Large deviation principle at fixed time in Glauber evolutions

Arnaud Le Ny†
Frank Redig‡

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Abstract: We consider the evolution of an asymptotically decoupled probability measure ν on Ising spin configurations under a Glauber dynamics. We prove that for any $t > 0$, ν_t is asymptotically decoupled and hence satisfies a large deviation principle with the relative entropy density as rate function.

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†Eurandom, LG 1.48, postbus 513, 5600 MB Eindhoven, The Netherlands. E-mail: leny@eurandom.tue.nl
‡Faculteit Wiskunde en Informatica, Technische Universiteit Eindhoven, postbus 513, 5600 MB Eindhoven, The Netherlands. E-mail: f.h.j.redig@tue.nl

1 Introduction

In the paper [1] it is proved that for a high-temperature Glauber dynamics started from a low-temperature Gibbs measure, the Gibbs property can be lost in the course of time. As long as the measure remains Gibbsian, we know that the empirical distribution satisfies a large deviation principle with the relative entropy density as a rate function. As soon as the Gibbs property is lost, we can a priori not even be sure of the existence of the relative entropy density $h(\mu|\nu_t)$, where ν_t is the measure at time t and μ any translation invariant probability measure. In [9], Pfister introduced the notion of asymptotically decoupled (AD) measures and proved a large deviation principle for this class. The AD measures constitute a class which is broader than the Gibbs measures since renormalization group transformations such as decimations, Kadanoff transformation, block spin averaging preserve AD, but do not in general preserve the Gibbs property [2, 3]. In this paper we obtain that for spin flip Glauber dynamics with local rates, the AD property is conserved in the course of time, and hence for any $t > 0$ the random measures

$$L^t_\Lambda(\sigma) = \frac{1}{|\Lambda|} \sum_{x\in\Lambda} \delta_{\tau_x \sigma_t} \tag{1.1}$$

satisfy the large deviation principle.

The plan of the paper is as follows: we start by defining AD probability measures, and introduce the type of dynamics we study in section 2. Section 3 is devoted to the proof of the conservation of the AD property.

2 Preliminaries

2.1 Configuration space, dynamics

We consider Ising spin systems on the lattice $\mathbb{Z}^d$, i.e., the configuration space is the compact metric space $\Omega = \{-1, +1\}^{\mathbb{Z}^d}$. The set of finite subsets of $\mathbb{Z}^d$ is denoted by $\mathcal{S}$ and for any $\Lambda \in \mathcal{S}$, we define its boundary to be $\partial\Lambda = \{x \in \Lambda : \exists y \in \Lambda^c, |x - y| = 1\}$. The cube $[-n, n]^d \cap \mathbb{Z}^d \in \mathcal{S}$ is denoted by Λ_n, for all $n \in \mathbb{N}$. For $A \subset \mathbb{Z}^d$, we denote $\mathcal{F}_A$ to be the sigma-field generated by the functions $\{\sigma : x \mapsto \sigma(x),\ x \in A\}$. We abbreviate $\mathcal{F}_{\mathbb{Z}^d}$ by $\mathcal{F}$. Functions $f : \Omega \to \mathbb{R}$ are called local (notation $f \in \mathcal{L}$) if there exists some finite $\Lambda \in \mathcal{S}$ such that f is $\mathcal{F}_\Lambda$-measurable. The minimal such Λ is called the dependence set of f and is denoted by D_f. Local functions are continuous and any continuous function is the uniform limit of local functions. The set of continuous functions $f : \Omega \to \mathbb{R}$ is written $C(\Omega)$. For $x \in \mathbb{Z}^d$, τ_x denotes translation over x, acting on elements of Ω by $\tau_x\sigma(y) = \sigma(y + x)$, on functions via $\tau_x f(\sigma) = f(\tau_x\sigma)$ and on measures via $\int f d\mu \circ \tau_x = \int \tau_x f d\mu$. The set of all probability measures on the Borel sigma-field of Ω is denoted by $\mathcal{M}^+_1$ and the translation invariant elements are collected in the set denoted by $\mathcal{M}^+_{1,\text{inv}}$. For $\Lambda \in \mathcal{S}$, we denote σ_Λ the restriction of σ to Λ, and for $\mu \in \mathcal{M}^+_1$, μ_Λ is the distribution of σ_Λ when σ is distributed according to μ. For all $x \in \mathbb{Z}^d$, σ^x denotes the spin configurations obtained from σ by flipping the spin at x: $\sigma^x(x) = -\sigma(x)$ and $\sigma^x(y) = \sigma(y)$ if $y \neq x$. As a dynamics we consider a Feller process with a generator of the type

