On the integral of the workload process of the single server queue

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Abstract

This paper is devoted to a study of the integral of the workload process of the single server queue, in particular during one busy period. Firstly, we find asymptotics of the area $\mathcal{A}$ swept under the workload process $W(t)$ during the busy period when the service time distribution has a regularly varying tail. We also investigate the case of a light-tailed service time distribution. Secondly, we consider the problem of obtaining an explicit expression for the distribution of $\mathcal{A}$. In the general $GI|G|1$ case, we use a sequential approximation to find the Laplace-Stieltjes transform of $\mathcal{A}$. In the $M|M|1$ case, this transform is obtained explicitly in terms of Whittaker functions. Thirdly, we consider moments of $\mathcal{A}$ in the $GI|G|1$ queue. Finally, we show asymptotic normality of $\int_0^t W(s)\,\mathrm{d}s$.

Keywords: workload process, area, regularly varying distribution, sequential approximation, covariance function.

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1 Introduction

This paper is devoted to a study of the integral of the workload process of the single server queue, in particular during one busy period. Besides its intrinsic interest, the tail behaviour of the area $\mathcal{A}$ has applications to various problems, like to the analysis of the load in TCP networks (TCP is the Transmission Control Protocol in the Internet). To analyse the performance of TCP networks one may create a simple model with the following features (to simplify considerations we assume infinite window size). If one has succeeded in sending $\mathcal{L}(t)$ bytes at time t (one obtains acknowledgements for $\mathcal{L}(t)$ bytes), then at time $t + \mathrm{d}t$ one tries to send $a\mathcal{L}(t)$ bytes for some constant a. Suppose that after some random time a transmission fails; then the demand randomly decreases. The amount of data sent in a busy period now equals the area under the line $\mathcal{L}(t)$. This area is equal to $\mathcal{A}$ in an appropriate $GI|G|1$ queue. Other applications are found in inventory theory and in risk theory. In risk theory Gerber [16], Embrechts and Schmidli [14] considered an insurance risk model, where the company is allowed to invest money. We assume that the force of interest for invested money and the premium income are constant. The company will obtain money earned from investment at the ruin epoch if this happens. Then the surplus process is proportional to the amount of money that the insurance company will obtain at the ruin time, and it is equal to $\mathcal{A}$ in an appropriate $GI|G|1$ queue.

In this paper, we make the following contributions to the analysis of the integral $\mathcal{A}$ in the $GI|G|1$ queue and related topics. Firstly, we find asymptotics of $\mathcal{A}$ when the service time distribution has a regularly varying tail. We also investigate the case of a light-tailed service time distribution. Secondly, we consider the problem of obtaining an explicit expression for the distribution of $\mathcal{A}$. Thirdly, we consider moments of $\mathcal{A}$ in the $GI|G|1$ queue. Finally, we show asymptotic normality of $\mathcal{A}_{[0,t]} := \int_0^t W(s)\,\mathrm{d}s$. We now discuss these topics in some more detail, also discussing the relevant literature.

Our first goal is to obtain tail asymptotics of $\mathcal{A}$. The $GI|G|1$ queue with regular varying tail distribution has been subject of many studies. Most of them focus on the tail of the waiting time distribution. We show in this paper that the occurrence of a large area is related to the occurrence of a large cycle maximum and we exploit asymptotic results for this random variable. These are known for the $GI|G|1$ queue with subexponential service times (see Asmussen [5]) or with light-tailed service time distribution (see Iglehart [23]). In Theorem 3.1 we give the following asymptotics:

$$\mathbb{P}(\mathcal{A} > x) \sim \mathbb{E}H\ V\left(\sqrt{2(1-\rho)x}\right) \qquad \text{as } x \to \infty, \tag{1.1}$$

where H is the number of customers served during the busy period, ρ is the offered traffic load, and $V(\cdot)$ denotes the tail of the service time distribution; $V(\cdot)$ is regularly varying at infinity and $f(x) \sim g(x)$ means that $\lim_{x\to\infty} f(x)/g(x) = 1$. We generalize (1.1) for the case in which the busy period starts with some service time and interarrival time that are not typical service and interarrival times. We also consider the case of light-tailed service times, proving that

$$\mathbb{P}(\mathcal{A} > x) \sim Ce^{-\eta\sqrt{2\frac{(1-\rho)(\tilde{\rho}-1)}{\tilde{\rho}-\rho}x}} \qquad \text{as } x \to \infty\,, \tag{1.2}$$

for given constant C and η, where $\tilde{\rho}$ is the traffic intensity in some related queueing system. The light-tailed case thus gives rise to Weibull-like asymptotics for the tail of the area.

Our second goal is to obtain an explicit expression for the Laplace-Stieltjes transform (LST) of the distribution of $\mathcal{A}$ in the $GI|G|1$ queue. We use a sequential approximation method for obtaining this transform, and we find the rate of convergence. For the $M|M|1$ queue we derive an explicit formula for the LST of $\mathcal{A}$:

$$\mathbb{E}e^{-s\mathcal{A}} = \int_0^\infty \mu e^{-\mu x} e^{-\frac{1}{4}sx^2} \frac{sx+\mu+\lambda}{\mu+\lambda} \frac{\mathrm{D}_{\frac{\lambda\mu}{s}}(\frac{sx+\mu+\lambda}{\sqrt{s}})}{\mathrm{D}_{\frac{\lambda\mu}{s}}(\frac{\mu+\lambda}{\sqrt{s}})} \mathrm{d}x \ , \tag{1.3}$$

where μ and λ are the intensities of the service and interarrival times, respectively, and $\mathrm{D}_\nu(z)$ is a parabolic cylinder function.

Our third goal is to study the moments of $\mathcal{A}$. The first moment of $\mathcal{A}$ is well-known (see Cohen [13]); it equals

$$\mathbb{E}\mathcal{A} = \mathbb{E}l\mathbb{E}W \ , \tag{1.4}$$

where W is the stationary workload and l denotes a busy cycle. The proof of (1.4) in Cohen [13] is based on regenerative arguments. Using level crossing arguments, Cohen [12] finds an explicit expression for $\mathbb{E}\mathcal{A}^2$ in the $M|G|1$ queue. Necessary condition for the first (respectively second) moment of the area $\mathcal{A}$ to be finite in the $M|G|1$ case is that the second (respectively fourth) moment of the service time is finite. In the present paper we generalize this result to the $GI|G|1$ case with general initial service and interarrival times. Under these conditions we prove a functional central limit theorem for the process $\{\int_0^t W(s)\ \mathrm{d}s, t \geq 0\}$. We also prove that in the stationary regime of $W(t)$ we have $\mathbb{E}(\int_0^t (W(s) - \mathbb{E}W(s))\ \mathrm{d}s)^2 \sim 2tR$ for $t \to \infty$ where $R = \int_0^\infty \rho(t)\ \mathrm{d}t$ and $\rho(t)$ is the covariance between $W(t)$ and $W(0)$. The necessary condition for the *asymptotic variance* R to be finite is that the fourth moment of the service time is finite; compare with an explicit expression for the $M|G|1$ queue in Abate and Whitt [1], Beneš [8], Ott [27].

The paper is organized as follows. In Section 2 we describe the model and give some preliminary results needed for proving Theorems 3.1 and 4.1. In Sections 3 and 4 we find asymptotics of $\mathbb{P}(\mathcal{A} > x)$ when the service time distribution is regularly varying and when it is light-tailed, respectively. In Section 5 we use a sequential approximation method to find the LST of $\mathcal{A}$ in the $GI|G|1$ queue. The LST of $\mathcal{A}$ in the $M|M|1$ queue is obtained in explicit form in Section 6. In Section 7 we determine the conditions for the first two moments of $\mathcal{A}$ in the $GI|G|1$ queue to be finite and we prove a functional central limit theorem for the process $\{\int_0^t W(s)\ \mathrm{d}s, t \geq 0\}$.

2 Preliminaries

Suppose that the first customer (with the number zero) enters an empty system at time 0 and that his service time is σ_0 with the tail distribution $V_0(\cdot)$. We denote the first interarrival time by τ_0. The service time of customer i ($i \geq 1$) is denoted by σ_i and the time between the arrivals of customers i and $i+1$ is denoted by τ_i. It is assumed that σ_i and τ_i ($i \geq 1$) are i.i.d. sequences and that both sequences are independent of each other and independent of (σ_0, τ_0). We assume that all service times and interarrival times have non-lattice distributions, finite mean and that the stability condition for the traffic load holds:

$$\rho := \frac{\mathbb{E}\sigma}{\mathbb{E}\tau} < 1 \ ,$$

where σ and τ are r.v.'s with generic distribution of σ_i and τ_i ($i \geq 1$), respectively. Denote by $V(\cdot)$ and $A(\cdot)$ the tails of the distribution of σ and τ, respectively. Let $W(t)$, $W(0-) = 0$ be the workload process, that is the amount of work at time t. Unless specified otherwise, we assume that σ_0 and τ_0 have the same distribution as σ and τ, respectively. If σ_0

and τ_0 have different distributions than σ and τ respectively, then we add the superscript (σ_0, τ_0) to each relevant quantity. Finally, $\mathbb{P}_x$ denotes the conditional probability under the condition that $W(0) = x$ and $\mathbb{E}_x$ denotes the expectation with respect to $\mathbb{P}_x$. Similarly, $\mathbb{P}^x(\cdot) = \mathbb{P}(\cdot|\sigma_0 = x)$ and $\mathbb{E}^x$ denotes the expectation with respect to $\mathbb{P}^x$. Denote by $V_0^*(\cdot)$ the tail of the distribution of $(\sigma_0 - \tau_0)^+$. The busy period is defined by

$$l := \inf\{t \geq 0 : W(t) = 0\} .$$

Let

$$S_0 := 0, \qquad S_{k+1} := \sum_{i=0}^{k} \xi_i \ , \tag{2.1}$$

where $\xi_i = \sigma_i - \tau_i$. Then the number of customers served during the busy period can be defined in the following way:

$$H := \min\{k \geq 1 : S_k \leq 0\} .$$

Note that

$$\mathbb{E}H = \exp\{\sum_{n=1}^{\infty} \mathbb{P}(S_n > 0)\}, \tag{2.2}$$

and when interarrival times are exponential then $\mathbb{E}H = \frac{1}{1-\rho}$; see Asmussen [4]. A random variable which plays a crucial role in the next sections is the cycle maximum (height) given by

$$h := \sup\{W(t), 0 \leq t \leq l\} .$$

By $F(\cdot)$ we denote the distribution of h. Define

$$l_1 := \inf\{t \geq 0 : W(t) = h\}, \qquad l_2 := l - l_1 .$$

First we prove a preliminary result which will be often used in the next sections, viz.: After passing a high level, the workload behaves according to the law of large numbers, decreasing almost linearly with the slope $1 - \rho$.

