

Cyclic Queueing Networks with Subexponential Service Times

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Abstract

For a K -stage cyclic queueing model with N customers and general service times we give an explicit expression for the n^{th} departure time from each stage. Starting from this expression we analyze the asymptotic tail behaviour of cycle times and waiting times given that at least one service time distribution is subexponential. Furthermore, we show that the tail of the residual of a subexponential service time seen by an arriving customer is of the same order as the service time itself where the asymptotic constant depends on the queue length on arrival.

KEYWORDS: CYCLE TIME, CLOSED NETWORK, SUBEXPONENTIAL ASYMPTOTICS, HARRIS MARKOV CHAIN

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1 Introduction

We consider a cyclic K -stage ($K \geq 2$) queueing system as shown in Figure 1. There is a single server at station i ($i = 1, \dots, K$) and the service discipline at all stations is First Come First

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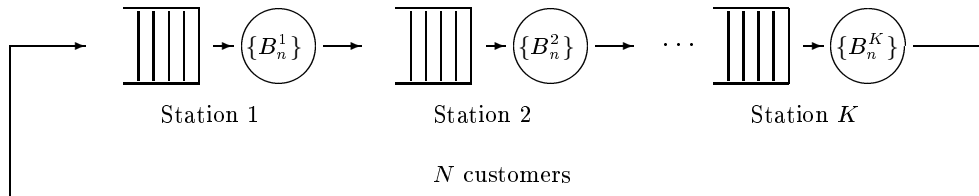


Figure 1: Cyclic queueing network K queues

Served (FCFS). The capacity of the buffer between two consecutive stations is infinite. There are N customers in the system, who cyclically visit station 1 to station K . We assume that at time zero there are N_i customers in front of station i , $i = 1, \dots, K$. Hence, $\sum_{i=1}^K N_i = N$. Service times at station i are independent and identically distributed random variables $\{B_n^i\}$ with distribution function $B_i(\cdot)$. and the sequences $\{B_n^1\}, \dots, \{B_n^K\}$ are assumed to be mutually independent. Furthermore, we assume that there exists a subexponential distribution $F(\cdot)$ ($F \in \mathcal{S}$) and there exist constants $c_i \in [0, \infty)$ with $\sum_{i=1}^K c_i > 0$ such that for all $i = 1, \dots, K$

$$\lim_{x \rightarrow \infty} \frac{\overline{B}_i(x)}{\overline{F}(x)} = c_i.$$

The tail behaviour of the cycle time (the time between successive departures of the same customer from a given station i , $i = 1, \dots, K$), waiting time at station i ($i = 1, \dots, K$) and residual service time at station i ($i = 1, \dots, K$) seen by an arriving customer are main objects of our interest. Relatively few mathematical methods are available for treating these quantities in closed queueing networks. The reason for this is that a customer passing through the system experiences the whole space-time structure of the network state process. One can use general methods for closed Gordon-Newell networks [18], Laplace transform techniques [8, 11, 24], reversibility arguments [12], standard embedded Markov chain techniques [15], or the theory of point processes [16]. All of these results are valid under the assumption that service times are exponential random variables. Boxma [8] derives an expression for the Laplace-Stieltjes transform of the distribution of the stationary cycle time in a two-stage cyclic queueing network with one exponential and one general station. This result reveals a surprising phenomenon: in general, the distribution of the cycle time

depends on the order in which two stations are visited. As Boxma [8] noted, this is because of the dependence between successive response times at two queues. As far as the open networks with subexponential service times are concerned, Baccelli, Schlegel and Schmidt [4] consider the tail behaviour of stationary response times in open $(\max,+)$ linear systems. In a similar paper, Huang and Sigman [19] focus on the asymptotics of sojourn times and queue lengths in tandem queues and split-match queues. More recently Baccelli and Foss [5] compute upper and lower bounds for the tail asymptotics of the stationary maximal dater in more general monotone-separable stochastic networks. Moreover, they obtain exact asymptotics for various special cases of these networks.

In this paper we provide an explicit representation for the n^{th} departure epoch from any given station (for similar results in open networks see e.g. [4]). This in turn allows us to obtain the subexponential asymptotics of the n^{th} cycle time at station i which is denoted by C_n^i . In particular, we have

$$\mathbb{P}(C_n^i > x) \sim N \sum_{\ell=1}^K c_\ell \bar{F}(x) \quad \text{as } x \rightarrow \infty, \forall n \geq N \quad (1)$$

where $f(x) \sim g(x)$ as $x \rightarrow \infty$ means that $\lim_{x \rightarrow \infty} f(x)/g(x) = 1$ for two functions $f(x)$ and $g(x)$. Note that the cycle time asymptotics is the same for each station. This is different for the cycle time distribution, which indeed depends on the station where the cycle starts; see [8]. Next we study the tail behaviour of the distribution of the n^{th} waiting time at station i for $i = 1, \dots, K$. In order to give an intuitive explanation for the behaviour of the network, we also analyze the tail behaviour of the distribution of a residual service seen by an arriving customer given that he sees k customers in front of him. We show that this tail is of the same order as the tail of the service time distribution at the corresponding station. This is in contrast to open systems where the residual service time distribution is "heavier" than the service time distribution itself (see for example [20]). Using Harris recurrence methods we prove that if the service times at each station have infinite support, then there exists a random variable C^i such that

$$C_n^i \xrightarrow{\mathcal{D}} C^i \quad (2)$$

holds as $n \rightarrow \infty$ where $\xrightarrow{\mathcal{D}}$ denotes convergence in distribution. The sequence C_n^i couples with C^i in finite time for all initial conditions N_1, N_2, \dots, N_K such that $\sum_{i=1}^K N_i = N$. Using (1) and (2)

we obtain the subexponential asymptotics for C^i :

$$\mathbb{P}(C^i > x) \sim N \sum_{\ell=1}^K c_\ell \bar{F}(x) \quad \text{as } x \rightarrow \infty. \quad (3)$$

Results similar to the one in (3) are also obtained for stationary waiting times and stationary residual service times. To the best of our knowledge, our results are the first of this kind for cyclic queues with more than two stations and non-exponential service times.

The paper is organized as follows. In Section 2, in order to provide the reader with a better understanding of our results we consider the two stage cyclic queueing network. In Section 3, we derive an explicit expression for the n^{th} departure time from station i . We use this result in Section 4 to get subexponential asymptotics for the n^{th} cycle time C_n^i . In Section 5, we obtain asymptotics for the n^{th} waiting time at station i . Asymptotic tail behaviour of the n^{th} residual service time at station i is considered in Section 6. In Section 7, we prove (3) and also obtain the results on tail behaviour of stationary waiting times and stationary residual service times. Finally, in the appendix we first present the proof of a technical lemma and then recall some properties of subexponential distributions used in this paper.

2 Two-stage Cyclic Queue

In this section, we analyze the case of two stations and two customers (i.e. $K = 2$ and $N = 2$). In particular we consider the n^{th} cycle time at station i which is the time between the $(n + N)^{\text{th}}$ and n^{th} departures from station i . Recall that C_n^i denotes the n^{th} cycle time for station i and let X_n^i denote the n^{th} departure time from station i ($i = 1, 2$).

The location of the customers in the network at time 0 is called the initial marking. We will see that the stationary regime does not depend on this initial marking. In fact, the influence of the initial marking already vanishes after two iterations when $K = 2$ and $N = 2$. Without loss of generality we assume that no customers are being served at time 0. Throughout the paper we use the following notation

$$[j] = \begin{cases} K & \text{if } j \bmod K = 0 \\ j \bmod K & \text{if } j \bmod K \neq 0 \end{cases}$$

where K is the number of stations as defined above. Furthermore, we use the symbol \oplus for maximization, and the symbol \otimes for addition. Thus, we write $a \oplus b$ for $\max\{a, b\}$, and $a \otimes b$ (or shortly ab) for $a + b$ where a and b are real numbers. This is the standard $(\max, +)$ algebra notation (see [3] for more details on this formalism). Note that as in conventional algebra \otimes has precedence over \oplus . Even though cyclic tandem queue is an example of a $(\max, +)$ linear system, general $(\max, +)$ linear systems and $(\max, +)$ algebra are beyond the scope of this paper. We use these symbols simply for notational convenience. Without loss of generality we set

$$X_0^i = 0.$$

One can easily see that in any cyclic queue for all $1 = 1, \dots, K$

$$X_n^i = B_n^i X_{n-1}^i \oplus B_n^i X_{n-N_i}^{[i-1]}. \quad (4)$$

For the tandem queue with two stations and two customers, we focus on C_n^1 . We first consider the case that initially there is one customer at each station (i.e., $N_1 = N_2 = 1$). Then,

$$X_1^1 = B_1^1, \quad X_1^2 = B_1^2$$

and

$$\begin{aligned} C_1^1 &= X_3^1 - X_1^1 = B_3^1(B_1^1 \oplus B_1^2)(B_2^1 \oplus B_2^2) - B_1^1 \\ &= (0 \oplus (B_1^2 - B_1^1))(B_2^1 \oplus B_2^2)B_3^1, \\ C_2^1 &= X_4^1 - X_2^1 = (B_3^1 \oplus B_3^2)(B_2^1 \oplus B_2^2)B_4^1 - B_2^1 \\ &= (0 \oplus (B_2^2 - B_2^1))(B_3^1 \oplus B_3^2)B_4^1. \end{aligned}$$

In general, for all $n \geq 1$

$$\begin{aligned} C_n^1 &= X_{n+2}^1 - X_n^1 \\ &= (0 \oplus (B_n^2 - B_n^1))(B_{n+1}^1 \oplus B_{n+1}^2)B_{n+2}^1. \end{aligned} \quad (5)$$

Thus from Lemmas 8.4, 8.5, and 8.6 we have

$$\mathbb{P}(C_n^1 > x) \sim 2(c_1 + c_2)\overline{F}(x), \quad x \rightarrow \infty. \quad (6)$$

Next, we initially place both customers at station 1 (i.e., $N_1 = 2, N_2 = 0$) and again consider C_n^1 . Then,

$$X_1^1 = B_1^1, \quad X_2^1 = B_2^1 B_1^1, \quad X_1^2 = B_1^2 B_1^1.$$

Note that at the completion of the first service time at station 1 (i.e., after B_1^1 has elapsed) we are in the first scenario. Thus, the n^{th} cycle time for this initial marking is the same as the $(n - 1)^{\text{th}}$ cycle time for the first initial marking with subscripts of B_i^1 increased by one. We have from (5) for all $n \geq 2$

$$\begin{aligned} C_n^1 &= X_{n+2}^1 - X_n^1 \\ &= (0 \oplus (B_{n-1}^2 - B_n^1))(B_n^2 \oplus B_{n+1}^1)B_{n+2}^1, \end{aligned}$$

which again gives (6). Finally, we consider the case that both customers are at station 2 at time 0 (i.e., $N_1 = 0, N_2 = 2$). At the completion of the first service time at station 1 (i.e. after B_1^2 has elapsed) we are in the first scenario. Thus, the n^{th} cycle time for this initial marking is the same as the n^{th} cycle time of the first initial marking but now with subscripts of B_i^2 increased by one. We have for all $n \geq 1$

$$\begin{aligned} C_n^1 &= X_{n+2}^1 - X_n^1 \\ &= (0 \oplus (B_{n+1}^2 - B_n^1))(B_{n+1}^1 \oplus B_{n+2}^2)B_{n+2}^1 \end{aligned}$$

and therefore again (6). So for the case of two queues and two customers, C_n^i does not depend on the initial marking for all $n \geq 2$. This follows since after at most one initial service (namely in the case that both customers are at the same station at time 0), services at station 1 and station 2 always start at the same time.