$$Lf(\sigma) = \sum_{x\in\mathbb{Z}^d} c_x(\sigma)\nabla_x f(\sigma) \tag{2.1}$$

where for all $x \in \mathbb{Z}^d$, for all $f \in \mathcal{F}$ bounded, for all $\sigma \in \Omega$

$$\nabla_x f(\sigma) = f(\sigma^x) - f(\sigma).$$

For the flip rates c_x we make the following assumptions:

1. Nearest neighbor dependence: For all $x \in \mathbb{Z}^d$, c_x is a local function such that

$$D_{c_x} = \{y : |y - x| \leq 1\}.$$

2. Strict positivity:

$$\exists \epsilon > 0, \forall x \in \mathbb{Z}^d,\ 0 < \epsilon = \min_\sigma c_x(\sigma) < \max_\sigma c_x(\sigma) < \infty.$$

3. Translation invariance:

$$\forall x \in \mathbb{Z}^d,\ c_x = \tau_x c_0.$$

The restriction to nearest neighbor dependence of the rates is for convenience only; it can be replaced by finite range.

In [6] it is proved that there corresponds a semigroup $S(t)$ on $\mathcal{C}(\Omega)$ and a unique Feller process to the generator L. We denote by $\mathbb{P}_\sigma$ its path space measure started at $\sigma_0 = \sigma$, and $\mathbb{E}_\sigma$ denotes the corresponding expectation. The semi group acts on functions: for all $t > 0$, for all $f \in C(\Omega)$, for all $\sigma \in \Omega$,

$$S(t)f(\sigma) = \mathbb{E}_\sigma[f(\sigma_t].$$

For a probability measure ν on Ω, we define $\nu S(t)$ by

$$\int f\ d\nu S(t) = \int S(t) f\ d\nu. \tag{2.2}$$

2.2 Asymptotically decoupled measures

Definition 2.3 *A probability measure $\nu \in \mathcal{M}^+_{1,\text{inv}}$ is called asymptotically decoupled (AD) if there exists sequences $d(n), c(n)$ such that*

$$\lim_{n\to\infty} \frac{c(n)}{|\Lambda_n|} = 0, \quad \lim_{n\to\infty} \frac{d(n)}{n} = 0$$

and for all $A \in \mathcal{F}_{\Lambda_n}$, $B \in \mathcal{F}^c_{\Lambda_{n+d(n)}}$ with $\nu(A)\nu(B) \neq 0$:

$$e^{-c(n)} \leq \frac{\nu(A \cap B)}{\nu(A)\nu(B)} \leq e^{c(n)}. \tag{2.4}$$

Important examples of AD measures are Gibbs measures with a translation invariant absolutely summable interaction [4]. In this case we can choose $d(n) = 0$ and $c(n) = \circ(|\Lambda_n|)$, and in the case of finite range potentials $c(n) = \circ(|\partial\Lambda_n|)$. Examples of non-Gibbsian AD measures are renormalization group transformations such as decimation, block spin averaging and Kadanoff transformation of Gibbs measures [2, 3]. Indeed, it is clear from the definition that if, for a finite set W, we consider a transformation

$$T : \Omega \to W^{\mathbb{Z}^d}$$

such that $T\sigma(x)$ depends only on $\{\sigma(y) : y \in B_x\}$ where for $x \neq x'$, $B_x \cap B_{x'} = \emptyset$ and such that $\max_x |B_x| < R \in \mathbb{R}_+$, then for ν AD, $\nu \circ T$ is AD. The simplest example of such a T is $T\sigma(x) = \sigma(kx)$ (decimation). Let us denote by $\mathcal{A}$ the set of all translation invariant asymptotically decoupled probability measures. The following theorem is proved in [9].