Lemma 2.1 *For any $\delta > 0$ and $\epsilon > 0$ there exists a sufficiently large M and $L \geq M$ such that for any $x \geq L$ we have*

$$\mathbb{P}^{(\sigma_0,\tau_0)}(B_1|h > x) \geq 1 - \delta, \tag{2.3}$$

and

$$\mathbb{P}^{(\sigma_0,\tau_0)}(B_2|h > x) \geq 1 - \delta \ , \tag{2.4}$$

where

$$B_1 = \{h - W(l_1 + u) \leq M + u(1-\rho)(1+\epsilon); 0 \leq u \leq l_2\} \ ,$$
$$B_2 = \{h - W(l_1 + u) \geq -M + u(1-\rho)(1-\epsilon); 0 \leq u \leq l_2\} \ .$$

Proof. We prove (2.3). Inequality (2.4) can be proved in a very similar way. By B^c we denote the complement of set B. To prove (2.3) it is enough to prove that

$$\mathbb{P}^{(\sigma_0,\tau_0)}(B_1^c|h > x) \to 0, \qquad \text{as } x, M \to \infty.$$

Denote by $X(u)$ the renewal process with the time intervals distributed according to τ and the jump size distributed according to σ. Without loss of generality we assume that $l_1 > 0$. If $l_1 = 0$, that is the cycle maximum is attained at the first service time, then we consider the delayed renewal process $X(u)$ with the delay τ_0. Let $\overline{X}(t) = \max_{0\le u\le t} X(u)$. Note that the distribution of $h - W(l_1+u)$ coincides with the conditional distribution of $X(u)$ under the condition

$$\overline{X}(l_2) = 0 \ . \tag{2.5}$$

We have

$$\begin{aligned}
\mathbb{P}^{(\sigma_0,\tau_0)}(B_1^c|h>x) &= \mathbb{P}^{(\sigma_0,\tau_0)}\left(\min_{u\le l_2}(M+u(1-\rho)(1+\epsilon)-X(u))<0|h>x,\overline{X}(l_2)=0\right)\\
&= \frac{\mathbb{P}^{(\sigma_0,\tau_0)}(\min_{u\le l_2}(M+u(1-\rho)(1+\epsilon)-X(u))<0,\overline{X}(l_2)=0|h>x)}{\mathbb{P}^{(\sigma_0,\tau_0)}(\overline{X}(l_2)=0|h>x)}\\
&\le \frac{\mathbb{P}^{(\sigma_0,\tau_0)}(\min_{u\ge 0}(M+u(1-\rho)(1+\epsilon)-X(u))<0|h>x)}{\mathbb{P}^{(\sigma_0,\tau_0)}(\overline{X}(\infty)=0)}\\
&= \frac{\mathbb{P}^{(\sigma_0,\tau_0)}(\min_{u\ge 0}(u(1-\rho)(1+\epsilon)-X(u))<-M)}{\mathbb{P}^{(\sigma_0,\tau_0)}(\overline{X}(\infty)=0)}
\end{aligned}$$

which goes to zero as M goes to ∞ by the law of large numbers:

$$\frac{X(u)}{u} \to 1-\rho \qquad \text{a.s.} \qquad \text{as } u\to\infty \ .$$

□

Denote by $\mathcal{A}_h$ the area swept under the workload process after l_1.

Corollary 2.1 *For any $\delta,\kappa>0$ and sufficiently large x the following holds*

$$\mathbb{P}^{(\sigma_0,\tau_0)}\left(1-\kappa\le\frac{\mathcal{A}_h}{h^2/(2(1-\rho))}\le 1+\kappa|h>x\right)\ge 1-\delta \ .$$

Proof. From Lemma 2.1 for any $\delta>0$ we can fix M large and find L such that for all $x\ge L$ we have

$$\mathbb{P}^{(\sigma_0,\tau_0)}\left(\frac{(h-M)^2}{2(1+\epsilon)(1-\rho)}\le\mathcal{A}_h\le\frac{(h+M)^2}{2(1-\epsilon)(1-\rho)}|h>x\right)>1-\delta \ ,$$

from which we obtain the assertion of the corollary.

□

Corollary 2.2 *For any $\delta,\kappa>0$ and sufficiently large x the following holds*

$$\mathbb{P}^{(\sigma_0,\tau_0)}\left(\left|\frac{h}{l_2}-(1-\rho)\right|<\kappa|h>x\right)\ge 1-\delta \ .$$

If random variables X and Y have the same distribution, then we write $X\stackrel{D}{=}Y$. We consider in this paper the following three cases of regularly varying distributions $V(\cdot)$ and $V_0(\cdot)$:

(R.1) $V(\cdot)$ and $V_0(\cdot)$ are regularly varying and $V_0(\cdot)=V(\cdot)$ and $\tau_0\stackrel{D}{=}\tau$,

(R.2) $V(\cdot)$ and $V_0(\cdot)$ are regularly varying and $\lim_{x\to\infty}V_0(x)/V(x)=c_0<\infty$,

(R.3) $V_0(\cdot)$ is regularly varying and $\lim_{x\to\infty} V(x)/V_0(x) = 0$.

If one of these conditions holds, then we say that condition **(R)** holds. Write $\overline{G}(x) = 1 - G(x)$ for $G(\cdot)$. Observe that below, $\overline{F}(x) = \mathbb{P}(h > x)$ but $\overline{V}_0^*(y) = \mathbb{P}((\sigma_0 - \tau_0)^+ \leq y)$.

Proposition 2.1 *(i) (Asmussen [5], Heath et al. [20]) If* **(R.1)** *holds, then*

$$\overline{F}(x) \sim \mathbb{E}HV(x) \ .$$

(ii) If **(R.2)** *holds, then*

$$\overline{F}(x) \sim \left(c_0 + \int_0^\infty \mathbb{E}H(y)\ \mathrm{d}\overline{V}_0^*(y)\right) V(x) \ ,$$

where $H(y)$ is the number of customers served during a busy period, counting from the customer who found y work upon arrival.

(iii) If **(R.3)** *holds, then*

$$\overline{F}(x) \sim V_0(x) \ .$$

Remark 2.1 Note that $\mathbb{E}(\sigma_0 - \tau_0)^+ \leq \mathbb{E}\sigma_0 < \infty$. Moreover, from the renewal theorem $\mathbb{E}H(y) \leq \frac{1}{\mathbb{E}(\sigma-\tau)}(1+\epsilon)y$ for some $\epsilon > 0$ and large y. Hence $\int_0^\infty \mathbb{E}H(y)\ \mathrm{d}\overline{V}_0^*(y) < \infty$.

Remark 2.2 If **(R)** holds, then the distribution of the cycle maximum $F(\cdot)$ also has a regularly varying tail.

Before we prove Proposition 2.1 (ii)-(iii), we generalize Lemma 4.2 of Asmussen and Møller [6] and Theorem 3.3 of Grainer *et al.* [17] in Lemma 2.2. Let

$$\beta(y) := \min\{k \geq 1 : S_k > y\} \ .$$

From Asmussen and Møller [6], p.162, we have that for large x there exists a constant c such that

$$\mathbb{P}_y(\beta(x) \leq H(y)) \leq cV(x) \ . \tag{2.6}$$

Lemma 2.2 *Assume that $V(\cdot)$ is regularly varying. Then for any y we have*

$$\lim_{x_0\to\infty} \limsup_{x\to\infty} \frac{\mathbb{P}_y(\beta(x_0) < \beta(x) < H(y))}{V(x)} = 0 \ .$$

Proof. First we prove that:

$$\lim_{u\to\infty} \limsup_{x\to\infty} \frac{\mathbb{P}_y(\beta(x) < H(y), u < S_{\beta(x)-1})}{V(x)} = 0 \ . \tag{2.7}$$

Define $\beta_1 := \inf\{n \geq 0 : S_n \leq y\}$ and $\beta_k := \inf\{n \geq \beta_{k-1} : S_n \leq S_{\beta_{k-1}}\}$. Now,

$$\begin{aligned}
\mathbb{P}_y&(\beta(x) < H(y), u < S_{\beta(x)-1}) \leq \mathbb{P}_y(\beta(x) < \beta_1, u < S_{\beta(x)-1}) \\
&+ \sum_{k=1}^\infty \mathbb{P}_y(\beta_k \leq \beta(x) < \beta_{k+1}, \beta_k < H(y), u < S_{\beta(x)-1}) \\
&= \mathbb{P}_y(\beta(x) < \beta_1, u < S_{\beta(x)-1}) \\
&+ \sum_{k=1}^\infty \mathbb{E}_y\left(\mathbb{P}_{S_{\beta_k}}(\beta(x) < \beta_1, u < S_{\beta(x)-1}) \mathbb{I}\,(\beta_k \leq \beta(x) < \beta_{k+1}, \beta_k < H(y))\right) \\
&\leq (1 + \mathbb{E}H(y))\mathbb{P}_0(\beta(x-y) < H, u - y < S_{\beta(x-y)-1}) \ .
\end{aligned}$$

Hence (2.7) follows from Lemma 2.3 of Asmussen [5]. To prove the lemma, we argue by contradiction. Assume that

$$\lim_{x_0\to\infty}\limsup_{x\to\infty}\frac{\mathbb{P}_y(\beta(x_0)<\beta(x)<H(y))}{V(x)}>\epsilon\ .$$

From (2.7), it follows that we can choose u (independent of x_0) such that

$$\limsup_{x\to\infty}\frac{\mathbb{P}_y(\beta(x_0)<\beta(x)<H(y),u<S_{\beta(x)-1})}{V(x)}<\frac{\epsilon}{2}\ .$$

Thus,

$$\lim_{x_0\to\infty}\limsup_{x\to\infty}\frac{\mathbb{P}_y(\beta(x_0)<\beta(x)<H(y),S_{\beta(x)-1}\le u)}{V(x)}\ge\frac{\epsilon}{2}\ .$$

Conditioning on the value of $S_{\beta(x)-1}$, we obtain

$$\begin{aligned}&\mathbb{P}_y(\beta(x_0)<\beta(x)<H(y),S_{\beta(x)-1}\le u)\\&\quad\le\mathbb{P}_y(\beta(x_0)<H(y))(1+\mathbb{E}H(y))\mathbb{P}_0(\beta(x-u)<H)\end{aligned}$$

and then by (2.6)

$$\lim_{x_0\to\infty}\limsup_{x\to\infty}\frac{\mathbb{P}_y(\beta(x_0)<\beta(x)<H(y),S_{\beta(x)-1}\le u)}{V(x)}=0\ ,$$

which yields the contradiction and proves the lemma.

□

Using the method of cutting interarrival times given in the proof of Corollary 2.2 of Heath *et al.* [20] one can prove that

$$\mathbb{P}_y(h>x)\sim\mathbb{P}_y(\max_{k\le H(y)}S_k>x)$$

for regularly varying distribution tail $V(\cdot)$. Now, using Lemma 2.2 as a generalization of Lemma 2.3 of Asmussen [5] it is easy to prove the following generalization of Lemma 2.2 of Asmussen [5].

Corollary 2.3 *Assume that $V(\cdot)$ is regularly varying. Then for any y we have*

$$\mathbb{P}_y(h>x)\sim\mathbb{P}_y(\beta(x)\le H(y))\sim\mathbb{E}H(y)V(x)\ .$$

Corollary 2.4 *Assume that* **(R)** *holds. Then*

$$\lim_{x_0\to\infty}\limsup_{x\to\infty}\frac{\mathbb{P}^{(\sigma_0,\tau_0)}(\beta(x_0)<\beta(x)<H)}{\overline{F}(x)}=0\ .$$

Proof. The assertion is a simple corollary of Lemma 2.2 if condition **(R.1)** holds. Note that for some constant c we have

$$\begin{aligned}&\mathbb{P}^{(\sigma_0,\tau_0)}(\beta(x_0)<\beta(x)<H)\le\mathbb{P}^{(\sigma_0,\tau_0)}(\beta(x_0)=1,1<\beta(x)<H)\\&\quad+\mathbb{P}^{(\sigma_0,\tau_0)}(1<\beta(x_0)<\beta(x)<H)\le V_0(x_0)\int_0^x\mathbb{P}_y(\beta(x)\le H(y))\,\mathrm{d}\overline{V}_0^*(y)\\&\quad+\int_0^{x_0}\mathbb{P}_y(\beta(x_0)<\beta(x)<H(y))\,\mathrm{d}\overline{V}_0^*(y)\ .\end{aligned}$$

Now, in case of conditions **(R.2)** and **(R.3)** the assertion of the corollary follows from Lemma 2.2, Corollary 2.3 and the dominated convergence theorem.

