1 Introduction

A germ-grain model may provide a good description for a very irregular pattern observed in microscopy materials science, biology and analysis of images. Perhaps the best known model is the Boolean model (Matheron [7]) formalizing a configuration of independent, randomly placed particles. A Boolean model is formed by placing random balls centered at the points of a Poisson process and taking the union of these balls. The points of the Poisson process are sometimes called the germs and the associated balls the grains. In a natural generalization of the Boolean model the Poisson process of grains is replaced by a general point process and balls by any compact sets or even more general objects. If we take these objects as a mark at the point of point process of germs, then such a marked point process $N(\cdot,\cdot)$ we will call a *marked point process (abbreviated by m.p.p.) driving the germ-grain model.*

The simulation of a Boolean model within a compact set T falls into the following stages. First, the number of points is determined by simulating a Poisson random variable J with the parameter $\lambda|T|$, where λ is the intensity of the Poisson process and $|T|$ the volume of the set T. Then J independent random points are simulated in T according to the Bernoulli process. Next, we generate J i.i.d. copies of radius r. Finally, the Boolean model is constructed by

$$\mathcal{A}(T) = \sum_{t_i \in T} (t_i + B(r_i)) \ ,$$

where $\oplus$ is Minkowski addition and $B(R)$ is the ball with radius R.

We want to simulate the so-called rare event A for Boolean model $\mathcal{A}(T)$, or more generally for a germ-grain model. That is, $\mathbb{P}_N(A)$ is "small" (typically of order 10^{-6}). Using the so-called Crude Monte Carlo (MC) method of simulation in this case is inefficient. Precisely, let n be a size of a sample and $\mathbb{I}(A_1), \mathbb{I}(A_2), \ldots, \mathbb{I}(A_n)$ replicants of $\mathbb{I}(A)$. Then we estimate $p = \mathbb{P}_N(A)$ by

$$\hat{p} = \frac{1}{n} \sum_{i=1}^{n} \mathbb{I}(A_i) \ .$$

By

$$\hat{\sigma}^2 = \hat{p}(1-\hat{p})$$

we denote an empirical variance. According to the standard Central Limit Theorem

$$\sqrt{n}(\hat{p} - p) \xrightarrow{D} N(0, \sigma^2) \ ,$$

where $\sigma^2 = p(1-p)$. Hence

$$\hat{p} \pm \frac{1.96}{\sqrt{n}} \hat{\sigma}$$

is an asymptotic 95% confidence interval. Note that although the absolute error $\hat{\sigma}^2 := \hat{p}(1-\hat{p})$ is small, the relative error is high:

$$\mathrm{Re}(p) := \frac{\hat{\sigma}}{\hat{p}} \sim \frac{1}{\sqrt{\hat{p}}} \to +\infty, \qquad \text{as} \quad \hat{p} \to 0.$$

In other words, a confidence interval of width 10^{-4} may look small, but if the point estimate $\hat{p}$ is of order 10^{-6}, then this estimation is in fact impossible.

Therefore we will use the Importance Sampling (IS) method. The main idea is to compute $\mathbb{P}_N(A)$ by simulating a germ-grain model from a probability measure $\mathbb{P}_{\tilde{N}}$, such that $\mathbb{P}_N$ is absolutely continuous with respect to it. By Theorem 2.1 this is the case when driving m.p.p.'s in both germ-grain models are absolutely continuous. We also find the Radon-Nikodym derivative $L(\cdot)$, that is

$$\frac{d\mathbb{P}_N}{d\mathbb{P}_{\tilde{N}}}(A) = L(A) \ .$$

Then

$$\mathbb{P}_N(A) = \mathbb{E}_{\tilde{N}} L(A) \mathbb{1}(A) \ , \tag{1.1}$$

where the expectation is with respect to measure $\mathbb{P}_{\tilde{N}}$. Hence to estimate $\mathbb{P}_N(A)$ we generate $(\mathbb{1}(A_1), L_1), (\mathbb{1}(A_2), L_2), \ldots, (\mathbb{1}(A_n), L_n)$ from the measure $\mathbb{P}_{\tilde{N}}$ and construct the estimator

$$\hat{p}_{IS} = \frac{1}{n} \sum_{i=1}^{n} L_i \mathbb{1}(A_i) \ . \tag{1.2}$$

The 95% confidence interval is then

$$\hat{p}_{IS} \pm \frac{1.96}{\sqrt{n}} \hat{\sigma}_{IS} \ , \tag{1.3}$$

where

$$\hat{\sigma}^2_{IS} = \frac{1}{n} \sum_{i=1}^{n} L_i^2 \mathbb{1}(A_i) - \hat{p}^2_{IS} \ . \tag{1.4}$$

We choose the measure $\mathbb{P}_{\tilde{N}}$ (that is, the parameters of the new germ-grain model) in such a way that the event A is observed frequently. In other words, under a good choice of the parameters we decrease the relative error.

In applications sometimes we consider events on a random set τ. This concept of localization plays a fundamental role in simulation. Our goal is to prepare an appropriate framework for it. Therefore, similarly like for a process on the real line, we introduce the notion of a set indexed martingale and a stopping set τ. The likelihood process $L(\cdot)$ is a mean one set indexed martingale. Then we can simulate an event A up to a stopping set τ. Namely, we use Wald's formula:

$$\mathbb{P}(A) = \tilde{\mathbb{E}}[L(\tau); A] \ .$$

We analyze in detail the Boolean model, where the point process is a Poisson process and balls are i.i.d. distributed. We consider the event A such that balls form a chain (all circles in this chain are connected), which joins the origin with the border of box T with side length K. That is , we simulate the probability that the radius of the occupied component of the origin is greater than K. The problem of finding $\mathbb{P}_N(A)$ is relevant in industry when we apply the electrodes to the plates of the dielectric materials. Because of the manufacturing process small holes arise in the electrodes. A chain of small holes crossing from origin to the border of the box means a diminished value of the capacitance. Typically the parameters of the model: size of T, intensity of Poisson process λ and distribution of radius of ball ν are such that $\mathbb{P}_N(A)$ is "small". We prove that the IS scheme for an appropriate choice of a new intensity of the Poisson process is logarithmic efficient which implies improvement. We also give some numerical results.

