

A characterization of the periods of periodic points of 1-norm nonexpansive maps

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1 Introduction

Pioneering research on the iterative behaviour of 1-norm nonexpansive maps was done by Akcoglu and Krengel in [1]. They observed that the asymptotic behaviour of the iterates of 1-norm nonexpansive maps is periodic. Indeed they proved the following theorem (compare [1]).

Theorem 1.1. *Let $X \subset \mathbb{R}^n$ be closed and let $f : X \rightarrow X$ be a 1-norm nonexpansive map. If there exists $x' \in X$ such that $(\|f^k(x')\|_1)_k$ is bounded, then for each $x \in X$ there exist an integer $p_x = p \geq 1$ and a point $\xi_x = \xi \in X$ such that*

- (i) *the sequence $(f^{kp}(x))_k$ converges to ξ ;*
- (ii) *the point ξ is a periodic point of f of minimal period p , that is $f^p(\xi) = \xi$ and $f^j(\xi) \neq \xi$ for $0 < j < p$.*

In later work Theorem 1.1 has been generalized by Weller [22] to norms on \mathbb{R}^n that have a polyhedral unit ball; examples of such norms are the 1-norm and the sup-norm. Furthermore Misiurewicz [10] showed for the 1-norm that the integer p_x in Theorem 1.1 is at most $n!2^n$ for all $x \in X$. Other upper bounds for the integer p_x for different polyhedral norms have been derived in: [2], [7], [8], [9], [11], and [21]. In particular, Nussbaum [11] has conjectured for the sup-norm that the optimal upper bound for the integer p_x is 2^n . At present this conjecture is proved solely for $n = 1, 2$, and 3 (see [8]).

These results motivate the following question: given a polyhedral norm on \mathbb{R}^n and a subset X of \mathbb{R}^n , can one determine the finite set of integers $p \geq 1$ for which there exist a nonexpansive map $f : X \rightarrow X$ and a periodic point of f of minimal period p , explicitly? In this paper this question is answered if $X = \mathbb{R}^n$ and the polyhedral norm is the 1-norm. It should be remarked that this case is different from the case where X can be an arbitrary subset of \mathbb{R}^n , because a 1-norm nonexpansive map $f : X \rightarrow X$, with $X \subset \mathbb{R}^n$, may not have a 1-norm nonexpansive extension to the whole of \mathbb{R}^n . Some results for the case where X can be an arbitrary subset of \mathbb{R}^n can be found in [7] and [10].

In [12], [15], [16], [17], and [19] one has examined the set $P^*(n)$, consisting of integers $p \geq 1$ for which there exist a 1-norm nonexpansive map $f : \mathbb{K}^n \rightarrow \mathbb{K}^n$, with $f(0) = 0$, and a periodic point of f of minimal period p . Here \mathbb{K}^n denotes the positive cone in \mathbb{R}^n . The main motivation for studying these maps is that they can be used as models for diffusion processes on a finite state space (see [1] and [14]). Surprisingly the set $P^*(n)$ allows a characterization by arithmetical and combinatorial constraints. Indeed Nussbaum, Scheutzow, and Verduyn Lunel showed in [15] and [16] that $P^*(n)$ is precisely the set of periods of admissible arrays on n symbols. Here an admissible array is defined as follows.

Definition 1.1. Let $(L, <)$ be a finite totally ordered set and let $\Sigma = \{1, \dots, n\}$. For each $\lambda \in L$ let $\vartheta_\lambda : \mathbb{Z} \rightarrow \Sigma$ be a map. The sequence $\vartheta = (\vartheta_\lambda : \mathbb{Z} \rightarrow \Sigma \mid \lambda \in L)$ is called an *admissible array on n symbols* if the maps ϑ_λ satisfy

- (i) for each $\lambda \in L$ there exists an integer p_λ with $1 \leq p_\lambda \leq n$ such that the map $\vartheta_\lambda : \mathbb{Z} \rightarrow \Sigma$ is periodic with period p_λ , and moreover $\vartheta_\lambda(s) \neq \vartheta_\lambda(t)$ for all $1 \leq s < t \leq p_\lambda$,
- (ii) if $\lambda_1 < \lambda_2 < \dots < \lambda_{r+1}$ is a sequence of distinct points in L and

$$\vartheta_{\lambda_i}(s_i) = \vartheta_{\lambda_{i+1}}(t_i) \quad \text{for } 1 \leq i \leq r,$$

then

$$\sum_{i=1}^r (t_i - s_i) \not\equiv 0 \pmod{\rho}, \quad \text{where } \rho = \gcd(\{p_{\lambda_i} : 1 \leq i \leq r+1\}).$$

Here $\gcd(S)$ denotes the greatest common divisor of the elements of S . The *period* of an admissible array is said to be $\text{lcm}(\{p_\lambda : \lambda \in L\})$, that is the least common multiple of the periods p_λ . Thus, if one defines for $n \geq 1$ the set

$$Q(n) = \{p \geq 1 : p \text{ is the period of an admissible array on } n \text{ symbols}\}, \quad (1)$$

then the characterization of the set $P^*(n)$ is given by the following equality.

Theorem 1.2 ([16], Theorem 3.1). $P^*(n) = Q(n)$ for all $n \geq 1$.

In [16, Section 4] Nussbaum, Scheutzow, and Verduyn Lunel have asked if a similar characterization can be found for the set $R(n)$, consisting of integers $p \geq 1$ for which there exist a 1-norm nonexpansive map $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ and a periodic point of f of minimal period p . It was proved in [13] and [20] that $R(n) \subset P^*(2n)$ for $n \geq 1$, so that Theorem 1.2 yields the inclusion $R(n) \subset Q(2n)$. A sharpening of this inclusion was obtained by Lemmens in [6]. Indeed it was shown there that each element of $R(n)$ is the period of an admissible array on $2n$ symbols that has some additional properties. As a result of this sharpening the set $R(n)$ has been computed for $n = 1, 2, 3, 4, 6, 7$, and 10. It remained however, an open problem

to decide if the upper bound for $R(n)$ was tight. In particular, it was not known if 18 is in $R(5)$, 90 is in $R(8)$, and 126 is in $R(9)$. In this paper the notion of a *restricted admissible array on $2n$ symbols* is introduced, and it is shown that $R(n)$ is precisely the set of possible periods of these arrays. The restricted admissible arrays are admissible arrays with one additional property. This additional property refines the properties that were obtained in [6]. As a consequence we derive that $18 \notin R(5)$, $90 \notin R(8)$, and $126 \notin R(9)$.

A combination of Theorem 1.2 with the following lemma suggests that an additional property for admissible arrays on $2n$ symbols can be found, such that their periods characterize the set $R(n)$.

Lemma 1.1 ([20]). *Let $\mathbb{E}^{2n} = \{(x, y) \in \mathbb{K}^n \times \mathbb{K}^n : x_i y_i = 0 \text{ for } 1 \leq i \leq n\}$. An integer $p \geq 1$ is an element of $R(n)$ if and only if there exist a 1-norm nonexpansive map $f : \mathbb{K}^{2n} \rightarrow \mathbb{K}^{2n}$, with $f(0) = 0$, and a periodic point $\xi \in \mathbb{K}^{2n}$ of f of minimal period p such that $f^j(\xi) \in \mathbb{E}^{2n}$ for all $j \geq 0$.*

Our proof of the characterization of $R(n)$ relies on this lemma and therefore we give a proof of it in the appendix. In the next section some basic definitions are collected, and a precise formulation of the characterization of $R(n)$ is given.

2 The characterization of $R(n)$

On \mathbb{R}^n the 1-norm is given by $\|x\|_1 = \sum_i |x_i|$ for $x = (x_1, \dots, x_n)$ in \mathbb{R}^n . A map $f : X \rightarrow \mathbb{R}^n$, where $X \subset \mathbb{R}^n$, is called *1-norm nonexpansive* or simply *1-nonexpansive* if

$$\|f(x) - f(y)\|_1 \leq \|x - y\|_1 \quad \text{for all } x, y \in X. \quad (2)$$

The map f is said to be a *1-isometry* if equality holds in (2) for all x and y in X . A point $x \in X$ is called a *periodic point* of $f : X \rightarrow X$ if there exists an integer $p \geq 1$ such that $f^p(x) = x$, and p is called a *period* of x . The smallest such integer $p \geq 1$ is said to be the *minimal period* of x .

On \mathbb{R}^n a partial ordering \leq is given by $x \leq y$ if $x_i \leq y_i$ for $1 \leq i \leq n$. We say that x and y are *comparable* if $x \leq y$ or $y \leq x$. We write $x < y$ if $x \leq y$ and $x \neq y$. The *positive cone* in \mathbb{R}^n is said to be $\mathbb{K}^n = \{x \in \mathbb{R}^n : x \geq 0\}$. Further for $x, y \in \mathbb{R}^n$ we let $x \wedge y$ be the vector with coordinates $(x \wedge y)_i = \min\{x_i, y_i\}$ for $1 \leq i \leq n$. Similarly, $x \vee y$ denotes the vector in \mathbb{R}^n with coordinates $(x \vee y)_i = \max\{x_i, y_i\}$ for $1 \leq i \leq n$. A map $f : X \rightarrow \mathbb{R}^n$, with $X \subset \mathbb{R}^n$, is called *order-preserving* if $x, y \in X$ and $x \leq y$ implies $f(x) \leq f(y)$. The map $f : X \rightarrow \mathbb{R}^n$ is said to be *integral-preserving* if $\sum_i f(x)_i = \sum_i x_i$ for all $x \in X$.