3 Departure Times

In this section we derive an explicit expression for the n^{th} departure time from station i , $i = 1, \dots, K$. Throughout our developments we set $B_j^k = 0$ for $j \leq 0$. Let $\lfloor u \rfloor$ denote the largest integer less than or equal to u and let $\lceil u \rceil$ denote the smallest integer greater than or equal to u .

Proposition 3.1 *The n^{th} departure time X_n^i from station i , $i = 1, \dots, K$, is given by*

$$X_n^i = \bigotimes_{k=1}^n B_k^i \oplus \bigoplus_{j=1}^{\infty} \bigoplus_{j_1=Nj-(\sum_{k=1}^{K-1} N_{[i-k]}-1)}^n \bigoplus_{r=2}^{Kj} \bigoplus_{j_r=Nj-(f(r)-1)}^{j_{r-1}-N_{[i-r+2]}} \left(\bigotimes_{i_1=j_1}^n B_{i_1}^i \bigotimes_{q=2}^{Kj} \bigotimes_{i_q=j_q}^{j_{q-1}-N_{[i-q+2]}} B_{i_q}^{[i-q+1]} \bigotimes_{l=1}^{j_{Kj}-N_{[i-K+1]}} B_l^i \right) \quad (7)$$

where

$$f(r) = \begin{cases} \lfloor \frac{r}{K} \rfloor N_i + \lceil \frac{r}{K} \rceil \sum_{k=1}^{K-1} N_{[i-k]} & \text{for } [r] = 1 \\ \lceil \frac{r}{K} \rceil \sum_{k=[r]-1}^K N_{[i-k]} + (\lceil \frac{r}{K} \rceil + 1) \sum_{k=1}^{[r]-2} N_{[i-k]} & \text{for } [r] = 2, \dots, K, \end{cases}$$

with the convention that \bigoplus over an empty set is $-\infty$ and \bigotimes over an empty set is 0.

More explicitly (7) can be written as

$$\begin{aligned} X_n^i = & \bigotimes_{k=1}^n B_k^i \oplus \bigoplus_{j=1}^{\infty} \bigoplus_{j_1=Nj-(\sum_{k=1}^{K-1} N_{[i-k]}-1)}^n \bigoplus_{j_2=Nj-(\sum_{k=2}^K N_{[i-k]}-1)}^{j_1-N_i} \\ & \bigoplus_{j_3=Nj-(2N_{[i-1]}+\sum_{k=2}^K N_{[i-k]}-1)}^{j_2-N_{[i-1]}} \bigoplus_{j_4=Nj-(2\sum_{k=1}^2 N_{[i-k]}+\sum_{k=3}^K N_{[i-k]}-1)}^{j_3-N_{[i-2]}} \dots \\ & \dots \bigoplus_{j_{K+2}=Nj-(2\sum_{k=1}^K N_{[i-k]}-1)}^{j_{K+1}-N_{[i-K]}} \bigoplus_{j_{K+3}=Nj-(3N_{[i-1]}+2\sum_{k=2}^K N_{[i-k]}-1)}^{j_{K+2}-N_{[i-1]}} \dots \\ & \dots \bigoplus_{j_{2K+2}=Nj-(3\sum_{k=1}^K N_{[i-k]}-1)}^{j_{2K+1}-N_{[i-K]}} \dots \bigoplus_{j_{Kj}=1}^{j_{Kj-1}-N_{[i-(K-2)]}} \left(\bigotimes_{i_1=j_1}^n B_{i_1}^i \bigotimes_{i_2=j_2}^{j_1-N_i} B_{i_2}^{[i-1]} \right. \\ & \bigotimes_{i_3=j_3}^{j_2-N_{[i-1]}} B_{i_3}^{[i-2]} \bigotimes_{i_4=j_4}^{j_3-N_{[i-2]}} B_{i_4}^{[i-3]} \dots \bigotimes_{i_K=j_K}^{j_{K-1}-N_{[i-(K-2)]}} B_{i_K}^{[i-K+1]} \\ & \left. \bigotimes_{i_{K+1}=j_{K+1}}^{j_{Kj}-N_{[i-(K-1)]}} B_{i_{K+1}}^i \dots \bigotimes_{i_{Kj}=j_{Kj}}^{j_{Kj-1}-N_{[i-(K-2)]}} B_{i_{Kj}}^{[i-K+1]} \bigotimes_{l=1}^{j_{Kj}-N_{[i-(K-1)]}} B_l^i \right). \quad (8) \end{aligned}$$

We will use (8) in our developments since this more explicit representation will provide the reader with more illustrative proofs that are not difficult to follow. Before giving the proof of Proposition 3.1, we present a lemma which provides an equivalent representation for departure times. The proof is given in the appendix.

Lemma 3.1 *The expression for departure times given in (7) is equivalent to*

$$X_n^i = \bigoplus_{j=1}^{\infty} \bigoplus_{j_1=Nj-(\sum_{k=1}^K N_{[i-k]}-1)}^n \bigoplus_{r=2}^{Kj-1} \bigoplus_{j_r=Nj-(g(r)-1)}^{j_{r-1}-N_{[i-r+2]}}$$

$$\left(\bigotimes_{i_1=j_1}^n B_{i_1}^i \bigotimes_{q=2}^{Kj-1} \bigotimes_{i_q=j_q}^{j_{q-1}-N_{[i-q+2]}} B_{i_q}^{[i-q+1]} \bigotimes_{l=1}^{j_{Kj-1}-N_{[i-K+2]}} B_l^{[i-K+1]} \right) \quad (9)$$

where

$$g(r) = \begin{cases} \lceil \frac{r}{K} \rceil \sum_{k=1}^K N_{[i-k]} & \text{for } [r] = 1 \\ \lceil \frac{r}{K} \rceil \sum_{k=\lceil \frac{r}{K} \rceil}^{K-1} N_{[i-k]} + (\lceil \frac{r}{K} \rceil + 1) (\sum_{k=0}^{\lceil \frac{r}{K} \rceil - 2} N_{[i-k]}) & \text{for } [r] = 2, \dots, K. \end{cases}$$

Proof of Proposition 3.1

We use induction. Without loss of generality we assume that $X_n^i = 0$, for all $n \leq 0$ and $i = 1, \dots, K$.

We first show that (8) is correct for $n = 1$. First let i be a station such that $N_i > 0$. Such an i always exists since $N > 0$. In this case, (8) reduces to

$$X_1^i = \bigotimes_{k=1}^1 B_k^i = B_1^i. \quad (10)$$

On the other hand, it follows from (4) and our assumptions on initial conditions that $X_1^i = B_1^i$.

Next assume i is such that $N_i = 0$ and $N_{[i-1]} > 0$. Then (8) reduces to

$$X_1^i = \bigotimes_{k=1}^1 B_k^i \oplus \bigoplus_{j_1=1}^1 \bigoplus_{j_2=1}^{j_1} \left(\bigotimes_{i_1=j_1}^1 B_{i_1}^i \bigotimes_{i_2=j_2}^{j_1} B_{i_2}^{[i-1]} \right) = B_1^i B_1^{[i-1]}.$$

Similarly, from (4), our assumptions on initial conditions and the fact that $X_1^{[i-1]} = B_1^{[i-1]}$ (which follows from (10)) we have

$$X_1^i = B_1^i \oplus B_1^i B_1^{[i-1]} = B_1^i B_1^{[i-1]}.$$

Continuing in this fashion, from (4) we have

$$X_1^i = B_1^i B_1^{[i-1]} B_1^{[i-2]} \dots B_1^{[i-k]}$$

for i with $N_i = N_{[i-1]} = \dots = N_{[i-(k-1)]} = 0$ and $N_{[i-k]} > 0$, and any $k = 1, \dots, K-1$. In this case (8) also reduces to

$$\begin{aligned} X_1^i &= \bigotimes_{k=1}^1 B_k^i \oplus \bigoplus_{j_1=1}^1 \bigoplus_{j_2=1}^{j_1} \dots \bigoplus_{j_{k+1}=1}^{j_k} \left(\bigotimes_{i_1=j_1}^1 B_{i_1}^i \bigotimes_{i_2=j_2}^{j_1} B_{i_2}^{[i-1]} \dots \bigotimes_{i_{k+1}=j_{k+1}}^{j_k} B_{i_{k+1}}^{[i-k]} \right) \\ &= B_1^i B_1^{[i-1]} \dots B_1^{[i-k]} \end{aligned}$$

which completes the proof for $n = 1$. Now suppose that (8) holds for $m \leq n-1$. Thus, it follows from (9) that for all $m \leq n-1$

$$X_m^i = \bigoplus_{j=1}^{\infty} \bigoplus_{j_1=Nj - (\sum_{k=1}^K N_{[i-k]-1})}^m \bigoplus_{j_2=Nj - (2N_i + \sum_{k=1}^{K-1} N_{[i-k]-1})}^{j_1 - N_i}$$

$$\begin{array}{c}
\begin{array}{ccc}
j_2 - N_{[i-1]} & & j_3 - N_{[i-2]} \\
\oplus & & \oplus \\
j_3 = Nj - (2N_i + 2N_{[i-1]} + \sum_{k=2}^{K-1} N_{[i-k]} - 1) & j_4 = Nj - (2N_i + 2 \sum_{k=1}^2 N_{[i-k]} + \sum_{k=3}^{K-1} N_{[i-k]} - 1) & \dots
\end{array} \\
\begin{array}{ccc}
j_{K+1} - N_i & & j_{K+2} - N_{[i-1]} \\
\oplus & & \oplus \\
\dots & & \dots
\end{array} \\
\begin{array}{ccc}
j_{K+2} = Nj - (3N_i + 2 \sum_{k=1}^{K-1} N_{[i-k]} - 1) & j_{K+3} = Nj - (3N_i + 3N_{[i-1]} + 2 \sum_{k=2}^{K-1} N_{[i-k]} - 1) & \\
\dots & & \\
j_{2K+1} - N_i & & j_{Kj-2} - N_{[i+3]} \\
\oplus & & \oplus \\
j_{2K+2} = Nj - (4N_i + 3 \sum_{k=1}^{K-1} N_{[i-k]} - 1) & j_{Kj-1} = 1 & \left(\bigotimes_{i_1=j_1}^m B_{i_1}^i \bigotimes_{i_2=j_2}^{j_1-N_i} B_{i_2}^{[i-1]} \right)
\end{array} \\
\begin{array}{ccc}
j_2 - N_{[i-1]} & j_3 - N_{[i-2]} & j_{K-1} - N_{[i+2]} \\
\bigotimes_{i_3=j_3} B_{i_3}^{[i-2]} & \bigotimes_{i_4=j_4} B_{i_4}^{[i-3]} & \dots \bigotimes_{i_K=j_K} B_{i_K}^{[i+1]} \\
j_K - N_{[i+1]} & j_{Kj-2} - N_{[i+3]} & j_{Kj-1} - N_{[i+2]} \\
\bigotimes_{i_{K+1}=j_{K+1}} B_{i_{K+1}}^i & \dots \bigotimes_{i_{Kj-1}=j_{Kj-1}} B_{i_{Kj-1}}^{[i+2]} & \bigotimes_{l=1} B_l^{[i+1]}
\end{array}
\end{array} \tag{11}$$