□

Proof of Proposition 2.1. Note that

$$\mathbb{P}^{(\sigma_0,\tau_0)}(h > x) = V_0(x) + \int_0^x \mathbb{P}_y(h > x)\, \mathrm{d}\overline{V}_0^*(y)\ . \tag{2.8}$$

This completes the proof of (ii) in view of Corollary 2.3. In case (iii) from (2.8) we have the lower bound:

$$\mathbb{P}^{(\sigma_0,\tau_0)}(h > x) \geq V_0(x)$$

and the upper bound:

$$\mathbb{P}^{(\sigma_0,\tau_0)}(h > x) \leq V_0(x) + \int_0^\infty \mathbb{E}H(y)\, \mathrm{d}\overline{V}_0(y)V(x) = V_0(x) + \mathrm{o}(V_0(x))\ . \tag{2.9}$$

This proves the proposition.

□

Using Corollary 2.4 as a generalization of Lemma 4.2 of Asmussen and Møller [6] in the proof of Theorem 5.3.1 of Zwart [32], one can get the following lemma.

Lemma 2.3 *If* **(R)** *holds, then*

$$\mathbb{P}^{(\sigma_0,\tau_0)}(l > x) \sim \mathbb{P}^{(\sigma_0,\tau_0)}(h > x(1-\rho)) = \overline{F}(x(1-\rho))\ .$$

Corollary 2.5 *Assume that* **(R.1)** *holds. Then there exists a constant c such that*

$$\mathbb{P}(l > x) \leq cV(x)\ ;$$

see Prop. 5.2.1 of Zwart [32].

Now, we prove that a large busy period is always related to the occurrence of a large cycle maximum.

Corollary 2.6 *Assume that* **(R)** *holds. Then for any* $b > \epsilon > 0$,

$$\mathbb{P}^{(\sigma_0,\tau_0)}(l > x(1+\epsilon)|h \in [x(1-\rho), x(1-\rho)(1+b)]) \to 0 \qquad \textit{as } x \to \infty. \tag{2.10}$$

Proof. Assume that (2.10) is not true, i.e. there exist $b > 0$, $\epsilon > 0$ ($b > \epsilon$) and a sequence $x \to \infty$ such that

$$\mathbb{P}^{(\sigma_0,\tau_0)}\left(l > x|h \in \left[x\frac{(1-\rho)}{1+\epsilon}, x\frac{(1-\rho)(1+b)}{1+\epsilon}\right]\right) > p > 0\ . \tag{2.11}$$

Then for any $\delta > 0$ and $\kappa > 0$ we have

$$\begin{aligned}
\mathbb{P}^{(\sigma_0,\tau_0)}(l > x) &\geq \mathbb{P}^{(\sigma_0,\tau_0)}(l > x, h > x(1-\rho+\delta)) \\
&\quad + \mathbb{P}^{(\sigma_0,\tau_0)}(l > x, x(1-\rho)(1-\kappa) \leq h \leq x(1-\rho+\delta)) \\
&\geq \mathbb{P}^{(\sigma_0,\tau_0)}(l_2 > x, h > x(1-\rho+\delta)) \\
&\quad + \mathbb{P}^{(\sigma_0,\tau_0)}(l > x, x(1-\rho)(1-\kappa) \leq h \leq x(1-\rho+\delta)) = I_1 + I_2\ .
\end{aligned}$$

By Corollary 2.2

$$I_1 \geq \mathbb{P}^{(\sigma_0,\tau_0)}(h > x(1-\rho+\delta)) + \mathrm{o}(\overline{F}(x)) = \overline{F}(x(1-\rho+\delta)) + \mathrm{o}(\overline{F}(x)) \ . \tag{2.12}$$

Let $1-\kappa = 1/(1+\epsilon)$. Then for some sequence $x \to \infty$ and δ such that $(1-\rho)(1-\kappa)(1+b) = 1-\rho+\delta$ we have from (2.11)

$$\begin{aligned} I_2 &\geq p\mathbb{P}^{(\sigma_0,\tau_0)}(x(1-\rho)(1-\kappa) \leq h < x(1-\rho+\delta)) \\ &= p\left(\overline{F}(x(1-\rho)(1-\kappa)) - \overline{F}(x(1-\rho+\delta))\right) \ . \end{aligned} \tag{2.13}$$

Hence from (2.12)-(2.13) we have

$$\begin{aligned} \mathbb{P}^{(\sigma_0,\tau_0)}(l > x) &\geq \overline{F}(x(1-\rho+\delta)) + \mathrm{o}(\overline{F}(x)) \\ &\quad + p\left(\overline{F}(x(1-\rho)(1-\kappa)) - \overline{F}(x(1-\rho+\delta))\right) \\ &= (1-p)\overline{F}(x(1-\rho+\delta)) + p\overline{F}(x(1-\rho)(1-\kappa)) \geq (1+\tilde{p})\overline{F}(x(1-\rho)) \end{aligned}$$

for a sequence $x \to \infty$ and $\tilde{p} > 0$. This contradicts Lemma 2.3.

□

Corollary 2.6 implies that the epoch of reaching the cycle maximum l_1 is small in comparison to the busy period l.

Corollary 2.7 *Assume that* **(R)** *holds. Then for any $b > \gamma/2 > 0$,*

$$\mathbb{P}^{(\sigma_0,\tau_0)}\left(l_1 > \gamma x | h \in [x(1-\rho), x(1-\rho)(1+b)]\right) \to 0 \qquad \text{as } x \to \infty.$$

Proof. Note that from Corollary 2.2 for any $\delta, \kappa > 0$ and sufficiently large x

$$\mathbb{P}^{(\sigma_0,\tau_0)}\left(l_2 \leq \frac{h}{(1-\rho+\kappa)} | h \in [x(1-\rho), x(1-\rho)(1+b)]\right) \leq \delta \ .$$

Choosing $\tilde{\kappa}$ such that $1-\tilde{\kappa} = (1-\rho)/(1-\rho+\kappa)$ we have

$$\begin{aligned} &\mathbb{P}^{(\sigma_0,\tau_0)}\left(l_1 > \gamma x | h \in [x(1-\rho), x(1-\rho)(1+b)]\right) \\ &\quad \leq \mathbb{P}^{(\sigma_0,\tau_0)}\left(l_1 > \gamma x, l_2 > (1-\tilde{\kappa})x | h \in [x(1-\rho), x(1-\rho)(1+b)]\right) + \mathrm{o}(1) \\ &\quad \leq \mathbb{P}^{(\sigma_0,\tau_0)}\left(l > (1-\tilde{\kappa}+\gamma)x | h \in [x(1-\rho), x(1-\rho)(1+b)]\right) + \mathrm{o}(1) \to 0 \end{aligned}$$

as $x \to \infty$ for $\tilde{\kappa} \leq \gamma/2$ by Corollary 2.6.

□

3 Regular varying asymptotics of $\mathbb{P}^{(\sigma_0,\tau_0)}(\mathcal{A} > x)$

The main theorem of this section is the following.

Theorem 3.1 *Assume that* **(R)** *holds. Then we have the following asymptotics:*

$$\mathbb{P}^{(\sigma_0,\tau_0)}(\mathcal{A} > x) \sim \overline{F}\left(\sqrt{2(1-\rho)x}\right) \qquad \text{as } x \to \infty \ . \tag{3.1}$$

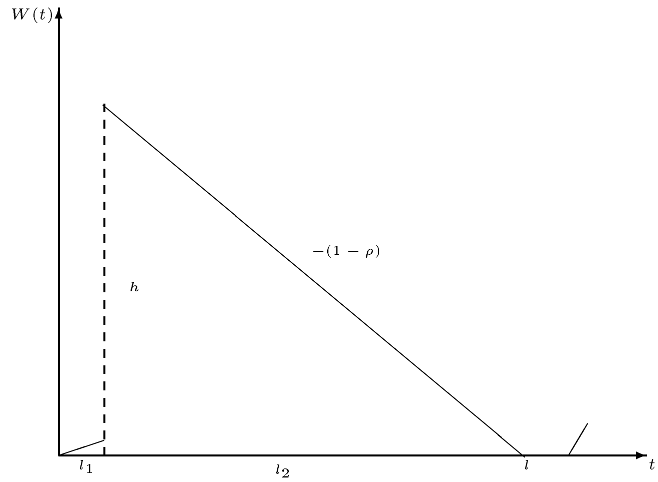

Figure 1: The heavy-tailed case

Before giving the proof of Theorem 3.1 we provide some heuristic arguments. Roughly, in the regularly varying case the most likely way for the area $\mathcal{A}$ to be large is that one early big service time occurs and apart from this, everything in the cycle develops normally. Using the law of large numbers and ignoring random fluctuations, this leads to the conclusion that the workload goes to zero with negative rate $-(1-\rho)$ (see Figure 1). Thus the area $\mathcal{A}$ exceeds level x iff the area of the triangle with the sides h and $l \sim h/(1-\rho)$ is greater than x, hence when

$$\frac{1}{2}\frac{h^2}{1-\rho} > x \ ,$$

which gives the assertion of the main theorem. The proof will be divided over the two Theorems 3.2 and 3.3.

Theorem 3.2 *Assume that* **(R)** *holds. Then we have the following asymptotics:*

$$\liminf_{x\to\infty} \frac{\mathbb{P}^{(\sigma_0,\tau_0)}(\mathcal{A} > \frac{1}{2}\frac{x^2}{1-\rho})}{\overline{F}(x)} \geq 1 \ . \tag{3.2}$$

Proof. The following lower bound holds:

$$\begin{aligned} &\mathbb{P}^{(\sigma_0,\tau_0)}\left(\mathcal{A} > \frac{1}{2}\frac{x^2}{1-\rho}\right) \\ &\quad \geq \mathbb{P}^{(\sigma_0,\tau_0)}\left(\mathcal{A}_h > \frac{1}{2}\frac{x^2}{1-\rho} | h > x(1+\epsilon)\right)\mathbb{P}^{(\sigma_0,\tau_0)}(h > x(1+\epsilon)) \ , \end{aligned} \tag{3.3}$$

for some $\epsilon > 0$. From Corollary 2.1 for any $\kappa, \delta > 0$ and sufficiently large x we have

$$\mathbb{P}^{(\sigma_0,\tau_0)}\left(\mathcal{A}_h > \frac{1}{2}\frac{x^2}{1-\rho}(1+\epsilon)^2(1-\kappa) | h > x(1+\epsilon)\right) \geq 1-\delta \ .$$

For small κ,

$$\frac{1}{2}\frac{x^2}{1-\rho}(1+\epsilon)^2(1-\kappa) \geq \frac{1}{2}\frac{x^2}{1-\rho}$$

and hence

$$\mathbb{P}^{(\sigma_0,\tau_0)}\left(\mathcal{A}_h > \frac{1}{2}\frac{x^2}{1-\rho}|h > x(1+\epsilon)\right) \to 1 \qquad \text{as } x \to \infty.$$

The proof of the theorem follows from (3.3) by letting $\epsilon \to 0$.