The paper is organized as follows. In Section 2 we find sufficient conditions under which the two germ-grain models are absolutely continuous. Section 3 deals with a wire frame model. In Section 4 we consider simulation on a general random stopping set. Finally, in Section 5 we analyze some numerical example.

2 Germ-grain model

We start with a formal definition of a germ-grain model. We will define a marked point process as a point process on a product space of locations and marks with the additional property that the marginal location process is itself a well-defined point process. By $\mathcal{B}(X)$ we denote the Borel σ-field of X. Let $(T, \mathcal{B}(T))$ and $(E, \mathcal{B}(E))$ be subspaces of some vector Polish space $(W, +)$ with binary operator $+$. Let $N^E(\cdot)$ constitute a simple point process on T, that is integer valued random function such that $N_t = 1$ or $N_t = 1$ for any $t \in T$. Denote by $\{t_n\}$ the points of $N^E(\cdot)$. Then $N(\cdot, \cdot) = \{[t_n, m_n]\}$ is a marked point process, where m_n is the mark corresponding to t_n. In other words,

$$N(\omega, B, M) = \sum_{n \geq 1} \epsilon_{(t_n(\omega), m_n(\omega))}(B, M) \tag{2.1}$$

for $B \in \mathcal{B}(T)$, $M \in \mathcal{B}(E)$ and $\epsilon_{t_n, m_n}(\cdot, \cdot)$ is a Dirac measure.

Example 2.1. In the case of the Boolean model, T and E are subspaces of $\mathbb{R}^d$. The space T is a location space and E is a mark space, that is a space of balls.

Let $\mu(\mathrm{d}t, \mathrm{d}m)$ be a mean measure of N:

$$\mathbb{E}N(B, M) = \mu(B, M), \qquad B \in \mathcal{B}(T), M \in \mathcal{B}(E) \ .$$

Similarly, let $\lambda(\mathrm{d}t)$ be a mean measure of point process $N^E(\cdot)$:

$$\mathbb{E}N^E(B) = \lambda(B), \qquad B \in \mathcal{B}(T) \ .$$

It can be shown that $\mu(\mathrm{d}t, \mathrm{d}m)$ is absolutely continuous with respect to $\lambda(\mathrm{d}t)$, that is from the Radon-Nikodym theorem there exists density $\nu_t(\mathrm{d}m)$ such that

$$\mu(\mathrm{d}t, \mathrm{d}m) = \lambda(\mathrm{d}t)\nu_t(\mathrm{d}m) \ , \tag{2.2}$$

where $\nu_t(\mathrm{d}m)$ can be interpreted as the distribution of the mark of the point t.

Example 2.2. If $N^E(\cdot)$ is a Cox process with i.i.d. marking, then $\mu(\mathrm{d}t, \mathrm{d}m) = \lambda(\mathrm{d}t)F(\mathrm{d}m)$, where $F(\cdot)$ is a distribution of mark.

Remark 2.1. In the classical theory of marked processes on the real line, it is well known that under certain conditions on the probability space and filtration, the mean measure of a marked point process determines its distribution (Jacod [5]). As we shall see, this is not true for processes on general spaces. Consider the Poisson process $N^E(\cdot) = N(\cdot, E)$ on $T = [0, 1]^2$ with mean measure equal to Lebesgue measure $\lambda(\mathrm{d}t) = \mathrm{d}t$. However, the Lebesgue measure is also the mean measure of the following process. Let $L_0, L_1, \ldots$ be i.i.d. unit rate Poisson processes on $[0, 1]$. Denote the time of the k^{th} jump of L_i, by T_k^i. Now, let locations of the points be $\{(T_i^0, T_k^i), i, k \geq 1\}$.

Now, writing $B \oplus M = \{b + m : b \in B, m \in M\}$ for the Minkowski addition of B and M, we define a germ-grain model by the union:

$$\mathcal{A}(N) = \cup_{n \geq 1}(t_n \oplus m_n) \ .$$

Denote $S = T \oplus E$.

Consider two marked points N and $\tilde{N}$ on $T \times E$ having the mean measures μ and $\tilde{\mu}$, respectively. Let $\mathbb{P}_N$ and $\mathbb{P}_{\tilde{N}}$ be the distribution of the germ-grain model driven by m.p.p's N and $\tilde{N}$, respectively. Denote $\mathcal{N} = \{n : \mathbb{P}(N(T \times E) = n) > 0\}$ and $\tilde{\mathcal{N}} = \{n : \mathbb{P}(\tilde{N}(T \times E) = n) > 0\}$. For $n \in \mathcal{N}$ define

$$N_{|n}(B, M) := \mathbb{E}\left[N(B, M) | N(T, E) = n\right], \qquad B \in \mathcal{B}(T), M \in \mathcal{B}(E) \ .$$

Note, that $N_{|n}$ is also a marked point process on $T \times E$. Let $\mu_{|n}(\mathrm{d}t, \mathrm{d}m)$ be its mean measure. Similarly we define $\tilde{N}_{|n}(\cdot, \cdot)$ and $\tilde{\mu}_{|n}(\cdot, \cdot)$.