First assume that i is such that $N_i > 0$. Then we have

$$\begin{array}{c}
X_n^i = \left(\bigotimes_{k=1}^n B_k^i \oplus \bigoplus_{j=1}^{\infty} \begin{array}{ccc}
n-1 & & j_1 - N_i \\
\oplus & & \oplus \\
j_1 = Nj - (\sum_{k=1}^{K-1} N_{[i-k]} - 1) & j_2 = Nj - (\sum_{k=1}^K N_{[i-k]} - 1) & \\
j_2 - N_{[i-1]} & & j_3 - N_{[i-2]} \\
\oplus & & \oplus \\
j_3 = Nj - (2N_{[i-1]} + \sum_{k=2}^K N_{[i-k]} - 1) & j_4 = Nj - (2 \sum_{k=1}^2 N_{[i-k]} + \sum_{k=3}^K N_{[i-k]} - 1) & \dots \\
j_{K+1} - N_i & & j_{K+2} - N_{[i-1]} \\
\oplus & & \oplus \\
\dots & & \dots \\
j_{K+2} = Nj - (2 \sum_{k=1}^K N_{[i-k]} - 1) & j_{K+3} = Nj - (3N_{[i-1]} + 2 \sum_{k=2}^K N_{[i-k]} - 1) & \\
\dots & & \\
j_{2K+1} - N_i & & j_{Kj-1} - N_{[i+2]} \\
\oplus & & \oplus \\
j_{2K+2} = Nj - (3 \sum_{k=1}^K N_{[i-k]} - 1) & j_{Kj} = 1 & \left(\bigotimes_{i_1=j_1}^n B_{i_1}^i \bigotimes_{i_2=j_2}^{j_1-N_i} B_{i_2}^{[i-1]} \right) \\
j_2 - N_{[i-1]} & j_3 - N_{[i-2]} & j_{K-1} - N_{[i+2]} \\
\bigotimes_{i_3=j_3} B_{i_3}^{[i-2]} & \bigotimes_{i_4=j_4} B_{i_4}^{[i-3]} & \dots \bigotimes_{i_K=j_K} B_{i_K}^{[i+1]} \\
j_K - N_{[i+1]} & j_{Kj-1} - N_{[i+2]} & j_{Kj} - N_{[i+1]} \\
\bigotimes_{i_{K+1}=j_{K+1}} B_{i_{K+1}}^i & \dots \bigotimes_{i_{Kj}=j_{Kj}} B_{i_{Kj}}^{[i+1]} & \bigotimes_{l=1} B_l^i \bigoplus
\end{array} \right) \oplus \\
\begin{array}{ccc}
\bigoplus_{j=1}^{\infty} & n - N_i & j_1 - N_{[i-1]} \\
\oplus & \oplus & \oplus \\
j_1 = Nj - (\sum_{k=1}^K N_{[i-1-k]} - 1) & j_2 = Nj - (2N_{[i-1]} + \sum_{k=1}^{K-1} N_{[i-1-k]} - 1) & \\
j_2 - N_{[i-2]} & & j_3 - N_{[i-3]} \\
\oplus & & \oplus \\
j_3 = Nj - (2N_{[i-1]} + 2N_{[i-2]} + \sum_{k=2}^{K-1} N_{[i-1-k]} - 1) & j_4 = Nj - (2N_{[i-1]} + 2 \sum_{k=1}^2 N_{[i-1-k]} + \sum_{k=3}^{K-1} N_{[i-1-k]} - 1) & \dots \\
j_{K+1} - N_{[i-1]} & & j_{K+2} - N_{[i-2]} \\
\oplus & & \oplus \\
\dots & & \dots \\
j_{K+2} = Nj - (3N_{[i-1]} + 2 \sum_{k=1}^{K-1} N_{[i-1-k]} - 1) & j_{K+3} = Nj - (3N_{[i-1]} + 3N_{[i-2]} + 2 \sum_{k=2}^{K-1} N_{[i-1-k]} - 1) &
\end{array}
\end{array}$$

$$\begin{aligned}
& \dots \quad \oplus_{j_{2K+1}=Nj-(4N_{[i-1]}+3\sum_{k=1}^{K-1}N_{[i-1-k]}-1)}^{j_{2K+1}-N_{[i-1]}} \quad \dots \quad \oplus_{j_{Kj-1}=1}^{j_{Kj-2}-N_{[i+2]}} \left(B_n^i \otimes_{i_1=j_1}^{n-N_i} B_{i_1}^{[i-1]} \otimes_{i_2=j_2}^{j_1-N_{[i-1]}} B_{i_2}^{[i-2]} \right. \\
& \quad \otimes_{i_3=j_3}^{j_2-N_{[i-2]}} B_{i_3}^{[i-3]} \otimes_{i_4=j_4}^{j_3-N_{[i-3]}} B_{i_4}^{[i-4]} \dots \otimes_{i_K=j_K}^{j_{K-1}-N_{[i+1]}} B_{i_K}^i \\
& \quad \left. \otimes_{i_{K+1}=j_{K+1}}^{j_{K-1}-N_i} B_{i_{K+1}}^{[i-1]} \dots \otimes_{i_{Kj-1}=j_{Kj-1}}^{j_{Kj-2}-N_{[i+2]}} B_{i_{Kj-1}}^{[i+1]} \otimes_{l=1}^{j_{Kj-1}-N_{[i+1]}} B_l^i \right)
\end{aligned}$$

$$\begin{aligned}
& = \left(\otimes_{k=1}^n B_k^i \oplus \oplus_{j=1}^{\infty} \oplus_{j_1=Nj-(\sum_{k=1}^{K-1}N_{[i-k]}-1)}^{n-1} \oplus_{j_2=Nj-(\sum_{k=1}^K N_{[i-k]}-1)}^{j_1-N_i} \right. \\
& \quad \oplus_{j_3=Nj-(2N_{[i-1]}+\sum_{k=1}^K N_{[i-k]}-1)}^{j_2-N_{[i-1]}} \oplus_{j_4=Nj-(2\sum_{k=1}^2 N_{[i-k]}+\sum_{k=3}^K N_{[i-k]}-1)}^{j_3-N_{[i-2]}} \dots \\
& \quad \oplus_{j_{K+1}=Nj-(\sum_{k=1}^{K-1}N_{[i-k]}-1)}^{j_{K+1}-N_{[i-1]}} \oplus_{j_{K+2}=Nj-(3N_{[i-1]}+2\sum_{k=2}^K N_{[i-k]}-1)}^{j_{K+2}-N_{[i-2]}} \dots \\
& \quad \oplus_{j_{2K+1}=Nj-(3\sum_{k=1}^K N_{[i-k]}-1)}^{j_{2K+1}-N_{[i-1]}} \oplus_{j_{Kj}=1}^{j_{Kj-1}-N_{[i+2]}} \left(\otimes_{i_1=j_1}^n B_{i_1}^i \otimes_{i_2=j_2}^{j_1-N_i} B_{i_2}^{[i-1]} \right. \\
& \quad \otimes_{i_3=j_3}^{j_2-N_{[i-1]}} B_{i_3}^{[i-2]} \otimes_{i_4=j_4}^{j_3-N_{[i-2]}} B_{i_4}^{[i-3]} \dots \otimes_{i_K=j_K}^{j_{K-1}-N_{[i+2]}} B_{i_K}^{[i+1]} \\
& \quad \left. \otimes_{i_{K+1}=j_{K+1}}^{j_K-N_{[i+1]}} B_{i_{K+1}}^i \dots \otimes_{i_{Kj}=j_{Kj}}^{j_{Kj-1}-N_{[i+2]}} B_{i_{Kj}}^{[i+1]} \otimes_{l=1}^{j_{Kj}-N_{[i+1]}} B_l^i \right) \oplus
\end{aligned}$$

$$\begin{aligned}
& \oplus_{j=1}^{\infty} \oplus_{j_2=Nj-(\sum_{k=1}^K N_{[i-k]}-1)}^{n-N_i} \oplus_{j_3=Nj-(2N_{[i-1]}+\sum_{k=1}^K N_{[i-k]}-1)}^{j_2-N_{[i-1]}} \oplus_{j_4=Nj-(2N_{[i-1]}+2\sum_{k=2}^3 N_{[i-k]}+\sum_{k=4}^K N_{[i-k]}-1)}^{j_3-N_{[i-2]}} \dots \\
& \quad \oplus_{j_4=Nj-(2N_{[i-1]}+2N_{[i-2]}+\sum_{k=3}^K N_{[i-k]}-1)}^{j_4-N_{[i-2]}} \oplus_{j_5=Nj-(2N_{[i-1]}+2\sum_{k=2}^3 N_{[i-k]}+\sum_{k=4}^K N_{[i-k]}-1)}^{j_5-N_{[i-3]}} \dots \\
& \quad \oplus_{j_{K+3}=Nj-(3N_{[i-1]}+2\sum_{k=2}^K N_{[i-k]}-1)}^{j_{K+3}-N_{[i-1]}} \oplus_{j_{K+4}=Nj-(3N_{[i-1]}+3N_{[i-2]}+2\sum_{k=3}^K N_{[i-k]}-1)}^{j_{K+4}-N_{[i-2]}} \dots \\
& \quad \oplus_{j_{2K+3}=Nj-(4N_{[i-1]}+3\sum_{k=1}^K N_{[i-k]}-1)}^{j_{2K+3}-N_{[i-1]}} \oplus_{j_{Kj}=1}^{j_{Kj-1}-N_{[i+2]}} \left(B_n^i \otimes_{i_2=j_2}^{n-N_i} B_{i_2}^{[i-1]} \otimes_{i_3=j_3}^{j_2-N_{[i-1]}} B_{i_3}^{[i-2]} \right. \\
& \quad \otimes_{i_4=j_4}^{j_3-N_{[i-2]}} B_{i_4}^{[i-3]} \otimes_{i_5=j_5}^{j_4-N_{[i-3]}} B_{i_5}^{[i-4]} \dots \otimes_{i_{K+1}=j_{K+1}}^{j_K-N_{[i+1]}} B_{i_{K+1}}^i
\end{aligned}$$

$$\begin{aligned}
& \bigotimes_{i_{K+2}=j_{K+2}}^{j_{K+1}-N_i} B_{i_{K+2}}^{[i-1]} \cdots \bigotimes_{i_{Kj}=j_{Kj}}^{j_{Kj-1}-N_{[i+2]}} B_{i_{Kj}}^{[i+1]} \bigotimes_{l=1}^{j_{Kj}-N_{[i+1]}} B_l^i \\
&= \bigotimes_{k=1}^n B_k^i \oplus \bigoplus_{j=1}^{\infty} \bigoplus_{j_1=Nj-(\sum_{k=1}^{K-1} N_{[i-k]}-1)}^n \bigoplus_{j_2=Nj-(\sum_{k=1}^K N_{[i-k]}-1)}^{j_1-N_i} \bigoplus_{j_3=N_{[i-1]}}^{j_2-N_{[i-1]}} \bigoplus_{j_4=N_{[i-2]}}^{j_3-N_{[i-2]}} \cdots \\
& \quad j_3=Nj-(2N_{[i-1]}+\sum_{k=2}^K N_{[i-k]}-1) \quad j_4=Nj-(2\sum_{k=1}^2 N_{[i-k]}+\sum_{k=3}^K N_{[i-k]}-1) \\
& \quad \cdots \quad \bigoplus_{j_{K+1}=N_{[i-1]}}^{j_{K+1}-N_i} \bigoplus_{j_{K+2}=N_{[i-1]}}^{j_{K+2}-N_{[i-1]}} \cdots \\
& \quad j_{K+2}=Nj-(2\sum_{k=1}^K N_{[i-k]}-1) \quad j_{K+3}=Nj-(3N_{[i-1]}+2\sum_{k=2}^K N_{[i-k]}-1) \\
& \quad \cdots \quad \bigoplus_{j_{2K+1}=N_{[i-1]}}^{j_{2K+1}-N_i} \quad \cdots \quad \bigoplus_{j_{Kj}=1}^{j_{Kj-1}-N_{[i+2]}} \left(\bigotimes_{i_1=j_1}^n B_{i_1}^i \bigotimes_{i_2=j_2}^{j_1-N_i} B_{i_2}^{[i-1]} \right. \\
& \quad j_{2K+2}=Nj-(3\sum_{k=1}^K N_{[i-k]}-1) \quad \bigotimes_{i_3=j_3}^{j_2-N_{[i-1]}} B_{i_3}^{[i-2]} \bigotimes_{i_4=j_4}^{j_3-N_{[i-2]}} B_{i_4}^{[i-3]} \cdots \bigotimes_{i_K=j_K}^{j_{K-1}-N_{[i+2]}} B_{i_K}^{[i+1]} \\
& \quad \bigotimes_{i_{K+1}=j_{K+1}}^{j_{K+1}-N_{[i+1]}} B_{i_{K+1}}^i \cdots \bigotimes_{i_{Kj}=j_{Kj}}^{j_{Kj-1}-N_{[i+2]}} B_{i_{Kj}}^{[i+1]} \bigotimes_{l=1}^{j_{Kj}-N_{[i+1]}} B_l^i \big)
\end{aligned}$$

where we used (4), the induction hypothesis and the representation in (11) for $X_{n-N_i}^{[i-1]}$ to obtain the first equality, we relabeled the indices to obtain the second equality, and the last equality follows by putting all terms together. On the other hand, if i is such that $N_i = 0$ and $N_{[i-1]} > 0$, then the expression for X_n^i can be obtained in the same way since we now have an expression for X_n^i for all i with $N_i > 0$, and thus in particular for $X_n^{[i-1]}$. Continuing in this fashion one can obtain X_n^i for i with $N_i = N_{[i-1]} = \cdots = N_{[i-(k-1)]} = 0$ and $N_{[i-k]} > 0$, for any $k = 1, \dots, K-1$. \square

