx$$

$$+(V_{n,t}^{-\gamma^-(t)}-1)\int_{V_{n,t}}^{\infty}W(C_{t,x})x^{\gamma^-(t)-1}dx$$

$$+(\sqrt{k}(V_{n,t}^{-\gamma^-(t)}-1) + \gamma^-(t)W(C_{t,1}))\int_{V_{n,t}}^{\infty}x^{\gamma^-(t)-2}dx$$

$$+(\sqrt{k}\int_{V_{n,t}}^{1}x^{\gamma^-(t)-2}dx + W(C_{t,1}))$$

$$-\int_1^{V_{n,t}}W(C_{t,x})x^{\gamma^-(t)-1}dx + \gamma^-(t)W(C_{t,1})\int_1^{V_{n,t}}x^{\gamma^-(t)-2}dx. \tag{31}$$

From Theorem 2.1 we obtain for the first term on the right in (31)

$$\sup_{t\in[0,1]}V_{n,t}^{-\gamma^-(t)}\left|\int_{V_{n,t}}^{\infty}(w_n(t,x) - W(C_{t,x}))x^{\gamma^-(t)-1}dx\right|$$

$$\leq \sup_{t\in[0,1]}V_{n,t}^{-\gamma^-(t)}\cdot\sup_{t\in[0,1],\ x\geq V_{n,t}}x^{\beta}|w_n(t,x) - W(C_{t,x})|$$

$$\cdot\sup_{t\in[0,1]}\int_{V_{n,t}}^{\infty}y^{\gamma^-(t)-1-\beta}dy. \tag{32}$$

Now it follows from Theorem 2.1 with β *positive* (this is crucial) and Corollary 2.2, that the right hand side of (32) converges to 0 in probability. It readily follows from Corollary 2.2 that the 5 other terms in the right hand side of (31) converge to 0 in probability. So we have

(33)
$$\sup_{0\le t\le 1}\left|\sqrt{k}\left(\frac{1}{k}\sum_{i=0}^{k-1}\frac{\left(\frac{\zeta_{n-i,n}(t)}{\zeta_{n-k,n}(t)}\right)^{\gamma^-(t)}-1}{\gamma^-(t)}-\frac{1}{1-\gamma^-(t)}\right)-\mathcal{P}(t)\right|\xrightarrow{P}0,\ \text{as}\, n\to\infty.$$

For the remainder term of

$$\frac{M_n^{(1)}(t)}{a_t(\zeta_{n-k,n}(t))/U_t(\zeta_{n-k,n}(t))}$$

in (30), note that we obtain from Lemma 3.2 in de Haan and Lin (2002) that for $0\le\varepsilon<1$

$$\sup_{0\le t\le 1}\left|\frac{1}{k}\sum_{i=0}^{k-1}\left(\frac{\zeta_{n-i,n}(t)}{\zeta_{n-k,n}(t)}\right)^{\varepsilon}-\frac{1}{1-\varepsilon}\right|\xrightarrow{P}0,\ \text{as}\, n\to\infty. \tag{34}$$

It can be derived from the second order condition (14) and Corollary 2.2 that

$$\sup_{0\le t\le 1}\left|\frac{A_t\left(\frac{n}{k}\right)}{A_t\left(\zeta_{n-k,n}(t)\right)}-1\right|\xrightarrow{P}0.$$

Using this in combination with (15) and (34) we see that the remainder term in (30) is negligible, so we obtain that

$$\sup_{0\le t\le 1}\left|\sqrt{k}\left(\frac{M_n^{(1)}(t)}{a_t(\zeta_{n-k,n}(t))/U_t(\zeta_{n-k,n}(t))}-\frac{1}{1-\gamma^-(t)}\right)-\mathcal{P}(t)\right|\xrightarrow{P}0 \tag{35}$$

as $n\to\infty$. Similarly

(36)
$$\sup_{0\le t\le 1}\left|\sqrt{k}\left(\frac{M_n^{(2)}(t)}{(a_t(\zeta_{n-k,n}(t))/U_t(\zeta_{n-k,n}(t)))^2}-\frac{2}{(1-\gamma^-(t))(1-2\gamma^-(t))}\right)-\mathcal{Q}(t)\right|\xrightarrow{P}0$$

as $n\to\infty$. Hence we get

$$\sup_{0\le t\le 1}\left|\sqrt{k}\left(\hat{\gamma}_n^-(t)-\gamma^-(t)\right)-\mathcal{M}(t)\right|\xrightarrow{P}0 \tag{37}$$

as $n\to\infty$, where

$$\mathcal{M}(t)=-2(1-\gamma^-(t))^2(1-2\gamma^-(t))\mathcal{P}(t)+\frac{1}{2}(1-\gamma^-(t))^2(1-2\gamma^-(t))^2\mathcal{Q}(t).$$