Proposition 2.1. *Marked point process $N(\cdot, \cdot)$ is absolutely continuous with respect to marked point process $\tilde{N}(\cdot, \cdot)$ ($N << \tilde{N}$) iff $\mu_{|n} << \tilde{\mu}_{|n}$ for $n \in \mathcal{N}$ and $N(T \times E) << \tilde{N}(T \times E)$.*

Proof. If $N << \tilde{N}$, then $N(T \times E) << \tilde{N}(T \times E)$ and $N_{|n} << \tilde{N}_{|n}$, hence also $\mu_{|n} << \tilde{\mu}_{|n}$. We prove the converse implication. We use the notation $(\boldsymbol{t}, \boldsymbol{m}) = (t_1, m_1, \ldots, t_k, m_k)$ for the $(T \times E)^k$-valued vectors ($k = 1, 2, \ldots$). The k^{th} order factorial measure $\alpha^k_{|n}$ of $N_{|n}$ is a measure on $(T \times E)^k$ defined by

$$\alpha^k_{|n}(\mathrm{d}(\boldsymbol{t}, \boldsymbol{m})) = \mathbb{E} N^k_{|n}(\mathrm{d}(\boldsymbol{t}, \boldsymbol{m})) \ ,$$

where

$$N^k_{|n}(\mathrm{d}(\boldsymbol{t}, \boldsymbol{m})) = N_{|n}(\mathrm{d}(t_1, m_1))(N_{|n} - \epsilon_{(t_1, m_1)})(\mathrm{d}(t_2, m_2)) \ldots (N_{|n} - \sum_{i=1}^{k-1} \epsilon_{(t_i, m_i)})(\mathrm{d}(t_k, m_k)) \ .$$

Thus

$$\int f(\boldsymbol{t}, \boldsymbol{m}) \alpha^k_{|n}(\mathrm{d}(\boldsymbol{t}, \boldsymbol{m})) = \int \sum_{(\boldsymbol{t}, \boldsymbol{m})}^{\#} f(\boldsymbol{t}, \boldsymbol{m}) N_{|n}(\mathrm{d}(\boldsymbol{t}, \boldsymbol{m})) \ , \tag{2.3}$$

where the summation $\sum^{\#}$ is over all n-tuples of distinct points in T. We prove that $\alpha^k_{|n}$ is absolutely continuous with respect to $\tilde{\alpha}^k_{|n}$ for each $n \in \mathcal{N} \cap \tilde{\mathcal{N}}$ and $k \leq n$. We use induction. For $k = 1$ the assertion is satisfied, since we have $\alpha^1_{|n}(\mathrm{d}t, \mathrm{d}m) = \mu_{|n}(\mathrm{d}t, \mathrm{d}m)$ and $\tilde{\alpha}^1_{|n}(\mathrm{d}t, \mathrm{d}m) = \tilde{\mu}_{|n}(\mathrm{d}t, \mathrm{d}m)$. Assume that the assertion is satisfied for $k - 1$. Let

$$L_{k-1}(\boldsymbol{t}, \boldsymbol{m}) = \frac{\mathrm{d}\alpha^{k-1}_{|n}}{\mathrm{d}\tilde{\alpha}^{k-1}_{|n}}(\boldsymbol{t}, \boldsymbol{m}) \tag{2.4}$$

and

$$L(t, m) = \frac{\mathrm{d}\mu_{|n}}{\mathrm{d}\tilde{\mu}_{|n}}(t, m) \ . \tag{2.5}$$

Then

$$\alpha^k_{|n}(B_1 \times M_1, \dots, B_k \times M_k) = \int_{T\times E}\int_{(T\times E)^{k-1}} \sum^{\#}_{t_1,\dots,t_{k-1}} \sum_{t_k \neq t_i, i\leq k-1}$$

$$\epsilon_{t_1,m_1}(B_1 \times M_1)\dots\epsilon_{t_k,m_k}(B_k \times M_k)$$

$$\mathrm{d}\alpha^{k-1}_{|n}(t_1,m_1,\dots,t_{k-1},m_{k-1})\mathrm{d}N_{|n}(t_k,m_k)$$

$$= \int_{T\times E}\int_{(T\times E)^{k-1}} \sum^{\#}_{t_1,\dots,t_{k-1}} \sum_{t_k \neq t_i, i\leq k-1} \epsilon_{t_1,m_1}(B_1 \times M_1)\dots\epsilon_{t_k,m_k}(B_k \times M_k)$$

$$\mathrm{d}\alpha^{k-1}_{|n}(t_1,m_1,\dots,t_{k-1},m_{k-1})\mathrm{d}\mu_{|n}(t_k,m_k)$$

$$= \int_{T\times E}\int_{(T\times E)^{k-1}} \sum^{\#}_{t_1,\dots,t_{k-1}} \sum_{t_k \neq t_i, i\leq k-1} L_{k-1}(t_1,m_1,\dots,t_{k-1},m_{k-1})L(t_k,m_k)$$

$$\epsilon_{t_1,m_1}(B_1 \times M_1)\dots\epsilon_{t_k,m_k}(B_k \times M_k)$$

$$\mathrm{d}\tilde{\alpha}^{k-1}_{|n}(t_1,m_1,\dots,t_{k-1},m_{k-1})\mathrm{d}\tilde{\mu}_{|n}(t_k,m_k)$$

$$= \int_{(T\times E)^k} \sum^{\#}_{t_1,\dots,t_k} L_k(\mathbf{t},\mathbf{m})\epsilon_{t_1,m_1}(B_1 \times M_1)\dots\epsilon_{t_k,m_k}(B_k \times M_k)\tilde{\alpha}^k_{|n}(\mathrm{d}t,\mathrm{d}m)\ ,$$