4 Cycle Times

Recall that C_n^i denotes the n^{th} cycle time at station i ($i = 1, \dots, K$). By a cycle time we mean the time between two successive departures of the same customer from a given station. Thus, the n^{th} cycle time at station i is computed as

$$C_n^i = X_{N+n}^i - X_n^i$$

Proposition 4.1 For $i \in \{1, \dots, K\}$ and $n \geq N - N_{[i+1]}$, we have

$$\mathbb{P}(C_n^i > x) \sim N \sum_{k=1}^K c_k \bar{F}(x) \quad \text{as } x \rightarrow \infty. \quad (12)$$

Proof We first derive a lower bound for C_n^i . By summing up all service times that appear in the expression of X_n^i , we get

$$\begin{aligned} X_n^i &\leq B_n^i \dots B_1^i \otimes B_{n-N_i}^{[i-1]} \dots B_1^{[i-1]} \\ &\quad \otimes B_{n-N_i-N_{[i-1]}}^{[i-2]} \dots B_1^{[i-2]} \otimes \dots \otimes B_{n-(N-N_{[i+1]})}^{[i-K+1]} \dots B_1^{[i-K+1]}. \end{aligned} \quad (13)$$

Note that X_n^i is the maximum of sums of service times. In order to get a lower bound for X_n^i , for each k , $k = 1, \dots, K$, we pick that sum of service times which contains a maximal number of service times with upper index k . Each of these sums is bounded from below by considering only the service times with upper index k and dropping all the rest. This yields

$$\begin{aligned} X_n^i &\geq B_n^i \dots B_1^i \oplus B_{n-N_i}^{[i-1]} \dots B_1^{[i-1]} \oplus B_{n-N_i-N_{[i-1]}}^{[i-2]} \dots B_1^{[i-2]} \\ &\quad \oplus B_{n-N_i-N_{[i-1]}-N_{[i-2]}}^{[i-3]} \dots B_1^{[i-3]} \oplus \dots \oplus B_{n-(N-N_{[i+1]})}^{[i-K+1]} \dots B_1^{[i-K+1]}. \end{aligned} \quad (14)$$

From (13) and (14),

$$\begin{aligned} C_n^i &= X_{N+n}^i - X_n^i \\ &\geq \left(B_{N+n}^i \dots B_{n+1}^i - \left(B_{n-N_i}^{[i-1]} \dots B_1^{[i-1]} \otimes B_{n-N_i-N_{[i-1]}}^{[i-2]} \dots B_1^{[i-2]} \right. \right. \\ &\quad \left. \left. \otimes \dots \otimes B_{n-(N-N_{[i+1]})}^{[i-K+1]} \dots B_1^{[i-K+1]} \right) \right) \\ &\quad \oplus \left(B_{N+n-N_i}^{[i-1]} \dots B_{n+1-N_i}^{[i-1]} - \left(B_n^i \dots B_1^i \otimes B_{n-N_i-N_{[i-1]}}^{[i-2]} \dots B_1^{[i-2]} \right. \right. \\ &\quad \left. \left. \otimes \dots \otimes B_{n-(N-N_{[i+1]})}^{[i-K+1]} \dots B_1^{[i-K+1]} \right) \right) \\ &\quad \oplus \left(B_{N+n-N_i-N_{[i-1]}}^{[i-2]} \dots B_{n+1-N_i-N_{[i-1]}}^{[i-2]} \right. \\ &\quad \left. - \left(B_n^i \dots B_1^i \otimes B_{n-N_i}^{[i-1]} \dots B_1^{[i-1]} \otimes B_{n-N_i-N_{[i-1]}-N_{[i-2]}}^{[i-3]} \dots B_1^{[i-3]} \right. \right. \\ &\quad \left. \left. \otimes \dots \otimes B_{n-(N-N_{[i+1]})}^{[i-K+1]} \dots B_1^{[i-K+1]} \right) \right) \\ &\quad \oplus \dots \oplus \left(B_{N+n-(N-N_{[i+1]})}^{[i-K+1]} \dots B_{n+1-(N-N_{[i+1]})}^{[i-K+1]} \right. \\ &\quad \left. - \left(B_n^i \dots B_1^i \otimes B_{n-N_i}^{[i-1]} \dots B_1^{[i-1]} \otimes \dots \otimes B_{n-(N-N_{[i+1]}-N_{[i+2]})}^{[i-K+2]} \dots B_1^{[i-K+2]} \right) \right) \\ &\geq \left(B_{N+n}^i \dots B_{n+1}^i \oplus B_{n+N-N_i}^{[i-1]} \dots B_{n+1-N_i}^{[i-1]} \right. \end{aligned}$$

$$\begin{aligned} & \left(\oplus \dots \oplus B_{n+N-(N-N_{[i+1]})}^{[i-K+1]} \dots B_{n+1-(N-N_{[i+1]})}^{[i-K+1]} \right) \\ & - \left(B_n^i \dots B_1^i \otimes B_{n-N_i}^{[i-1]} \dots B_1^{[i-1]} \otimes \dots \otimes B_{n-(N-N_{[i+1]})}^{[i-K+1]} \dots B_1^{[i-K+1]} \right). \end{aligned}$$

Therefore, by Lemmas 8.4, 8.5 and 8.6

$$\liminf_{x \rightarrow \infty} \frac{\mathbb{P}(C_n^i > x)}{\overline{F}(x)} \geq N \sum_{k=1}^K c_k. \quad (15)$$

We now obtain an upper bound on C_n^i . Note that it follows from (8) that X_{n+N}^i can be expressed in the following form

$$X_{n+N}^i = \bigoplus_{k=1}^{\kappa} \beta_k$$

where β_k is the sum of various combinations of service times that belong to the set

$$\Theta = \{B_1^i, \dots, B_{n+N}^i, B_1^{[i-1]}, \dots, B_{n+N-N_i}^{[i-1]}, B_1^{[i-2]}, \dots, B_{n+N-N_i-N_{i-1}}^{[i-2]}, \dots, B_1^{[i-K+1]}, \dots, B_{n+N-(N-N_{[i+1]})}^{[i-K+1]}\}$$

and κ is an integer determined by $\max\{j : Nj - (\sum_{k=1}^{K-1} N_{[i-k]} - 1) \leq n + N\}$. Similarly, X_n^i can be expressed as

$$X_n^i = \bigoplus_{l=1}^{\eta} \gamma_l$$

where γ_l is the sum of various combinations of service times that belong to the set

$$\Upsilon = \{B_1^i, \dots, B_n^i, B_1^{[i-1]}, \dots, B_{n-N_i}^{[i-1]}, B_1^{[i-2]}, \dots, B_{n-N_i-N_{i-1}}^{[i-2]}, \dots, B_1^{[i-K+1]}, \dots, B_{n-(N-N_{[i+1]})}^{[i-K+1]}\}$$

and $\eta \leq \kappa$ is an integer determined by $\max\{j : Nj - (\sum_{k=1}^{K-1} N_{[i-k]} - 1) \leq n\}$. Consider $\beta_k \in \{\beta_1, \dots, \beta_{\kappa}\}$. We know that β_k is written as a sum of the elements of some $\Upsilon_k \subset \Upsilon$ and some $\Theta_k \subset \Theta \setminus \Upsilon$. However, it then follows from (8) that there exists a $\gamma_{(k)} \in \{\gamma_1, \dots, \gamma_{\eta}\}$ which also contains all the elements of Υ_k . Then

$$X_{n+N}^i - X_n^i \leq \bigoplus_{k=1}^{\kappa} (\beta_k - \gamma_{(k)})$$

where $\gamma_{(k)}$, $k = 1, \dots, \kappa$, is defined as above. Since for $k = 1, \dots, \kappa$

$$\beta_k - \gamma_{(k)} \leq \bigotimes_{j=1}^N B_{n+j}^i \bigotimes_{j=1}^N B_{n+j-N_i}^{[i-1]} \dots \bigotimes_{j=1}^N B_{n+j-(N-N_{[i-1]})}^{[i-K+1]},$$

$$C_n^i \leq \bigotimes_{j=1}^N B_{n+j}^i \bigotimes_{j=1}^N B_{n+j-N_i}^{[i-1]} \cdots \bigotimes_{j=1}^N B_{n+j-(N-N_{[i-1]})}^{[i-K+1]}.$$

Thus, from Lemma 8.6,

$$\limsup_{x \rightarrow \infty} \frac{\mathbb{P}(C_n^i > x)}{\overline{F}(x)} \leq N \sum_{k=1}^K c_k$$

which together with (15) completes the proof. \square

Remark 4.1 *The cycle time asymptotics is the same for each station. Thus, it does not matter at which station a cycle starts. This is different for the cycle time distribution, for which indeed one gets different results depending on where the cycle starts; see [8].*

Remark 4.2 *Let $c_{i_0} > 0$ and $c_i = 0$ for all $i \neq i_0$. Then*

$$\mathbb{P}(C_n^i > x) \sim N c_{i_0} \overline{F}(x) \sim N \overline{B}_{i_0}(x) \quad \text{as } x \rightarrow \infty.$$

Thus the asymptotics depends only on the service time distribution with the heaviest tail.

5 Waiting Times

Let W_n^i denote the n^{th} waiting time at station i ($i = 1, \dots, K$). Thus, W_n^i is the time from the arrival of the n^{th} customer at station i to the beginning of his service at this station and it is computed as

$$W_n^i = \left(X_{n-1}^i - X_{n-N_i}^{[i-1]} \right) \oplus 0$$

with the convention that $X_n^i = 0$ for all $n \leq 0$ and for all $i = 1, \dots, K$.