Theorem 2.5 *For all $\nu \in \mathcal{A}$, we have the following:*

1. *For any $\mu \in \mathcal{M}^+_{1,\text{inv}}$, the relative entropy density*

$$h(\mu|\nu) = \lim_{n\to\infty} \frac{1}{|\Lambda_n|} \int d\mu_{\Lambda_n} \log \frac{d\mu_{\Lambda_n}}{d\nu_{\Lambda_n}} \tag{2.6}$$

exists.

2. *For any $f \in \mathcal{C}(\Omega)$, the pressure*

$$P_\nu(f) = \lim_{n\to\infty} \frac{1}{|\Lambda_n|} \log \int \exp(\sum_{x\in\Lambda_n} \tau_x f)\ d\nu \tag{2.7}$$

exists.

3. *$P_\nu(\cdot)$ and $h(\cdot|\nu)$ are conjugate convex functions, i.e.,*

$$\begin{aligned} h(\mu|\nu) &= \sup_{f\in\mathcal{C}(\Omega)} \left(\int f\ d\mu - P_\nu(f)\right) \\ P_\nu(f) &= \sup_{\mu\in\mathcal{M}^+_{1,\text{inv}}} \left(\int f\ d\mu - h(\mu|\nu)\right) \end{aligned} \tag{2.8}$$

4. *Under ν, the empirical measures*

$$L_n(\sigma) = \sum_{x\in\Lambda_n} \frac{1}{|\Lambda_n|} \delta_{\tau_x\sigma} \tag{2.9}$$

satisfy the large deviation principle with rate function $I(\cdot) = h(\cdot|\nu)$, extended to $\mathcal{M}^+_1$ by putting $I(\mu) = \infty$ for $\mu \notin \mathcal{M}^+_{1,\text{inv}}$.

3 Result

In this section we prove

Theorem 3.1 *Let $\nu \in \mathcal{A}$ and let $S(t)$ be the semi group of the generator (2.1), then $\nu S(t) \in \mathcal{A}$. In particular, for any $t > 0$, the random measures*

$$L^t_n = \frac{1}{|\Lambda_n|} \sum_{x\in\Lambda_n} \delta_{\tau_x\sigma_t}$$

satisfy the large deviation principle with rate function $h(\cdot|\nu S(t))$.

Proof. First notice that by the semi group property, it suffices to show that for some $t_0 > 0$, $\{\nu S(t) : \nu \in \mathcal{A}, t \le t_0\} \subset \mathcal{A}$. The t_0 should depend only on the rates c_x and not on the initial measure $\nu \in \mathcal{A}$. Fix $n \in \mathbb{N}$ and choose $\nu \in \mathcal{A}$ with corresponding $c(n), d(n)$ of Definition (2.3). We will prove that there exists $t_0 > 0$, $c_1 > 0$ depending only on the rates such that for all $A \in \mathcal{F}_{\Lambda_n}$, $B \in \mathcal{F}_{\Lambda^c_{n+d(n+4)+4}}$, and for all $t \le t_0$:

$$e^{-c(n)} e^{-c_1|\partial\Lambda_n|} \le \frac{\nu S(t)(A\cap B)}{\nu S(t)(A)\nu S(t)(B)} \le e^{c(n)} e^{c_1|\partial\Lambda_n|} \tag{3.2}$$