To formulate the characterization of $R(n)$ we need the idea of a restricted admissible array on $2n$ symbols. Before this idea is explained it is convenient to introduce the following notion.

Definition 2.1. Let $\vartheta = (\vartheta_\lambda : \mathbb{Z} \rightarrow \Sigma \mid \lambda \in L)$ be an admissible array and let p_λ denote the period of ϑ_λ for $\lambda \in L$. A symbol $a \in \Sigma$ is called *permitted for $q \in \mathbb{Z}$* if

- (i) $a = \vartheta_\lambda(q)$ for some $\lambda \in L$, or
- (ii) there exist distinct $\lambda_1 < \lambda_2 < \dots < \lambda_{r+1}$ in L such that

$$\vartheta_{\lambda_i}(s_i) = \vartheta_{\lambda_{i+1}}(t_i) \quad \text{for } 1 \leq i \leq r,$$

and there exists $\delta \in \mathbb{Z}$ such that $a = \vartheta_{\lambda_1}(q - \delta)$ and

$$\sum_{i=1}^r (t_i - s_i) \equiv \delta \pmod{\rho}, \quad \text{where } \rho = \gcd(\{p_{\lambda_i} : 1 \leq i \leq r+1\}).$$

For an admissible array ϑ and $q \in \mathbb{Z}$ the set of permitted symbols is denoted by

$$\mathcal{P}(q, \vartheta) = \{a \in \Sigma : a \text{ is permitted for } q\}. \quad (3)$$

If a is a symbol in Σ and $\Sigma = \{1, \dots, 2n\}$, then we write $a^+ = a + n \pmod{2n}$. A restricted admissible array on $2n$ symbols is now defined as follows.

Definition 2.2. An admissible array $\vartheta = (\vartheta_\lambda : \mathbb{Z} \rightarrow \Sigma \mid \lambda \in L)$ on $2n$ symbols, where $\Sigma = \{1, \dots, 2n\}$, is called a *restricted admissible array on $2n$ symbols* if

$$\{a, a^+\} \not\subset \mathcal{P}(q, \vartheta) \quad \text{for all } a \in \Sigma \text{ and } q \in \mathbb{Z}.$$

Remark that if ϑ is an admissible array and ϑ has period p , then $\mathcal{P}(q, \vartheta) = \mathcal{P}(q + p, \vartheta)$ for each $q \in \mathbb{Z}$. Therefore one can decide in finite time if an admissible array on $2n$ symbols is a restricted admissible array.

For $n \geq 1$ we set

$$Q'(2n) = \{p \geq 1 : p \text{ is the period of a restricted admissible array on } 2n \text{ symbols}\}. \quad (4)$$

Then the characterization of $R(n)$ is given by the following equality.

Theorem 2.1. $R(n) = Q'(2n)$ for all $n \geq 1$.

The proof of Theorem 2.1 is split in two parts. First it is shown in Section 3 that $R(n) \subset Q'(2n)$. To prove this inclusion we build on results from [15] and [19]. Subsequently in Section 4 the other inclusion, $R(n) \supset Q'(2n)$, is proved. To establish this inclusion ideas from [16] are used. In Section 5 some remarks are made about the computation of $R(n)$ for small n , and analysis of the largest element of $R(n)$ is given. We conclude with an appendix in which a proof of Lemma 1.1 is given, and a list of elements of $R(n)$ for $1 \leq n \leq 10$ is displayed.

3 The left inclusion: $\mathbf{R}(n) \subset \mathbf{Q}'(2n)$

If $f : \mathbb{K}^n \rightarrow \mathbb{K}^n$ is a 1-nonexpansive map, with $f(0) = 0$, then Nussbaum and Scheutzwow have constructed in [15] for each periodic point of f of minimal period p , an admissible array on n symbols with period p . In this section we exploit this construction to prove the inclusion $\mathbf{R}(n) \subset \mathbf{Q}'(2n)$. More precisely, it is shown that if $f : \mathbb{K}^{2n} \rightarrow \mathbb{K}^{2n}$ is 1-nonexpansive, with $f(0) = 0$, and ξ is a periodic point of f of minimal period p , with $f^j(\xi) \in \mathbb{E}^{2n} = \{(x, y) \in \mathbb{K}^n \times \mathbb{K}^n : x_i y_i = 0 \text{ for } 1 \leq i \leq n\}$ for all $j \geq 0$, then the associated admissible array is a restricted admissible array on $2n$ symbols with period p . A combination of this result with Lemma 1.1 then yields the inclusion $\mathbf{R}(n) \subset \mathbf{Q}'(2n)$.

So, let us explain the construction of the admissible arrays. The first step is to relate to each periodic point of a 1-nonexpansive map $f : \mathbb{K}^n \rightarrow \mathbb{K}^n$, with $f(0) = 0$, a so-called lower semilattice homomorphism. A set $V \subset \mathbb{R}^n$ is called a *lower semilattice* if $x \wedge y \in V$ for all x and y in V . A map $g : V \rightarrow V$, where V is a lower semilattice, is called a *lower semilattice homomorphism* if $g(x \wedge y) = g(x) \wedge g(y)$ for all $x, y \in V$. If S is a subset of \mathbb{R}^n , then V_S denotes the smallest (in the sense of inclusion) lower semilattice that contains S , and V_S is said to be the *lower semilattice generated by S* . The connection between these notions and 1-nonexpansive maps is given in the following lemma (compare [19, Lemma 3.2]).

Lemma 3.1. *Let $f : \mathbb{K}^n \rightarrow \mathbb{K}^n$ be a 1-nonexpansive map, with $f(0) = 0$, and $\xi \in \mathbb{K}^n$ be a periodic point of f of minimal period p . If $V \subset \mathbb{K}^n$ is the lower semilattice generated by $\{f^j(\xi) : 0 \leq j < p\}$, then the restriction of f to V is a lower semilattice homomorphism that maps V onto itself.*

The next step is to construct for each periodic point of a lower semilattice homomorphism $g : V \rightarrow V$, where $V \subset \mathbb{R}^n$, an admissible array on n symbols. To do this it is useful to introduce the following notions. Let V be a finite lower semilattice on \mathbb{R}^n . If $A \subset V$ and there exists $\beta \in V$ such that $\alpha \leq \beta$ for all $\alpha \in A$, then A is said to be *bounded above in V* , and β is called an *upper bound of A in V* . Likewise, we say that A is *bounded below in V* if there exist $\beta \in V$ with $\alpha \geq \beta$ for all $\alpha \in A$, and β is called a *lower bound of A in V* . If A is bounded above in V , then there exists a unique upper bound α of A in V such that $\gamma < \alpha$ implies that γ is not an upper bound of A in V . The point α is called the *supremum of A in V* , and it is denoted by $\sup_V(A)$. Analogously, the *infimum of A in V* is said to be the unique lower bound α of A in V , so that there exists no lower bound β of A in V with $\alpha < \beta$. This element is denoted by $\inf_V(A)$.

For $x \in V$ the *height* is defined by

$$h_V(x) = \sup\{k \geq 0 : \text{there exist } y^0, \dots, y^k \in V \text{ such that } y^k = x \text{ and } y^j < y^{j+1} \text{ for } 0 \leq j < k\}. \quad (5)$$

In particular, we put $h_V(x) = 0$ if no $y \in V$ exists with $y < x$. For every $x \in V$ we let $S_x = \{y \in V : y < x\}$. An element $x \in V$ is called *irreducible* in V if either

S_x is empty or

$$x > \sup_V(S_x). \quad (6)$$

If $x \in V$ is irreducible in V and S_x is nonempty, then we define

$$I_V(x) = \{i : x_i > \sup_V(S_x)_i\}. \quad (7)$$

We put $I_V(x) = \{1, \dots, n\}$ if S_x is empty. Observe that if x is an irreducible element in a finite lower semilattice $V \subset \mathbb{R}^n$, and S_x is nonempty, then

$$x_i > 0 \quad \text{for all } i \in I_V(x), \quad (8)$$

since $x_i > \sup_V(S_x)_i \geq 0$ for all $i \in I_V(x)$. This inequality will be useful to us later. Using these notions the following lemma can be proved (see [15, Lemma 1.1]).