Proposition 5.1 *For $i \in \{1, \dots, K\}$ with $B_i \in \mathcal{S}$ and $n \geq N$, we have*

$$\mathbb{P}(W_n^i > x) \sim (N-1) \overline{B}_i(x) \sim (N-1) c_i \overline{F}(x) \quad \text{as } x \rightarrow \infty. \quad (16)$$

Proof Using the bounds on departure times given in (13) and (14), for $n \geq N$ we have

$$\begin{aligned} W_n^i &\geq \left(B_{n-1}^i \cdots B_1^i - \left(B_{n-N_i}^{[i-1]} \cdots B_1^{[i-1]} \otimes B_{n-N_i-N_{[i-1]}}^{[i-2]} \cdots B_1^{[i-2]} \right. \right. \\ &\quad \left. \left. \otimes \cdots \otimes B_{n-(N-N_{[i+1]})}^{[i-K+1]} \cdots B_1^{[i-K+1]} \otimes B_{n-N}^i \cdots B_1^i \right) \right) \oplus 0 \\ &\geq B_{n-1}^i \cdots B_{n-N+1}^i - \left(B_{n-N_i}^{[i-1]} \cdots B_1^{[i-1]} \otimes B_{n-N_i-N_{[i-1]}}^{[i-2]} \cdots B_1^{[i-2]} \right. \\ &\quad \left. \otimes \cdots \otimes B_{n-(N-N_{[i-1]})}^{[i-K+1]} \cdots B_1^{[i-K+1]} \right), \end{aligned}$$

and employing Lemma 8.4 and Lemma 8.6 we obtain

$$\liminf_{x \rightarrow \infty} \frac{\mathbb{P}(W_n^i > x)}{\overline{B}_i(x)} \geq (N - 1).$$

For an upper bound, we proceed in the same way as for the cycle times. Since X_{n-1}^i is expressed as the maximum of sums of service times that belong to the set

$$\{B_1^i, \dots, B_{n-1}^i, B_1^{[i-1]}, \dots, B_{n-1-N_i}^{[i-1]}, B_1^{[i-2]}, \dots, B_{n-1-N_i-N_{[i-1]}}^{[i-2]}, \dots, B_1^{[i-K+1]}, \dots, B_{n-1-(N-N_{[i+1]})}^{[i-K+1]}\}$$

and $X_{n-N_i}^{[i-1]}$ is expressed as the maximum of sums of service times that belong to the set

$$\{B_1^{[i-1]}, \dots, B_{n-N_i}^{[i-1]}, B_1^{[i-2]}, \dots, B_{n-N_i-N_{[i-1]}}^{[i-2]}, B_1^{[i-3]}, \dots, B_{n-N_i-N_{[i-1]}-N_{[i-2]}}^{[i-3]}, \dots, B_1^{[i-K]}, \dots, B_{n-N}^{[i-K]}\},$$

employing the arguments used in the derivation of the upper bound for cycle times we conclude that

$$W_n^i \leq \bigotimes_{j=1}^{N-1} B_{n-j}^i.$$

Thus,

$$\limsup_{x \rightarrow \infty} \frac{\mathbb{P}(W_n^i > x)}{\overline{B}_i(x)} \leq (N - 1).$$

Since $\overline{B}_i(x) \sim c_i \overline{F}(x)$ as $x \rightarrow \infty$, the assertion follows. \square

6 Residual Service Times

In this section we analyze the asymptotic tail behaviour of the distribution of a residual service seen by an arriving customer. In contrast to an open system where the arrival process is independent of the service process, such a residual service time $B^{i,\text{res}}$ does not follow the residual service time distribution $B_i^s(x) = \int_0^x \overline{B}_i(y) dy / \beta_i$ where β_i is the mean of $B_i(\cdot)$, but, up to a constant, has the same tail behaviour as $B_i(\cdot)$ given that the latter distribution is subexponential. Let $B_{n,k}^{i,\text{res}}$ denote the residual service time seen by the n^{th} arriving customer at station i , $i = 1, \dots, K$, given that he sees k customers in front of him. Then

$$B_{n,k}^{i,\text{res}} = 0 \oplus \left(X_{n-k+N_i}^i - X_n^{[i-1]} \right)$$

with the convention that $X_n^i = 0$ for all $n \leq 0$ and for all $i = 1, \dots, K$.

Proposition 6.1 For $i \in \{1, \dots, K\}$ with $B_i \in \mathcal{S}$, $1 \leq k \leq N - 1$, and $n \geq N - N_i$,

$$\mathbb{P}(B_{n,k}^{i,\text{res}} > x) \sim (N - k)\overline{B}_i(x) \sim (N - k)c_i\overline{F}(x) \quad \text{as } x \rightarrow \infty. \quad (17)$$

Proof From the bounds in (13) and (14), for $n \geq N - N_i$

$$\begin{aligned} & B_{n,k}^{i,\text{res}} \\ & \geq B_{n-k+N_i}^i \dots B_1^i - \left(B_n^{[i-1]} \dots B_1^{[i-1]} \otimes \dots \otimes B_{n-(N-N_i)}^i \dots B_1^i \right) \\ & = B_{n-k+N_i}^i \dots B_{n-(N-N_i)+1}^i \\ & \quad - \left(B_n^{[i-1]} \dots B_1^{[i-1]} \otimes \dots \otimes B_{n-(N-N_{[i-K+1]}-N_{[i-K]})}^{[i-K+1]} \dots B_1^{[i-K+1]} \right), \end{aligned}$$

and thus from Lemmas 8.4 and 8.6

$$\liminf_{x \rightarrow \infty} \frac{\mathbb{P}(B_{n,k}^{i,\text{res}} > x)}{\overline{B}_i(x)} \geq (N - k).$$

To get an upper bound, we again proceed in the same way as for the cycle times. Note that

$X_{n-k+N_i}^i$ is written as the maximum of sums of service times that belong to the set

$$\{B_1^i, \dots, B_{n-k+N_i}^i, B_1^{[i-1]}, \dots, B_{n-k}^{[i-1]}, B_1^{[i-2]}, \dots, B_{n-k-N_{[i-1]}}^{[i-2]}, \dots, B_1^{[i-K+1]}, \dots, B_{n-k-(N-N_i-N_{[i+1]})}^{[i-K+1]}\}$$

and $X_n^{[i-1]}$ is written as the maximum of sums of service times that belong to the set

$$\{B_1^{[i-1]}, \dots, B_n^{[i-1]}, B_1^{[i-2]}, \dots, B_{n-N_{[i-1]}}^{[i-2]}, B_1^{[i-3]}, \dots, B_{n-N_{[i-1]}-N_{[i-2]}}^{[i-3]}, \dots, B_1^{[i-K]}, \dots, B_{n-(N-N_i)}^{[i-K]}\}.$$

Employing the arguments used in the derivation of the upper bound for cycle times yields

$$B_{n,k}^{i,\text{res}} \leq \bigotimes_{j=k}^{N-1} B_{n+N_i-j}^i.$$

Therefore, from Lemma 8.6

$$\limsup_{x \rightarrow \infty} \frac{\mathbb{P}(B_{n,k}^{i,\text{res}} > x)}{\overline{B}_i(x)} \leq (N - k).$$

Again, since $\overline{B}_i(x) \sim c_i\overline{F}(x)$, this completes the proof. \square

Notice that the asymptotic constant in (17) is decreasing with respect to the number of customers an arriving customer sees in front of him. Thus, an intuitive explanation for the result of Proposition 6.1 could be as follows: the more customers there are waiting at a queue, the more

likely it is that the ongoing service has started quite a while ago giving rise to the build up of a long queue, and is thus more likely to be finished soon. Similarly, seeing a short queue may indicate that the ongoing service has started not too long ago and therefore is more likely to have a longer residual service time.

Remark 6.1 *Proposition 6.1 provides an alternative way to derive upper bounds for cycle and waiting times. Consider the worst case scenario, namely that a customer always sees $N - 1$ customers in front of him when arriving at a station, with one service going on. Then,*

$$C_n^i \leq \sum_{k=1}^K \left(\sum_{j=1}^{N-1} B_j^{[i+k]} + B_{n, N-1}^{[i+k], \text{res}} \right),$$

and thus

$$\limsup_{x \rightarrow \infty} \frac{\mathbb{P}(C_n^i > x)}{\overline{F}(x)} \leq N \sum_{k=1}^K c_k.$$

Similarly,

$$\limsup_{x \rightarrow \infty} \frac{\mathbb{P}(W_n^i > x)}{\overline{F}(x)} \leq (N - 1)c_i.$$

Note that these upper bounds agree with those obtained in the proofs of Propositions 4.1 and 5.1.

7 Stationary Results

In this section we assume that all service time distributions $B_i(\cdot)$, $1 \leq i \leq K$, have infinite support. Note that this assumption is always fulfilled for subexponential distributions. We observe the state of the cyclic network at departure epochs from station i . Let

$$\mathbf{Y}^i(n) = (Y_1^i(n), \dots, Y_K^i(n), Y_{K+1}^i(n), \dots, Y_{2K}^i(n))$$

be the system state vector, where $Y_\ell^i(n)$ and $Y_{K+\ell}^i(n)$ ($\ell = 1, \dots, K$) represent, respectively, the number of customers and the past service time of the ongoing service at station ℓ at the moment of the n^{th} departure from station i . Hence, if $Y_\ell^i(n) = 0$, then $Y_{K+\ell}^i(n) = 0$. Moreover, let $\mathbf{Y}^i(0) = (N_1, \dots, N_K, 0, \dots, 0)$ which is consistent with the initial condition defined in Sections 1 and 2. Note that $\{\mathbf{Y}^i(n), n = 0, 1, 2, \dots\}$ is a Markov chain. Let $R = \{\mathbf{Y}^0\}$ be the one-point set such that \mathbf{Y}^0 is a $\{0, 1, \dots, N\}^K \times \mathbb{R}^K$ vector fulfilling

$$Y_{i+1}^0 = 1, \quad Y_i^0 = N - 1, \quad Y_\ell^0 = 0 \quad (\ell \neq i, i + 1), \quad Y_{K+\ell}^0 = 0 \quad (\ell = 1, \dots, K).$$

The event R means that at a departure epoch from station i the departing customer leaves all the other $N - 1$ customers behind at station i . The one-step transition $\mathbf{Y}^i(n) \rightarrow \mathbf{Y}^i(n+1) = \mathbf{Y}^0$ takes place if and only if

$$X_{n+1}^i > X_{n+N-N_i}^{[i-1]}.$$

Hence, the set R is a regeneration set such that, for any initial marking,

$$\mathbb{P}(\mathbf{Y}^i(n+1) \in R) = \mathbb{P}(X_{n+1}^i > X_{n+N-N_i}^{[i-1]}) \geq \mathbb{P}(B_{n+1}^i > X_{n+N-N_i}^{[i-1]} - X_{n+1-N_i}^{[i-1]}),$$

where the last step follows from (4). Now, analogous to the proof of Proposition 4.1,

$$X_{n+N-N_i}^{[i-1]} - X_{n+1-N_i}^{[i-1]} \leq \bigotimes_{j=1}^{N-1} B_{n+1-N_i+j}^{[i-1]} \bigotimes_{j=1}^{N-1} B_{n+1-N_i+j-N_{[i-1]}}^{[i-2]} \cdots \bigotimes_{j=1}^{N-1} B_{n+1-N_i+j-(N-N_i)}^i,$$

and therefore

$$\begin{aligned} & \mathbb{P}(X_{n+1}^i > X_{n+N-N_i}^{[i-1]}) \\ & \geq \mathbb{P}\left(B_{n+1}^i > \bigotimes_{j=1}^{N-1} B_{n+1-N_i+j}^{[i-1]} \bigotimes_{j=1}^{N-1} B_{n+1-N_i+j-N_{[i-1]}}^{[i-2]} \cdots \bigotimes_{j=1}^{N-1} B_{n+1-N_i+j-(N-N_i)}^i\right) > 0. \end{aligned} \quad (18)$$

Thus, $\mathbf{Y}^i(n)$ is a Harris ergodic Markov chain and has a unique stationary distribution. Namely, from (18) and Theorem 3.6 and Proposition 3.13 of [1] (see also [2, 3, 6]) we have the following theorem.