which clearly implies that $\nu S(t) \in \mathcal{A}$ for $t \leq t_0$. The idea is to "artificially decouple" A and B by introducing in the process a "corridor" of independently flipping spins. This requires a modification of the process in a region of the order of the boundary of Λ_n. Then via a Girsanov's formula and a cluster expansion we control the "price to pay" for this modification. It is because of the cluster expansion technique used that we first have to restrict to small t. Fix $A \in \mathcal{F}_{\Lambda_n}$ and $B \in \mathcal{F}_{\Lambda^c_{n+d(n+4)+4}}$ and denote $R_n = \Lambda_{n+4} \setminus \Lambda_n$, and $R'_n = \Lambda_{n+d(n+4)+4} \setminus \Lambda_{n+d(n+4)}$. Notice that since $d(n)/n \to 0$ as $n \to \infty$ the regions R_n, R'_n satisfy $|R_n \cup R'_n| = \circ(|\Lambda_n|)$. Introduce the following "decoupled generator":

$$L_n = \sum_{x \in \Lambda_n} c_x \nabla_x + \sum_{x \in R_n \cup \ R'_n} \nabla_x + \sum_{x \in \mathbb{Z}^d \setminus (\Lambda_n \cup R_n \cup R'_n)} c_x \nabla_x. \tag{3.3}$$

In the process with generator L_n, the spins in $R'_n \cup R_n$ flip on the event times of independent rate one Poisson processes. By the nearest neighbor character of the flip rates c_x, this implies that under the path space measure $\mathbb{P}^n_\sigma$, the random variables $\{\sigma_s(x) : x \in \Lambda^c_{n+d(n+4)+4},\ 0 \leq s \leq t\}$ and $\{\sigma_s(x) : x \in \Lambda_n,\ 0 \leq s \leq t\}$ are independent. Moreover, again by the nearest neighbor character of the rates, for $A \in \mathcal{F}_{\Lambda_n}$, $S_n(t)(1_A) \in \mathcal{F}_{\Lambda_{n+4}}$ and for $B \in \mathcal{F}_{\Lambda^c_{n+4+d(n+4)}}$, $S_n(t)(1_B)$ is $\mathcal{F}_{\Lambda^c_{n+d(n)}}$ measurable. Therefore we have

$$\int S_n(t)(1_A 1_B) d\nu = \int S_n(t)(1_A) S_n(t)(1_B) d\nu \tag{3.4}$$

and

$$e^{-c(n)} \leq \frac{\int S_n(t)(1_A) S_n(t)(1_B) d\nu}{\int S_n(t)(1_A) d\nu \int S_n(t)(1_B) d\nu} \leq e^{c(n)}. \tag{3.5}$$

Notice that we used here a consequence of (2.4), namely that for any non-negative $f \in \mathcal{F}_{\Lambda_n}$, $g \in \mathcal{F}_{\Lambda^c_{n+d(n)}}$,

$$e^{-c(n)} \leq \frac{\int (fg) d\nu}{\int f d\nu \int g d\nu} \leq e^{c(n)}.$$

This follows immediately from the definition (2.4) together with the fact that such f, g can be approximated in $L^1(\nu)$ by linear combinations of indicator functions with positive coefficients. In order to prove (3.2), it is sufficient to have the existence of $t_0 > 0$, $\xi(n) = O(|\partial \Lambda_n|)$ such that all $t \leq t_0$,

$$e^{-\xi(n)} \leq \inf_{C \in \mathcal{F}} \frac{\nu S(t)(1_C)}{\nu S_n(t)(1_C)} \leq \sup_{C \in \mathcal{F}} \frac{\nu S(t)(1_C)}{\nu S_n(t)(1_C)} \leq e^{\xi(n)} \tag{3.6}$$

i.e., the measures $\nu S_n(t)$ and $\nu S(t)$ are absolutely continuous with a density that is uniformly bounded from below by $e^{-\xi(n)}$ and from above by $e^{\xi(n)}$. This can be expected from the fact that we modified our process only in a corridor, i.e. the generator L_n differs from L in the region $R_n \cup R'_n$ only. To obtain (3.6), it is in turn sufficient to check it on cylinder events and to prove that

$$e^{-\xi(n)} \leq \frac{\mathbb{E}_\eta \left(I(\eta_t(x) = \sigma(x),\ \forall x \in \Lambda_N) \right)}{\mathbb{E}^n_\eta \left(I(\eta_t(x) = \sigma(x),\ \forall x \in \Lambda_N) \right)} \leq e^{\xi(n)} \tag{3.7}$$

where this inequality holds for all σ, η, N with the same ξ. In order to obtain (3.7), we approximate by finite volume processes. For $M \in \mathbb{N}$, introduce the generator

$$L_M = \sum_{x \in \Lambda^c_M} \nabla_x + \sum_{x \in \Lambda_M} c_x \nabla_x. \tag{3.8}$$

