Variational principle and almost quasilocality for renormalized measures

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Abstract: We study the variational principle for some non-Gibbsian measures. We give a necessary and sufficient condition for the validity of the implication "zero relative entropy density implies common version of conditional probabilities" (so-called "second part of the variational principle"). Applying this to noisy decimations of the low-temperature phases of the Ising model, we obtain almost sure quasilocality for these measures and the second part of the variational principle. For the projection of low temperature Ising phases on a one-dimensional layer, we also obtain the second part of the variational principle.

Keywords: Gibbs vs non-Gibbs, renormalization group, almost quasilocality, variational principle.

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1 Introduction

Non-Gibbsian measures were initially detected as "pathologies" of renormalization group transformations of low-temperature Gibbs measures [7, 8]. Initial efforts were directed towards the construction of a sufficiently rich catalogue of examples and mathematical mechanisms leading to non-Gibbsianness [4]. Afterwards, Dobrushin lanced a program of "restoration of Gibbsianness" mainly consisting in two parts: (i) determination of weaker notions of Gibbsianness which are preserved by the transformations of interest, and (ii) extension of the thermodynamic formalism to these broader class of measures. The first issue motivated the notions of weak Gibbsianness and almost quasilocality [1, 2, 11, 12, 13, 14, 15]. A measure is called almost quasilocal if it admits a version of its conditional probabilities which is continuous on a set of measure one. A measure is called weakly Gibbs if there exists a potential which is (absolutely) convergent such that the conditional probabilities can be written in "Gibbsian form" on a set of measure one. If one does not insist on *absolute* convergence and translation invariance of the potential, then almost quasilocal implies weakly Gibbs, and the converse is not true, see [14, 11].

Most efforts in the realization of the Dobrushin program have been directed towards showing that transformations of low-temperature Gibbs measures are weakly Gibbs, see [1, 2, 14]. However, it is not clear at this point whether "weakly Gibbsian" measures do not constitute a class that is too general in order to represent a reasonable generalization of the classical Gibbs measures. One test a possible generalization of Gibbs measures should pass is the existence of a reasonable "thermodynamic formalism". For weakly Gibbsian measures e.g., the notion of "physical equivalence" -two interactions sharing a phase have the same set of Gibbs measures- has not been proved. Neither any general statement on the existence of thermodynamic quantities such as pressure, free energy has been obtained in the context of weakly Gibbsian measures. In fact, recent counterexamples in the context of disordered spin systems show that two weakly Gibbsian measures with different interactions (i.e., with no common version of their conditional probabilities) can have the same free energy (relative entropy density zero). This shows that in complete generality, a reasonable thermodynamic formalism for weakly Gibbsian measures cannot be expected [9].

In this paper we show that in the context of FKG, "almost quasilocal measures" admit a reasonable thermodynamic formalism. Concretely, for the decimation of the low-temperature Ising model, we first remark existence of relative entropy density (as a corollary of the recent large deviation formalism of [16]), and next we show that if a measure has zero relative entropy density w.r.t. the decimation of the plus phase of the Ising model then it has the same conditional probabilities. In applying this to the decimation of the minus phase, we obtain that the set of points of discontinuity of the conditional probabilities of the decimated Ising model is of measure zero, thereby solving an open problem in [5]. More general transformations can be considered, as long as we do not leave the FKG context. A further criterion for "zero relative entropy density" implies "common version of conditional probabilities" is obtained, which is then applied to the projection of the d-dimensional Ising model on a one-dimensional layer.

The main message of this paper is that within the context of "almost quasilocality" a generalization of the thermodynamic formalism is possible. In particular, regularity properties of the specification rather than "almost sure existence of Hamiltonians" can be a key to a physically acceptable generalization of Gibbs measures giving naturally important aspects of the variational principle. Since we use the FKG property, our proofs are short and do not rely on laborious techniques such as cluster expansion, multi-scale analysis. However, the results in this particular context are stronger than what can be obtained by the former techniques [1, 10, 14, 15], and hold up to the critical point.