where $L_k(\mathbf{t},\mathbf{m}) = L_{k-1}(t_1,m_1,\dots,t_{k-1},m_{k-1})L(t_k,m_k)$. Thus $\alpha^k_{|n} << \tilde{\alpha}^k_{|n}$ for $k \leq n$. Suppose now that $\tilde{N}(B\times M) = 0$. That is,

$$\sum_{n\in\tilde{\mathcal{N}}} \mathbb{P}(\tilde{N}(T\times E) = n)\int_{(T\times E)^n} \mathbb{I}_{B\times M}\tilde{N}^n_{|n}(\mathrm{d}(\mathbf{t},\mathbf{m}))$$

$$= \sum_{n\in\tilde{\mathcal{N}}} \mathbb{P}(\tilde{N}(T\times E) = n)\int_{(B\times M)^n} \tilde{\alpha}^n_{|n}(\mathrm{d}(\mathbf{t},\mathbf{m})) = 0\ .$$

Hence all terms must be zero and for all $n \in \tilde{\mathcal{N}} \supseteq \mathcal{N}$ we have

$$\int_{(B\times M)^n} \tilde{\alpha}^n_{|n}(\mathrm{d}(\mathbf{t},\mathbf{m})) = 0\ . \tag{2.6}$$

Thus from the above considerations for all $n \in \mathcal{N}$

$$\int_{(B\times M)^n} \alpha^n_{|n}(\mathrm{d}(\mathbf{t},\mathbf{m})) = 0 \tag{2.7}$$

yielding $N(B\times M) = 0$. This completes the proof.

□

Remark 2.2. For a marked Poisson process with intensity λ and a mark independent of a position with a distribution measure $\nu(\cdot)$, we have

$$\mathbb{P}(N(T\times E) = n) = \frac{(\lambda|T|)^n}{n!}e^{-\lambda|T|}\ ,$$

where $|B|$ is the volume of a set B, and

$$\alpha^n_{|n}(\mathrm{d}(\mathbf{t},\mathbf{m})) = \mathrm{d}t_1\dots\mathrm{d}t_n\mathrm{d}\nu(m_1)\dots\mathrm{d}\nu(m_n)\ .$$

Remark 2.3. Note that from the assumptions of the Proposition 2.1 it follows that μ is absolutely continuous with respect to $\tilde{\mu}$. In fact, let $B \times M$ be such that $\tilde{\mu}(B \times M) = 0$. We have $\mathcal{N} \subseteq \tilde{\mathcal{N}}$. Then

$$\sum_{n \in \tilde{\mathcal{N}}} \mathbb{P}(\tilde{N}(T \times E) = n)\tilde{\mu}_{|n}(B \times M) = 0$$

and for $n \in \tilde{\mathcal{N}}$ we have $\tilde{\mu}_{|n}(B \times M) = 0$. Hence also $\mu_{|n}(B \times M) = 0$ for $n \in \mathcal{N}$ and finally $\mu(B \times M) = 0$. The converse statement in general is not true. In fact, consider two points processes $N^E(\cdot)$ and $\tilde{N}^E(\cdot)$ on $T = [0,1]^2$ given by $N^E(\cdot) = \delta_{(1/2,1/2)}(\cdot)$ and $\tilde{N}^E(\cdot) = \delta_{(1/2,1/2)}(\cdot) + \delta_{(1/4,1/4)}(\cdot)$.

Theorem 2.1. *Let T and E be compact subsets of some vector Polish space W. If $N << \tilde{N}$, then $\mathbb{P}_N << \mathbb{P}_{\tilde{N}}$.*

Remark 2.4. The same assertion for a Boolean model driven by a Poisson process marked by i.i.d. balls was proved recently by Van Lieshout [13].

Remark 2.5. Note that if N and $\tilde{N}$ are Cox processes, then by Remark 2.2 the assumptions of Theorem 2.1 are satisfied.

Proof. Suppose that $\mathbb{P}_{\tilde{N}}(B) = 0$ for some $B \in \mathcal{B}(S)$. That is,

$$\sum_{n \in \tilde{\mathcal{N}}} \mathbb{P}(\tilde{N}(T \times E) = n) \int_{(T \times E)^n} \mathbb{I}_B(\sum_{i=1}^n (t_i \oplus m_i))\tilde{N}^n_{|n}(\mathrm{d}(\mathbf{t}, \mathbf{m}))$$
$$= \sum_{n \in \tilde{\mathcal{N}}} \mathbb{P}(\tilde{N}(T \times E) = n) \int_{(T \times E)^n} \mathbb{I}_B(\sum_{i=1}^n (t_i \oplus m_i))\tilde{\alpha}^n_{|n}(\mathrm{d}(\mathbf{t}, \mathbf{m})) = 0 \ .$$

Hence all terms must be zero and for all $n \in \mathcal{N}$ we have

$$\int_{(T \times E)^n} \mathbb{I}_B(\sum_{i=1}^n (t_i \oplus m_i))\tilde{\alpha}^n_{|n}(\mathrm{d}(\mathbf{t}, \mathbf{m})) = 0 \ . \tag{2.8}$$