Since the rates c_x are nearest neighbor, L_M generates a process on $\Omega_{M+1} = \{-1,+1\}^{\Lambda_{M+1}}$ and by the Trotter-Kurtz theorem ([6], chp 1) for the associated semi group $S_M(t)$ we have $S_M(t)f \longrightarrow S(t)f$ uniformly on compacts as M goes to infinity. Similarly, the finite volume approximation of L_M is introduced by

$$L^n_M = \sum_{x\in R_n\cup R'_n} \nabla_x + \sum_{x\in\Lambda_M\setminus(R_n\cup R'_n)} c_x\nabla_x + \sum_{x\in\Lambda^c_M} \nabla_x. \tag{3.9}$$

Therefore, in order to obtain (3.7), we have to prove

$$e^{-\xi(n)} \leq \frac{\mathbb{E}^M_\eta\left(I(\eta_t(x)=\sigma(x),\ \forall x\in\Lambda_N)\right)}{\mathbb{E}^{n,M}_\eta\left(I(\eta_t(x)=\sigma(x),\ \forall x\in\Lambda_N)\right)} \leq e^{\xi(n)} \tag{3.10}$$

where $\mathbb{E}^M_\eta$ denotes expectation in the process with generator L_M, $\mathbb{E}^{n,M}_\eta$ expectation in the process with generator L^n_M and where the ξ of the inequality (3.10) does not depend on η,σ,N and $M>N>n$. Finally introduce the generator of the independent spin flip process:

$$L^o = \sum_x \nabla_x \tag{3.11}$$

and $\mathbb{P}^o_\eta$, $\mathbb{E}^o_\eta$ for corresponding path space measure and expectation. By Girsanov's formula, the ratio in (3.10) can be rewritten as a quotient of expectations in the process of independent spin flips, and we are led to show that

$$e^{-\xi(n)} \leq \frac{\mathbb{E}^o_\eta\left(e^{\sum_{x\in\Lambda_M}\Psi^t_x}\prod_{x\in\Lambda_N} I(\eta_t(x)=\sigma(x))\right)}{\mathbb{E}^o_\eta\left(e^{\sum_{x\in\Lambda_M\setminus(R_n\cup R'_n)}\Psi^t_x}\prod_{x\in\Lambda_N} I(\eta_t(x)=\sigma(x))\right)} \leq e^{\xi(n)} \tag{3.12}$$

where ξ does not depend on σ,η and M and where

$$\Psi^t_x = \int_0^t \log(c_x(\sigma_s))dN^x_s - \int_0^t (c_x(\sigma_s)-1)ds \tag{3.13}$$

and N^x_s denotes the number of flips at x in the time interval $[0,s]$. Let us denote by $\mathbb{P}^o_{M,t,\eta,\sigma}$ the "bridge between η and σ", i.e. the measure $\mathbb{P}^o_\eta$ conditioned on the event $\{\eta_t(x)=\sigma(x),\ \forall x\in\Lambda_M\}$. We can rewrite the ratio of (3.12) as follows

$$\frac{\mathbb{E}^o_{M,t,\eta,\sigma}\left(e^{\sum_{x\in\Lambda_M}\Psi^t_x}\right)}{\mathbb{E}^o_{M,t,\eta,\sigma}\left(e^{\sum_{x\in\Lambda_M\setminus(R_n\cup R'_n)}\Psi^t_x}\right)}. \tag{3.14}$$