2 Preliminaries

2.1 Configuration space

We work with Ising spins on the lattice $\mathbb{Z}^d$, i.e., configurations are elements of the product space $\Omega = \{-1, 1\}^{\mathbb{Z}^d}$. The set of finite subsets of $\mathbb{Z}^d$ is denoted by $\mathcal{S}$ and for $\sigma \in \Omega$, $\Lambda \in \mathcal{S}$, σ_Λ denotes the restriction of σ to Λ. The cube $[-n, n]^d \cap \mathbb{Z}^d$ is denoted by Λ_n. For $A \subset \mathbb{Z}^d$, $\mathcal{F}_A$ denotes the sigma field generated by the mappings $\sigma \to \sigma(x)$, $x \in A$. $\mathcal{F}_{\mathbb{Z}^d}$ is abbreviated by $\mathcal{F}$ and is the Borel sigma field on Ω.

A function $f : \Omega \to \mathbb{R}$ is local if it is $\mathcal{F}_\Lambda$ measurable for some $\Lambda \in \mathcal{S}$. The set of local functions is denoted by $\mathcal{L}$. $\mathcal{L}$ is a uniformly dense subalgebra of $\mathcal{C}(\Omega)$, the set of all continuous functions.

Translations τ_x are defined on Ω via $\tau_x\sigma(y) = \sigma(y + x)$, on functions via $\tau_x f(\sigma) = f(\tau_x\sigma)$ and on probability measures μ on $(\Omega, \mathcal{F})$ via $\int \tau_x f d\mu = \int f d(\tau_x\mu)$. A probability measure μ on $(\Omega, \mathcal{F})$ is translation invariant if for all $x \in \mathbb{Z}^d$, $\tau_x\mu = \mu$. The set of all translation invariant probability measures is denoted by $\mathcal{M}^+_{1,\mathrm{inv}}$.

For later purposes we need the notion of directional continuity. For $\theta \in \Omega$, a function $f : \Omega \to \mathbb{R}$ is called continuous in the direction θ if for any $\sigma \in \Omega$:

$$\lim_{\Lambda \uparrow \mathbb{Z}^d} f(\sigma_\Lambda \theta_{\Lambda^c}) = f(\sigma). \tag{2.1}$$

If f is continuous, then f is continuous in every direction. The converse is not true, since continuity requires continuity in every direction, uniformly in the direction. The set of all functions which are continuous in the direction θ is denoted by $\mathcal{C}_\theta(\Omega)$. When θ is the plus (resp. minus) configuration, i.e. $\theta(x) = +1$ (resp. -1) $\forall x \in \mathbb{Z}^d$, continuity in the direction θ is called right-continuity (resp. left-continuity).

On Ω we have the natural order $\eta \leq \zeta$ if for all $x \in \mathbb{Z}^d$, $\eta(x) \leq \zeta(x)$. A function is called monotone if it preserves the order, i.e., if $\eta \leq \zeta$ implies $f(\eta) \leq f(\zeta)$.

2.2 Specification

Definition 2.2 *A specification on $(\Omega, \mathcal{F})$ is a family $\{\gamma_\Lambda, \Lambda \in \mathcal{S}\}$ of stochastic kernels such that*

1. *For every $A \in \mathcal{F}$, $\gamma_\Lambda(A|\cdot)$ is a $\mathcal{F}_{\Lambda^c}$-measurable function.*

2. *For every $\omega \in \Omega$, $\gamma_\Lambda(\cdot|\omega)$ is a probability measure on $(\Omega, \mathcal{F})$.*

3. *The kernels are proper, i.e., for every $B \in \mathcal{F}_{\Lambda^c}$:*

$$\gamma_\Lambda(B|\omega) = 1_B(\omega)$$

4. *The kernels are consistent, i.e., for $\Lambda \subset \Lambda'$:*

$$\gamma_{\Lambda'}\gamma_\Lambda = \gamma_{\Lambda'}$$

i.e.,

$$\gamma_{\Lambda'}(B|\eta) = \int \gamma_\Lambda(B|\omega)\gamma_{\Lambda'}(d\omega|\eta),\ \forall B \in \mathcal{F},\ \eta \in \Omega.$$

For a specification γ and f a bounded measurable function, we denote

$$\gamma_\Lambda(f)(\omega) = \int \gamma_\Lambda(d\sigma|\omega) f(\sigma_\Lambda \omega_{\Lambda^c})$$

Definition 2.3 *A probability measure μ on $(\Omega, \mathcal{F})$ is called consistent with a specification γ (notation $\mu \in \mathcal{G}(\gamma)$) if for any f bounded measurable:*

$$\int \gamma_\Lambda(f) d\mu = \int f d\mu$$

If $\mu \in \mathcal{G}(\gamma)$, then the specification defines a version of the conditional probabilities of μ ,i.e., μ-a.s.

$$\gamma_\Lambda(A|\cdot) = \mathbb{E}_\mu(1_A|\mathcal{F}_{\Lambda^c})$$

We write $\mathcal{G}_{\mathrm{inv}}(\gamma) = \mathcal{G}(\gamma) \cap \mathcal{M}^+_{1,\mathrm{inv}}$ for the translation invariant elements of $\mathcal{G}(\gamma)$.