□

Theorem 3.3 *Assume that* **(R)** *holds. Then we have the following asymptotics:*

$$\limsup_{x\to\infty} \frac{\mathbb{P}^{(\sigma_0,\tau_0)}(\mathcal{A} > \frac{1}{2}\frac{x^2}{1-\rho})}{\overline{F}(x)} \leq 1\ . \tag{3.4}$$

Proof. Let $\epsilon > 0$. Note that

$$\begin{aligned}
&\mathbb{P}^{(\sigma_0,\tau_0)}\left(\mathcal{A} > \frac{1}{2}\frac{x^2}{1-\rho}\right) \leq \mathbb{P}^{(\sigma_0,\tau_0)}\left(h > (1-\epsilon)x\right) \\
&+\mathbb{P}^{(\sigma_0,\tau_0)}\left(\mathcal{A} > \frac{1}{2}\frac{x^2}{1-\rho}, \epsilon x < h \leq (1-\epsilon)x\right) + \mathbb{P}^{(\sigma_0,\tau_0)}\left(\mathcal{A} > \frac{1}{2}\frac{x^2}{1-\rho}, h \leq \epsilon x\right) \\
&= R_1 + R_2 + R_3\ .
\end{aligned}$$

Term R_1 gives the needed asymptotics by letting $\epsilon \to 0$. We prove that the other two terms, when divided by $\overline{F}(x)$, tend to 0 as $x \to \infty$. We start with R_2. Choose $\gamma > 0$. Observe now that

$$\begin{aligned}
R_2 &= \mathbb{P}^{(\sigma_0,\tau_0)}\left(\mathcal{A} > \frac{1}{2}\frac{x^2}{1-\rho}, l_1 > \gamma x, \epsilon x < h \leq (1-\epsilon)x\right) \\
&+\mathbb{P}^{(\sigma_0,\tau_0)}\left(\mathcal{A} > \frac{1}{2}\frac{x^2}{1-\rho}, l_1 \leq \gamma x, \epsilon x < h \leq (1-\epsilon)x\right) \\
&\leq \mathbb{P}^{(\sigma_0,\tau_0)}\left(l_1 > \gamma x, \epsilon x < h \leq (1-\epsilon)x\right) \\
&+\mathbb{P}^{(\sigma_0,\tau_0)}\left(\mathcal{A} > \frac{1}{2}\frac{x^2}{1-\rho}, l_1 \leq \gamma x, \epsilon x < h \leq (1-\epsilon)x\right) = R_{2a} + R_{2b}\ .
\end{aligned}$$

From Corollary 2.7 we get that for small $\gamma > 0$ the term R_{2a} divided by $\overline{F}(x)$ tends to 0 as $x \to \infty$.

Denote by $\mathcal{A}_{[0,t]} := \int_0^t W(s)\ \mathrm{d}s$ the area swept under the workload process up to time t. To deal with R_{2b} note that

$$\begin{aligned}
&\mathbb{P}^{(\sigma_0,\tau_0)}\left(\mathcal{A} > \frac{1}{2}\frac{x^2}{1-\rho}, l_1 \leq \gamma x, \epsilon x < h \leq (1-\epsilon)x\right) \\
&\leq \mathbb{P}^{(\sigma_0,\tau_0)}\left(\mathcal{A} > \frac{1}{2}\frac{x^2}{1-\rho}, \mathcal{A}_{[0,l_1]} \leq \gamma x^2, \epsilon x < h \leq (1-\epsilon)x\right) \\
&\leq \mathbb{P}^{(\sigma_0,\tau_0)}\left(\mathcal{A}_h \geq x^2(\frac{1}{2(1-\rho)} - \gamma), h \leq (1-\epsilon)x | h > \epsilon x\right) \mathbb{P}^{(\sigma_0,\tau_0)}(h > \epsilon x)\ .
\end{aligned}$$

From Corollary 2.1 for any small $\kappa, \delta > 0$ and sufficiently large x we have that

$$\begin{aligned}
&\mathbb{P}^{(\sigma_0,\tau_0)}\left(\mathcal{A}_h \geq x^2(\frac{1}{2(1-\rho)} - \gamma), h \leq (1-\epsilon)x | h > \epsilon x\right) \\
&\leq \mathbb{P}^{(\sigma_0,\tau_0)}\left(\mathcal{A}_h \geq x^2(\frac{1}{2(1-\rho)} - \gamma), \mathcal{A}_h \leq x^2\frac{1}{2(1-\rho)}(1-\epsilon)^2(1+\kappa), h \leq (1-\epsilon)x | h > \epsilon x\right) + \delta\ .
\end{aligned}$$

For sufficiently small $\gamma > 0$ and $\kappa > 0$ we have

$$\frac{1}{2(1-\rho)} - \gamma > \frac{1}{2(1-\rho)}(1-\epsilon)^2(1+\kappa) \ ,$$

and hence $R_{2b} \leq \delta \mathbb{P}^{(\sigma_0,\tau_0)}(h > \epsilon x)$. This makes R_{2b} divided by $\overline{F}(x)$ tend to 0 as $x \to \infty$ by letting $\delta \to 0$.

Now, we have

$$R_3 \leq \mathbb{P}^{(\sigma_0,\tau_0)}\left(lh > \frac{1}{2}\frac{x^2}{1-\rho}, h \leq \epsilon x\right) \leq \mathbb{P}^{(\sigma_0,\tau_0)}\left(l > \frac{1}{2}\frac{x}{\epsilon(1-\rho)}\right) \ .$$

Hence from Lemma 2.3 for some constant c we have

$$R_3 \leq c\overline{F}\left(\frac{1}{2\epsilon}x\right) \ .$$

The proof of the theorem now follows by letting $\epsilon \to 0$.

□

4 Light-tailed asymptotics of $\mathbb{P}^{(\sigma_0,\tau_0)}(\mathcal{A} > x)$

In this section we consider a service time σ that is light-tailed. Let $\tau_0 \overset{D}{=} \tau$. We assume that the distributions are such that there exists $\eta > 0$ fulfilling

$$\mathbb{E}e^{\eta\sigma_0} < \infty, \quad \mathbb{E}e^{\eta(\sigma-\tau)} = 1 \quad \text{and} \quad \mathbb{E}|\sigma-\tau|e^{\eta(\sigma-\tau)} < \infty \ . \tag{4.1}$$

Note that if η exists, then it is unique (see Asmussen [4], p. 258 and Miller [26] for further discussion on (4.1)). Introduce

$$T_k^{(0)} := \sigma_0 + (\sigma_1 - \tau_0) + \ldots + (\sigma_{k+1} - \tau_k) = \sigma_0 + T_{k+1} \ , \tag{4.2}$$

where

$$T_0 := 0, \quad T_k := \sum_{i=1}^{k}(\sigma_i - \tau_{i-1}) \tag{4.3}$$

is a random walk. Define the new probability space $(\tilde{\mathbb{P}}, \{\mathcal{F}_n\})$ by the exponential change of measure with the likehood ratio up to time n:

$$L_n := \frac{\mathrm{d}\mathbb{P}_{|\mathcal{F}_n}}{\mathrm{d}\tilde{\mathbb{P}}_{|\mathcal{F}_n}} = e^{-\eta T_n} \ .$$

By Proposition 6.3 of Palmowski and Rolski [28] (see also Asmussen [4], p. 258 and 263) on the new probability space the service times $\{\tilde{\sigma}_i\}$ $(i \geq 1)$ and interarrival times $\{\tilde{\tau}_i\}$ $(i \geq 0)$ have generic distribution tails

$$\tilde{V}(x) := 1 - \int_0^x e^{\eta y} \,\mathrm{d}\overline{V}(y)/\mathbb{E}e^{\eta\sigma}, \tag{4.4}$$

$$\tilde{A}(x) := 1 - \int_0^x e^{-\eta y} \,\mathrm{d}\overline{A}(y)/\mathbb{E}e^{-\eta\tau}, \tag{4.5}$$

respectively. We show that for large x conditioning on event $\{h > x\}$ the system up to the time of reaching the cycle maximum behaves like the new system with the service and the interarrival time distribution tails given in (4.4) and (4.5), respectively (see Asmussen [3] for the similar result when we are conditioning on the event $\{\tau(x) < \infty\}$, where $\tau(x) := \inf\{t \geq 0 : W(t) > x\}$). By $\tilde{\mathbb{E}}$ we mean the expectation with respect to $\tilde{\mathbb{P}}$. Let

$$\tilde{\rho} := \tilde{\mathbb{E}}\sigma / \tilde{\mathbb{E}}\tau \tag{4.6}$$

be the traffic load in the new system. From Asmussen [4], p. 259, a drift of the workload process $\{W(t), t \geq 0\}$ is positive, that is $\tilde{\rho} > 1$. Note that in the $M|M|1$, creating the new system corresponds to interchanging the service and the interarrival intensities, leading to a traffic load equal to $\tilde{\rho} = 1/\rho > 1$. We define

$$D_1 := \{W(u) \leq \sqrt{x} + u(\tilde{\rho} - 1)(1 + \epsilon); 0 \leq u \leq l_1\} ,$$

$$D_2 := \{W(u) \geq -\sqrt{x} + u(\tilde{\rho} - 1)(1 - \epsilon); 0 \leq u \leq l_1\} .$$

Let

$$\alpha(x) := \min\{k \geq 1 : T_k > x\}, \quad \alpha_H(x) := \min\{\alpha(x), H - 1\} .$$

Introducing the overshoot $\xi(x) := T_{\alpha(x)} - x$, we get using the renewal theorem (see Theorem 2.1 of Asmussen [4]) that $\xi(x) \xrightarrow{D} \xi(\infty)$ and (with $1 - V_0(y) = \overline{V}_0(y) = \mathbb{P}(\sigma_0 \leq y)$):

$$\begin{aligned}
\mathbb{P}^{(\sigma_0,\tau_0)}(h > x) &= \int_0^\infty \mathbb{P}^y(\alpha_H(x-y) = \alpha(x-y) < \infty)\, \mathrm{d}\overline{V}_0(y) \\
&= \int_0^\infty \tilde{\mathbb{E}}^y \left[L_{\alpha(x-y)}; T_{\alpha_H(x-y)} > x\right] \mathrm{d}\overline{V}_0(y) \\
&= e^{-\eta x} \int_0^\infty e^{\eta y} \tilde{\mathbb{E}}^y \left[e^{-\eta \xi(x)}; \alpha(x-y) < H - 1\right] \mathrm{d}\overline{V}_0(y) \sim C e^{-\eta x} ,
\end{aligned} \tag{4.7}$$

where

$$C = \mathbb{E} e^{\eta \sigma_0} \tilde{\mathbb{E}} \left[e^{-\eta \xi(\infty)}\right] \tilde{\mathbb{P}}(H = \infty) , \tag{4.8}$$

and the last steps follow from the dominated convergence theorem and assumptions (4.1). This result is a classical one in the light-tailed case and was obtained by Iglehart [23] (see also Bartfai [7], Cramér's estimate [15], Kelly [24], Kingman [25] and Ross [30] for the uniform bounds in x). Note that $D_i \in \mathcal{F}_{\alpha_H(x)}$. Hence,

$$\begin{aligned}
\mathbb{P}^{(\sigma_0,\tau_0)}(D_i^c; h > x) &= \int_0^\infty \mathbb{P}^y(D_i^c; \alpha_H(x-y) = \alpha(x-y) < \infty)\, \mathrm{d}\overline{V}_0(y) \\
&= \int_0^\infty \tilde{\mathbb{E}}^y \left[L_{\alpha(x-y)}; D_i^c \cap \{T_{\alpha_H(x-y)} > x\}\right] \mathrm{d}\overline{V}_0(y) \\
&\leq e^{-\eta x} \int_0^\infty \tilde{\mathbb{P}}^y \left(D_i^c \cap \{\alpha(x-y) < H - 1\}\right) e^{\eta y}\, \mathrm{d}\overline{V}_0(y) .
\end{aligned} \tag{4.9}$$

From the law of large numbers for the process $\{W(t), t \geq 0\}$ in the new system for any $\delta > 0$ and for sufficiently large x we have

$$\tilde{\mathbb{P}}^y(D_i^c \cap \{\alpha(x) < H - 1\}) \leq \delta . \tag{4.10}$$

Thus from (4.9), the assumptions (4.1) and the dominated convergence theorem for any $\delta > 0$ and sufficiently large x we get the following inequality:

$$\mathbb{P}^{(\sigma_0,\tau_0)}(D_i^c; h > x) \leq \delta e^{-\eta x} . \tag{4.11}$$

Hence from (4.7) and (4.11) we have the following lemma.