Lemma 3.2. *Suppose that $j \in \mathbb{Z}$, V is a finite lower semilattice in \mathbb{R}^n , and $f : V \rightarrow V$ is a bijective lower semilattice homomorphism.*

- (i) *If $y \in V$ and $f^j(y) \neq y$, then y and $f^j(y)$ are not comparable, and moreover $h_V(y) = h_V(f^j(y))$.*
- (ii) *If y is irreducible in V , then $f^j(y)$ is irreducible in V .*
- (iii) *If y and y' are two irreducible elements in V that are not comparable, then $I_V(y) \cap I_V(y') = \emptyset$.*
- (iv) *If $y \in V$ is irreducible in V and y is a periodic point of f of minimal period p , then $1 \leq p \leq n$.*

The following technical definition forms the basis from which the admissible arrays are constructed.

Definition 3.1. Let W be a lower semilattice in \mathbb{R}^n , let $g : W \rightarrow W$ be a lower semilattice homomorphism, and let $\xi \in W$ be a periodic point of g of minimal period p . Let V denote the lower semilattice generated by $\{g^j(\xi) : j \geq 0\}$ and let f be the restriction of g to V . A finite sequence $(y^i)_{i=1}^m \subset V$ is called a *complete sequence for ξ* , if the elements satisfy

- (i) $y^i \leq \xi$ for $1 \leq i \leq m$,
- (ii) y^i is irreducible in V for $1 \leq i \leq m$,
- (iii) if p_i is the minimal period of y^i under f , then $p = \text{lcm}(\{p_i : 1 \leq i \leq m\})$,
- (iv) $h_V(y^i) \leq h_V(y^{i+1})$ for $1 \leq i < m$,
- (v) the sets $\{f^k(y^i) : k \geq 0\}$ and $\{f^k(y^j) : k \geq 0\}$ are disjoint for $1 \leq i < j \leq m$,
- (vi) y^i and y^j are not comparable for $1 \leq i < j \leq m$.

The following proposition ensures that every periodic point of a lower semilattice homomorphism has a complete sequence (compare [15, Proposition 1.1]).

Proposition 3.1. *If W is a lower semilattice in \mathbb{R}^n , $g : W \rightarrow W$ is a lower semilattice homomorphism, and $\xi \in W$ is a periodic point of g , then there exists a complete sequence for ξ .*

From a complete sequence an admissible array can be constructed in the following manner. Suppose that $g : W \rightarrow W$ is a lower semilattice homomorphism, where W is a lower semilattice in \mathbb{R}^n . Assume that $\xi \in W$ is a periodic point of g of minimal period p . Let V be the lower semilattice generated by $\{g^j(\xi) : j \geq 0\}$, and let f be the restriction of g to V . Remark that f is a bijective lower semilattice homomorphism that maps the finite lower semilattice V onto itself, and its inverse is f^{p-1} . By Proposition 3.1 there exists a complete sequence $(y^i)_{i=1}^m$ for ξ . Let p_i denote the minimal period of y^i under f for $1 \leq i \leq m$. It follows from property (ii) in Definition 3.1 and Lemma 3.2 that $f^j(y^i)$ is irreducible in V for $1 \leq i \leq m$ and $j \in \mathbb{Z}$. Therefore the set $I_V(f^j(y^i))$ is not empty, and hence we can select for $1 \leq i \leq m$ and $1 \leq j \leq p_i$ an integer $a_{ij} \in I_V(f^j(y^i))$. Further we define for general $j \in \mathbb{Z}$ the integer a_{ij} by

$$a_{ij} = a_{ik}, \quad \text{where } 1 \leq k \leq p_i \text{ and } j \equiv k \pmod{p_i}.$$

Observe that $a_{ij} \in I_V(f^j(y^i))$ for all $1 \leq i \leq m$ and $j \in \mathbb{Z}$, because y^i has period p_i under f . The resulting semi-infinite matrix (a_{ij}) , where $1 \leq i \leq m$ and $j \in \mathbb{Z}$, is called an *array of ξ* . There exists the following connection with the admissible arrays (compare [15, Proposition 1.2]).

Proposition 3.2. *Let W be a lower semilattice in \mathbb{R}^n , $g : W \rightarrow W$ be a lower semilattice homomorphism, and $\xi \in W$ be a periodic point of g of minimal period p . Let (a_{ij}) , where $1 \leq i \leq m$ and $j \in \mathbb{Z}$, be an array of ξ . Put $\Sigma = \{1, \dots, n\}$ and let $L = \{1, \dots, m\}$ be equipped with the usual ordering. If $\vartheta = (\vartheta_\lambda : \mathbb{Z} \rightarrow \Sigma \mid \lambda \in L)$ is defined by*

$$\vartheta_\lambda(j) = a_{\lambda j} \quad \text{for } \lambda \in L \text{ and } j \in \mathbb{Z}, \quad (9)$$

then ϑ is an admissible array on n symbols with period p .

We now consider arrays of periodic points of lower semilattice homomorphisms $g : W \rightarrow W$, where $W \subset \mathbb{E}^{2n}$, and show that they give rise to restricted admissible arrays on $2n$ symbols.

Proposition 3.3. *Let W be a lower semilattice in \mathbb{E}^{2n} , $g : W \rightarrow W$ be a lower semilattice homomorphism, and $\xi \in W$ be a periodic point of g of minimal period p . Let (a_{ij}) , where $1 \leq i \leq m$ and $j \in \mathbb{Z}$, be an array of ξ . Put $\Sigma = \{1, \dots, 2n\}$ and let $L = \{1, \dots, m\}$ be equipped with the usual ordering. If $\vartheta = (\vartheta_\lambda : \mathbb{Z} \rightarrow \Sigma \mid \lambda \in L)$ is defined by*

$$\vartheta_\lambda(j) = a_{\lambda j} \quad \text{for } \lambda \in L \text{ and } j \in \mathbb{Z}, \quad (10)$$

then ϑ is a restricted admissible array on $2n$ symbols with period p .

Proof. Let $g : W \rightarrow W$ be a lower semilattice homomorphism, where W is a lower semilattice in \mathbb{E}^{2n} . Suppose that $\xi \in W$ is a periodic point of g of minimal period p . Let V be the lower semilattice generated by $\{g^j(\xi) : j \geq 0\}$ and let f be the restriction of g to V . Assume that (a_{ij}) , where $1 \leq i \leq m$ and $j \in \mathbb{Z}$, is an array of ξ , and let $(y^i)_{i=1}^m$ be a complete sequence for ξ that induces this array. Further let p_i denote the minimal period of y^i under f for $1 \leq i \leq m$. Now set $\Sigma = \{1, \dots, 2n\}$ and let $L = \{1, \dots, m\}$ be equipped with the usual ordering. Define $\vartheta = (\vartheta_\lambda : \mathbb{Z} \rightarrow \Sigma \mid \lambda \in L)$ by (10).

It follows from Proposition 3.2 that ϑ is an admissible array on $2n$ symbols with period p . Therefore it remains to be shown that

$$\{a, a^+\} \not\subset \mathcal{P}(q, \vartheta) \quad \text{for all } a \in \Sigma \text{ and } q \in \mathbb{Z}. \quad (11)$$

There are two cases: $|L| = 1$ and $|L| > 1$. If $|L| = 1$, then it follows from Definition 2.1 that $|\mathcal{P}(q, \vartheta)| = 1$, and hence (11) is true for this case.

To obtain (11) for the second case we show the following claim.

Claim. If $|L| > 1$ and $a \in \mathcal{P}(q, \vartheta)$, then there exists a point $y \in V$ such that $y_a > 0$ and $y \leq f^q(\xi)$.

It is sufficient to prove this claim, since $\{a, a^+\} \subset \mathcal{P}(q, \vartheta)$ implies that there exist elements y and y' in V , with $y_a > 0$ and $y'_{a^+} > 0$, such that $y \leq f^q(\xi)$ and $y' \leq f^q(\xi)$. This of course contradicts the fact that $f^q(\xi) \in \mathbb{E}^{2n}$.

So suppose that $|L| > 1$ and $a \in \mathcal{P}(q, \vartheta)$. As $|L| > 1$ it follows from property (vi) in Definition 3.1 that $y^\lambda > \inf_V(V)$ for all $\lambda \in L$. This implies that $S_{f^j(y^\lambda)}$ is not empty for $\lambda \in L$ and $j \in \mathbb{Z}$, because f is order-preserving and y^λ is periodic. Therefore we know by (ii) in Lemma 3.2 and (8) that for all $\lambda \in L$ and $j \in \mathbb{Z}$:

$$f^j(y^\lambda)_i > 0 \quad \text{for all } i \in I_V(f^j(y^\lambda)). \quad (12)$$

According to Definition 2.1 there are two possibilities. We begin with the first one: $a = \vartheta_\lambda(q)$ for some $\lambda \in L$. By construction $a = \vartheta_\lambda(q) = a_{\lambda q} \in I_V(f^q(y^\lambda))$. Hence by (12) we find that $f^q(y^\lambda)_a > 0$. Moreover property (i) in Definition 3.1 says that $y^\lambda \leq \xi$. As f is a lower semilattice homomorphism it is order-preserving, so that $f^q(y^\lambda) \leq f^q(\xi)$. This completes the proof of the claim for this case.