Thus from Proposition 2.1 for $n \in \mathcal{N}$

$$\int_{(T \times E)^n} \mathbb{I}_B(\sum_{i=1}^n (t_i \oplus m_i))\alpha^n_{|n}(\mathrm{d}(\mathbf{t}, \mathbf{m})) = 0 \tag{2.9}$$

yielding $\mathbb{P}_N(B) = 0$. This completes the proof. □

The likelihood ratio $\mathrm{d}\mathbb{P}_{\tilde{N}}/\mathrm{d}\mathbb{P}_N$ is

$$L(A) := \frac{\mathrm{d}\mathbb{P}_N}{\mathrm{d}\mathbb{P}_{\tilde{N}}}(A) = \frac{\sum_{n \in \mathcal{N}} \mathbb{P}(N(T \times E) = n) \int_{(T \times E)^n} \mathbb{I}_A(\sum_{i=1}^n (t_i \oplus m_i))\alpha^n_{|n}(\mathrm{d}(\mathbf{t}, \mathbf{m}))}{\sum_{n \in \tilde{\mathcal{N}}} \mathbb{P}(\tilde{N}(T \times E) = n) \int_{(T \times E)^n} \mathbb{I}_A(\sum_{i=1}^n (t_i \oplus m_i))\tilde{\alpha}^n_{|n}(\mathrm{d}(\mathbf{t}, \mathbf{m}))} \ . \tag{2.10}$$

Example 2.3. [Poisson cluster process] Let $N(\cdot,\cdot)$ be a Poisson process with the intensity λ marked by a point process $N_i(\cdot)$ at position t_i, where $N_i(\cdot)$ are conditionally independent given the realization of the parent Poisson process. Then $\overline{N}(\cdot) = \sum_{i=1}^{\infty}(t_i \oplus N_i(\cdot))$ is a cluster Poisson process (see [6]). Assume that $N_i(\cdot)$ are absolutely continuous with respect to the unit Poisson process on S with the offspring intensity measure $\nu(\cdot)$. Let $\tilde{\overline{N}}(\cdot)$ be the Poisson cluster process with intensity $\tilde{\lambda}$ and the offspring measure $\tilde{\nu}$. Then

$$\frac{\mathrm{d}\overline{N}}{\mathrm{d}\tilde{\overline{N}}}(A) = L(A) = \frac{\sum_{n=1}^{\infty} \frac{\lambda^n}{n!} e^{-\lambda|T|} e^{|S|(1-n)} \sum_{\phi} \int_{S^n} \prod_{i=1}^{n} \nu(t_{\phi^{-1}(i)} - s_i)\mathrm{d}s_1 \dots \mathrm{d}s_n}{\sum_{n=1}^{\infty} \frac{\tilde{\lambda}^n}{n!} e^{-\tilde{\lambda}|T|} e^{|S|(1-n)} \sum_{\phi} \int_{S^n} \prod_{i=1}^{n} \tilde{\nu}(t_{\phi^{-1}(i)} - s_i)\mathrm{d}s_1 \dots \mathrm{d}s_n} , \quad (2.11)$$

where the sum is over all ordered partitions ϕ of A; see Van Lieshout [12]. If each parent point has a single daughter point with displacement densities $\nu(\cdot)$ and $\tilde{\nu}(\cdot)$, then (2.11) reduces to

$$L(A) = e^{(\tilde{\lambda}-\lambda)|T|} \left(\frac{\lambda}{\tilde{\lambda}}\right)^{H(A)} \prod_{i=1}^{H(A)} \frac{\int_S \nu(t_i - s)\mathrm{d}s}{\int_S \tilde{\nu}(t_i - s)\mathrm{d}s} ,$$

where $H(A)$ denotes the number of points in configuration A.

Sometimes the expression (2.10) can be simplified. This is the case for a wire frame model.

3 Wire frame model

The germ-grain model in which there is one-to-one correspondence between the driving m.p.p. and the germ-grain model itself, we will call 'wire frame model'. The classical example is a Boolean disc model on $\mathbb{R}^d$, in which marks are discs centered at the location points $\{t_i\}$ of the point process $N^E(\cdot)$ with diameters $\{r_i\}$. That is, $\mathcal{A}(N) = \bigcup_{i=1}^{\infty} S_i$, where $S_i = \{t \in \mathbb{R}^d : ||t - t_i|| = r_i\}$ and $||\cdot||$ means Euclidean norm. Denote by $\pi : T \oplus E \to T \times E$ the $1-1$ mapping such that $\pi(\mathcal{A}(N)) = N$. Then from (2.10) we have

$$\frac{\mathrm{d}\mathbb{P}_N}{\mathrm{d}\mathbb{P}_{\tilde{N}}}(A) = L(A) = \frac{\mathrm{d}N}{\mathrm{d}\tilde{N}}(\pi(A)) . \quad (3.1)$$

Consider the marked Cox processes $N(\cdot,\cdot)$ and $\tilde{N}(\cdot,\cdot)$ with intensity measures $\lambda(\cdot)$ and $\tilde{\lambda}(\cdot)$, respectively, and with independent marking with measures $\nu(\cdot)$ and $\tilde{\nu}(\cdot)$. Let $\lambda << \tilde{\lambda}$ and $\nu << \tilde{\nu}$. Then by Prop. 6.10, p. 233 of Karr [6], $N << \tilde{N}$ and hence also $\mathbb{P}_N << \mathbb{P}_{\tilde{N}}$. In this case

$$\frac{\mathrm{d}\mathbb{P}_N}{\mathrm{d}\mathbb{P}_{\tilde{N}}}(A) = \exp\left\{\int_{\pi(A)} \log\left(\frac{\mathrm{d}\lambda}{\mathrm{d}\tilde{\lambda}}(t)\right) \mathrm{d}\tilde{N}^E(t)\right\} \exp\left\{\int_T \left(1 - \frac{\mathrm{d}\lambda}{\mathrm{d}\tilde{\lambda}}(t)\right) \mathrm{d}\lambda(t)\right\} \prod_{i=1}^{H(\pi(A))} \frac{\mathrm{d}\nu}{\mathrm{d}\tilde{\nu}}(t_i) , \quad (3.2)$$