Lemma 4.1 *For any δ there exists L such that for all $x \geq L$ we have*

$$\mathbb{P}^{(\sigma_0,\tau_0)}(D_1 \cap D_2 | h > x) \geq 1 - \delta \ . \tag{4.12}$$

Corollary 4.1 *For any $\delta, \kappa > 0$ and sufficiently large x the following holds*

$$\mathbb{P}^{(\sigma_0,\tau_0)}\left(1 - \kappa \leq \frac{\mathcal{A}_{[0,l_1]}}{h^2/(2(\tilde{\rho}-1))} \leq 1 + \kappa | h > x\right) \geq 1 - \delta \ .$$

From (4.7) and Corollaries 2.1 and 4.1 we get the main theorem of this section.

Theorem 4.1 *Assume that (4.1) holds. Then we have the following asymptotics:*

$$\mathbb{P}^{(\sigma_0,\tau_0)}(\mathcal{A} > x) \sim C e^{-\eta\sqrt{2\frac{(1-\rho)(\tilde{\rho}-1)}{\tilde{\rho}-\rho}x}} \qquad \text{as } x \to +\infty \ , \tag{4.13}$$

where C is given in (4.8), η solves

$$\mathbb{E}e^{\eta(\sigma-\tau)} = 1 \tag{4.14}$$

and $\tilde{\rho}$ is the traffic load in the new system with the service and the interarrival distribution tails given in (4.4) and (4.5), respectively.

Remark 4.1 Note that only constant C in the asymptotics (4.13) depends on the distribution of σ_0.

Thus the tail of the probability survivor function $\mathbb{P}^{(\sigma_0,\tau_0)}(\mathcal{A} > x)$ does not exhibit an exponential decay, but r.v. $\mathcal{A}$ has a Weibull-like tail. Note that the area $\mathcal{A}$ is large as a consequence of the big cycle maximum which is realized, by the large deviation principle, along "the most likely path", that is the workload develops along the line with a slope $\tilde{\rho} - 1$. After that, everything in the cycle develops normally. Hence by the law of large numbers and ignoring random fluctuations, the workload goes to zero with negative rate $-(1-\rho)$ (see Figure 2). Thus the area $\mathcal{A}$ exceeds the level x iff the area of the triangle with the height h and the side $l \sim h(1/(1-\rho) + 1/(\tilde{\rho}-1))$ is greater than x, hence when

$$\frac{1}{2}h^2\frac{\tilde{\rho}-\rho}{(1-\rho)(\tilde{\rho}-1)} > x \ ,$$

which gives the assertion of the main theorem in view of (4.7).

5 Sequential approximation

Let $\sigma_0 \stackrel{D}{=} \sigma$ and $\tau_0 \stackrel{D}{=} \tau$. At an arrival epoch, let $\mathcal{A}(x)$ be the remaining area under the workload process in the $GI|G|1$ queue in the busy period under the condition that the arriving customer sees an amount of work x. That is, we assume that $W(0) = x + \sigma_0$. Note that $\mathcal{A} = \mathcal{A}(0)$. Denote by

$$\psi(s,x) := \mathbb{E}e^{-s\mathcal{A}(x)} \ , \qquad s \geq 0 \ .$$

In this section we will find a sequential approximation for $\psi(s,x)$. Let

$$\begin{aligned}\omega_1(s,x) &:= \mathbb{E}\left[e^{-s\mathcal{A}(x)}; x + \sigma_0 < \tau_0\right] = \mathbb{E}\left[e^{-\frac{s}{2}(x+\sigma_0)^2}; x + \sigma_0 < \tau_0\right] \\ &= \int_{u_0=0}^{\infty} A(x+u_0)e^{-\frac{s}{2}(x+u_0)^2}\, \mathrm{d}\overline{V}(u_0) \ .\end{aligned}$$

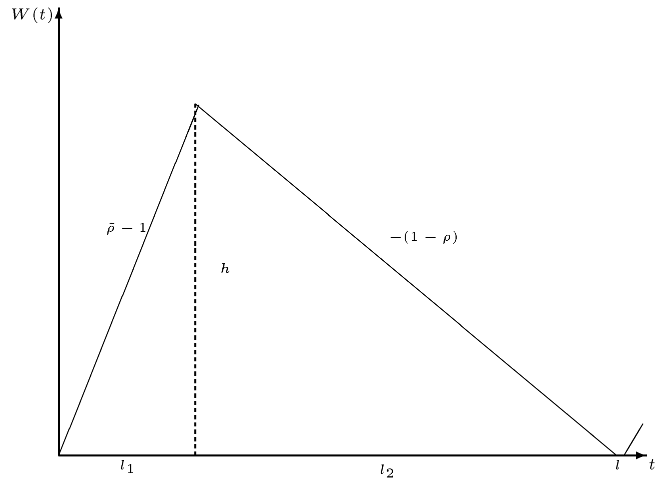

Figure 2: The light-tailed case

Remark 5.1 In the $M|M|1$ queue with arrival rate λ and service rate μ we have

$$\omega_1(s,x) = \mu e^{\mu x} e^{\frac{(\lambda+\mu)^2}{2s}} \int_0^\infty e^{-\frac{s}{2}(x+u_0+\frac{\lambda+\mu}{s})^2} du_0 = \sqrt{2\pi}\mu e^{\mu x} e^{\frac{(\lambda+\mu)^2}{2s}} \mathbb{P}\left(U > x + \frac{\lambda+\mu}{s}\right) ,$$

where $U \overset{D}{=} \mathrm{N}(0, 1/\sqrt{s})$.

Conditioning on whether the busy period ends after $\sigma_0 + x$ or not, it is readily seen that $\psi(s,x)$ satisfies the following integral equation:

$$\begin{aligned}\psi(s,x) &= \mathbb{E}\left[e^{-s\mathcal{A}(x)}; x+\sigma_0 < \tau_0\right] + \mathbb{E}\left[e^{-s\{(x+\sigma_0)\tau_0 - \frac{1}{2}\tau_0^2 + \mathcal{A}(x+\sigma_0-\tau_0)\}}; x+\sigma_0 \geq \tau_0\right] \\ &= \omega_1(s,x) + \int\int_{x+u_0\geq t_1} \mathrm{d}\overline{A}(t_1)\,\mathrm{d}\overline{V}(u_0)\omega_2(s,x,u_0,t_1)\psi(s,x+u_0-t_1) \ , \qquad (5.1)\end{aligned}$$

with

$$\omega_2(s,x,w,t) := e^{-s\{(x+w)t-\frac{1}{2}t^2\}} \ .$$

Notice that $(x+u_0)t_1 - \frac{1}{2}t_1^2$ is the area under the workload process for the interarrival interval of length t_1 when $x + u_0 \geq t_1$. We iterate (5.1), starting with $\psi_0(s,x) = 0$ and writing

$$\psi_{n+1}(s,x) = \omega_1(s,x) + \int\int_{x+u_0\geq t_1} \mathrm{d}\overline{A}(t_1)\,\mathrm{d}\overline{V}(u_0)\omega_2(s,x,u_0,t_1)\psi_n(s,x+u_0-t_1) \ . \quad (5.2)$$

Then

$$\begin{aligned}\psi_0(s,x) &= 0, \\ \psi_1(s,x) &= \omega_1(s,x), \\ \psi_2(s,x) &= \omega_1(s,x) + \int\int_{x+u_0\geq t_1} \mathrm{d}\overline{A}(t_1)\,\mathrm{d}\overline{V}(u_0)\omega_2(s,x,u_0,t_1)\psi_1(s,x+u_0-t_1), \\ \psi_3(s,x) &= \omega_1(s,x) + \int\int_{x+u_0\geq t_1} \mathrm{d}\overline{A}(t_1)\,\mathrm{d}\overline{V}(u_0)\omega_2(s,x,u_0,t_1)\psi_1(s,x+u_0-t_1) \\ &+ \int\int_{x+u_0\geq t_1} \mathrm{d}\overline{A}(t_1)\,\mathrm{d}\overline{V}(u_0)\omega_2(s,x,u_0,t_1)\int\int_{x+u_0-t_1+u_1\geq t_2} \mathrm{d}\overline{A}(t_2)\,\mathrm{d}\overline{V}(u_1) \\ &\quad \omega_2(s,x+u_0-t_1,u_1,t_2)\psi_1(s,x+u_0-t_1+u_1-t_2),\end{aligned}$$

and so on.

Theorem 5.1 *We have*

$$\lim_{n\to\infty} \psi_n(s,x) = \psi(s,x) \quad \textit{for all } s,x \geq 0 \ . \tag{5.3}$$

In particular, if **(R.1)** *holds, then there exists a constant* c_1 *such that*

$$|\psi(s,x) - \psi_n(s,x)| \leq c_1 V\left((\mathbb{E}\tau - \mathbb{E}\sigma)n\right) \quad \textit{for } s \geq 0. \tag{5.4}$$

If σ *is light-tailed, that is, there exists a* $\theta > 0$ *such that* $\vartheta(\theta) = \mathbb{E}e^{\theta(\sigma-\tau)} < \infty$*, then there exists a constant* c_2 *such that*

$$|\psi(s,x) - \psi_n(s,x)| \leq \frac{c_2}{n^{3/2}} e^{-n\vartheta} \quad \textit{for } s \geq 0, \tag{5.5}$$

where $\vartheta = -\log \inf_{\theta \geq 0} \vartheta(\theta)$.

Proof. Notice that $\psi_n(s,x)$ is a sum of n terms. A crucial observation is that these terms can be interpreted in the following way:

$$\psi_n(s,x) = \sum_{j=1}^{n} Q_j(s,x) \ , \tag{5.6}$$

where

$$Q_j(s,x) = \mathbb{E}\left[e^{-s\mathcal{A}(x)}; H(x) = j\right], \qquad j = 1,2,\ldots \ .$$

Under the assumption that for any fixed x, $\mathbb{E}\sigma < \mathbb{E}\tau$, we have $\mathbb{P}(H(x) < \infty) = 1$ and $\mathbb{E}H(x) < \infty$; see Asmussen [4], Prob. 2.3, p. 171. Consider

$$|\psi(s,x) - \psi_n(s,x)| = \left|\mathbb{E}\left[e^{-s\mathcal{A}(x)}; H(x) > n\right]\right| \leq \mathbb{P}(H(x) > n) \ .$$

Hence (5.3) holds. The convergence rate is bounded by the rate at which $\mathbb{P}(H(x) > n) \to 0$ for $n \to \infty$. Note that

$$\mathbb{P}(H(x) > n) \leq \mathbb{P}(\overline{Y}_n \leq x) \ ,$$

where $\overline{Y}_n = \max_k(-S_k)$ for random walk S_k defined in (2.1). The inequalities (5.4)-(5.5) now follow from Theorems 2.5-2.8 of Borovkov [11].

□

In fact, we can sharpen (5.4), proving geometric convergence whenever $s > 0$ or $\mathbb{P}(x + \sigma > \tau) < 1$.