To prove the claim for the second case we assume that there exist distinct $\lambda_1 < \lambda_2 < \dots < \lambda_{r+1}$ in L such that

$$\vartheta_{\lambda_i}(s_i) = \vartheta_{\lambda_{i+1}}(t_i) \quad \text{for } 1 \leq i \leq r,$$

and there exists $\delta \in \mathbb{Z}$ such that $a = \vartheta_{\lambda_1}(q - \delta)$ and

$$\sum_{i=1}^r (t_i - s_i) \equiv \delta \pmod{\rho}, \quad \text{where } \rho = \gcd(\{p_{\lambda_i} : 1 \leq i \leq r+1\}).$$

Observe that there exist constants A_1, A_2, \dots, A_{r+1} such that $\rho = \sum_i A_i p_{\lambda_i}$. As $\sum_{i=1}^r (t_i - s_i) \equiv \delta \pmod{\rho}$ we can find constants B_1, B_2, \dots, B_{r+1} such that

$$\sum_{i=1}^r (t_i - s_i) - \sum_{i=1}^{r+1} B_i p_{\lambda_i} = \delta. \quad (13)$$

Since $\lambda_i < \lambda_{i+1}$ we know by (iv) in Definition 3.1 that $h_V(y^{\lambda_i}) \leq h_V(y^{\lambda_{i+1}})$, so that (i) in Lemma 3.2 implies:

$$h_V(f^{s_i}(y^{\lambda_i})) \leq h_V(f^{t_i}(y^{\lambda_{i+1}})) \quad \text{for } 1 \leq i \leq r. \quad (14)$$

By construction we have that

$$\vartheta_{\lambda_i}(s_i) = a_{\lambda_i, s_i} \in I_V(f^{s_i}(y^{\lambda_i})) \quad \text{and} \quad \vartheta_{\lambda_{i+1}}(t_i) = a_{\lambda_{i+1}, t_i} \in I_V(f^{t_i}(y^{\lambda_{i+1}})).$$

Therefore the equality $\vartheta_{\lambda_i}(s_i) = \vartheta_{\lambda_{i+1}}(t_i)$ and (iii) in Lemma 3.2 imply that $f^{s_i}(y^{\lambda_i})$ and $f^{t_i}(y^{\lambda_{i+1}})$ are comparable, so that (14) yields:

$$f^{s_i}(y^{\lambda_i}) \leq f^{t_i}(y^{\lambda_{i+1}}) \quad \text{for } 1 \leq i \leq r. \quad (15)$$

As f is order-preserving and y^λ has period p_λ under f , we can deduce from (15) that

$$y^{\lambda_i} \leq f^{t_i - s_i - B_i p_{\lambda_i}}(y^{\lambda_{i+1}}) \quad \text{for } 1 \leq i \leq r. \quad (16)$$

Applying (16) iteratively gives:

$$y^{\lambda_1} \leq f^\nu(y^{\lambda_{r+1}}) \quad \text{where } \nu = \sum_{i=1}^r (t_i - s_i) - \sum_{i=1}^r B_i p_{\lambda_i}. \quad (17)$$

Now set $\mu = -B_{r+1} p_{\lambda_{r+1}}$. As $f^\mu(y^{\lambda_{r+1}}) = y^{\lambda_{r+1}}$ it follows from (13) and (17) that

$$y^{\lambda_1} \leq f^{\nu + \mu}(y^{\lambda_{r+1}}) = f^\delta(y^{\lambda_{r+1}}). \quad (18)$$

Finally we use property (i) of Definition 3.1 and the fact that f is order-preserving to deduce from (18) that

$$f^{q-\delta}(y^{\lambda_1}) \leq f^q(y^{\lambda_{r+1}}) \leq f^q(\xi).$$

Moreover it follows from (12) that $f^{q-\delta}(y^{\lambda_1})_a > 0$, because

$$a = \vartheta_{\lambda_1}(q - \delta) = a_{\lambda_1, q-\delta} \in I_V(f^{q-\delta}(y^{\lambda_1})).$$

This shows the claim for the second case, and hence the proof of the proposition is complete. \square

The results from this section yield the following inclusion.

Corollary 3.1. $R(n) \subset Q'(2n)$ for all $n \geq 1$.

Proof. Let p be an element of $R(n)$. By Lemma 1.1 there exist a 1-nonexpansive map $f : \mathbb{K}^{2n} \rightarrow \mathbb{K}^{2n}$, with $f(0) = 0$, and a periodic point ξ of f of minimal period p , such that $f^j(\xi) \in \mathbb{E}^{2n}$ for all $j \geq 0$. Let V be the lower semilattice generated by $\{f^j(\xi) : j \geq 0\}$, and let g be the restriction of f to V . It follows from Lemma 3.1 that g is a lower semilattice homomorphism that maps V onto itself. Further observe that V is a lower semilattice in \mathbb{E}^{2n} , since \mathbb{E}^{2n} is a lower semilattice. Therefore we can apply Propositions 3.1 and 3.3 to conclude that p is the period of a restricted admissible array on $2n$ symbols. \square

In the next section we discuss the other inclusion $R(n) \supset Q'(2n)$.

4 The right inclusion: $R(n) \supset Q'(2n)$

To prove the equality $P^*(n) = Q(n)$ Nussbaum, Scheutzow, and Verduyn Lunel have constructed in [16] for each admissible array ϑ on n symbols with period p , a 1-nonexpansive map $f_\vartheta : \mathbb{K}^n \rightarrow \mathbb{K}^n$, with $f_\vartheta(0) = 0$, and a periodic point $\xi \in \mathbb{K}^n$ of f_ϑ of minimal period p . With the results from the previous section in mind it is interesting to know if for a restricted admissible array on $2n$ symbols, this construction yields a periodic point $\xi \in \mathbb{K}^{2n}$ such that $f_\vartheta^j(\xi) \in \mathbb{E}^{2n}$ for all $j \geq 0$. If this is the case, then Lemma 1.1 implies the inclusion $R(n) \supset Q'(2n)$. In this section we prove the inclusion in this manner. To explain our arguments it is good to recall some of the results from [16].

The 1-nonexpansive maps $f_\vartheta : \mathbb{K}^n \rightarrow \mathbb{K}^n$ that appear in the construction in [16] are so-called *sand-shift maps*, which were introduced by Nussbaum in [14, Example 2]. These maps describe certain diffusion processes on a finite state space, and they can be conveniently introduced in the following way. Consider n containers C_1, \dots, C_n each with an amount of sand x_i , and put $x = (x_1, \dots, x_n)$. Suppose that to each container C_i a sequence of buckets b_{i1}, b_{i2}, \dots is associated. Let v_{im} denote the volume of bucket b_{im} , and assume that

$$\sum_{m=1}^{\infty} v_{im} = \infty \quad \text{for } 1 \leq i \leq n.$$

Now start the following procedure to pour sand from the containers into the buckets. For each container C_i pour sand into bucket b_{i1} until either b_{i1} is full or C_i is empty. If b_{i1} is full, then pour the remaining sand of C_i into bucket b_{i2} until either b_{i2} is full or C_i is empty. Continue in this manner until C_i is empty. Observe that the amount of sand in bucket b_{ik} after this procedure is given by

$$M_{ik}(x) = \min\{v_{ik}, \max\{x_i - \sum_{m=1}^{k-1} v_{im}, 0\}\}. \quad (19)$$

We use a *rule* $\gamma : \{1, \dots, n\} \times \mathbb{N} \rightarrow \{1, \dots, n\}$ to redistribute the sand among the containers. The contents of each bucket b_{ik} is poured into container $C_{\gamma(i,k)}$. The new distribution of sand in the containers is given by $y = (y_1, \dots, y_n)$, where

$$y_j = \sum_{\gamma(i,k)=j} M_{ik}(x) \quad \text{for } 1 \leq j \leq n.$$

This diffusion process is described by a map $f : \mathbb{K}^n \rightarrow \mathbb{K}^n$ that is given by

$$f(x)_j = \sum_{\gamma(i,k)=j} M_{ik}(x) \quad \text{for } x \in \mathbb{K}^n \text{ and } 1 \leq j \leq n. \quad (20)$$

Observe that $f(0) = 0$, and that f only depends on the volume of the buckets and the rule γ . To see that the sand-shift maps defined by (20) are 1-nonexpansive one can use a result of Crandall and Tartar [3] that says: If $X \subset \mathbb{R}^n$ is such that $x \vee y \in X$ for all $x, y \in X$ and $f : X \rightarrow \mathbb{R}^n$ is integral-preserving, then f is 1-nonexpansive if and only if f is order-preserving. Next we associate with each admissible array ϑ on n symbols a sand-shift map $f_\vartheta : \mathbb{K}^n \rightarrow \mathbb{K}^n$.