Definition 2.4 *A specification is called*

1. *Quasilocal (or Feller) if for any $f \in \mathcal{C}(\Omega)$, $\Lambda \in \mathcal{S}$, $\gamma_\Lambda(f) \in \mathcal{C}(\Omega)$.*
2. *Quasilocal in the direction θ if for any $f \in \mathcal{C}(\Omega)$, $\Lambda \in \mathcal{S}$, $\gamma_\Lambda(f) \in \mathcal{C}_\theta(\Omega)$.*
3. *Monotone if for every f monotone, and $\Lambda \in \mathcal{S}$, $\gamma_\Lambda(f)$ is monotone.*
4. *Translation invariant if for any $\Lambda \in \mathcal{S}$, $\omega \in \Omega$, $A \in \mathcal{F}$:*

$$\gamma_\Lambda(A|\omega) = \gamma_{\Lambda+x}(\tau_x A|\tau_x(\omega))$$

Important examples of quasilocal specifications are given by so-called Gibbs specifications, i.e., where

$$\gamma_\Lambda(\sigma|\omega) = \frac{\exp(-H^\omega_\Lambda(\sigma))}{Z^\omega_\Lambda}$$

with

$$H^\omega_\Lambda(\sigma) = \sum_{A\cap\Lambda\neq\emptyset} U(A, \sigma_\Lambda\omega_{\Lambda^c})$$

is the Hamiltonian corresponding to a translation invariant absolutely summable potential $U \in \mathcal{B}_1$. This means that $U(A,\cdot)$ are $\mathcal{F}_A$ measurable functions, that $U(A+x,\cdot) = \tau_x U(A,\cdot)$ and that

$$\|U\|_1 = \sum_{A\ni 0} \sup_\sigma |U(A,\sigma)| < \infty$$

The measures consistent with such Gibbs specifications are the usual Gibbs measures of equilibrium statistical mechanics.

For a general specification we denote by Ω_γ the set of its points of continuity, i.e., the set of those $\sigma \in \Omega$ such that for all $A \in \mathcal{F}$, $\Lambda \in \mathcal{S}$ the map $\omega \mapsto \gamma_\Lambda(A|\omega)$ is continuous at σ. For quasilocal specifications, we have of course that $\Omega_\gamma = \Omega$. For a specification which is quasilocal in some direction θ, the set Ω_γ is non-empty, see e.g. [12].

Definition 2.5 *A probability measure μ is called almost quasilocal if there exists a specification γ such that $\mu \in \mathcal{G}(\gamma)$ and $\mu(\Omega_\gamma) = 1$.*

2.3 Relative entropy

For $\mu, \nu \in \mathcal{M}^+_{1,\text{inv}}$ we denote

$$H_\Lambda(\mu|\nu) = \sum_{\sigma_\Lambda \in \Omega_\Lambda} \mu(\sigma_\Lambda) \log \frac{\mu(\sigma_\Lambda)}{\nu(\sigma_\Lambda)}$$

with the convention $H_\Lambda = \infty$ if μ_Λ is not absolutely continuous with respect to ν_Λ. We denote

$$h(\mu|\nu) = \lim_{n \uparrow \infty} \frac{1}{|\Lambda_n|} H_{\Lambda_n}(\mu|\nu) \tag{2.6}$$

the relative entropy density of μ w.r.t. ν, provided the limit exists. The existence of the limit in (2.6) is a non-trivial problem, and besides the context of Gibbs measures with a translation invariant $\mathcal{B}_1$-potential, and the context of asymptotically decoupled measures in [16], there is no general existence result.

2.4 Variational Principle

Let γ be a translation invariant specification, $\nu \in \mathcal{G}(\gamma)$ a translation invariant probability measure and $\mathcal{M}_\gamma$ a subset of $\mathcal{M}^+_{1,\text{inv}}$.