where $H(\pi(A))$ is the number of points of the Cox process constructing a set A. In particular, if $N(\cdot,\cdot)$ and $\tilde{N}(\cdot,\cdot)$ are marked Poisson processes, then

$$\frac{\mathrm{d}\mathbb{P}_N}{\mathrm{d}\mathbb{P}_{\tilde{N}}}(A) = e^{(\tilde{\lambda}-\lambda)|T|} \left(\frac{\lambda}{\tilde{\lambda}}\right)^{H(\pi(A))} \prod_{i=1}^{H(\pi(A))} \frac{\mathrm{d}\nu}{\mathrm{d}\tilde{\nu}}(t_i) .$$

For example, if the diameters in both Boolean disc models have the same distribution, then

$$\frac{\mathrm{d}\mathbb{P}_N}{\mathrm{d}\mathbb{P}_{\tilde{N}}}(A) = e^{(\tilde{\lambda}-\lambda)|T|} \left(\frac{\lambda}{\tilde{\lambda}}\right)^{H(\pi(A))} . \quad (3.3)$$

4 Random stopping

In this section, we will index m.p.p.'s $N(\cdot,\cdot)$ and $\tilde{N}(\cdot,\cdot)$ defined via (2.1) by a semilattice $\mathcal{T}$ of a compact subset of $T\times E$, where $B\wedge C = B\cap C$ for $B, C \in T\times E$. Let W be a locally compact space. We assume that $\emptyset \in \mathcal{T}$, that $\mathcal{T}$ is closed under arbitrary intersections, and that $\overline{\cup_{B\in\mathcal{T}} B} = T$ ($\overline{(\cdot)}$ denotes the closure of set). The probability space $(\Omega, \mathcal{F}, \mathbb{P}_N)$ is equipped with a right-continuous filtration $\{\mathcal{F}_B, B \in \mathcal{T}\}$ that satisfies $\mathcal{F}_B \subseteq \mathcal{F}_C$ for $B \subseteq C$ and $\mathcal{F}_\emptyset$ contains all the null sets. Let $\mathcal{F} = \bigvee_B \mathcal{F}_B$.

We will denote the class $\mathcal{C}(u)$ of finite unions of sets in class $\mathcal{C}$. Let "$\subset$" mean strict inclusion and $(\cdot)^o$ denote the interior of a set. We make the following two assumptions:

SHAPE If $B, C \in \mathcal{T}$ and $B, C \neq \emptyset$, then $B\cap C \neq \emptyset$. Also, if $B_1, \ldots, B_k \in \mathcal{T}$ and $B \subseteq \sum_{i=1}^k B_i$, then there exists index i $(1 \le i \le k)$ such that $B \subseteq B_i$.

Separability from above There exists an increasing sequence of finite subsemilattices $\mathcal{T}_n$ such that $\mathcal{T}_n \subseteq \mathcal{T}$, $\emptyset \in \mathcal{T}_n$, $\wedge_{\mathcal{T}_n} = \cap$, $\forall n$, and sequence of functions $g_n : \mathcal{T} \to \mathcal{T}_n(u) \cup \{T\}$ preserving all intersections, satisfying $B \subseteq (g_n(B))^o$ $\forall B \in \mathcal{T}$, such that $g_n(B) \subseteq g_m(B)$ if $n \ge m$, and $B = \cap_n g_n(B)$, $\forall B \in \mathcal{T}$. Also, if $B \neq \emptyset$, $C \in \mathcal{T}$ and $B \subset C$, then $B \subset g_n(B) \cap C$ $\forall n$.

The condition SHAPE imposes a restriction on the geometric shapes of sets in $\mathcal{T}$.

Definition 4.1. $\tau : \Omega \to \mathcal{T}$ *is called a stopping set if, for any set* $B \in \mathcal{T}$, $\{\omega : B \subseteq \tau(\omega)\} \in \mathcal{F}_B$, $\{\omega : \tau(\omega) = \emptyset\} \in \mathcal{F}_\emptyset$.

Sometimes τ is also called a simple stopping set.

Example 4.1. The classical example of set T and lattice $\mathcal{T}$ fulfilling above conditions is the case $T = E = [0, D]^d$ for some $D > 0$ and $\mathcal{T} = \{[0,t] \times [0,s] : t \in T, s \in E\}$. For a filtration $\{\mathcal{F}_B\}$ one may take $\mathcal{F}_B = \cup_n \mathcal{F}^o_{g_n(B)}$, where $\mathcal{F}^o_B$ is generated by the driving m.p.p. on a set $B \in \mathcal{T}$. As an example of stopping set τ for a Boolean model one can consider the smallest (in a sense of inclusion) box $[0,(n,n,\ldots,n)]$ $(n \in \mathbb{N})$ such that there exists a path through the balls joining origin with the corner $(n,n,\ldots,n)$ if it exists, otherwise $\tau = T \times E$. For more complicated examples, where e.g. W is a function space, see Merzbach [4].