This expression has the form of the ratio of two 'abstract' partition functions of different volumes, i.e., a quotient of the form

$$\frac{Z_{\Lambda_M}}{Z_{\Lambda_M\setminus R_n\cup R'_n}}. \tag{3.15}$$

If for the logarithm of Z_Λ we can write a convergent cluster expansion, then it is clear that the ratio is of the order $e^{|R_n\cup R'_n|}$. The natural parameter which has to be chosen small in order to ensure convergence of the cluster expansion will be the time t. However, since we are working with a conditioned expectation $\mathbb{E}^o_{M,t,\eta,\sigma}$, we cannot expect that the exponential $e^{\sum_x \Psi_x^t}$ is close to one as t tends to zero. Indeed for the lattice sites $x \in \Lambda_M$ such that $\sigma(x) \neq \eta(x)$, at least one jump took place in the conditioned measure $\mathbb{P}^o_{M,t,\eta,\sigma}$, and the integral $\int \log(c_x(\sigma_s))dN_s^x$ equals $\log(c_x(\eta^x)/c_x(\sigma))$ if precisely one jump of N^x took place in the interval $[0,t]$. To remedy this problem, we will subtract from Ψ_x^t the value it takes in the limit $t \to 0$ in the conditioned measure $\mathbb{P}^o_{M,t,\eta,\sigma}$, which is

$$\varphi_x(\eta,\sigma) = \log \frac{c_x(\eta^x)}{c_x(\sigma)} I(\eta(x) \neq \sigma(x)). \tag{3.16}$$

By an expansion of the exponential function around zero, we will then prove that the logarithm of the expectation

$$\mathbb{E}^o_{M,t,\eta,\sigma}\left(e^{\sum_x(\Psi_x^t - \varphi_x)}\right) \tag{3.17}$$

can be given by a convergent cluster expansion for small t. In order to set up the expansion, remind that the rates c_x depend on $\sigma(y)$ for $|y-x| \leq 1$. Under the measure $\mathbb{P}^o_{M,t,\eta,\sigma}$, the spins at different lattice sites evolve independently. Therefore, if x, y are more than one lattice distant apart, the random variables Ψ_x^t and Ψ_y^t are independent. For a set $A \in \mathcal{S}$, we denote

$$\bar{A} = \{y \in \mathbb{Z}^d : d(y,A) \leq 1\}.$$

Two connected subsets A and B are called compatible if $\bar{A} \cap \bar{B} = \emptyset$ With this notation, we write

$$\begin{aligned}
\mathbb{E}^o_{M,t,\eta,\sigma}\left(e^{\sum_x(\Psi_x^t - \varphi_x)}\right) &= \mathbb{E}^o_{M,t,\eta,\sigma}\left(\prod_{x \in \Lambda_M}\left((e^{\Psi_x^t - \varphi_x} - 1) + 1\right)\right) \\
&= 1 + \sum_{k=1}^{\infty} \frac{1}{k!} \sum_{\gamma_1,\ldots,\gamma_k \subset \Lambda_M} \prod_{i=1}^{k} w_{t,\eta,\sigma}(\gamma_i)
\end{aligned} \tag{3.18}$$

where the sum over γ_i is over compatible collections of nearest neighbor connected subsets. The polymer weights are given by

$$w_{t,\eta,\sigma}(\gamma) = \mathbb{E}^o_{M,t,\eta,\sigma}\left(\prod_{x \in \gamma}(e^{\Psi_x^t - \varphi_x} - 1)\right). \tag{3.19}$$