Theorem 5.2 *Assume that* $C := \mathbb{E}[e^{-\frac{s}{2}\tau^2}; x + \sigma > \tau] < 1$*. Then*

$$|\psi(s,x) - \psi_n(s,x)| \leq C^{n-1} \ . \tag{5.7}$$

Proof. Define operator $f \to Kf$ by

$$(Kf)(s,x) := \int_{x+w\geq t} \omega_2(s,x,w,t)\ \mathrm{d}\overline{A}(t)\ \mathrm{d}\overline{V}(w) f(s, x+w-t) \ .$$

Note that

$$\omega_1(s,x) \leq 1 \tag{5.8}$$

and

$$\omega_2(s,x,w,t) \le e^{-\frac{s}{2}t^2} \quad \text{for } x+w \ge t. \tag{5.9}$$

Hence, if $|f(s,x)| \le L$, then

$$|(Kf)(s,x)| \le L \int_{x+w\ge t} e^{-\frac{s}{2}t^2} \mathrm{d}\overline{A}(t)\, \mathrm{d}\overline{V}(w) \le LC\ . \tag{5.10}$$

The iteration (5.2) with $\psi_0(s,x)=0$ then yields:

$$\begin{aligned}\psi_{n+1}(s,x) &= \omega_1(s,x) + (K\omega_1)(s,x) \\ &\quad +(K^2\omega_1)(s,x) + \ldots + (K^n\omega_1)(s,x)\ ,\end{aligned}$$

where $(K^1 f) := Kf$ and $K^n f := K(K^{n-1}f)$ for some function $f(\cdot,\cdot)$. Note that

$$\psi_{n+1}(s,x) - \psi_n(s,x) = (K^n\omega_1)(s,x)\ .$$

Now, the assertion follows from (5.8) and (5.10).

□

6 $M|M|1$ queue

In this section we restrict ourselves to the special case of the $M|M|1$ queue. In this case, we can find an explicit expression for the LST of the joint distribution of $\mathcal{A}$ and l. Let the service time σ and the interarrival time τ be exponentially distributed with parameters μ and λ, respectively. Denote

$$\phi(s,r,x) := \mathbb{E}_x e^{-s\mathcal{A}-rl}\ , \qquad r,s \ge 0\ .$$

Then $\phi(s) = \mathbb{E}e^{-s\mathcal{A}} = \mu\int_0^\infty \phi(s,0,x)e^{-\mu x}\,\mathrm{d}x$. Let $\rho = \lambda/\mu < 1$. Considering infinitesimal changes we can write the equation:

$$\begin{aligned}\phi(s,r,x+\mathrm{d}x) &= (1-\lambda\mathrm{d}x)\mathbb{E}_x e^{-s(\mathcal{A}+x\mathrm{d}x+o(\mathrm{d}x))-r(l+\mathrm{d}x+o(\mathrm{d}x))} \\ &\quad +\lambda\mathrm{d}x\int_0^\infty \mu e^{-\mu y}\mathbb{E}_{x+y}e^{-s\mathcal{A}}\,\mathrm{d}y\ ,\end{aligned}$$

which is equivalent to

$$\phi(s,r,x+\mathrm{d}x) = (1-\lambda\mathrm{d}x)(1-sx\mathrm{d}x - r\mathrm{d}x)\phi(s,r,x) + \lambda\mathrm{d}x\int_0^\infty \mu e^{-\mu y}\phi(s,r,x+y)\,\mathrm{d}y\ .$$

This gives the following integro-differential equation:

$$\frac{\partial}{\partial x}\phi(s,r,x) + (\lambda+r+sx)\phi(s,r,x) = \lambda\mu e^{\mu x}\int_x^\infty e^{-\mu z}\phi(s,r,z)\,\mathrm{d}z\ ,$$

or

$$\frac{\partial^2}{\partial x^2}\phi(s,r,x) + (\lambda-\mu+r+sx)\frac{\partial}{\partial x}\phi(s,r,x) - (s\mu x + r\mu - s)\phi(s,r,x) = 0\ . \tag{6.1}$$

According to Abramowitz and Stegun [2], the general solution of equation (6.1) is:

$$\begin{aligned}\phi(s,r,x) &= e^{-\frac{1}{4}(2\lambda-2\mu+2r+sx)}\left\{ C_1(s)\frac{\mathrm{WW}(\frac{1}{4}\frac{s+2\lambda\mu}{s},\frac{1}{4},\frac{1}{2}\frac{(sx+\mu+\lambda+r)^2}{s})}{\sqrt{\frac{sx+\lambda+\mu+r}{\sqrt{s}}}}\right. \\ &\quad \left. + C_2(s)\frac{\mathrm{WM}(\frac{1}{4}\frac{s+2\lambda\mu}{s},\frac{1}{4},\frac{1}{2}\frac{(sx+\mu+\lambda+r)^2}{s})}{\sqrt{\frac{sx+\lambda+\mu+r}{\sqrt{s}}}}\right\}\ ,\end{aligned}$$

where the constants $C_1(s)$ and $C_2(s)$ do not depend on x and $\mathrm{WM}(a,b,z)$, $\mathrm{WW}(a,b,z)$ are Whittaker's functions (see Abramowitz and Stegun [2], p. 505). Note that

$$\mathrm{WM}(a,b,z) = e^{-\frac{1}{2}z} z^{\frac{1}{2}+b} \mathrm{M}(\frac{1}{2}+b-a, 1+2b, z), \tag{6.2}$$

$$\mathrm{WW}(a,b,z) = e^{-\frac{1}{2}z} z^{\frac{1}{2}+b} \mathrm{U}(\frac{1}{2}+b-a, 1+2b, z)\ , \tag{6.3}$$

where $\mathrm{M}(a,b,z)$ and $\mathrm{U}(a,b,z)$ are Kummer's functions. From Abramowitz and Stegun [2], p. 504 we have the following asymptotics:

$$\mathrm{M}(a,b,z) \sim \frac{\Gamma(b)}{\Gamma(a)} e^z z^{a-b}, \qquad \mathrm{U}(a,b,z) \sim z^{-a}, \qquad \text{as } z \to \infty. \tag{6.4}$$

Thus from (6.2)-(6.4) we have

$$\lim_{x\to\infty} e^{-\frac{1}{4}(2\lambda-2\mu+2r+sx)} \frac{\mathrm{WW}(\frac{1}{4}\frac{s+2\lambda\mu}{s}, \frac{1}{4}, \frac{1}{2}\frac{(sx+\mu+\lambda+r)^2}{s})}{\sqrt{\frac{sx+\lambda+\mu+r}{\sqrt{s}}}} = 0\ ,$$

and

$$\lim_{x\to\infty} e^{-\frac{1}{4}(2\lambda-2\mu+2r+sx)} \frac{\mathrm{WM}(\frac{1}{4}\frac{s+2\lambda\mu}{s}, \frac{1}{4}, \frac{1}{2}\frac{(sx+\mu+\lambda+r)^2}{s})}{\sqrt{\frac{sx+\lambda+\mu+r}{\sqrt{s}}}} = +\infty\ . \tag{6.5}$$

Note that the function $x \to \phi(s,r,x)$ is bounded for all x. Hence $C_2(s) = 0$. Also, $\phi(s,r,0) = 1$ for all s, thus

$$C_1(s) = e^{\frac{1}{2}(\lambda-\mu+r)} \frac{\sqrt{\lambda+\mu+r}}{s^{\frac{1}{4}}\mathrm{WW}(\frac{1}{4}\frac{s+2\lambda\mu}{s}, \frac{1}{4}, \frac{1}{2}\frac{(r+\mu+\lambda)^2}{s})}\ ,$$

and finally using the representation $U(\frac{1}{2}-\frac{1}{2}\nu, \frac{3}{2}, \frac{1}{2}z^2) = z^{\frac{1}{2}-\frac{1}{2}\nu} e^{z^2/4} \mathrm{D}_\nu(z)$ we have

$$\phi(s,r,x) = e^{-\frac{1}{4}sx} \left(\frac{sx+\mu+\lambda+r}{\mu+\lambda+r}\right)^{1+\frac{\lambda\mu}{s}} \frac{\mathrm{D}_{\frac{\lambda\mu}{s}}(\frac{sx+\mu+\lambda+r}{\sqrt{s}})}{\mathrm{D}_{\frac{\lambda\mu}{s}}(\frac{\mu+\lambda+r}{\sqrt{s}})}\ , \tag{6.6}$$

where

$$\mathrm{D}_{-\nu}(z) = e^{-z^2/4} 2^{-\nu/2}\sqrt{\pi}\left\{\frac{1}{\Gamma((\nu+1)/2)}\left(1+\sum_{k=1}^{+\infty}\frac{\nu(\nu+2)\ldots(\nu+2k-2)}{3\cdot5\cdots(2k-1)k!}\left(\frac{z^2}{2}\right)^k\right)\right.$$
$$\left. -\frac{z\sqrt{2}}{\Gamma(\nu/2)}\left(1+\sum_{k=1}^{+\infty}\frac{(\nu+1)(\nu+3)\ldots(\nu+2k-1)}{3\cdot5\cdots(2k+1)k!}\left(\frac{z^2}{2}\right)^k\right)\right\}$$

is a parabolic cylinder function; see Abramowitz and Stegun [2]. From (6.6) one can obtain numerically e.g. the first two moments of the area $\mathcal{A}$ in the $M|M|1$ queue and the covariance $\rho(\mathcal{A}, l)$ between $\mathcal{A}$ and l . Unfortunately, the expression (6.6) is too complicated to obtain moments of $\mathcal{A}$ in explicit form. This can be found in a more general setting using other methods. Applying the regenerative method and the Wald identity, Cohen [13] proved that for the $GI|G|1$ queue

$$\mathbb{E}\mathcal{A} = \frac{\mu_2}{2(1-\rho)^2}\ , \tag{6.7}$$

where $\mu_k = \mathbb{E}\sigma^k$. Using level crossing arguments, Cohen [12] found the second moment of $\mathcal{A}$ for the $M|G|1$ queue:

$$\mathbb{E}\mathcal{A}^2 = \frac{\mu_1^4}{4(1-\rho)^3}\frac{\mu_4}{\mu_1^4} + \frac{4}{3}\frac{\rho\mu_1^4}{(1-\rho)^4}\frac{\mu_2\mu_3}{\mu_1^5} + \frac{5}{4}\frac{\rho^2\mu_1^4}{(1-\rho)^5}\frac{\mu_2^3}{\mu_1^6} . \tag{6.8}$$

Note that for the first (second) moment of the area $\mathcal{A}$ to be finite one needs the second (fourth) moment of the service time to be finite. In the next section we generalize both formulas to the $GI|G|1$ case with general initial service and interarrival time distributions.

7 Central limit theorem

In this section we consider general tail distributions of σ_0 and σ. We prove the following theorem.

Theorem 7.1 *If there exists $\epsilon > 0$ such that $\mathbb{E}\sigma^{2k+\epsilon} < \infty$, then $\mathbb{E}\mathcal{A}^k < \infty$. Moreover, if* **(R)** *is satisfied, then the following equivalence holds*

$$\mathbb{E}^{(\sigma_0,\tau_0)}\mathcal{A}^k < \infty \qquad \Longleftrightarrow \qquad \mathbb{E}\sigma^{2k} < \infty \quad \textit{and} \quad \mathbb{E}\sigma_0^{2k} < \infty . \tag{7.1}$$

Proof. We prove the first statement. The second one (7.1) can be proved in a similar way. If $\mathbb{E}\sigma^{2k+\epsilon} < \infty$, then $\lim_{x\to\infty} x^{2k+\epsilon}V(x) = 0$. Hence

$$V(x) \le V^{(u)}(x) := x^{-2k-\epsilon} ,$$

for large x. Considering the modified queue with regularly varying tail $V^{(u)}(\cdot)$ of the service time, Theorem 3.1 implies that $\mathbb{E}(\mathcal{A}^{(u)})^k < \infty$. Note that $\mathbb{E}(\mathcal{A}^{(u)})^k \ge \mathbb{E}\mathcal{A}^k$. Thus $\mathbb{E}\mathcal{A}^k < \infty$.