Consider the wire frame models $\mathbb{P}_N$ and $\mathbb{P}_{\tilde{N}}$ such that $\mathbb{P}_N << \mathbb{P}_{\tilde{N}}$, that is (3.1) holds. Note that $L(B)$ is a mean one martingale, that is $\mathbb{E}[L(B)|\mathcal{F}_C] = L(C)$ for $C \subseteq B$ and $B, C \in \mathcal{T}$. Then from Th. 2.5 of Merzbach [3], we have Wald's formula

$$\mathbb{P}_N(A) = \mathbb{E}_{\tilde{N}}[L(\tau); A] \tag{4.1}$$

for $A \in \mathcal{F}_\tau$, where $\mathcal{F}_\tau = \{F \in \mathcal{F} : F \cap \{\tau \subseteq B\} \in \mathcal{F}_B \ \forall B \in \mathcal{T}\}$. Then the IS estimator of $\mathbb{P}_N(A)$ in a wire frame model is

$$p_{IS} = \sum_{i=1}^{n} L_i(\tau)\mathbb{I}(A_i) \ . \tag{4.2}$$

5 Radius of the occupied component of the origin

In this section we consider the Boolean disc model driven by the marked Poisson process $N(\cdot,\cdot)$ with the intensity λ and marks being i.i.d. discs. Marked Poisson process $\tilde{N}(\cdot,\cdot)$ has the intensity $\tilde{\lambda}$ and marks being discs distributed like in a Boolean disc model governed by $\mathbb{P}_N$. We will consider a rare event A on T for which $\lim_{\lambda\to 0}\mathbb{P}_N(A)=0$. By (3.3) we have

$$L(A)=e^{(\tilde{\lambda}-\lambda)|T|}\left(\frac{\lambda}{\tilde{\lambda}}\right)^{H(\pi(A))} .$$

We will give now an example of A for which the IS scheme work well, that is, it reduces the relative error.

Definition 5.1. *We say that the IS scheme is logarithmic efficient if*

$$\lim_{\lambda\to 0}\frac{\log \mathbf{Var}\hat{p}_{IS}}{\log p^2}\geq 1 .$$

If

$$\lim_{\lambda\to 0}\frac{\mathbf{Var}\hat{p}_{IS}}{p}=0 ,$$

then the IS scheme is an improvement over MC simulation.

Logarithmic efficiency implies improvement (see Asmussen [1]).

Theorem 5.1. *Let A be such an event that*

$$\lim_{\lambda\to 0}-\frac{1}{\beta(\lambda)}\log \mathbb{P}_\lambda(A)=1 \tag{5.1}$$

for some positive scaling function $\beta(\cdot)$. Let $\tilde{\lambda}>\lambda$. If

$$\lim_{\lambda\to 0}\frac{\beta(\frac{\lambda^2}{\tilde{\lambda}})}{2\beta(\lambda)}\geq 1 , \tag{5.2}$$

then IS scheme is logarithmic efficient. If $\beta(\cdot)$ is strictly decreasing such that

$$\lim_{\lambda\to 0}\left(\beta(\frac{\lambda^2}{\tilde{\lambda}})-\beta(\lambda)\right)=\infty , \tag{5.3}$$

then the IS scheme is an improvement.

Proof. Let $\overline{\lambda}=\frac{\lambda^2}{\tilde{\lambda}}$. Denote by $\mathbb{P}_\beta$ and $\mathbb{E}_\beta$ the probability and the expectation with respect to the Poisson process with the intensity β, respectively. Note that

$$\begin{aligned}\mathbb{E}_{\tilde{\lambda}}[L^2(T);A]&=e^{(\tilde{\lambda}-\lambda)|T|}\mathbb{E}_\lambda\left[\left(\frac{\lambda}{\tilde{\lambda}}\right)^{H(T)};A\right]\\&=e^{(\tilde{\lambda}-\lambda)|T|}e^{(\overline{\lambda}-\lambda)|T|}\mathbb{E}_{\overline{\lambda}}\left[\left(\frac{\lambda}{\overline{\lambda}}\right)^{H(T)}\left(\frac{\lambda}{\tilde{\lambda}}\right)^{H(T)};A\right]\\&=e^{\frac{1}{\tilde{\lambda}}(\tilde{\lambda}-\lambda)^2|T|}\mathbb{P}_{\overline{\lambda}}(A) .\end{aligned}$$

Thus

$$\lim_{\lambda\to 0}\log \mathbb{E}_{\tilde{\lambda}}[L^2(T);A]=\lim_{\lambda\to 0}\log \mathbb{P}_{\overline{\lambda}}(A)=-\beta(\overline{\lambda})$$

and $\lim_{\lambda\to 0}\log \mathbb{P}_\lambda(A) = -\beta(\lambda)$, which completes the proof in view of Definition 5.1.

□

Assume from now, that the radius of discs is equal to $r = 1$. Let $T = T(K) := [-K, K]^d \subseteq \mathbb{R}^d$ $(d \geq 2)$ for fixed $K > 1$. We shall apply the above theorem to the event $A_0 := \{0 \longleftrightarrow \partial T(K)\}$ that there exists a path through balls of the Boolean model joining 0 with the surface ∂T of T:

$$\partial T(K) = \{t = (t_1, \ldots, t_d) \in \mathbb{R}^d : \max_i |t_i| = K\} \ .$$

In other words, in the wire frame model, the origin is inside a disc which is connected through a chain of discs with the surface ∂T.

Theorem 5.2. *The IS scheme for A_0 is logarithmic efficient.*

Before we give the proof we prove the following lemmas.

Lemma 5.1. *There exists a decreasing positive function $\phi(\lambda)$ such that*

$$\lim_{\lambda\to 0} -\frac{1}{K\phi(\lambda)} \log \mathbb{P}_\lambda(0 \longleftrightarrow \partial T(K)) = 1 \ . \tag{5.4}$$

Proof. Using version BK and FKG inequalities for continuous percolation (see Th. 2.2 and Th. 2.3 of Meester and Roy [8]) we obtain (6.24) in Grimmett [2]. Then using the subadditive inequality limit theorem one can mimic the proof of Th. 6.10 of Grimmett [2] to obtain its version for the continuous percolation. That is, there exist strictly positive constants ρ and σ, independent of λ, and a decreasing positive function $\phi(\lambda)$ such that

$$\rho K^{1-d} e^{-K\phi(\lambda)} \leq \mathbb{P}_\lambda(0 \longleftrightarrow \partial T(K)) \leq \sigma K^{d-1} e^{-K\phi(\lambda)}$$

for all $K > 1$. This completes the proof.