In order to write down a convergent cluster expansion of the logarithm of the right hand site of (3.18), we use the Kotecký-Preiss criterion, i.e., we have to prove an estimate of the type

$$|w_{t,\eta,\sigma}(\gamma)| \leq e^{-c_t|\gamma|} \tag{3.20}$$

where the constant c_t does not depend on η, σ and where $c_t \to \infty$ as $t \to 0$. Indeed if that estimate holds, then for t small enough the weights will beat the entropy and we can write

$$\log \mathbb{E}^o_{M,t,\eta,\sigma}\left(e^{\sum_x(\Psi_x^t - \varphi_x)}\right) = \sum_{\Gamma \subset \Lambda_M} a(\Gamma) w_{t,\eta,\sigma}(\Gamma) \tag{3.21}$$

where the sum is over clusters, i.e., multi-indices of compatible polymers, see e.g., [5, 8, 10]. In order to obtain (3.20), remind that the rates c_x are bounded away from zero and infinity, and hence we can estimate

$$\mathbb{E}[|e^{\Psi_x^t - \varphi_x} - 1|] \leq \mathbb{E}\big[(e^{Bt}e^{AN_x^t I(N_x^t \geq 2)} - 1)I(\eta(x) \neq \sigma(x)) + (e^{Bt}e^{AN_x^t} - 1)I(\eta(x) = \sigma(x))\big] \tag{3.22}$$

where we used that, by the choice of φ_x, the integral $\int_0^t \log c_x(\sigma_s) dN_s^x = \varphi_x$ if the Poisson process N^x made exactly one jump in the time interval $[0,t]$. Arrived at this point, the weight can be estimated as follows

$$\begin{aligned} & |w_{t,\eta,\sigma}(\gamma)| \\ \leq & \prod_{x\in\gamma\cap\Delta(\eta,\sigma)} \frac{\mathbb{E}\left(I(N_t^x \in 2\mathbb{N}+1)(e^{Bt}e^{AN_x^t I(N_x^t\geq 2)}-1)\right)}{\mathbb{E}(I(N_t^x\in 2\mathbb{N}+1)} \\ \times & \prod_{x\in\gamma\cap\Delta^c(\eta,\sigma)} \frac{\mathbb{E}\left(I(N_t^x\in 2\mathbb{N})(e^{Bt}e^{AN_x^t}-1)\right)}{\mathbb{E}(I(N_t^x\in 2\mathbb{N})} \end{aligned} \tag{3.23}$$

where $\mathbb{E}$ denotes expectation w.r.t. independent mean one Poisson processes, and where

$$\Delta(\eta,\sigma) = \{x\in\mathbb{Z}^d : \sigma(x)\neq\eta(x)\} \tag{3.24}$$

i.e., the sites where the spin flipped an uneven number of times. A straightforward computation using that N_t^x is Poisson gives

$$\frac{\mathbb{E}\left(I(N_t^x \in 2\mathbb{N}+1)(e^{Bt}e^{AN_x^t I(N_x^t\geq 2)}-1)\right)}{\mathbb{E}(I(N_t^x\in 2\mathbb{N}+1))} \leq \left((e^{Bt}-1)\frac{t}{\sinh(t)} + \frac{O(t^3)}{\sinh(t)}\right) \tag{3.25}$$

and

$$\frac{\mathbb{E}\left(I(N_t^x\in 2\mathbb{N})(e^{Bt}e^{AN_x^t}-1)\right)}{\mathbb{E}(I(N_t^x\in 2\mathbb{N}))} \leq \frac{e^{Bt}\cosh(te^A)-\cosh(t)}{\cosh(t)}. \tag{3.26}$$

Since both expressions do not depend on σ,η and converge to zero as $t\to 0$, we obtain the estimate (3.20). This implies that we can write

$$\begin{aligned} \frac{\mathbb{E}_\eta^M\left(I(\eta_t(x)=\sigma(x),\ \forall x\in\Lambda_N)\right)}{\mathbb{E}_\eta^{n,M}\left(I(\eta_t(x)=\sigma(x),\ \forall x\in\Lambda_N)\right)} &= \exp\left(\sum_{\Gamma\cap(R_n\cup R_n')\neq\emptyset} a(\Gamma)w_{t,\eta,\sigma}(\Gamma)\right)\prod_{x\in R_n\cup R'n} e^{\varphi_x} \\ &\leq \exp\left(C|R_n\cup R_n'|\right) \end{aligned} \tag{3.27}$$

which concludes the proof of the theorem. ∎

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