□

Introduce $a_l := \mathbb{E}l$, $\sigma_l^2 := \mathbb{E}(l - a_l)^2$ and $a_{\mathcal{A}} := \mathbb{E}\mathcal{A}$, $\sigma_{\mathcal{A}}^2 := \mathbb{E}(\mathcal{A} - a_{\mathcal{A}})^2$. Note that

$$\mathbb{E}(\mathcal{A} - \frac{a_{\mathcal{A}}}{a_l}l)^2 = \left(\frac{a_{\mathcal{A}}}{a_l}\sigma_l\right)^2 + \sigma_{\mathcal{A}}^2 - 2\frac{a_{\mathcal{A}}}{a_l}\sigma_l\sigma_{\mathcal{A}}\rho(\mathcal{A}, l) , \tag{7.2}$$

where $\rho(\mathcal{A}, l)$ is the correlation between $\mathcal{A}$ and l. Denote by $l^{(0)}$ our first cycle period and by $\{l^{(k)}\}_{k=1}^{\infty}$ the next i.i.d. cycle periods. Let $\mathcal{A}^{(0)} := \int_0^{l^{(0)}} W(s)\ \mathrm{d}s$ and $\mathcal{A}^{(k)} := \int_{l^{(k-1)}}^{l^{(k)}} W(s)\ \mathrm{d}s$, $k = 1, 2, 3, \ldots$. Define also

$$\mathcal{A}_0 := 0, \qquad \mathcal{A}_n := \sum_{k=1}^{n} \mathcal{A}^{(k)} .$$

The main theorem of this section is the following.

Theorem 7.2 *Assume that there exists $\epsilon > 0$ such that $\mathbb{E}\sigma_0^{3+\epsilon} < \infty$ and $\mathbb{E}\sigma^{4+\epsilon} < \infty$. Then $\mathcal{A}_{[0,t]}^{(\sigma_0,\tau_0)}$ is asymptotically normal with parameters*

$$\left(\frac{ta_{\mathcal{A}}}{a_l}, t\frac{1}{a_l}\left(\sigma_{\mathcal{A}}^2 + \left(\frac{a_{\mathcal{A}}}{a_l}\sigma_l\right)^2 - 2\frac{a_{\mathcal{A}}}{a_l}\sigma_l\sigma_{\mathcal{A}}\rho(\mathcal{A}, l)\right)\right) . \tag{7.3}$$

Remark 7.1 The assertion of Theorem 7.2 is satisfied if $V(\cdot)$ and $V_0(\cdot)$ are regularly varying and $\mathbb{E}\sigma_0^3 < \infty$ and $\mathbb{E}\sigma^4 < \infty$.

Remark 7.2 Under assumption that $\sigma_{\mathcal{A}} < \infty$ the same assertion is given in Theorem 4.1 of Iglehart [22]. It should be noted that the proof in [22] has a flaw, because it uses the assumption that the time at which the random walk $\{\mathcal{A}_n\}$ is stopped is a stopping time with respect to the filtration generated by $\{\mathcal{A}_n\}$, which is not satisfied in this case.

The proof of Theorem 7.2 is based on Lemmas 7.1-7.3 below. By $h^{(0)}$ we mean the cycle maximum during the first cycle.

Lemma 7.1 *Assume that $V_0(\cdot)$ and $V(\cdot)$ are regularly varying. Then*

$$\mathbb{P}^{(\sigma_0,\tau_0)}(h^{(0)} > x) \le C_1 \left(V_0\left(\frac{x}{2}\right) + V(x) \right), \tag{7.4}$$

$$\mathbb{P}^{(\sigma_0,\tau_0)}(l^{(0)} > t) \le C_2 \left(V_0\left(\frac{t}{2}\right) + V(t) \right), \tag{7.5}$$

for some constants C_1 and C_2 and large x and t.

Proof. The inequality (7.4) follows from (2.8) and Corollary 2.3. To prove (7.5), note that

$$\begin{aligned}\mathbb{P}^{(\sigma_0,\tau_0)}(l^{(0)} > t) &\le V_0\left(\frac{t}{2}\right) \\ &+ \int_{u=0}^{t/2} \int_{s=0}^{u} \mathbb{P}(\sigma_0 \in \mathrm{d}u, \tau_0 \in \mathrm{d}s) \mathbb{P}_u(l^{(0)} > t-s) \;.\end{aligned}$$

Next, we use the representation $l^{(0)} = \sigma_1 + \ldots + \sigma_{H(u)}$. Theorem 43.3 of Borovkov [10] guarantees the existence of a constant c_1 such that $\mathbb{P}(H(u) > n) \le c_1 V(n)$. Theorem 42.2 of Borovkov [10] now gives the existence of constants c_2, c_3 such that $\mathbb{P}_u(l^{(0)} > t-s) \le c_2 V(t-s) \le c_3 V(t)$. This completes the proof of (7.5).

□

By $l_t^{(0)}$ we mean $l_t^{(0)} := \min\{l^{(0)}, t\}$.

Lemma 7.2 *Assume that there exists $\epsilon > 0$ such that $\mathbb{E}\sigma_0^{3+\epsilon} < \infty$ and $\mathbb{E}\sigma^{3+\epsilon} < \infty$. Then*

$$\mathbb{E}^{(\sigma_0,\tau_0)} \left(\int_0^{l_t^{(0)}} W(s)\, \mathrm{d}s \right)^2 = \mathrm{o}(t) \;.$$

Proof. As in the proof of Theorem 7.1, considering dominating tails, we can assume without loss of generality that $V_0(\cdot)$ and $V(\cdot)$ are regularly varying and $\mathbb{E}\sigma_0^3 < \infty$ and $\mathbb{E}\sigma^3 < \infty$. Note that in this case $u^3 V_0(u) \to 0$, $u^3 V(u) \to 0$ as $u \to \infty$ and $\int_t^\infty u^2 V_0(u)\, \mathrm{d}u \to 0$ and $\int_t^\infty u^2 V(u)\, \mathrm{d}u \to 0$ as $t \to \infty$. Hence by Proposition 2.1 also $u^3 \overline{F}(u) \to 0$ as $u \to \infty$ and $\int_t^\infty u^2 \overline{F}(u)\, \mathrm{d}u \to 0$ as $t \to \infty$. Note that

$$\begin{aligned}
&\mathbb{E}^{(\sigma_0,\tau_0)} \left(\int_0^{l_t^{(0)}} W(s)\, \mathrm{d}s \right)^2 \le \mathbb{E}(h^{(0)} l_t^{(0)})^2 \\
&\le 2 \int_0^{t^2} v \mathbb{P}(h^{(0)} l_t^{(0)} > v)\, \mathrm{d}v + 2 \int_{t^2}^{\infty} v \mathbb{P}(h^{(0)} > v/t)\, \mathrm{d}v \\
&\le 2 \int_0^{t^2} v \mathbb{P}(h^{(0)} > \sqrt{v})\, \mathrm{d}v + 2 \int_0^{t^2} v \mathbb{P}(l_t^{(0)} > \sqrt{v})\, \mathrm{d}v + 2 \int_{t^2}^{\infty} v \mathbb{P}(h^{(0)} > v/t)\, \mathrm{d}v \\
&\le 4 \int_0^t u^3 \mathbb{P}(h^{(0)} > u)\, \mathrm{d}u + 4 \int_0^t u^3 \mathbb{P}(l^{(0)} > u)\, \mathrm{d}u + 2t^2 \int_t^\infty u \mathbb{P}(h^{(0)} > u)\, \mathrm{d}u \;,
\end{aligned}$$

which completes the proof in view of the assumptions that were made and Lemma 7.1.

□

Define the random walk

$$Z_0 := 0, \qquad Z_n := \sum_{k=1}^{n} l^{(k)} \ .$$

Let $\nu^{(0)}(t) := 0$ and $\chi(t) := 0$ if $l^{(0)} > t$ and otherwise

$$\nu^{(0)}(t) := \min\{k \geq 1 : l^{(0)} + Z_k > t\}, \qquad \chi(t) := l^{(0)} + Z_{\nu^{(0)}(t)} - t \ .$$

Lemma 7.3 *Assume that there exists $\epsilon > 0$ such that $\mathbb{E}\sigma^{4+\epsilon} < \infty$. Then*

$$\mathbb{E}\left(\int_t^{t+\chi(t)} W(s)\ \mathrm{d}s\right)^2 = \mathrm{o}(t) \ .$$

Proof. As in the previous lemma, by considering dominating tails, we can assume without loss of generality that $V(\cdot)$ is regularly varying and $\mathbb{E}\sigma^4 < \infty$. In this case $u^4 V(u) \to 0$ as $u \to \infty$ and $\int_t^\infty u^3 V(u)\ \mathrm{d}u \to 0$ as $t \to \infty$. By Corollary 2.2 for a small constant $c_1 << 1/\sqrt{1-\rho}$, large v and some constant c_2 we have:

$$\begin{aligned} &\mathbb{P}(lh > v) \leq \mathbb{P}(l > c_1\sqrt{v}) + \mathbb{P}(h > \sqrt{v}/c_1, l \leq c_1\sqrt{v}) \\ &\leq \mathbb{P}(l > c_1\sqrt{v}) + \mathbb{P}(h > \sqrt{v}/c_1, l_2 \leq c_1\sqrt{v}) \leq \mathbb{P}(l > c_1\sqrt{v}) + \mathrm{o}(V(\sqrt{v})) \leq c_2\mathbb{P}(l > \sqrt{v}) \ . \end{aligned}$$

Thus for a suitable constant c_3 we have

$$\begin{aligned} &\mathbb{P}(\int_t^{t+\chi(t)} W(s)\ \mathrm{d}s > v) \leq c_2 \int_0^t \sum_{k=1}^{\infty} \mathbb{P}(Z_k \in \mathrm{d}u)\mathbb{P}(l > \max\{\sqrt{v}, t-u\}) \\ &\leq c_3 \int_0^t \mathbb{P}(l > \max\{\sqrt{v}, u\})\mathrm{d}u \ . \end{aligned}$$

Using Corollary 2.5 and Karamata's Theorem (see Bingham *et al.* [9], Section I.5) this is bounded from above by

$$c_4\left[\sqrt{v}V(\sqrt{v}) + \int_{\sqrt{v}}^t \mathbb{P}(l > u)\mathrm{d}u\right] \leq c_4\left[\sqrt{v}V(\sqrt{v}) + \int_{\sqrt{v}}^\infty \mathbb{P}(l > u)\mathrm{d}u\right] \leq c_5\sqrt{v}V(\sqrt{v}) \ ,$$

for $v < t^2$ and by

$$c_6 t V(\sqrt{v}) \ ,$$

for $v \geq t^2$ and some constants c_4, c_5 and c_6. Note that

$$\begin{aligned} &\mathbb{E}\left(\int_t^{t+\chi(t)} W(s)\ \mathrm{d}s\right)^2 \leq 2\int_0^\infty v\mathbb{P}\left(\int_t^{t+\chi(t)} W(s)\ \mathrm{d}s > v\right)\mathrm{d}v \\ &\leq 2\int_0^{t^2} v\mathbb{P}\left(\int_t^{t+\chi(t)} W(s)\ \mathrm{d}s > v\right)\mathrm{d}v + 2\int_{t^2}^\infty v\mathbb{P}\left(\int_t^{t+\chi(t)} W(s)\ \mathrm{d}s > v\right)\mathrm{d}v \ . \end{aligned}$$

Using the previous considerations and moment assumptions we have

$$\begin{aligned} &\int_0^{t^2} v\mathbb{P}(\int_t^{t+\chi(t)} W(s)\ \mathrm{d}s > v)\mathrm{d}v \leq c_5 \int_0^{t^2} v^{3/2} V(\sqrt{v})\mathrm{d}v \\ &= 2c_5 \int_0^t u^4 V(u)\mathrm{d}u = \mathrm{o}(t) \ , \end{aligned}$$

and

$$\int_{t^2}^{\infty} v\mathbb{P}(\int_t^{t+\chi(t)} W(s)\,\mathrm{d}s > v)\mathrm{d}v \le c_6 t\int_{t^2}^{\infty} vV(\sqrt{v})\mathrm{d}v$$
$$\le 2c_6 t\int_t^{\infty} u^3 V(u)\mathrm{d}u = \mathrm{o}(t)\ ,$$

which completes the proof.