□

Lemma 5.2. *Let $0 \leq f(x) \leq 1$. Then*

$$\lim_{\lambda\to 0} \frac{\phi(\lambda)\log(\lambda)}{\phi(\lambda f(\lambda))\log(\lambda f(\lambda))} \geq 1 \ . \tag{5.5}$$

Proof. The main idea of the proof is to approximate the continuous problem by site percolation problems on a special lattice constructed by partinioning $\mathbb{R}^d$ into small cubes. Let κ be a positive integer and $\mathbb{Z}^d_\kappa = \frac{1}{\kappa}\mathbb{Z}^d$. We partition $\mathbb{R}^d$ into cubes whose centers are the points of $\mathbb{Z}^d_\kappa$, defining $B_\kappa(t) = \times_{i=1}^d [t_i - \frac{1}{2\kappa}, t_i + \frac{1}{2\kappa}]$ for $t \in \mathbb{Z}^d_\kappa$. We turn $\mathbb{Z}^d_\kappa$ into a lattice $\mathcal{G}_\kappa$ by defining the adjacency relation $\sim$ on $\mathbb{Z}^d_\kappa$ with the rule that $x \sim y$ iff there exist points $u \in B_\kappa(X)$ and $v \in B_\kappa(y)$ such that $\rho(u, v) \leq 2$ where $\rho(\cdot,\cdot)$ is the Euclidean distance. We shall consider site percolation on the ensuing lattice $\mathcal{G}_\kappa$. We declare a vertex x of $\mathcal{G}_\kappa$ to be open if there exist one or more points of the Poisson process within the cube $B_\kappa(x)$, and closed otherwise. The states of different vertices are independent random variables and the probability $p_\kappa(\lambda)$ that any given vertex is open is given by

$$p_\kappa(\lambda) = 1 - e^{-\lambda\kappa^{-d}} \ . \tag{5.6}$$

Let $\gamma_\kappa = (1 + \frac{1}{\kappa}\sqrt{d})^d$. From the rescaling property of a Boolean disc model and the considerations made by Meester and Roy [8], p. 60 or Grimmett [2], Sec. 12.10, for sufficiently small λ we have

$$\mathbb{P}^s_{p_\kappa(\lambda\gamma_\kappa)}(0 \longleftrightarrow \partial T(K)) \leq \mathbb{P}_\lambda(0 \longleftrightarrow \partial T(K)) \leq \mathbb{P}^s_{p_\kappa(\lambda)}(0 \longleftrightarrow \partial T(K)) \ , \tag{5.7}$$

where $\mathbb{P}^s_{p_\kappa}(\cdot)$ is a law of site percolation on $\mathcal{G}_\kappa$ defined by the adjacency relation $\sim$ for p_κ. Thus by Th. 2.38 of Grimmett [2] applied to site percolation we have

$$\begin{aligned}
&\frac{\log \mathbb{P}_{\lambda f(\lambda)}(0 \longleftrightarrow \partial T(K)) \log \lambda}{\log \mathbb{P}_{\lambda}(0 \longleftrightarrow \partial T(K)) \log \lambda f(\lambda)} \\
&\qquad \geq \frac{\log \mathbb{P}_{p_\kappa(\lambda f(\lambda))}(0 \longleftrightarrow \partial T(K)) \log \lambda}{\log \mathbb{P}_{p_\kappa(\lambda \gamma_\kappa)}(0 \longleftrightarrow \partial T(K)) \log \lambda f(\lambda)} \\
&\qquad \geq \frac{\log p_\kappa(\lambda f(\lambda)) \log \lambda}{\log p_\kappa(\lambda \gamma_\kappa) \log \lambda f(\lambda)} \ .
\end{aligned}$$

Note that $\lim_{\alpha \to 0} \frac{\log p_\kappa(\alpha)}{\log \alpha} = 1$. This completes the proof.

□

Proof of Theorem 5.2 Let $f(\lambda) = \frac{\lambda}{\tilde\lambda} < 1$. Then from Lemmas 5.1 - 5.2 we have

$$\begin{aligned}
\lim_{\lambda \to 0} \frac{\log \mathbb{P}_{\tilde\lambda}(0 \longleftrightarrow \partial T(K))}{2 \log \mathbb{P}_{\lambda}(0 \longleftrightarrow \partial T(K))} &= \lim_{\lambda \to 0} \frac{\phi(\lambda f(\lambda))}{2\phi(\lambda)} \\
&\geq \lim_{\lambda \to 0} \frac{\log(\lambda f(\lambda))}{2 \log \lambda} = 1 \ ,
\end{aligned}$$

which completes the proof of the theorem.

□

We make also some numerical analysis using the IS scheme described in this section. We made 10000 simulations for $K = 64$, $d = 2$, $r = 1$ and $\tilde\lambda = 1,7$.

λ	p_{IS}
1.6	$7.5 * 10^{-6}$
1.5	$2,5 * 10^{-6}$
1.45	$1,2 * 10^{-6}$

Table 1: Simulation of the event $0 \longleftrightarrow \partial T(K)$

Acknowledgment

The author is indebted to M. van den Berg for the help in deriving Table 1.

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