Theorem 7.1 *If service time distributions at every station have infinite support, then there exists a random vector \mathbf{Y}^i such that the distribution of $\mathbf{Y}^i(n)$ converges in total variation to the distribution of \mathbf{Y}^i . The Markov chain $\{\mathbf{Y}^i(n)\}$ couples with \mathbf{Y}^i in finite time for any initial marking N_1, N_2, \dots, N_K .*

Note that the n^{th} cycle at station i , $C_n^i = X_{N+n}^i - X_n^i$, can be expressed in terms of $\mathbf{Y}^i(n)$ and the completed services taking place in the time interval $[X_n^i, X_{n+N}^i)$. At the k^{th} departure epoch from station i , the number of completed services since time 0 at the individual stations depends on the initial marking. Each station l has completed at least $\max\{k - N_i, 0\} = (k - N_i)^+ \geq (k - N)^+$ and at most $k + N - N_i - 1 \leq k + N - 1$ services. We define the following sequence of random vectors for $n \geq N$

$$\mathbf{B}^\ell(n) = (B_{n-N+1}^\ell, B_{n-N+2}^\ell, \dots, B_{n+2N-1}^\ell), \quad \ell = 1, 2, \dots, K.$$

Since \oplus and \otimes are continuous operators, we have

$$C_n^i = f_1(\mathbf{Y}^i(n), \mathbf{B}^1(n), \dots, \mathbf{B}^K(n))$$

for $n \geq N$ and some continuous function $f_1 : \{0, 1, \dots, N\}^K \times \mathbb{R}^{3NK} \rightarrow \mathbb{R}_+$. Thus, since convergence in total variation implies weak convergence, from the continuous mapping theorem and Theorem 7.1 we obtain the following result.

Theorem 7.2 *If service time distributions at each station have infinite support, then there exists a random variable $C^i = f_1(\mathbf{Y}^i, \mathbf{B}^1, \dots, \mathbf{B}^K)$ such that C_n^i converges in distribution to C^i as $n \rightarrow \infty$. The sequence C_n^i couples with C^i in finite time for any initial marking N_1, N_2, \dots, N_K .*

It follows from Proposition 4.1 that for all $n \geq N - N_{[i+1]}$ and any $\varepsilon > 0$ there exists L such that for all $x > L$ we have

$$(1 - \varepsilon) N \sum_{k=1}^K c_k \bar{F}(x) \leq \mathbb{P}(C_n^i > x) \leq (1 + \varepsilon) N \sum_{k=1}^K c_k \bar{F}(x),$$

for $i \in \{1, \dots, K\}$, and $c_i > 0$. Taking limits as $n \rightarrow \infty$ and $\varepsilon \downarrow 0$ we obtain the following corollary.

Corollary 7.1 *If service times at each station have infinite support, then for $i \in \{1, \dots, K\}$ we have that*

$$\mathbb{P}(C^i > x) \sim N \sum_{k=1}^K c_k \bar{F}(x) \quad \text{as } x \rightarrow \infty.$$

Note also that

$$W_n^i = f_2(\mathbf{Y}^i(n), \mathbf{B}^1(n), \dots, \mathbf{B}^K(n))$$

for some continuous function $f_2 : \{0, 1, \dots, N\}^K \times \mathbb{R}^{3NK} \rightarrow \mathbb{R}_+$. Thus, from Theorem 7.1 and Proposition 5.1 we can obtain the tail asymptotics of the distribution of the limiting waiting time at station i (W^i).

Corollary 7.2 *For $i \in \{1, \dots, K\}$ with $B_i \in \mathcal{S}$ we have*

$$\mathbb{P}(W^i > x) \sim (N - 1) \bar{B}_i(x) \sim (N - 1) c_i \bar{F}(x) \quad \text{as } x \rightarrow \infty.$$

Similarly, we can obtain from Theorem 7.1 and Proposition 6.1 the asymptotics of the limiting residual distribution $Y_{K+i}^{[i-1]}$ given that $Y_i^{[i-1]} = k$, which we denote by $B_k^{i, \text{res}}$.

Corollary 7.3 For $i \in \{1, \dots, K\}$ with $B_i \in \mathcal{S}$, $1 \leq k \leq N-1$,

$$\mathbb{P}(B_k^{i, \text{res}} > x) \sim (N-k)\overline{B}_i(x) \sim (N-k)c_i\overline{F}(x) \quad \text{as } x \rightarrow \infty.$$

8 Appendix

In this appendix we first give the proof of Lemma 3.1 and then recall some properties of the class \mathcal{S} of subexponential distributions introduced in [13].

8.1 Proof of Lemma 3.1

First note that more explicitly (9) can be written as

$$\begin{aligned}
X_n^i = & \bigoplus_{j=1}^{\infty} \bigoplus_{j_1=Nj-(\sum_{k=1}^K N_{[i-k]}-1)}^n \bigoplus_{j_2=Nj-(2N_{[i-K]}+\sum_{k=1}^{K-1} N_{[i-k]}-1)}^{j_1-N_i} \\
& \bigoplus_{j_2-N_{[i-1]}} \bigoplus_{j_3-N_{[i-2]}} \dots \\
& j_3=Nj-(2N_{[i-K]}+2N_{[i-1]}+\sum_{k=2}^{K-1} N_{[i-k]}-1) \quad j_4=Nj-(2N_{[i-K]}+2\sum_{k=1}^2 N_{[i-k]}+\sum_{k=3}^{K-1} N_{[i-k]}-1) \\
& \dots \quad \bigoplus_{j_{K+1}-N_{[i-K]}} \bigoplus_{j_{K+2}-N_{[i-1]}} \dots \\
& j_{K+2}=Nj-(3N_{[i-K]}+2\sum_{k=1}^{K-1} N_{[i-k]}-1) \quad j_{K+3}=Nj-(3N_{[i-K]}+3N_{[i-1]}+2\sum_{k=2}^{K-1} N_{[i-k]}-1) \\
& \dots \quad \bigoplus_{j_{2K+1}-N_{[i-K]}} \quad \dots \quad \bigoplus_{j_{Kj-2}-N_{[i-(K-3)]}} \quad \left(\bigotimes_{i_1=j_1}^n B_{i_1}^i \quad \bigotimes_{i_2=j_2}^{j_1-N_i} B_{i_2}^{[i-1]} \right) \\
& j_{2K+2}=Nj-(4N_{[i-K]}+3\sum_{k=1}^{K-1} N_{[i-k]}-1) \quad j_{Kj-1}=1 \quad \dots \quad \bigotimes_{i_3=j_3}^{j_2-N_{[i-1]}} B_{i_3}^{[i-2]} \quad \bigotimes_{i_4=j_4}^{j_3-N_{[i-2]}} B_{i_4}^{[i-3]} \quad \dots \quad \bigotimes_{i_K=j_K}^{j_{K-1}-N_{[i-(K-2)]}} B_{i_K}^{[i-K+1]} \\
& \bigotimes_{i_{K+1}=j_{K+1}}^{j_K-N_{[i-(K-1)]}} B_{i_{K+1}}^i \quad \dots \quad \bigotimes_{i_{Kj-1}=j_{Kj-1}}^{j_{Kj-2}-N_{[i-(K-3)]}} B_{i_{Kj-1}}^{[i-K+2]} \quad \bigotimes_{l=1}^{j_{Kj-1}-N_{[i-(K-2)]}} B_l^{[i-K+1]} \Big). \tag{19}
\end{aligned}$$

Thus, we will show that (8) is equivalent to (19). Let

$$\begin{aligned}
j^* &= \max\{j \in \mathbb{N} : Nj - \left(\sum_{k=1}^{K-1} N_{[i-k]} - 1\right) \leq n\}, \text{ and} \\
j_0^* &= \max\{j \in \mathbb{N} : Nj - \left(\sum_{k=1}^K N_{[i-k]} - 1\right) \leq n\}.
\end{aligned}$$

Note that j_0^* always exists. However, if $n < N_i + 1$, j^* does not exist. We start with this case.

Suppose j^* does not exist which implies that $j_0^* = 1$ and which is only possible if $N_i > 0$. Then (8)

reduces to $\bigotimes_{k=1}^n B_k^i$. On the other hand in (19) j_2, j_3, \dots, j_{K-1} vanish since

$$j_2 \leq j_1 - N_i < 1.$$

Moreover, the maximum in (19) is achieved for $j_1 = 1$ which yields $\bigotimes_{i_1=1}^n B_{i_1}^i$. For the rest of the proof we assume that j^* exists, then in (8), $j = 1, \dots, j^*$ and in (19), $j = 1, \dots, j_0^*$. Note that $0 \leq j_0^* - j^* \leq 1$. Now suppose $N_i > 0$ and $j^* = j_0^*$. We start with $\bigotimes_{k=1}^n B_k^i$ and $j = 1$ term in (8),

$$\begin{aligned} & \bigotimes_{k=1}^n B_k^i \oplus \bigoplus_{j_1=N_i+1}^n \bigoplus_{j_2=1}^{j_1-N_i} \bigoplus_{j_3=1}^{j_2-N_{[i-1]}} \bigoplus_{j_4=1}^{j_3-N_{[i-2]}} \dots \bigoplus_{j_K=1}^{j_{K-1}-N_{[i-(K-2)]}} \left(\bigotimes_{i_1=j_1}^n B_{i_1}^i \bigotimes_{i_2=j_2}^{j_1-N_i} B_{i_2}^{[i-1]} \dots \right. \\ & \quad \left. \dots \bigotimes_{i_K=j_K}^{j_{K-1}-N_{[i-(K-2)]}} B_{i_K}^{[i-K+1]} \bigotimes_{l=1}^{j_K-N_{[i-(K-1)]}} B_l^i \right) \\ & = \bigoplus_{j_1=1}^n \bigoplus_{j_2=1}^{j_1-N_i} \dots \bigoplus_{j_K=1}^{N_{[i-K+1]}} \left(\bigotimes_{i_1=j_1}^n B_{i_1}^i \bigotimes_{i_2=j_2}^{j_1-N_i} B_{i_2}^{[i-1]} \dots \bigotimes_{i_K=1}^{j_{K-1}-N_{[i-(K-2)]}} B_{i_K}^{[i-K+1]} \right) \oplus \end{aligned} \quad (20)$$

$$\begin{aligned} & \bigoplus_{j_1=N+1}^n \bigoplus_{j_2=N-N_i+1}^{j_1-N_i} \bigoplus_{j_3=N-N_i-N_{[i-1]}+1}^{j_2-N_{[i-1]}} \dots \bigoplus_{j_K=N_{[i-K+1]}+1}^{j_{K-1}-N_{[i-(K-2)]}} \left(\bigotimes_{i_1=j_1}^n B_{i_1}^i \bigotimes_{i_2=j_2}^{j_1-N_i} B_{i_2}^{[i-1]} \dots \right. \\ & \quad \left. \dots \bigotimes_{i_K=j_K}^{j_{K-1}-N_{[i-(K-2)]}} B_{i_K}^{[i-K+1]} \bigotimes_{l=1}^{j_K-N_{[i-(K-1)]}} B_l^i \right). \end{aligned} \quad (21)$$