Definition 2.7 *We say that the variational principle holds for* $(\gamma, \mathcal{M}_\gamma, \nu)$ *if*

0. *For any* $\mu \in \mathcal{M}_\gamma$, $h(\mu|\nu)$ *exists.*

1. $\mu \in \mathcal{G}_{\text{inv}}(\gamma) \cap \mathcal{M}_\gamma$ *implies* $h(\mu|\nu) = 0$.

2. *If* $\mu \in \mathcal{M}_\gamma$ *is such that* $h(\mu|\nu) = 0$ *then* $\mu \in \mathcal{G}_{\text{inv}}(\gamma)$.

Point 1 of the definition is usually called 'first part of the variational principle' and corresponds physically speaking to the fact that the Gibbs measures minimize the free energy. Point 2 is called the 'second part of the variational principle' and corresponds to the fact that a minimizer of the free energy is a Gibbs measure. In the context of Gibbs measures with a translation invariant $\mathcal{B}_1$-potential, the variational principle holds for every $\nu \in \mathcal{G}_{\text{inv}}(\gamma)$ with $\mathcal{M}_\gamma = \mathcal{M}^+_{1,\text{inv}}$. In more general contexts, $\mathcal{M}_\gamma$ will be a set of measures concentrating on configurations which behave properly. It is clear that without any locality requirements on the specification, there is no hope to have a variational principle. E.g., in [6] an example of a specification is given such that every Bernoulli measure $\nu_p \in \mathcal{G}_{\text{inv}}(\gamma)$, and hence (1) is not satisfied. On the other hand the examples of transformations of low-temperature Gibbs measures show that some extension of the class of specifications for which the variational principle holds is needed, since for these measures we expect a reasonable thermodynamic formalism.

3 Decimation of the Ising model

For $\Lambda \in \mathcal{S}$, $\eta \in \Omega$, the Hamiltonian of the Ising model in Λ with boundary condition η outside Λ is given by

$$H^\eta_\Lambda(\sigma) = -\beta \sum_{x,y \in \Lambda, |x-y|=1} \sigma(x)\sigma(y) - \beta \sum_{x \in \Lambda, y \notin \Lambda, |x-y|=1} \eta(y)\sigma(x).$$

The corresponding specification is given by

$$\gamma_\Lambda^\beta(\sigma|\eta) = \frac{\exp(-H_\Lambda^\omega(\sigma))}{Z_\Lambda^\omega}$$

We consider $d \geq 2$ and $\beta > \beta_c$ such that $|\mathcal{G}(\gamma^\beta)| > 1$. We call μ^+ (μ^-) the plus (minus) phase, i.e., the weak limits of $\gamma_\Lambda^\beta(\cdot|+)$ ($\gamma_\Lambda^\beta(\cdot|-)$) as $\Lambda \uparrow \mathbb{Z}^d$. For $b \in \mathbb{Z}^+$ we denote by T_b the decimation defined by

$$T_b\sigma(x) = \sigma(bx)$$

and ν^+ (ν^-) denotes the image measure of μ^+ (μ^-) under T_b. In [5] the following results are proved:

Proposition 3.1 *For all $\beta > \beta_c$,*

1. *ν^+, ν^- are not quasilocal, i.e., there does not exist a quasilocal specification γ such that $\mathcal{G}(\gamma) \cap \{\nu^+, \nu^-\} \neq \emptyset$.*
2. *There exists a right continuous monotone specification γ^+ with $\nu^+ \in \mathcal{G}(\gamma^+)$, and a left continuous monotone specification γ^- with $\nu^- \in \mathcal{G}(\gamma^-)$. Moreover, for any f monotone, $\gamma_\Lambda^+(f) \geq \gamma_\Lambda^-(f)$.*
3. *Both specifications have the same set of points of continuity, i.e., $\Omega_{\gamma^+} = \Omega_{\gamma^-} := \Omega_\pm$*
4. *$\nu^- \in \mathcal{G}(\gamma^+)$ implies $\nu^+(\Omega_\pm) = 1$ (i.e., ν^+ is almost quasilocal), and $\nu^+ \in \mathcal{G}(\gamma^-)$ implies $\nu^-(\Omega_\pm) = 1$ (i.e., ν^- is almost quasilocal).*

We can now state our first result.