Note

$$\sqrt{k}\left(\hat\gamma_n^+(t)-\gamma^+(t)\right)-\gamma^+(t)\mathcal{P}(t)$$
$$=\frac{a_t(\zeta_{n-k,n}(t))}{U_t(\zeta_{n-k,n}(t))}\left(\sqrt{k}\left(\frac{M_n^{(1)}(t)}{a_t(\zeta_{n-k,n}(t))/U_t(\zeta_{n-k,n}(t))}-\frac{1}{1-\gamma^-(t)}\right)-\mathcal{P}(t)\right)$$
$$+\sqrt{k}\left(\frac{a_t(\zeta_{n-k,n}(t))}{U_t(\zeta_{n-k,n}(t))}-\gamma^+(t)\right)\frac{1}{1-\gamma^-(t)}+\left(\frac{a_t(\zeta_{n-k,n}(t))}{U_t(\zeta_{n-k,n}(t))}-\gamma^+(t)\right)\mathcal{P}(t).$$

It follows from the second order condition (14) and Corollary 2.2 that

$$\sup_{0\le t\le 1}\left|\frac{a_t\left(\frac{n}{k}\right)/U_t\left(\frac{n}{k}\right)}{a_t(\zeta_{n-k,n}(t))/U_t(\zeta_{n-k,n}(t))}-1\right|\xrightarrow{P}0.$$

Combining this with (16), we get (17). Finally, we obtain (18) from (17) and (37).

For (19) note

$$\sqrt{k}\frac{\hat U_t\left(\frac{n}{k}\right)-U_t\left(\frac{n}{k}\right)}{a_t\left(\frac{n}{k}\right)}$$
$$=\sqrt{k}\frac{\log U_t(\zeta_{n-k,n}(t))-\log U_t\left(\frac{n}{k}\right)}{a_t\left(\frac{n}{k}\right)/U_t\left(\frac{n}{k}\right)}\left(\log\left(\frac{\xi_{n-k,n}(t)}{U_t\left(\frac{n}{k}\right)}\right)\right)^{-1}\left(\frac{\xi_{n-k,n}(t)}{U_t\left(\frac{n}{k}\right)}-1\right),$$

and

$$\frac{\xi_{n-k,n}(t)}{U_t\left(\frac{n}{k}\right)}-1=\frac{\xi_{n-k,n}(t)-U_t\left(\frac{n}{k}\right)}{a_t\left(\frac{n}{k}\right)}\frac{a_t\left(\frac{n}{k}\right)}{U_t\left(\frac{n}{k}\right)}.$$

From Lemma 3.4 in de Haan and Lin (2002) we obtain

$$\lim_{n\to\infty}\sup_{0\le t\le 1}\left|\frac{a_t\left(\frac{n}{k}\right)}{U_t\left(\frac{n}{k}\right)}-\gamma^+(t)\right|=0.$$

Combining this with (10) yields

$$\sup_{0\le t\le 1}\left|\frac{\xi_{n-k,n}(t)}{U_t\left(\frac{n}{k}\right)}-1\right|\xrightarrow{P}0,\ as\, n\to\infty.$$

Hence

$$\sup_{0\le t\le 1}\left|\left(\log\left(\frac{\xi_{n-k,n}(t)}{U_t\left(\frac{n}{k}\right)}\right)\right)^{-1}\left(\frac{\xi_{n-k,n}(t)}{U_t\left(\frac{n}{k}\right)}-1\right)-1\right|\xrightarrow{P}0,\ as\, n\to\infty.$$

A proof similar to the one leading to (35) shows

$$\sup_{0\le t\le 1}\left|\sqrt{k}\frac{\log U_t(\zeta_{n-k,n}(t))-\log U_t(\frac{n}{k})}{a_t(\frac{n}{k})/U_t(\frac{n}{k})}-\mathcal{U}(t)\right|\xrightarrow{P}0,\ as\, n\to\infty.$$

So we have obtained (19).

For (20) note

$$\sqrt{k}\left(\frac{\hat{a}_t\left(\frac{n}{k}\right)}{a_t\left(\frac{n}{k}\right)}-1\right) = \sqrt{k}\frac{\hat{U}_t(\frac{n}{k})-U_t(\frac{n}{k})}{a_t(\frac{n}{k})}\hat{\gamma}_n^+(t)(1-\hat{\gamma}_n^-(t))$$
$$+\sqrt{k}\left(\frac{\hat{\gamma}_n^+(t)}{a_t(\frac{n}{k})/U_t(\frac{n}{k})}-\frac{1}{1-\gamma^-(t)}\right)(1-\hat{\gamma}_n^-(t))-\sqrt{k}\frac{\hat{\gamma}_n^-(t)-\gamma^-(t)}{1-\gamma^-(t)}.$$

From (19), (17), (37) and Theorem 1.2 we get (20). □

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