In order to obtain the above equality first note that for $j_1 < N_i + 1$, j_2, j_3, \dots, j_K vanish and for $j_1 = 1$ we obtain $\bigotimes_{k=1}^n B_k^i$ and for $2 \leq j_1 < N_i + 1$ we have

$$\bigotimes_{i_1=j_1}^n B_{i_1}^i \leq \bigotimes_{k=1}^n B_k^i.$$

We then consider $j_K \leq N_{[i-K+1]}$ and $j_K > N_{[i-K+1]}$ separately to obtain (20) and (21). If $j_K \leq N_{[i-K+1]}$, l vanishes and

$$\bigotimes_{i_K=1}^{j_{K-1}-N_{[i-(K-2)]}} B_{i_K}^{[i-K+1]} \geq \bigotimes_{i_K=j_K}^{j_{K-1}-N_{[i-(K-2)]}} B_{i_K}^{[i-K+1]}$$

for $1 < j_K < N_{[i-K+1]}$. Finally, to obtain (21) for $j_K \geq N_{[i-K+1]}$, we need $j_{K-1} \geq N_{[i-K+1]} + N_{[i-K+2]} + 1$, $j_{K-2} \geq N_{[i-K+1]} + N_{[i-K+2]} + N_{[i-K+3]} + 1$, \dots , $j_1 \geq N + 1$. Note that (20) is the $j = 1$ term in (19). We next consider $j = 2$ term in (8) which is equal to

$$\bigoplus_{j_1=N+N_i+1}^n \bigoplus_{j_2=N+1}^{j_1-N_i} \dots \bigoplus_{j_K=N_{[i-K]}+N_{[i-K+1]}+1}^{j_{K-1}-N_{[i-K+2]}} \bigoplus_{j_{K+1}=N_{[i-K]}+1}^{j_K-N_{[i-K+1]}} \bigoplus_{j_{K+2}=1}^{j_{K+1}-N_{[i-K]}} \bigoplus_{j_{K+3}=1}^{j_{K+2}-N_{[i-1]}} \dots \bigoplus_{j_{2K}=1}^{j_{2K-1}-N_{[i-K+2]}}$$

$$\left(\bigotimes_{i_1=j_1}^n B_{i_1}^i \bigotimes_{i_2=j_2}^{j_1-N_i} B_{i_2}^{[i-1]} \dots \bigotimes_{i_K=j_K}^{j_{K-1}-N_{[i-K+2]}} B_{i_2}^{[i-1]} \bigotimes_{i_{K+1}=j_{K+1}}^{j_K-N_{[i-K+1]}} B_{i_{K+1}}^{[i-K]} \dots \right. \\ \left. \dots \bigotimes_{i_{2K}=j_{2K}}^{j_{2K-1}-N_{[i-K+2]}} B_{i_{2K}}^{[i-K+1]} \bigotimes_{l=1}^{j_{2K}-N_{[i-K+1]}} B_l^i \right). \quad (22)$$

Putting (21) and (22) together yields

$$\begin{aligned} & \bigoplus_{j_1=N+1}^n \bigoplus_{j_2=N-N_i+1}^{j_1-N_i} \dots \bigoplus_{j_K=N_{[i-K+1]}+1}^{j_{K-1}-N_{[i-K+2]}} \bigoplus_{j_{K+1}=1}^{j_K-N_{[i-K+1]}} \bigoplus_{j_{K+2}=1}^{j_{K+1}-N_{[i-K]}} \bigoplus_{j_{K+3}=1}^{j_{K+2}-N_{[i-1]}} \dots \bigoplus_{j_{2K}=1}^{j_{2K-1}-N_{[i-K+2]}} \\ & \left(\bigotimes_{i_1=j_1}^n B_{i_1}^i \bigotimes_{i_2=j_2}^{j_1-N_i} B_{i_2}^{[i-1]} \dots \bigotimes_{i_K=j_K}^{j_{K-1}-N_{[i-K+2]}} B_{i_2}^{[i-1]} \bigotimes_{i_{K+1}=j_{K+1}}^{j_K-N_{[i-K+1]}} B_{i_{K+1}}^{[i-K]} \dots \right. \\ & \quad \dots \bigotimes_{i_{2K}=j_{2K}}^{j_{2K-1}-N_{[i-K+2]}} B_{i_{2K}}^{[i-K+1]} \bigotimes_{l=1}^{j_{2K}-N_{[i-K+1]}} B_l^i \left. \right) \\ & = \bigoplus_{j_1=N+1}^n \bigoplus_{j_2=N-N_i+1}^{j_1-N_i} \dots \bigoplus_{j_K=N_{[i-K+1]}+1}^{j_{K-1}-N_{[i-K+2]}} \bigoplus_{j_{K+1}=1}^{j_K-N_{[i-K+1]}} \bigoplus_{j_{K+2}=1}^{j_{K+1}-N_{[i-K]}} \bigoplus_{j_{K+3}=1}^{j_{K+2}-N_{[i-1]}} \dots \bigoplus_{j_{2K}=1}^{N_{[i-K+1]}} \\ & \left(\bigotimes_{i_1=j_1}^n B_{i_1}^i \bigotimes_{i_2=j_2}^{j_1-N_i} B_{i_2}^{[i-1]} \dots \bigotimes_{i_K=j_K}^{j_{K-1}-N_{[i-K+2]}} B_{i_2}^{[i-1]} \bigotimes_{i_{K+1}=j_{K+1}}^{j_K-N_{[i-K+1]}} B_{i_{K+1}}^{[i-K]} \dots \right. \\ & \quad \dots \bigotimes_{i_{2K}=1}^{j_{2K-1}-N_{[i-K+2]}} B_{i_{2K}}^{[i-K+1]} \left. \right) \oplus \quad (23) \\ & \bigoplus_{j_1=2N+1}^n \bigoplus_{j_2=2N-N_i+1}^{j_1-N_i} \dots \bigoplus_{j_K=N+N_{[i-K+1]}+1}^{j_{K-1}-N_{[i-K+2]}} \bigoplus_{j_{K+1}=N+1}^{j_K-N_{[i-K+1]}} \bigoplus_{j_{K+2}=N-N_i+1}^{j_{K+1}-N_{[i-K]}} \bigoplus_{j_{K+3}=N-N_i-N_{[i-1]}+1}^{j_{K+2}-N_{[i-1]}} \dots \\ & \dots \bigoplus_{j_{2K-1}=N+N_{[i-K+1]}+N_{[i-K+2]}}^{j_{2K-1}-N_{[i-K+3]}} \bigoplus_{j_{2K}=N_{[i-K+1]}+1}^{j_{2K-1}-N_{[i-K+2]}} \left(\bigotimes_{i_1=j_1}^n B_{i_1}^i \bigotimes_{i_2=j_2}^{j_1-N_i} B_{i_2}^{[i-1]} \dots \bigotimes_{i_K=j_K}^{j_{K-1}-N_{[i-K+2]}} B_{i_2}^{[i-1]} \right. \\ & \quad \bigotimes_{i_{K+1}=j_{K+1}}^{j_K-N_{[i-K+1]}} B_{i_{K+1}}^{[i-K]} \dots \bigotimes_{i_{2K}=1}^{j_{2K-1}-N_{[i-K+2]}} B_{i_{2K}}^{[i-K+1]} \bigotimes_{l=1}^{j_{2K}-N_{[i-K+1]}} B_l^i \left. \right) \quad (24) \end{aligned}$$

where the equality is obtained by considering $j_{2K} \leq N_{[i-K+1]}$ and $j_{2K} > N_{[i-K+1]}$ separately as is done above. Note that (23) is the $j = 2$ term in (19). Continuing in this fashion for $j = j^*$ we have

$$\begin{aligned} & \bigoplus_{j_1=N(j^*-1)+1}^n \bigoplus_{j_2=N(j^*-1)-N_i+1}^{j_1-N_i} \dots \bigoplus_{j_K=N(j^*-2)+N_{[i-K+1]}+1}^{j_{K-1}-N_{[i-K+2]}} \bigoplus_{j_{K+1}=N(j^*-2)+1}^{j_K-N_{[i-K+1]}} \bigoplus_{j_{K+2}=N(j^*-2)-N_i+1}^{j_{K+1}-N_{[i-K]}} \dots \\ & \dots \bigoplus_{j_{2K}=N(j^*-3)+N_{[i-K+1]}+1}^{j_{2K-1}-N_{[i-K+2]}} \bigoplus_{j_{2K+1}=N(j^*-3)+1}^{j_{2K}-N_{[i-K+1]}} \dots \bigoplus_{j_{K(j^*-1)+1}=1}^{j_{K(j^*-1)}-N_{[i-K+1]}} \bigoplus_{j_{K(j^*-1)+2}=1}^{j_{K(j^*-1)+1}-N_{[i-K]}} \dots \bigoplus_{j_{Kj^*}=1}^{j_{Kj^*-1}-N_{[i-K+2]}} \\ & \left(\bigotimes_{i_1=j_1}^n B_{i_1}^i \bigotimes_{i_2=j_2}^{j_1-N_i} B_{i_2}^{[i-1]} \dots \bigotimes_{i_K=j_K}^{j_{K-1}-N_{[i-K+2]}} B_{i_2}^{[i-1]} \bigotimes_{i_{K+1}=j_{K+1}}^{j_K-N_{[i-K+1]}} B_{i_{K+1}}^{[i-K]} \dots \right. \end{aligned}$$

$$\dots \bigotimes_{i_{j^*K}=j_{j^*K}}^{j_{j^*K-1}-N_{[i-K+2]}} B_{i_{j^*K}}^{[i-K+1]} \bigotimes_{l=1}^{j_{j^*K}-N_{[i-K+1]}} B_l^i. \quad (25)$$

In order to obtain (25) we combined the $j = j^*$ term of (8) with the remaining term from $j = j^* - 1$ (i.e. the part corresponding to $j_{(j^*-1)K} > N_{[i-K+1]}$). Since $n < Nj^* + 1$,

$$j_{j^*K} \leq n - (Nj^* - N_{[i-K+1]}) < N_{[i-K+1]} + 1.$$

Thus, $j_{j^*K} - N_{[i-K+1]} < 1$ and the l term vanishes in (25). Moreover, the maximum is achieved for $j_{Kj^*} = 1$ since this yields $\bigotimes_{i_{j^*K}=1}^{j_{j^*K-1}-N_{[i-K+2]}} B_{i_{j^*K}}^{[i-K+1]}$ which implies that (25) is equivalent to the $j = j_0^* = j^*$ term in (19). This completes the proof for $N_i > 0$ and $j^* = j_0^*$. If $N_i = 0$ then clearly $j^* = j_0$. The proof of $N_i = 0$ is the same as the one given above except in this case $\bigotimes_{k=1}^n B_k^i$ is redundant since for $j = 1$ in (8), the expression corresponding to $j_1 = 1$ and $j_2 = 1$ dominates $\bigotimes_{k=1}^n B_k^i$. We next consider $j_0^* = j^* + 1$ which is only possible if $N_i > 0$. The proof will be the same as the proof of $j^* = j_0^*$ except for $j = j^*$ we now again need to split (25) into two parts by considering $j_{j^*K} \leq N_{[i-K+1]}$ and $j_{j^*K} > N_{[i-K+1]}$. Thus, (25) is written as

$$\begin{aligned} & \bigoplus_{j_1=N(j^*-1)+1}^n \bigoplus_{j_2=N(j^*-1)-N_i+1}^{j_1-N_i} \dots \bigoplus_{j_K=N(j^*-2)+N_{[i-K+1]}+1}^{j_{K-1}-N_{[i-K+2]}} \bigoplus_{j_{K+1}=N(j^*-2)+1}^{j_K-N_{[i-K+1]}} \bigoplus_{j_{K+2}=N(j^*-2)-N_i+1}^{j_{K+1}-N_{[i-K]}} \dots \\ & \dots \bigoplus_{j_{2K}=N(j^*-3)+N_{[i-K+1]}+1}^{j_{2K-1}-N_{[i-K+2]}} \bigoplus_{j_{2K+1}=N(j^*-3)+1}^{j_{2K}-N_{[i-K+1]}} \dots \bigoplus_{j_{K(j^*-1)+1}=1}^{j_{K(j^*-1)}-N_{[i-K+1]}} \bigoplus_{j_{K(j^*-1)+2}=1}^{j_{K(j^*-1)+1}-N_{[i-K]}} \dots \bigoplus_{j_{Kj^*}=1}^{N_{[i-K+1]}} \\ & \left(\bigotimes_{i_1=j_1}^n B_{i_1}^i \bigotimes_{i_2=j_2}^{j_1-N_i} B_{i_2}^{[i-1]} \dots \bigotimes_{i_K=j_K}^{j_{K-1}-N_{[i-K+2]}} B_{i_K}^{[i-1]} \bigotimes_{i_{K+1}=j_{K+1}}^{j_K-N_{[i-K+1]}} B_{i_{K+1}}^{[i-K]} \dots \right. \\ & \left. \dots \bigotimes_{i_{j^*K}=1}^{j_{j^*K-1}-N_{[i-K+2]}} B_{i_{j^*K}}^{[i-K+1]} \right) \oplus \end{aligned} \quad (26)$$