Theorem 3.2 *For all $\beta > \beta_c$,*

1. *For every $\mu \in \mathcal{M}_{1,\mathrm{inv}}^+$, $h(\mu|\nu^+)$ exists and in particular $h(\nu^-|\nu^+) = 0$*
2. *$\nu^- \in \mathcal{G}_{\mathrm{inv}}(\gamma^+)$.*
3. *ν^+ is almost quasilocal.*
4. *If $h(\mu|\nu^+) = 0$ and $\mu(\Omega_\pm) = 1$, then $\mu \in \mathcal{G}_{\mathrm{inv}}(\gamma^+)$, and hence μ is almost quasilocal.*

The same results hold if $+$ is replaced by $-$.

Remark 3.3 *Statement 4 of the theorem has to be interpreted in the spirit of variational principle for unbounded spin systems: if a measure has zero relative entropy density w.r.t. to the 'Gibbs' measure and concentrates on a set of 'good configurations', then it is also 'Gibbs' (i.e. consistent with the same γ).*

For the proof of the theorem, we first state a general proposition.

Proposition 3.4 *Let γ be a specification that is quasilocal in the direction $\theta \in \Omega$ and $\nu \in \mathcal{G}_{\mathrm{inv}}(\gamma)$. For each $\Lambda \in \mathcal{S}$, $M \in \mathbb{N}$, $\Lambda \subset \Lambda_M$ and each local f, let $\gamma_\Lambda^{M,\theta}(f)$ denote the function $\omega \to \gamma_\Lambda(f|\omega_{\Lambda_M}\theta_{\mathbb{Z}^d\setminus\Lambda_M})$. Then, if $\mu \in \mathcal{M}_{1,\mathrm{inv}}^+$ is such that $h(\mu|\nu) = 0$, then the following two statements are equivalent*

1. $\mu \in \mathcal{G}_{\mathrm{inv}}(\gamma)$

2.

$$\nu\Big[\frac{d\mu_{\Lambda_M\setminus\Lambda}}{d\nu_{\Lambda_M\setminus\Lambda}}\Big(\gamma_\Lambda^{M,\theta}(f)-\gamma_\Lambda(f)\Big)\Big] \underset{M\to\infty}{\longrightarrow} 0 \tag{3.5}$$

for all $\Lambda \in \mathcal{S}$ *and* $f \in \mathcal{L}$.

The right-hand-side of (3.5) shows that consistency requires the concentration properties of $d\mu_{\Lambda_M\setminus\Lambda}/d\nu_{\Lambda_M\setminus\Lambda}$ to beat asymptotic divergences due to the lack of continuity of γ_Λ. This imposes some conditions on μ which are reminiscent of what happens for unbounded spin-systems. The analogy between unbounded spin systems and non-Gibbsian measures is an early remark from Dobrushin. Within approaches based on potentials (weak Gibbsianness) these conditions are defined and handled by cluster-expansion methods [14, 10]. As we discuss below, in favorable cases monotonicity arguments can be used instead.

Proof. The hypothesis $h(\mu|\nu) = 0$ implies that for n sufficiently large the $\mathcal{F}_{\Lambda_n}$-measurable function $g_{\Lambda_n} = d\mu_{\Lambda_n}/d\nu_{\Lambda_n}$ exists. For f local and $\Lambda \in \mathcal{S}$, pick M such that $\Lambda_M \supset \Lambda$ and g_{Λ_M} exist and write

$$\mu(\gamma_\Lambda f - f) \;=\; A_M + B_M + C_M \tag{3.6}$$

with

$$A_M \;=\; \mu\Big[\gamma_\Lambda(f) - \gamma_\Lambda^{M,\theta}(f)\Big]\;,\; B_M \;=\; \nu\Big[(g_{\Lambda_M} - g_{\Lambda_M\setminus\Lambda})\, f\Big] \tag{3.7}$$

and C_M is the right-hand side in (3.5). We shall prove that A_M and B_M converge to zero as M tends to infinity.

Indeed, $\lim_{M\to\infty} A_M = 0$ follows by dominated convergence, because γ is quasilocal in the direction θ and $|\gamma_\Lambda^M(f)| \le \|f\|_\infty$.