□

Proof of Theorem 7.2. Note that if $\mathbb{E}\sigma^{4+\epsilon} < \infty$ for some $\epsilon > 0$, then Theorem 7.1 implies that $\sigma_{\mathcal{A}}^2 < \infty$ and Corollary 2.5 implies that $\sigma_l^2 < \infty$. Hence by the Schwartz inequality we have $\rho(\mathcal{A}, l) \le \sigma_{\mathcal{A}}\sigma_l < \infty$. Note that by Lemmas 7.2 and 7.3 it is enough to prove a functional CLT for $\mathcal{A}_{\nu(n)}$ with parameters given in (7.3), where

$$\nu(n) := \min\{k \ge 1 : Z_k \ge n\}\ .$$

Although this can be derived from Theorem 2.2 of Gut [19] (see also Gut [18]) we give here another shorter proof using the method of sequential levels.

For $i \to \infty$ the joint distribution of $(\mathcal{A}_i, Z_i)$ is asymptotically normal. We denote it as

$$(\mathcal{A}_i, Z_i) \overset{D}{\sim} (a_{\mathcal{A}} i + \sigma_{\mathcal{A}}\sqrt{i}\zeta_1, a_l i + \sigma_l\sqrt{i}\zeta_2)\ , \tag{7.6}$$

where $\zeta_k \overset{D}{=} \mathrm{N}(0,1)$ $(k = 1,2)$ and the correlation between ζ_1 and ζ_2 is $\rho(\zeta_1, \zeta_2) = \rho(\mathcal{A}, l)$. Put

$$n_1 := n^{1/2+\delta}$$

for some small $\delta > 0$. We define the first level

$$m := \frac{n - n_1}{a_l}\ .$$

Let $A_n^{(1)} := \{|\zeta_2| < c_1 n^\delta\}$. Then $\mathbb{P}(A_n^{(1)}) \to 1$ as $n \to \infty$. Moreover,

$$\mathcal{A}_{\nu(n)} \overset{D}{=} \mathcal{A}_m + \mathcal{A}_{\nu(n-Z_m)} \tag{7.7}$$

and for a suitable constant c_1 in $A_n^{(1)}$ we have

$$\frac{1}{2}n_1 \le n - Z_m \sim n - \frac{a_l(n-n_1)}{a_l} - \sigma_l\sqrt{\frac{n-n_1}{a_l}}\zeta_2 \le \frac{3}{2}n_1\ . \tag{7.8}$$

We now fix ζ_2 and $n - Z_m$ and consider $\mathcal{A}_{\nu(n-Z_m)}$ to which we again apply decomposition (7.7). Put

$$\tilde{n} := n - Z_m, \quad \tilde{n}_1 := \tilde{n}^{1/2+\delta}, \quad \tilde{m} := \frac{\tilde{n} - \tilde{n}_1}{a_l}\ ,$$

and

$$\mathcal{A}_{\nu(\tilde{n})} \overset{D}{=} \mathcal{A}_{\tilde{m}} + \mathcal{A}_{\nu(\tilde{n}-Z_{\tilde{m}})}\ . \tag{7.9}$$

Similarly, for an independent copy $\zeta_2^{(1)}$ of ζ_2 we introduce the set $A_n^{(2)} := \{|\zeta_2^{(1)}| < c_2\tilde{n}^\delta\}$ such that $\mathbb{P}(A_n^{(2)}) \to 1$ as $n \to \infty$. Then for a suitable constant c_2 on this set we have

$$\frac{\tilde{n}_1}{2} \le \tilde{n} - Z_{\tilde{m}} \le \frac{3}{2}\tilde{n}_1\ .$$

Note that for small $\delta > 0$ by (7.8) we have

$$\begin{aligned}\tilde{n} &= n - Z_m = \mathrm{O}(n_1) = \mathrm{O}\left(n^{1/2+\delta}\right), \\ \tilde{\tilde{n}} &:= \tilde{n} - Z_{\tilde{m}} = \mathrm{O}(\tilde{n}_1) = \mathrm{O}\left(n_1^{1/2+\delta}\right) = \mathrm{O}\left(n^{(1/2+\delta)^2}\right) = \mathrm{o}(\sqrt{n}) \ .\end{aligned}$$

Let $A_n^{(3)} := \{|\zeta_1| \le c_3 n^\delta\}$. Then $\mathbb{P}(A_n^{(3)}) \to 1$ as $n \to \infty$. For a suitable constant c_3 in the set $A_n^{(3)}$, large n and some constants c_4 and c_5 on $A_n^{(2)} \cap A_n^{(3)}$ we have

$$\nu(\tilde{n} - Z_{\tilde{m}}) = \nu(\tilde{\tilde{n}}) < c_4 \tilde{\tilde{n}}, \quad |\mathcal{A}_{\nu(\tilde{\tilde{n}})}| < c_5 \tilde{\tilde{n}} = \mathrm{o}(\sqrt{n}) \ . \tag{7.10}$$

Thus up to a term of order $\mathrm{o}(\sqrt{n})$ on the set $\cap_{i=1}^3 A_n^{(i)}$ we have

$$\mathcal{A}_{\nu(n)} \stackrel{D}{=} \mathcal{A}_m + \mathcal{A}_{\tilde{m}} + \mathrm{o}(\sqrt{n}) \ , \tag{7.11}$$

where from (7.6)

$$\begin{aligned}\mathcal{A}_m &\stackrel{D}{\sim} a_{\mathcal{A}} m + \sigma_{\mathcal{A}} \zeta_1 \sqrt{m} = a_{\mathcal{A}} \frac{n - n_1}{a_l} + \sigma_{\mathcal{A}} \zeta_1 \sqrt{\frac{n - n_1}{a_l}} \ , \\ \mathcal{A}_{\tilde{m}} &\stackrel{D}{\sim} a_{\mathcal{A}} \frac{\tilde{n} - \tilde{n}_1}{a_l} + \sigma_{\mathcal{A}} \zeta_1^{(1)} \sqrt{\frac{\tilde{n} - \tilde{n}_1}{a_l}} \stackrel{D}{=} \frac{a_{\mathcal{A}}}{a_l}\left(n_1 - \sigma_l \zeta_2 \sqrt{\frac{n}{a_l}}\right) + \mathrm{o}(\sqrt{n}) \ ,\end{aligned}$$

and $\zeta_1^{(1)}$ is an independent copy of ζ_1. Thus

$$\mathcal{A}_m + \mathcal{A}_{\tilde{m}} \stackrel{D}{\sim} \frac{a_{\mathcal{A}}}{a_l} n + \sigma_{\mathcal{A}} \zeta_1 \sqrt{\frac{n}{a_l}} - \frac{a_{\mathcal{A}}}{a_l} \sigma_l \zeta_2 \sqrt{\frac{n}{a_l}} \stackrel{D}{=} \frac{a_{\mathcal{A}}}{a_l} n + \sigma \sqrt{\frac{n}{a_l}} \zeta \ ,$$

where $\zeta \stackrel{D}{=} \mathrm{N}(0,1)$ and

$$\sigma^2 = \sigma_{\mathcal{A}}^2 + \left(\frac{a_{\mathcal{A}}}{a_l}\sigma_l\right)^2 - 2\frac{a_{\mathcal{A}}}{a_l}\sigma_{\mathcal{A}}\sigma_l \rho(\mathcal{A}, l) \ .$$

The theorem is proved.

□

Corollary 7.1 *Assume that there exists $\epsilon > 0$ such that $\mathbb{E}\sigma_0^{3+\epsilon} < \infty$ and $\mathbb{E}\sigma^{4+\epsilon} < \infty$. Then the variance of $\mathcal{A}_{[0,t]}^{(\sigma_0,\tau_0)}$ asymptotically behaves as follows:*

$$\mathcal{D}\mathcal{A}_{[0,t]}^{(\sigma_0,\tau_0)} \sim \frac{t}{a_l} \mathbb{E}(\mathcal{A} - \frac{a_{\mathcal{A}}}{a_l} l)^2 \ . \tag{7.12}$$

Note that the asymptotics (7.12) hold for any σ_0 such that $\mathbb{E}\sigma_0^{3+\epsilon}$ for some $\epsilon > 0$. In particular, this holds when we choose (σ_0, τ_0) in such a way that we obtain the stationary workload process $\{W(t), t \ge 0\}$. Note that from Takács [31] recurrence formula, if $\mathbb{E}\sigma^{4+\epsilon} < \infty$ for some $\epsilon > 0$, then also $\mathbb{E}\sigma_0^{3+\epsilon} < \infty$. Let $a_W = \mathbb{E}W(0)$. Then

$$\begin{aligned}\mathcal{D}\mathcal{A}_{[0,t]}^{(\sigma_0,\tau_0)} &\sim \mathbb{E}\left[\int_0^t (W(s) - a_W)\,\mathrm{d}s\right]^2 = \\ &2\int_0^t \int_0^{t-u} \mathbb{E}(W(s) - a_W)(W(0) - a_W)\,\mathrm{d}s\,\mathrm{d}u \sim 2tR \ ,\end{aligned} \tag{7.13}$$

where

$$R := \int_0^\infty \rho(t)\,\mathrm{d}t, \quad \text{and} \quad \rho(t) := \mathbb{E}(W(t) - a_W)(W(0) - a_W) \ .$$

Thus we have proven:

Theorem 7.3

$$R = \int_0^\infty \rho(t)\ \mathrm{d}t = \frac{1}{2}\left(\left(\frac{a_{\mathcal{A}}}{a_l}\sigma_l\right)^2 + \sigma_{\mathcal{A}}^2 - 2\frac{a_{\mathcal{A}}}{a_l}\sigma_l\sigma_{\mathcal{A}}\rho(\mathcal{A}, l)\right) \ . \tag{7.14}$$

Furthermore, if $\mathbb{E}\sigma^{4+\epsilon} < \infty$ *for some* $\epsilon > 0$*, then* $R < \infty$*.*

Remark 7.3 For the $M|G|1$ queue with $\mathbb{E}\sigma = 1$, Abate and Whitt [1] obtain the following expression for R (notice that a factor 2 is missing in (53) of Abate and Whitt [1]):

$$R = \frac{\rho}{(1-\rho)^4}\left[\frac{(1-\rho)^2}{8}\frac{\mu_4}{\mu_1} + \frac{5}{12}(1-\rho)\rho\frac{\mu_2\mu_3}{\mu_1^2} + \frac{1}{4}\rho^2\frac{\mu_2^3}{\mu_1^3}\right] \ , \tag{7.15}$$

where μ_k is the kth moment of the distribution of the service time. In the $M|M|1$ case, $R = \rho(3-\rho)/(1-\rho)^4$. See also Beneš [8], Ott [27] for other results related with R in the $M|M|1$ queue.

Remark 7.4 We say that the stationary process $\{X(t), t \geq 0\}$ possesses a long range dependent structure if $\int_0^\infty R(t)\ \mathrm{d}t = \infty$ where $R(t)$ is the covariance function between $X(0)$ and $X(t)$ (see Heath *et al.* [21]) . Thus from Theorem 7.3 and Remark 7.1 the stationary waiting time in the $GI|G|1$ queue is long-range dependent iff service time is regularly varying and the fourth moment of it is infinite.

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