Suppose that $\vartheta = (\vartheta_\lambda : \mathbb{Z} \rightarrow \Sigma \mid \lambda \in L)$ is an admissible array on n symbols with period p . Let $R(\vartheta_\lambda) = \{\vartheta_\lambda(q) : q \in \mathbb{Z}\}$ denote the range of ϑ_λ . For each $a \in \Sigma$ let $\Lambda_a = \{\lambda \in L : a \in R(\vartheta_\lambda)\}$ and put $\rho(a) = |\Lambda_a|$. If $\rho(a) > 0$ we label the elements of Λ_a by $\lambda_1(a), \lambda_2(a), \dots, \lambda_{\rho(a)}(a)$ such that $\lambda_i(a) < \lambda_{i+1}(a)$ for $1 \leq i < \rho(a)$.

Now $f_\vartheta : \mathbb{K}^n \rightarrow \mathbb{K}^n$ is defined as the sand-shift map, where $v_{im} = 1$ for $1 \leq i \leq n$ and $m \geq 1$, and the rule $\gamma : \{1, \dots, n\} \times \mathbb{N} \rightarrow \{1, \dots, n\}$ is given by:

(a) if $1 \leq i \leq n$ and $1 \leq k \leq \rho(i)$, then

$$\gamma(i, k) = \vartheta_{\lambda_k(i)}(s + 1), \quad \text{where } i = \vartheta_{\lambda_k(i)}(s), \quad (21)$$

(b) if $1 \leq i \leq n$ and $k > \rho(i)$, then

$$\gamma(i, k) = i. \quad (22)$$

Observe that this rule γ is well-defined, as the maps ϑ_λ are periodic with period p_λ and $\vartheta_\lambda(s) \neq \vartheta_\lambda(t)$ for $1 \leq s < t \leq p_\lambda$. Thus, $f_\vartheta : \mathbb{K}^n \rightarrow \mathbb{K}^n$ is defined by

$$f_\vartheta(x)_j = \sum_{\gamma(i,k)=j} M_{ik}(x) \quad \text{for } x \in \mathbb{K}^n \text{ and } 1 \leq j \leq n, \quad (23)$$

where γ is given by (21) and (22), and

$$M_{ik}(x) = \min\{1, \max\{x_i - (k - 1), 0\}\}. \quad (24)$$

The next step is to produce for $f_\vartheta : \mathbb{K}^n \rightarrow \mathbb{K}^n$ a periodic point ξ of minimal period p . To do this it is convenient to introduce some auxiliary numbers $\xi_{a,\lambda}^q$.

Definition 4.1. Let $\vartheta = (\vartheta_\lambda : \mathbb{Z} \rightarrow \Sigma \mid \lambda \in L)$ be an admissible array on n symbols. For $q \in \mathbb{Z}$, $a \in \Sigma$, and $\lambda \in L$ the numbers $\xi_{a,\lambda}^q$ are defined by

$$\xi_{a,\lambda}^q = 1/2, \text{ if } a = \vartheta_\lambda(q);$$

$$\xi_{a,\lambda}^q = 1, \text{ if there exist distinct } \lambda_1 < \lambda_2 < \dots < \lambda_{r+1} \text{ in } L \text{ such that } \lambda = \lambda_1 \text{ and}$$

$$\vartheta_{\lambda_i}(s_i) = \vartheta_{\lambda_{i+1}}(t_i) \quad \text{for } 1 \leq i \leq r,$$

and there exists $\delta \in \mathbb{Z}$ such that $a = \vartheta_{\lambda_1}(q - \delta)$ and

$$\sum_{i=1}^r (t_i - s_i) \equiv \delta \pmod{\rho}, \quad \text{where } \rho = \gcd(\{p_{\lambda_i} : 1 \leq i \leq r+1\});$$

$$\xi_{a,\lambda}^q = 0, \text{ otherwise.}$$

These numbers are the same as the numbers in [16, Lemma 3.5], even though they are defined differently. Moreover they have the following properties.

Lemma 4.1. *If $\vartheta = (\vartheta_\lambda : \mathbb{Z} \rightarrow \Sigma \mid \lambda \in L)$ is an admissible array and the numbers $\xi_{a,\lambda}^q$ are defined as in Definition 4.1, then the following assertions are true:*

(i) *The numbers $\xi_{a,\lambda}^q$ are well-defined;*

(ii) *If $a \notin R(\vartheta_\lambda)$, then $\xi_{a,\lambda}^q = 0$;*

(iii) *If $\xi_{a,\lambda}^q > 0$, then $a \in \mathcal{P}(q, \vartheta)$;*

(iv) *If $\xi_{a,\lambda}^q > 0$, then $\xi_{a,\lambda'}^q = 1$ for all $\lambda' < \lambda$ with $a \in R(\vartheta_{\lambda'})$.*

Proof. To prove the first assertion we suppose, by way of contradiction, that simultaneously $\xi_{a,\lambda}^q = 1/2$ and $\xi_{a,\lambda}^q = 1$. Then $a = \vartheta_\lambda(q)$ and there exist distinct $\lambda_1 < \lambda_2 < \dots < \lambda_{r+1}$ in L such that $\lambda = \lambda_1$ and

$$\vartheta_{\lambda_i}(s_i) = \vartheta_{\lambda_{i+1}}(t_i) \quad \text{for } 1 \leq i \leq r,$$

and there exists $\delta \in \mathbb{Z}$ such that $a = \vartheta_{\lambda_1}(q - \delta)$ and

$$\sum_{i=1}^r (t_i - s_i) \equiv \delta \pmod{\rho}, \quad \text{where } \rho = \gcd(\{p_{\lambda_i} : 1 \leq i \leq r+1\}).$$

Observe that $\vartheta_\lambda(q - \delta) = \vartheta_{\lambda_1}(q - \delta) = a = \vartheta_\lambda(q)$. Since ϑ_λ is periodic with period p_λ and $\vartheta_\lambda(s) \neq \vartheta_\lambda(t)$ for $1 \leq s < t \leq p_\lambda$, we find that $\delta \equiv 0 \pmod{p_\lambda}$ and hence $\sum_{i=1}^r (t_i - s_i) \equiv 0 \pmod{\rho}$. This however contradicts the fact that ϑ is an admissible array.

The second and third assertion follow directly from Definition 4.1. To prove the last assertion we assume that $\xi_{a,\lambda}^q > 0$ and that $\lambda' \in L$ is such that $\lambda' < \lambda$ and

$a \in R(\vartheta_{\lambda'})$. If $\xi_{a,\lambda}^q = 1/2$, then $\vartheta_\lambda(q) = a$. Since $a \in R(\vartheta_{\lambda'})$ there exists $k \in \mathbb{Z}$ with $\vartheta_{\lambda'}(k) = a$. Now put $\delta = q - k$ and observe that $\vartheta_{\lambda'}(k) = \vartheta_\lambda(q)$ and $a = \vartheta_{\lambda'}(q - \delta)$. Moreover $q - k \equiv \delta \pmod{\rho}$, where $\rho = \gcd(p_\lambda, p_{\lambda'})$, and hence $\xi_{a,\lambda}^q = 1$. On the other hand if $\xi_{a,\lambda}^q = 1$, then there exist distinct $\lambda_1 < \lambda_2 < \dots < \lambda_{r+1}$ in L such that $\lambda = \lambda_1$ and

$$\vartheta_{\lambda_i}(s_i) = \vartheta_{\lambda_{i+1}}(t_i) \quad \text{for } 1 \leq i \leq r,$$

and there exists $\delta \in \mathbb{Z}$ such that $a = \vartheta_{\lambda_1}(q - \delta)$ and

$$\sum_{i=1}^r (t_i - s_i) \equiv \delta \pmod{\rho}, \quad \text{where } \rho = \gcd(\{p_{\lambda_i} : 1 \leq i \leq r+1\}).$$

Further $\vartheta_{\lambda'}(k) = a$ for some $k \in \mathbb{Z}$, since $a \in R(\vartheta_{\lambda'})$. Now put $\lambda_0 = \lambda'$, $s_0 = k$, $t_0 = q - \delta$, $\delta' = q - k$, and $\rho' = \gcd(\{p_{\lambda_i} : 0 \leq i \leq r+1\})$. Clearly

$$\vartheta_{\lambda_i}(s_i) = \vartheta_{\lambda_{i+1}}(t_i) \quad \text{for } 0 \leq i \leq r$$

and $a = \vartheta_{\lambda_0}(q - \delta')$. Moreover there exists $m \in \mathbb{Z}$ such that

$$\sum_{i=0}^r (t_i - s_i) = t_0 - s_0 + \delta + m\rho = q - \delta - k + \delta + m\rho = q - k + m\rho.$$

Since ρ' is a divisor of ρ we find that $\sum_{i=0}^r (t_i - s_i) \equiv q - k \equiv \delta' \pmod{\rho'}$, and hence we conclude that $\xi_{a,\lambda'}^q = 1$. This completes the proof of the lemma. \square

Now for $q \in \mathbb{Z}$ define $\xi^q \in \mathbb{K}^n$ by

$$\xi_i^q = \sum_{\lambda \in L} \xi_{i,\lambda}^q \quad \text{for } 1 \leq i \leq n. \quad (25)$$

It is shown in [16, Lemma 3.6] that if ϑ is an admissible array on n symbols with period p , then ξ^q as defined in (25) is a periodic point of f_ϑ of minimal period p . For the sake of completeness we give a proof of this result here (compare [16, Lemma 3.6]).