On the other hand, Csiszár's inequality [3],

$$H_\Delta(\mu|\nu) - H_{\Delta'}(\mu|\nu) \;\ge\; \frac{1}{2}\left[\int_\Omega \Big|g_\Delta(\omega) - g_{\Delta'}(\omega)\Big| d\nu(\omega)\right]^2 .$$

valid for $\Delta' \subset \Delta \in \mathcal{S}$, implies that

$$|B_M| \;\le\; \sqrt{2}\,\|f\|_\infty \Big[H_\Delta(\mu|\nu) - H_{\Delta\setminus\Lambda}(\mu|\nu)\Big]$$

for any $\Delta \supset \Lambda_M$. But the hypothesis $h(\mu|\nu) = 0$ implies that the difference in entropies in the right-hand side tends to zero as $\Delta \uparrow \mathbb{Z}^d$, as shown in [6] or [16]. Hence $B(M) \to 0$ as M goes to infinity. ∎

Proof of Theorem 3.2

1. The first assertion follows from the results in [16]. Indeed, the decimation of an asymptotically decoupled measure is asymptotically decoupled. More precisely, since ν^+ is the decimation of the Ising model we automatically have the inequalities

$$e^{-c_n}\nu^+(A)\nu^+(B) \le \nu^+(A \cap B) \le e^{+c_n}\nu^+(A)\nu^+(B) \tag{3.8}$$

for all $A \in \mathcal{F}_{\Lambda_n}$, $B \in \mathcal{F}_{\Lambda_n^c}$, and where $c_n = O(n^{d-1})$. This inequality follows immediately from the inequality

$$e^{-\kappa_n} \le \frac{\mu^+(A|\eta)}{\mu^+(A)} \le e^{\kappa_n} \tag{3.9}$$

with $\kappa_n = O(n^{d-1})$, uniformly in the boundary condition $\eta \in \{-1, 1\}^{\Lambda_n^c}$.
Pfister in [16] shows that (3.8) implies the existence of the relative entropy density $h(\mu|\nu^+)$ for any $\mu \in \mathcal{M}_{1,\text{inv}}^+$. Zero relative entropy $h(\nu^-|\nu^+) = 0$ follows from the locality of the transformation T_b (see [4]):

$$h(\nu^-|\nu^+) = h(T_b\mu^-|T_b\mu^+) \leq h(\mu^-|\mu^+) = 0.$$

2. It is enough to verify the right-hand side of (3.5) for monotone local functions f since linear combinations of these are uniformly dense in the set of continuous functions. Until the end of the proof of this theorem, we write the reference specification γ instead of γ^+. By proposition 3.4 we only have to show that

$$C_M \;=\; \nu^+\Big[g_{\Lambda_M\setminus\Lambda}\,\Big(\gamma_\Lambda^{M,+}(f) - \gamma_\Lambda(f)\Big)\Big] \;\underset{M\to\infty}{\longrightarrow}\; 0\,, \tag{3.10}$$

where $g_D = d\nu_D^-/d\nu_D^+$ for $D \subset \mathbb{Z}^d$. We first point out that

$$C_M \;\geq\; 0 \tag{3.11}$$

because γ is monotonicity preserving, while

$$\nu^+\Big(g_{\Lambda_M\setminus\Lambda}\,\gamma_\Lambda^{M,+}(f)\Big) \;=\; \nu^-\Big(\gamma_\Lambda^{M,+}(f)\Big) \tag{3.12}$$

because $\gamma_\Lambda^{M,+}f$ is $\mathcal{F}_{\Lambda_M\setminus\Lambda}$-measurable. On the other hand,

$$\nu^+\Big(g_{\Lambda_M\setminus\Lambda}\,\gamma_\Lambda(f)\Big) \;=\; \nu^+\Big[g_{\Lambda_M\setminus\Lambda}\,\nu^+(\gamma_\Lambda(f)|\mathcal{F}_{\Lambda_M\setminus\Lambda})\Big] \tag{3.13}$$

and thus by monotonicity,

$$\nu^+\Big(g_{\Lambda_M\setminus\Lambda}\,\gamma_\Lambda(f)\Big) \;\geq\; \nu^+\Big[g_{\Lambda_M\setminus\Lambda}\,\nu^-(\gamma_\Lambda(f)|\mathcal{F}_{\Lambda_M\setminus\Lambda})\Big] \;=\; \nu^-\Big(\gamma_\Lambda(f)\Big)\,. \tag{3.14}$$