$$\begin{aligned} & \bigoplus_{j_1=Nj^*+1}^n \bigoplus_{j_2=Nj^*-N_i+1}^{j_1-N_i} \dots \bigoplus_{j_K=N(j^*-1)+N_{[i-K+1]}+1}^{j_{K-1}-N_{[i-K+2]}} \bigoplus_{j_{K+1}=N(j^*-1)+1}^{j_K-N_{[i-K+1]}} \bigoplus_{j_{K+2}=N(j^*-1)-N_i+1}^{j_{K+1}-N_{[i-K]}} \dots \\ & \dots \bigoplus_{j_{2K}=N(j^*-2)+N_{[i-K+1]}+1}^{j_{2K-1}-N_{[i-K+2]}} \bigoplus_{j_{2K+1}=N(j^*-2)+1}^{j_{2K}-N_{[i-K+1]}} \dots \bigoplus_{j_{K(j^*-1)+1}=N+1}^{j_{K(j^*-1)}-N_{[i-K+1]}} \bigoplus_{j_{K(j^*-1)+2}=N-N_i+1}^{j_{K(j^*-1)+1}-N_{[i-K]}} \dots \\ & \bigoplus_{j_{Kj^*-1}-N_{[i-K+2]}}^{j_{Kj^*}-1-N_{[i-K+2]}} \left(\bigotimes_{i_1=j_1}^n B_{i_1}^i \bigotimes_{i_2=j_2}^{j_1-N_i} B_{i_2}^{[i-1]} \dots \bigotimes_{i_K=j_K}^{j_{K-1}-N_{[i-K+2]}} B_{i_K}^{[i-1]} \bigotimes_{i_{K+1}=j_{K+1}}^{j_K-N_{[i-K+1]}} B_{i_{K+1}}^{[i-K]} \dots \right. \\ & \left. \dots \bigotimes_{i_{j^*K}=j_{j^*K}}^{j_{j^*K-1}-N_{[i-K+2]}} B_{i_{j^*K}}^{[i-K+1]} \bigotimes_{l=1}^{j_{j^*K}-N_{[i-K+1]}} B_l^i \right). \end{aligned} \quad (27)$$

Note that (26) is the $j = j^*$ term in (19). On the other hand for $j = j_0^* = j^* + 1$ in (19) we have

$$\begin{aligned}
& \begin{array}{cccccccc}
\bigoplus^n & \bigoplus^{j_1-N_i} & \dots & \bigoplus^{j_{K-1}-N_{[i-K+2]}} & \bigoplus^{j_K-N_{[i-K+1]}} & \bigoplus^{j_{K+1}-N_{[i-K]}} & \dots & \\
j_1=Nj^*+1 & j_2=Nj^*-N_i+1 & & j_K=N(j^*-1)+N_{[i-K+1]}+1 & j_{K+1}=N(j^*-1)+1 & j_{K+2}=N(j^*-1)-N_i+1 & & \\
& \bigoplus^{j_{2K-1}-N_{[i-K+2]}} & & \bigoplus^{j_{2K}-N_{[i-K+1]}} & \bigoplus^{j_{K(j^*-1)}-N_{[i-K+1]}} & \bigoplus^{j_{K(j^*-1)+1}-N_{[i-K]}} & & \\
\dots & & & & & & & \\
& \bigoplus^{j_{2K}=N(j^*-2)+N_{[i-K+1]}+1} & \bigoplus^{j_{2K+1}=N(j^*-2)+1} & & \bigoplus^{j_{K(j^*-1)+1}=N+1} & \bigoplus^{j_{K(j^*-1)+2}=N-N_i+1} & & \\
j_{Kj^*-1}-N_{[i-K+2]} & j_{Kj^*-N_{[i-K+1]}} & \left(\bigotimes_{i_1=j_1}^n B_{i_1}^i \right. & \bigotimes_{i_2=j_2}^{j_1-N_i} B_{i_2}^{[i-1]} & \dots & \bigotimes_{i_K=j_K}^{j_{K-1}-N_{[i-K+2]}} B_{i_2}^{[i-1]} & \bigotimes_{i_{K+1}=j_{K+1}}^{j_K-N_{[i-K+1]}} B_{i_{K+1}}^{[i-K]} & \dots \\
j_{Kj^*}=N_{[i-K+1]}+1 & j_{Kj^*+1}=1 & & & & & & \\
& & & \bigotimes_{i_{j^*K}=j_{j^*K}}^{j_{j^*K-1}-N_{[i-K+2]}} & B_{i_{j^*K}}^{[i-K+1]} & \bigotimes_{i_{j^*K+1}=j_{j^*K+1}}^{j_{j^*K}-N_{[i-K+1]}} & B_{i_{j^*K+1}}^i & \left. \right). \quad (28)
\end{array}
\end{aligned}$$

The terms corresponding to $j_{Kj^*+2}, j_{Kj^*+3}, \dots, j_{Kj_0^*-1}$ disappear when $j = j_0^*$. Since $n < Nj^* + N_i + 1$, we have

$$j_{Kj^*+2} \leq j_{Kj^*+1} - N_i < n - Nj^* - N_i < 1.$$

Moreover, the maximum is achieved by setting $j_{Kj^*+1} = 1$ which implies that (28) is equivalent to (27). This completes the proof. \square

8.2 Subexponential distributions

As before, we write $f(x) \sim cg(x)$ to express that $\lim_{x \rightarrow \infty} f(x)/g(x) = c$ for two functions $f(x), g(x)$, and some constant $c \geq 0$.

Definition 8.1 A distribution function F on $\mathbb{R}_+ = [0, \infty)$ with $F(x) < 1$ for all $x > 0$ is called subexponential ($F \in \mathcal{S}$) if

$$\overline{F^{*2}}(x) \sim 2\overline{F}(x),$$

where $\overline{F}(x) = 1 - F(x)$ and F^{*2} denotes the convolution $F * F$.

The class \mathcal{S} has some very useful properties. Those which are used in this paper are the following ones.

Lemma 8.1 Let F, G be two distributions on \mathbb{R}_+ and assume that there exists a constant $c \in (0, \infty)$ with $\overline{G}(x) \sim c\overline{F}(x)$. Then, $F \in \mathcal{S}$ if and only if $G \in \mathcal{S}$.

Lemma 8.2 Let F, G, H be distributions on \mathbb{R}_+ such that $F \in \mathcal{S}$, $\overline{G}(x) \sim c_1 \overline{F}(x)$ and $\overline{H}(x) \sim c_2 \overline{F}(x)$, where $c_i \in (0, \infty)$ for $i = 1, 2$. Then,

$$\overline{G * H}(x) \sim (c_1 + c_2) \overline{F}(x) .$$

The results stated in Lemmas 8.1 and 8.2 are well-known. For a proof see for example [14] and [17].

Lemma 8.3 Let $F \in \mathcal{S}$ and let G be a distribution on \mathbb{R}_+ such that $\overline{G}(x) = o(\overline{F}(x))$ as $x \rightarrow \infty$. Then $\overline{F * G}(x) \sim \overline{F}(x)$.

Lemma 8.4 Let X and $Y \geq 0$ be independent random variables with distribution functions $F_X \in \mathcal{S}$ and F_Y , respectively. Then

$$\mathbb{P}(X - Y > x) \sim \mathbb{P}(X > x) \text{ as } x \rightarrow \infty .$$

For the proofs of Lemmas 8.3 and 8.4, see [22]. We further need the following results, which are easily derived from the above lemmas.

Lemma 8.5 Let $F \in \mathcal{S}$ and let G_1, \dots, G_n , $n \geq 1$, be distributions on \mathbb{R}_+ such that $\overline{G}_i(x) \sim c_i \overline{F}(x)$ as $x \rightarrow \infty$; $c_i \geq 0$. Then,

$$1 - \prod_{i=1}^n G_i(x) \sim \sum_{i=1}^n c_i \overline{F}(x) . \quad (29)$$

Proof For $n = 1$, the assertion is true. Now assume that (29) holds for some $n \geq 1$. Since

$$1 - \prod_{i=1}^{n+1} G_i(x) = \left(1 - \prod_{i=1}^n G_i(x)\right) G_{n+1}(x) + 1 - G_{n+1}(x) ,$$

the assumptions imply that

$$\lim_{x \rightarrow \infty} \frac{1 - \prod_{i=1}^{n+1} G_i(x)}{\overline{F}(x)} = \lim_{x \rightarrow \infty} \frac{1 - \prod_{i=1}^n G_i(x)}{\overline{F}(x)} G_{n+1}(x) + \lim_{x \rightarrow \infty} \frac{\overline{G}_{n+1}(x)}{\overline{F}(x)} = \sum_{i=1}^n c_i + c_{n+1} .$$

□

Lemma 8.6 Let $F \in \mathcal{S}$ and let F_1, \dots, F_n , $n \geq 1$, and G_1, \dots, G_m , $m \geq 1$, be distributions on \mathbb{R}_+ such that $\overline{F}_i(x) \sim c_i \overline{F}(x)$ with $c_i > 0$, $1 \leq i \leq n$, and $\overline{G}_i(x) = o(\overline{F}(x))$ for $1 \leq i \leq m$. Then,

$$\overline{F_1 * \dots * F_n * G_1 * \dots * G_m}(x) \sim \sum_{i=1}^n c_i \overline{F}(x) .$$

Proof By induction it follows immediately from Lemma 8.2 that

$$\overline{F_1 * \dots * F_n}(x) \sim \sum_{i=1}^n c_i \overline{F}(x).$$

Thus, by Lemma 8.1, $H \stackrel{\text{def}}{=} F_1 * \dots * F_n \in \mathcal{S}$. Since, for $1 \leq i \leq m$,

$$\lim_{x \rightarrow \infty} \frac{\overline{G}_i(x)}{\overline{H}(x)} = \lim_{x \rightarrow \infty} \frac{\overline{G}_i(x)}{\overline{F}(x)} \frac{\overline{F}(x)}{\overline{H}(x)} = 0,$$

(i.e., $\overline{G}_i(x) = o(\overline{H}(x))$) from Lemma 8.3 we have in particular $\overline{H * G_1}(x) \sim \overline{H}(x)$. Therefore the assertion is true for $m = 1$, because $\overline{H}(x) \sim \sum_{i=1}^n c_i \overline{F}(x)$. Assume that the assertion holds for some $m \in \mathbb{N}$. Then,

$$\lim_{x \rightarrow \infty} \frac{1 - G_{m+1}(x)}{1 - H * G_1 * \dots * G_m(x)} = \lim_{x \rightarrow \infty} \frac{1 - G_{m+1}(x)}{\overline{H}(x)} \frac{\overline{H}(x)}{\sum_{i=1}^n c_i \overline{F}(x)} \frac{\sum_{i=1}^n c_i \overline{F}(x)}{1 - H * G_1 * \dots * G_m(x)} = 0,$$

(i.e., $\overline{G}_{m+1}(x) = o(\overline{H * G_1 * \dots * G_m}(x))$). Thus, by Lemma 8.3 and the induction hypothesis, the assertion follows. \square

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