Lemma 4.2. *Let $\vartheta = (\vartheta_\lambda : \mathbb{Z} \rightarrow \Sigma \mid \lambda \in L)$ be an admissible array on n symbols with period p , and let $f_\vartheta : \mathbb{K}^n \rightarrow \mathbb{K}^n$ be given by (23). If $\xi^q \in \mathbb{K}^n$ is given by (25), then for each $q \in \mathbb{Z}$ we have that $f_\vartheta(\xi^q) = \xi^{q+1}$ and ξ^q is a periodic point of f_ϑ of minimal period p .*

Proof. The proof of this lemma is based on two claims.

Claim 1.

$$\xi_{\vartheta_\lambda(s),\lambda}^q = \xi_{\vartheta_\lambda(s+t),\lambda}^{q+t} \quad \text{for } q, s, t \in \mathbb{Z} \text{ and } \lambda \in L.$$

Claim 2.

$$f_{\vartheta}(\xi^q)_j = \sum_{(i,\lambda): \vartheta_{\lambda}(s)=i, \vartheta_{\lambda}(s+1)=j} \xi_{i,\lambda}^q \quad \text{for } q \in \mathbb{Z} \text{ and } 1 \leq j \leq n.$$

If we assume these claims for a moment we can complete the proof of the lemma in the following manner. It follows from the claims and (ii) of Lemma 4.1 that

$$\begin{aligned} f_{\vartheta}(\xi^q)_j &= \sum_{(i,\lambda): \vartheta_{\lambda}(s)=i, \vartheta_{\lambda}(s+1)=j} \xi_{i,\lambda}^q \\ &= \sum_{(i,\lambda): \vartheta_{\lambda}(s)=i, \vartheta_{\lambda}(s+1)=j} \xi_{\vartheta_{\lambda}(s+1),\lambda}^{q+1} \\ &= \sum_{\lambda \in L: j \in R(\vartheta_{\lambda})} \xi_{j,\lambda}^{q+1} \\ &= \xi_j^{q+1}, \end{aligned}$$

for $q \in \mathbb{Z}$ and $1 \leq j \leq n$. Therefore $f_{\vartheta}(\xi^q) = \xi^{q+1}$ for $q \in \mathbb{Z}$.

It follows from Claim 1 that if $j = \vartheta_{\lambda}(s)$, then

$$\xi_{j,\lambda}^q = \xi_{\vartheta_{\lambda}(s),\lambda}^q = \xi_{\vartheta_{\lambda}(s+p_{\lambda}),\lambda}^{q+p_{\lambda}} = \xi_{j,\lambda}^{q+p_{\lambda}}.$$

Moreover if j is not in $R(\vartheta_{\lambda})$, then (ii) in Lemma 4.1 gives $\xi_{j,\lambda}^q = \xi_{j,\lambda}^{q+p_{\lambda}} = 0$. Therefore $\xi^{q+p} = \xi^q$ for $q \in \mathbb{Z}$ and $p = \text{lcm}(\{p_{\lambda} : \lambda \in L\})$, and hence $f_{\vartheta}^p(\xi^q) = \xi^q$.

It remains to be shown that p is the minimal period of ξ^q under f_{ϑ} . To do this let $\mu \geq 1$ be the smallest integer with $f_{\vartheta}^{\mu}(\xi^q) = \xi^q$. Remark that it suffices to show that p_{λ} divides μ for all $\lambda \in L$, since $p = \text{lcm}(\{p_{\lambda} : \lambda \in L\})$. So, take $\lambda \in L$ and put $j = \vartheta_{\lambda}(q)$. It follows from (ii) and (iv) of Lemma 4.1 that λ is the only element of L with $\xi_{j,\lambda}^q = 1/2$ in the sum $\sum_{\lambda} \xi_{j,\lambda}^q$. Therefore ξ_j^q is not an integer, and hence $\xi_j^{q+\mu}$ is not an integer, as $\xi_j^{q+\mu} = f_{\vartheta}^{\mu}(\xi^q)_j = \xi_j^q$. This implies that there exists a unique $\lambda' \in L$ such that $\xi_{j,\lambda'}^{q+\mu} = 1/2$, and thus $\vartheta_{\lambda'}(q+\mu) = j$. It is clear from (iv) in Lemma 4.1 that if $\lambda' < \lambda$, then $\xi_j^{q+\mu} < \xi_j^q$. Likewise $\lambda' > \lambda$ implies $\xi_j^{q+\mu} > \xi_j^q$. As $\xi_j^{q+\mu} = \xi_j^q$ we conclude that $\lambda' = \lambda$, so that $\vartheta_{\lambda}(q+\mu) = \vartheta_{\lambda'}(q+\mu) = j = \vartheta_{\lambda}(q)$. Since ϑ_{λ} is periodic with period p_{λ} , and $\vartheta_{\lambda}(s) \neq \vartheta_{\lambda}(t)$ for $1 \leq s < t \leq p_{\lambda}$, we find that p_{λ} divides μ . This shows that ξ^q has minimal period p under f_{ϑ} .

To complete the proof of the lemma we need to show the claims. We begin with the first one. If $\xi_{\vartheta_{\lambda}(s),\lambda}^q = 1/2$, then $\vartheta_{\lambda}(s) = \vartheta_{\lambda}(q)$ and hence $\vartheta_{\lambda}(s+t) = \vartheta_{\lambda}(q+t)$. Therefore $\xi_{\vartheta_{\lambda}(s+t),\lambda}^{q+t} = 1/2 = \xi_{\vartheta_{\lambda}(s),\lambda}^q$. On the other hand if $\xi_{\vartheta_{\lambda}(s),\lambda}^q = 1$, then there exist distinct $\lambda_1 < \lambda_2 < \dots < \lambda_{r+1}$ in L such that $\lambda = \lambda_1$ and

$$\vartheta_{\lambda_i}(s_i) = \vartheta_{\lambda_{i+1}}(t_i) \quad \text{for } 1 \leq i \leq r,$$

and there exists $\delta \in \mathbb{Z}$ such that $\vartheta_{\lambda}(s) = \vartheta_{\lambda}(q - \delta)$ and

$$\sum_{i=1}^r (t_i - s_i) \equiv \delta \pmod{\rho}, \quad \text{where } \rho = \text{gcd}(\{p_{\lambda_i} : 1 \leq i \leq r+1\}).$$

As $\vartheta_\lambda(s) = \vartheta_\lambda(q - \delta)$ we find that $\vartheta_\lambda(s + t) = \vartheta_\lambda(q + t - \delta)$, and therefore $\xi_{\vartheta_\lambda(s+t), \lambda}^{q+t} = 1 = \xi_{\vartheta_\lambda(s), \lambda}^q$.

To establish the second claim remark that by (23), (24), and (25) we have

$$f_\vartheta(\xi^q)_j = \sum_{\gamma(i,k)=j} M_{ik}(\xi^q) \quad \text{for } 1 \leq j \leq n,$$

where $M_{ik}(\xi^q) = \min\{1, \max\{\sum_{\lambda \in L} \xi_{i,\lambda}^q - (k-1), 0\}\}$. It follows from (ii) and (iv) in Lemma 4.1 that

$$M_{ik}(\xi^q) = \begin{cases} \xi_{i,\lambda_k(i)}^q & \text{if } k \leq \rho(i), \\ 0 & \text{otherwise.} \end{cases}$$

Now by using the definition of γ (see (21) and (22)) we deduce

$$\begin{aligned} f_\vartheta(\xi^q)_j &= \sum_{(i,k): \gamma(i,k)=j} M_{ik}(\xi^q) \\ &= \sum_{(i,k): \gamma(i,k)=j} \xi_{i,\lambda_k(i)}^q \\ &= \sum_{(i,\lambda): \vartheta_\lambda(s)=i, \vartheta_\lambda(s+1)=j} \xi_{i,\lambda}^q. \end{aligned}$$

This completes the proof of the lemma. \square

Let us go back to the inclusion $R(n) \supset Q'(2n)$. Before we prove this inclusion we make the following observation.