where the last equality follows from $\mathcal{F}_{\Lambda_M\setminus\Lambda}$-measurability of $\nu^-(\gamma_\Lambda(f)|\mathcal{F}_{\Lambda_M\setminus\Lambda})(\,\cdot\,)$. From (3.11), (3.12) and (3.14) we conclude

$$0 \;\leq\; C_M \;\leq\; \nu^-\Big(\big|\gamma_\Lambda^{M,+}(f) - \gamma_\Lambda(f)\big|\Big)$$

and hence (3.10) follows from the right-continuity of γ and dominated convergence.
3. Follows from assertion 4 of proposition 3.1
4. We apply proposition (3.4) again with f a monotone local function. By monotonicity

$$\begin{aligned}
0 \;&\leq\; \nu^+\Big[g_{\Lambda_M\setminus\Lambda}\,\Big(\gamma_\Lambda^{M,+}(f) - \gamma_\Lambda(f)\Big)\Big]\\
&\leq\; \nu^+\Big[g_{\Lambda_M\setminus\Lambda}\,\Big(\gamma_\Lambda^{M,+}(f) - \gamma_\Lambda^{M,-}(f)\Big)\Big]\\
&=\; \mu\Big(\gamma_\Lambda^{M,+}(f) - \gamma_\Lambda^{M,-}(f)\Big)
\end{aligned}$$

where g_D now denotes the Radon-Nikodym density of μ_D with respect to ν_D^+, and the last equality follows from the $\mathcal{F}_{\Lambda_M\setminus\Lambda}$-measurability of $\gamma_\Lambda^{M,+}(f) - \gamma_\Lambda^{M,-}(f)$. The last line tends to zero with M by dominated convergence, because $\mu(\Omega_\pm) = 1$.

Remark 3.15 *A straightforward generalization of Theorem 3.2 consists in considering a noisy decimation, i.e., the transformation is given by*

$$T\sigma(x) = \sigma(bx)(-1)^{\epsilon_x}$$

where ϵ_x are i.i.d. bernoulli and independent of σ, with $P(\epsilon_x = 1) = p$. This means that the transformed spin at site x is with probability $(1-p)$ equal to the spin at site bx and "an error" is made with probability p. The case $b = 1$ corresponds to the single site Kadanoff transformation.

4 Projection on a layer

Theorem 3.2 does not apply to Schonmann's example –the projection of the two-dimensional low temperature plus phase of the Ising model on a line–. More precisely, consider the restriction

$$T : \Omega \to \{-1, +1\}^{\mathbb{Z}} : T\sigma(x) = \sigma(x, 0, \ldots, 0)$$

Denote ν_β^+, resp. ν_β^- the induced measure on $\{-1, 1\}^{\mathbb{Z}}$ by applying T to the low temperature plus phase (resp. minus phase) of the Ising model. It is known that ν_β^+ is not a Gibbs measure, and from [17] one can conclude that if $h(\nu^-|\nu^+)$ exists then it is not zero. The existence of $h(\mu|\nu^+)$ for a general $\mu \in \mathcal{M}_{1,\mathrm{inv}}^+$ cannot be derived from [16] since the projection has not been shown to be asymptotically decoupled. The existence of $h(\mu|\nu^+)$ for $\mu \in \mathcal{M}_{1,\mathrm{inv}}^+$ concentrating on a set of "good configurations" (with ν^+ measure one) has been proved in [14]. However, the assertions of [5] stated before Theorem 3.2 remain true for ν_β^+, ν_β^-. Here we prove

Theorem 4.1 *If $\beta > \beta_c$ is sufficiently large, then we have the implication $h(\mu|\nu_\beta^+) = 0$ implies $\mu^+ \in \mathcal{G}_{\mathrm{inv}}(\gamma^+)$.*

For the proof, we need some more notation.

For Λ a fixed finite volume a direction $\theta \in \Omega$, and f a local function, put

$$\delta_{\Lambda,M}^\theta(f) \;=\; \left|\gamma_\Lambda^{M,\theta}(f) - \gamma_\Lambda(f)\right| \tag{4.2}$$

and introduce for $\epsilon > 0$ the sets

$$A(\theta, \Lambda, f, \epsilon, M) \;=\; \{\eta \in \Omega : \delta_{\Lambda,M}^\theta(f) > \epsilon\}\,. \tag{4.3}$$

If γ is continuous in the direction θ, then $\delta_{\Lambda,M}^\theta$ tends to zero as M tends to infinity, and hence for any probability measure μ, $\mu[A(\theta, \Lambda, f, \epsilon, M)]$ tends to zero as M tends to infinity.