Lemma 4.3. *If ϑ is a restricted admissible array on $2n$ symbols and ξ^q is defined by (25), then $\xi^q \in \mathbb{E}^{2n}$.*

Proof. Seeking a contradiction, we suppose that $\xi_i^q > 0$ and $\xi_{i^+}^q > 0$. Since $\xi_j^q = \sum_{\lambda \in L} \xi_{j,\lambda}^q$ and $\xi_{j,\lambda}^q \geq 0$ for $1 \leq j \leq 2n$ and $\lambda \in L$, there exist λ and λ' in L such that $\xi_{i,\lambda}^q > 0$ and $\xi_{i^+,\lambda'}^q > 0$. Therefore it follows from (iii) in Lemma 4.1 that $\{i, i^+\} \subset \mathcal{P}(q, \vartheta)$. This however contradicts the fact that ϑ is a restricted admissible array on $2n$ symbols. \square

Corollary 4.1. *$R(n) \supset Q'(2n)$ for all $n \geq 1$.*

Proof. Let $p \in Q'(2n)$. Then there exists a restricted admissible array ϑ on $2n$ symbols with period p . Now let $f_\vartheta : \mathbb{K}^{2n} \rightarrow \mathbb{K}^{2n}$ be given by (23) and let $\xi^q \in \mathbb{K}^{2n}$ be as in (25) for $q \in \mathbb{Z}$. It follows from Lemma 4.2 that ξ^0 is a periodic point of f_ϑ of minimal period p . Moreover $f_\vartheta(\xi^q) = \xi^{q+1}$ for $q \in \mathbb{Z}$, so that Lemma 4.3 implies $f^j(\xi^0) \in \mathbb{E}^{2n}$ for all $j \geq 0$. Applying Lemma 1.1 now gives $p \in R(n)$. \square

5 Some remarks concerning the set $R(n)$

In this section the set $R(n)$ is determined for $1 \leq n \leq 10$. Moreover it is shown that the largest element $\psi(n)$ of $R(n)$ satisfies:

$$\log \psi(n) \sim \sqrt{2n \log n}.$$

The set $R(n)$ has been computed for $n = 1, 2, 3, 4, 6, 7$, and 10 by the first author in [5] and [6]. To obtain $R(n)$ for all $1 \leq n \leq 10$ it only remained to be decided whether 18 is in $R(5)$, 90 is in $R(8)$, and 126 is in $R(9)$. We will see that none of these integers can occur. To prove this the following notion is used.

Definition 5.1. A set $S \subset \{1, \dots, 2n\}$ of m elements is called *feasible for $2n$* , if there exists a restricted admissible array $\vartheta = (\vartheta_\lambda : \mathbb{Z} \rightarrow \Sigma \mid \lambda \in L)$ on $2n$ symbols such that $|L| = m$ and $S = \{p_\lambda : \lambda \in L\}$. Moreover a feasible set S is said to be *minimal* if $\text{lcm}(S') < \text{lcm}(S)$ for all $S' \subset S$ with $S' \neq S$.

Remark that if $\vartheta = (\vartheta_\lambda : \mathbb{Z} \rightarrow \Sigma \mid \lambda \in L)$ is a restricted admissible array on $2n$ symbols, and L' is a subset L with the ordering inherited from L , then $\vartheta' = (\vartheta_\lambda : \mathbb{Z} \rightarrow \Sigma \mid \lambda \in L')$ is also a restricted admissible array on $2n$ symbols. Using this remark we see that

$$Q'(2n) = \{\text{lcm}(S) : S \text{ is a feasible set for } 2n\} \quad (26)$$

$$= \{\text{lcm}(S) : S \text{ is a minimal feasible set for } 2n\}. \quad (27)$$

There exist several necessary conditions for feasible sets in the literature. We list some of them here. The proofs of these conditions can be found in [5, Section 3.5] and [6, Section 5]. We note that in [5] and [6] the conditions are formulated in terms of so-called strongly array admissible sets. However, as any feasible set is strongly array admissible, the conditions also hold for feasible sets.

Lemma 5.1. *A set $S \subset \{1, \dots, 2n\}$ is not feasible for $2n$ if one of the following conditions holds:*

(i) *there exist $p_1, \dots, p_k \in S$ such that $\gcd(p_i, p_j) = 1$ for all $1 \leq i < j \leq k$ and*

$$\sum_{i=1}^k 2 \lceil p_i/2 \rceil > 2n,$$

where $\lceil p \rceil$ is the smallest integer k with $p \leq k$;

(ii) *there exists $p \in S$ such that $\gcd(p, q) = 1$ for all $q \in S$ with $p \neq q$, and $S \setminus \{p\}$ is not feasible for $2m$, where $m = n - \lceil p/2 \rceil$;*

(iii) *there exist $p_1, p_2 \in S$ such that $p_1 + p_2 > 2n$ and $\gcd(p_1, p_2) = 2$;*

(iv) *there exist $p_1, p_2, p_3 \in S$ such that $p_1 + p_2 + p_3 > 3n$ and $\gcd(p_i, p_j) = 3$ for all $1 \leq i < j \leq 3$.*

To prove that $18 \notin R(5)$, $90 \notin R(8)$, and $126 \notin R(9)$ we need the following lemma.

Lemma 5.2. *The set $\{6, 9\}$ is not feasible for 10.*

Proof. Seeking a contradiction, we assume that $\{6, 9\}$ is feasible for 10. Then there exists a restricted admissible array $\vartheta = (\vartheta_\lambda : \mathbb{Z} \rightarrow \Sigma \mid \lambda \in L)$ on 10 symbols with period 18, where $|L| = \{1, 2\}$ and L is equipped with the usual ordering. Furthermore the periods p_λ of ϑ_λ either satisfy: $p_1 = 6$ and $p_2 = 9$, or $p_1 = 9$ and $p_2 = 6$.

In both cases there exist $i, j, k \in \mathbb{Z}$ such that

$$\vartheta_2(i) = \vartheta_1(j) = \vartheta_1(k)^+. \quad (28)$$

Since ϑ is an admissible array we know that $i - j \not\equiv 0 \pmod{3}$. Further we claim that $i - k \not\equiv 0 \pmod{3}$. As $i - k \equiv 0 \pmod{3}$ implies that there exist $m, m_1, m_2 \in \mathbb{Z}$ such that $i - k + m(m_1 p_1 + m_2 p_2) = 0$. Therefore $i + m m_2 p_2 = k - m m_1 p_1$ and hence $\{\vartheta_2(i + m m_2 p_2), \vartheta_1(k - m m_1 p_1)\} \subset \mathcal{P}(i + m m_2 p_2, \vartheta)$. Using (28) and the fact that ϑ_1 and ϑ_2 have period p_1 and p_2 , respectively, we find that $\{\vartheta_2(i), \vartheta_2(i)^+\} \subset \mathcal{P}(i + m m_2 p_2, \vartheta)$, which contradicts the fact that ϑ is a restricted admissible array. Furthermore $i - j \not\equiv i - k \pmod{3}$, because $i - j \equiv i - k \pmod{3}$ implies that $\vartheta_1(k) = \vartheta_1(i - (i - k)) \in \mathcal{P}(i, \vartheta)$, so that (28) gives $\{\vartheta_2(i), \vartheta_2(i)^+\} \subset \mathcal{P}(i, \vartheta)$. This again contradicts the fact that ϑ is a restricted admissible array.

This leaves us two cases: $i - j \equiv 1 \pmod{3}$, $i - k \equiv 2 \pmod{3}$, and $i - j \equiv 2 \pmod{3}$, $i - k \equiv 1 \pmod{3}$. We begin with the first one. By definition $\vartheta_1(j) \in \mathcal{P}(j, \vartheta)$. On the other hand, $j - k \equiv (j - i) + (i - k) \equiv -1 + 2 \equiv 1 \pmod{3}$, so that $i - j \equiv j - k \pmod{3}$. Thus, $\vartheta_1(k) = \vartheta_1(j - (j - k)) \in \mathcal{P}(j, \vartheta)$, and hence (28) gives $\{\vartheta_1(j), \vartheta_1(j)^+\} \subset \mathcal{P}(j, \vartheta)$, which is a contradiction. In the second case $\vartheta_1(j) \in \mathcal{P}(j, \vartheta)$ and $j - k \equiv (j - i) + (i - k) \equiv -2 + 1 \equiv 2 \pmod{3}$, so that $i - j \equiv j - k \pmod{3}$. This implies that $\vartheta_1(k) = \vartheta_1(j - (j - k)) \in \mathcal{P}(j, \vartheta)$, and therefore (28) gives $\{\vartheta_1(j), \vartheta_1(j)^+\} \subset \mathcal{P}(j, \vartheta)$, which is a contradiction. \square

This lemma has the following corollary.

Theorem 5.1. *We have that $18 \notin R(5)$, $90 \notin R(8)$, and $126 \notin R(9)$.*

Proof. To see that $18 \notin R(5)$ we look at the candidate minimal feasible sets for 10 that give period 18. There are two such sets: $S_1 = \{2, 9\}$ and $S_2 = \{6, 9\}$. However, S_1 is not feasible for 10 by (i) in Lemma 5.1, and S_2 is not feasible for 10 by Lemma 5.2. Thus we conclude from (27) and Theorem 2.1 that $18 \notin R(5)$.

For period 90 in dimension 8 there are five candidate minimal feasible sets: $S_1 = \{2, 5, 9\}$, $S_2 = \{10, 9\}$, $S_3 = \{2, 15, 9\}$, $S_4 = \{6, 5, 9\}$, and $S_5 = \{6, 15, 9\}$. It follows from (i) in Lemma 5.1 that S_1 , S_2 , and S_3 are not feasible for 16. Further (iv) in Lemma 5.1 tells us that S_5 is not feasible for 16. By combining (ii) in Lemma 5.1 with Lemma 5.2 we see that S_4 is also not feasible for 16, and hence $90 \notin R(8)$.

For period 126 in dimension 9 there are three candidate minimal feasible set: $S_1 = \{2, 7, 9\}$, $S_2 = \{14, 9\}$, and $S_3 = \{6, 7, 9\}$. The sets S_1 and S_2 are not feasible for 18 by (i) in Lemma 5.1. Furthermore it follows from (ii) in Lemma 5.1 and Lemma 5.2 that S_3 is not feasible for 18, and thus $126 \notin R(9)$. This completes the proof of the theorem. \square

The results in [6, Section 5] and Theorem 5.1 together yield a complete list of elements of $R(n)$ for $1 \leq n \leq 10$. This list is given in Table 1 in the appendix.

We conclude this section with a theorem for the largest element of $R(n)$. In this theorem the notation $f(n) \sim g(n)$ is used to say that $\lim_{n \rightarrow \infty} f(n)/g(n) = 1$.

Theorem 5.2. *Let $\psi(n) = \max\{p : p \in R(n)\}$. Then*

$$\log \psi(n) \sim \sqrt{2n \log n}. \quad (29)$$

Proof. Let $g(n)$ denote the maximal order of a permutation on n letters, and let $\pi(n)$ denote the number of primes at most n . We first prove the inequality

$$g(2n - \pi(2n)) \leq \psi(n) \quad \text{for all } n \geq 1. \quad (30)$$

To derive this inequality it is shown that for each order m of a permutation on $2n - \pi(2n)$ letters there exists a restricted admissible array on $2n$ symbols with period m . This is sufficient, as $R(n) = Q'(2n)$.

So, let m be the order of a permutation on $2n - \pi(2n)$ letters, and suppose that m has a prime factorization $\prod_{i=1}^k q_i^{\alpha_i}$. We know from elementary properties of permutations that there exists a permutation μ on $2n - \pi(2n)$ letters that has a disjoint cycle representation $\mu = \mu_1 \mu_2 \dots \mu_k$, where μ_i has order $q_i^{\alpha_i}$ for $1 \leq i \leq k$. Furthermore we have that $\sum_{i=1}^k q_i^{\alpha_i} \leq 2n - \pi(2n)$. Let D_i denote the domain of μ_i for $1 \leq i \leq k$, and put $D = \cup_i D_i$. Further for $S \subset \{1, \dots, 2n\}$ define $\text{cl}(S) = \{a : a \in S \text{ or } a^+ \in S\}$.

As $k \leq \pi(2n)$ we know that $\sum_{i=1}^k (q_i^{\alpha_i} + 1) \leq 2n$. Therefore we can rename the elements of D such that $D \subset \{1, \dots, 2n\}$ and the sets $\text{cl}(D_i)$ are pairwise disjoint. Now let a_i be the smallest element of D_i for $1 \leq i \leq k$, and define $\vartheta = (\vartheta_i : \mathbb{Z} \rightarrow \{1, \dots, 2n\} \mid 1 \leq i \leq k)$ by

$$\vartheta_i(j) = \mu_i^j(a_i) \quad \text{for } j \in \mathbb{Z} \text{ and } 1 \leq i \leq k.$$

As $\text{cl}(D_i)$ and $\text{cl}(D_j)$ are disjoint for distinct i and j the array ϑ is a restricted admissible array on $2n$ symbols. Moreover, ϑ_i has period $q_i^{\alpha_i}$ for $1 \leq i \leq k$, so that ϑ has period $m = \prod_{i=1}^k q_i^{\alpha_i}$. Hence the proof of (30) is complete.

Now let $\gamma(n)$ denote the largest element of $Q(n)$. Then it follows from (30) and the inclusion $R(n) \subset Q(2n)$ that

$$\frac{\log g(2n - \pi(2n))}{\sqrt{2n \log n}} \leq \frac{\log \psi(n)}{\sqrt{2n \log n}} \leq \frac{\log \gamma(2n)}{\sqrt{2n \log n}} \quad \text{for } n \geq 2. \quad (31)$$

It has been shown in [18] that

$$\log \gamma(n) \sim \sqrt{n \log n}, \quad (32)$$

and hence

$$\lim_{n \rightarrow \infty} \frac{\log \gamma(2n)}{\sqrt{2n \log n}} = 1. \quad (33)$$

On the other hand, Landau [4, pp. 222–229] has proved that

$$\log g(n) \sim \sqrt{n \log n}, \quad (34)$$

so that we can use the prime number theorem, which says that $\pi(n) \sim n/\log n$, to find that

$$\lim_{n \rightarrow \infty} \frac{\log g(2n - \pi(2n))}{\sqrt{2n \log n}} = 1. \quad (35)$$

The equations (31), (33), and (35) together yield:

$$\lim_{n \rightarrow \infty} \frac{\log \psi(n)}{\sqrt{2n \log n}} = 1,$$

and hence the proof of the theorem is complete. \square

Appendix

Proof of Lemma 1.1. Let p be in $R(n)$. Then there exist a 1-nonexpansive map $h : \mathbb{R}^n \rightarrow \mathbb{R}^n$ and a periodic point $\zeta \in \mathbb{R}^n$ of h of minimal period p . Define $D = \bigcap_{j=0}^{p-1} B(h^j(\zeta), d)$, where $B(h^j(\zeta), d)$ is the 1-norm ball with radius d around $h^j(\zeta)$, and d is the 1-norm diameter of the set $\{h^j(\zeta) : 0 \leq j < p\}$. Observe that $h[D] \subset D$, as h is 1-nonexpansive. Therefore the Brouwer fixed point theorem implies that there exists an $x^* \in D$ with $h(x^*) = x^*$.

Now let $g : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be given by $g(x) = h(x + x^*) - x^*$ for $x \in \mathbb{R}^n$. Clearly, g is 1-nonexpansive and $g(0) = 0$. Further let $J : \mathbb{R}^n \rightarrow \mathbb{E}^{2n}$ be defined by $J(x) = (x \vee 0, (-x) \vee 0)$ for $x \in \mathbb{R}^n$. It is easy to verify that J is a 1-isometry that maps \mathbb{R}^n onto \mathbb{E}^{2n} . Hence the inverse $J^{-1} : \mathbb{E}^{2n} \rightarrow \mathbb{R}^n$ is also a 1-isometry. Finally let $R : \mathbb{K}^{2n} \rightarrow \mathbb{E}^{2n}$ be the 1-nonexpansive retraction given by

$$R(x, y) = (x - (x \wedge y), y - (x \wedge y)) \quad \text{for } (x, y) \in \mathbb{K}^n \times \mathbb{K}^n.$$

Observe that $f : \mathbb{K}^{2n} \rightarrow \mathbb{K}^{2n}$ given by $f(z) = (J \circ g \circ J^{-1} \circ R)(z)$ for $z \in \mathbb{K}^{2n}$ is a 1-nonexpansive map with $f(0) = 0$. Set $\xi = J(\zeta - x^*)$ and remark that ξ is a periodic point of f of minimal period p . Moreover $f^j(\xi) \in \mathbb{E}^{2n}$ for all $j \geq 0$.

Conversely, if $f : \mathbb{K}^{2n} \rightarrow \mathbb{K}^{2n}$, with $f(0) = 0$, is a 1-nonexpansive map, and $\xi \in \mathbb{K}^{2n}$ is a periodic point of f of minimal period p , such that $f^j(\xi) \in \mathbb{E}^{2n}$ for all $j \geq 0$, then $h : \mathbb{R}^n \rightarrow \mathbb{R}^n$ given by $h(x) = (J^{-1} \circ R \circ f \circ J)(x)$ for $x \in \mathbb{R}^n$, is 1-nonexpansive and $J^{-1}(\xi)$ is a periodic point of h of minimal period p . \square

Table 1: The elements of $R(n)$ for $1 \leq n \leq 10$.

n	Elements of $R(n)$
1	1, 2
2	1, 2, 3, 4
3	1, 2, 3, 4, 5, 6
4	1, 2, 3, 4, 5, 6, 7, 8, 10, 12
5	1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 14, 15, 20
6	1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 14, 15, 18, 20, 21, 24, 28, 30
7	1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 18, 20, 21, 22, 24, 28, 30, 35, 36, 40, 42, 60
8	1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 18, 20, 21, 22, 24, 26, 28, 30, 33, 35, 36, 40, 42, 44, 45, 48, 56, 60, 70, 84
9	1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 20, 21, 22, 24, 26, 28, 30, 33, 35, 36, 39, 40, 42, 44, 45, 48, 52, 55, 56, 60, 63, 66, 70, 72, 84, 90, 105, 120, 140
10	1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 24, 26, 28, 30, 33, 34, 35, 36, 39, 40, 42, 44, 45, 48, 52, 55, 56, 60, 63, 65, 66, 70, 72, 77, 78, 80, 84, 88, 90, 105, 110, 120, 126, 132, 140, 168, 180, 210

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