Definition 4.4 *Let $\alpha_M \uparrow \infty$ be an increasing sequence of positive numbers and $\mu \in \mathcal{M}_{1,\mathrm{inv}}^+$. We say that the specification γ admits α_M as a μ-rate of continuity in the direction θ if for all $\epsilon > 0$, for all f local, and for all Λ:*

$$\limsup_{M\uparrow\infty} \frac{1}{\alpha_M} \log \mu[A(\theta, \Lambda, f, \epsilon, M)] \;<\; 0. \tag{4.5}$$

The following proposition shows that for a given rate of continuity, the condition of proposition 3.4 will be satisfied if the relative entropies tend to zero at the same rate.

Proposition 4.6 *Let $\nu \in \mathcal{G}_{\rm inv}(\gamma)$ and suppose that α_M is a ν-rate of θ-continuity. Suppose furthermore that $\mu \in \mathcal{M}^+_{1,\rm inv}$ is such that*

$$\lim_{M\uparrow\infty} \frac{1}{\alpha_M} H_{\Lambda_M}(\mu|\nu) \;=\; 0. \tag{4.7}$$

Then $\mu \in \mathcal{G}_{\rm inv}(\gamma)$.

Proof. Let us fix a local function f, a finite set Λ and some $\epsilon > 0$. We have

$$\nu\left[\frac{d\mu_{\Lambda_M\setminus\Lambda}}{d\nu_{\Lambda_M\setminus\Lambda}}\left(\gamma_\Lambda^{M,\theta}(f) - \gamma_\Lambda(f)\right)\right] \;\le\; \epsilon + 2\|f\|_\infty\, \widetilde{\mu}_M(A^M_\epsilon) \tag{4.8}$$

where A^M_ϵ denotes the set (4.3) and we abbreviated

$$\widetilde{\mu}_M(A^M_\epsilon) \;=\; \nu\left(\frac{d\mu_{\Lambda_M\setminus\Lambda}}{d\nu_{\Lambda_M\setminus\Lambda}}\, 1_{A^M_\epsilon}\right) . \tag{4.9}$$

By (4.5) there exists $c > 0$ such that for M large enough,

$$\nu(A^M_\epsilon) \;\le\; e^{-c\alpha_M} , \tag{4.10}$$

hence, for $0 < \delta < c$, and we can write the following inequalities:

$$\begin{aligned}
\widetilde{\mu}_M(A^M_\epsilon) \;&\le\; \frac{1}{\alpha_M\,\delta}\log\int \exp(\delta\,\alpha_M\, 1_{A^M_\epsilon})\, d\nu \;+\; \frac{1}{\alpha_M\,\delta} H(\tilde{\mu}_M|\nu) \\
&\le\; \frac{1}{\alpha_M\,\delta}\log\left(1 + e^{\alpha_M\,\delta}\,\nu(A^M_\epsilon)\right) \;+\; \frac{1}{\alpha_M\,\delta} H(\tilde{\mu}_M|\nu) \\
&\le\; \frac{1}{\alpha_M\,\delta} e^{\alpha_M(\delta - c)} \;+\; \frac{1}{\alpha_M\,\delta} H(\tilde{\mu}_M|\nu) \,.
\end{aligned} \tag{4.11}$$

By (4.7), the last line tends to zero as $M \to \infty$. By (4.8), and the fact that $\epsilon > 0$ is arbitrary, we conclude that condition (3.5) of proposition (3.4) is satisfied, which implies that $\mu \in \mathcal{G}(\gamma)$. ∎

The result of Theorem 4.1 now follows immediately from the estimates on the Kozlov potential in [14, equation (3.23)] which imply that γ^+ admits $\alpha_M = M$ as ν^+_β-rate of right-continuity for β large enough.

Remark 4.12 *For proposition 4.6 no monotonicity is needed, but in order to verify the rate of θ-continuity one needs some control on a potential for which the measure is weakly Gibbs. For one dimensional projections of low temperature phases of models in the realm of Pirogov-Sinai theory (e.g. Potts model, Ising antiferromagnet) it can be verified that $\alpha_M = M$ is a rate of θ-continuity, where θ is a suitably chosen vacuum (see [13]).*

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