Periods of order-preserving nonexpansive maps on strictly convex normed spaces

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1 Introduction

The purpose of this paper is to present a refinement of a result of Nussbaum [13, Theorem 3.1] that says: if X is the positive cone in $\mathbb{R}^n$ and $f : X \to X$, with $f(0) = 0$, is order-preserving and nonexpansive with respect to a strictly monotone norm, then there exists an integer $p \geq 1$ such that for each $x \in X$ the sequence of iterates $(f^{kp}(x))_k$ converges to a periodic point of f. Moreover p is a divisor of the least common multiple of $\{1, \ldots, n\}$. Later in [14] it has been shown that the possible minimal periods of periodic points of these maps are only periods of admissible arrays on n symbols. Further it was proved in [15] that in case of the 1-norm: $\|x\|_1 = \sum_i |x_i|$, the set of possible minimal periods of periodic points is precisely the set of periods of admissible arrays on n symbols. The question has been raised whether one can determine for other strictly monotone norms the set of possible minimal periods.

In this paper we consider the case where the norm is besides strictly monotone also strictly convex. For instance, classical p-norms, given by $\|x\|_p = (\sum_i |x_i|^p)^{1/p}$, where $1 < p < \infty$, but not the 1-norm or the sup-norm. For these norms we show that the minimal periods of periodic points are orders of permutations on n letters. More precisely, we prove the following theorem.

Theorem 1.1. *Suppose that $X \subset \mathbb{R}^n$ is a closed convex lattice with $0 \in X$. If $f : X \to X$, with $f(0) = 0$, is order-preserving and nonexpansive with respect to a strictly monotone and strictly convex norm, then there exists an integer $p \geq 1$ such that*

(i) for each $x \in X$ the sequence $(f^{kp}(x))_k$ converges to a periodic point of f of minimal period q, where q is a divisor of p;

(ii) p is the order of a permutation on n letters.

Although Theorem 1.1 is a sharpening of a special case of Nussbaum's result, its proof will not depend on it. The proof of Theorem 1.1 is based on the intuitive idea that the iterates of nonexpansive maps $f : X \to X$, with $f(0) = 0$, behave asymptotically like the composition of a nonexpansive projection and an isometry on the range of the projection. Therefore to understand the asymptotic behaviour of the iterates of f one has to study the iterates of the isometry on the range of the projection. In our case we will see that the range of the nonexpansive projection is a closed convex lattice, and that the isometry is a lattice homomorphism preserving convex combinations. These observations lie at the core of the proof of Theorem 1.1.

Related results for nonexpansive maps $f : X \to X$ that are not necessarily order-preserving, have been obtained for different domains $X \subset \mathbb{R}^n$ and different norms on $\mathbb{R}^n$. In particular, it is known that if X is a compact subset of $\mathbb{R}^n$ and $f : X \to X$ is nonexpansive with respect to a norm that has a polyhedral unit sphere, then for each $x \in X$ there exists an integer $p \geq 1$ such that $(f^{kp}(x))_k$ converges to a periodic point of f (see [1] and [18]). Moreover for each polyhedral norm Martus [10] proved that there exists an upper bound for the minimal periods of periodic points that only depends on the dimension of the ambient space. Finding the optimal upper bound for a given polyhedral norm, however, appears to be a difficult combinatorial geometric problem. Some partial results can be found in [2], [5], [9], [11], and [12]. Similar results for maps $f : \mathbb{R}^n \to \mathbb{R}^n$ that are nonexpansive with respect to a classical p-norm where $1 \leq p < \infty$ can be found in [4], [6], [16], and [17].

The remainder of the paper consists of four sections. In Section 2 some basic definitions are collected. Subsequently in Section 3 some results concerning nonexpansive projections and their ranges are proved. In Section 4 several results on isometries and lattice homomorphisms are given. Finally in Section 5 the results from Sections 3 and 4 are combined to prove Theorem 1.1.

2 Basic definitions

On $\mathbb{R}^n$ a partial ordering $\leq$ is defined by $x \leq y$ if $x_i \leq y_i$ for $1 \leq i \leq n$. We write $x < y$ if $x \leq y$ and $x \neq y$. The *positive cone* in $\mathbb{R}^n$ is given by $\mathbb{K}^n = \{x \in \mathbb{R}^n : x \geq 0\}$. A map $f : X \to \mathbb{R}^n$, with $X \subset \mathbb{R}^n$, is called *order-preserving* if $x \leq y$ implies $f(x) \leq f(y)$ for all $x, y \in X$. We say that $f : X \to \mathbb{R}^n$ is *positive* if $f(x) \geq 0$ for all $x \in X$ with $x \geq 0$.

Further for $x, y \in \mathbb{R}^n$ we let $x \wedge y$ denote the greatest lower bound of x and y in $\mathbb{R}^n$, so $(x \wedge y)_i = \min\{x_i, y_i\}$ for $1 \leq i \leq n$. Likewise, $x \vee y$ denotes the least upper bound of x and y in $\mathbb{R}^n$, so $(x \vee y)_i = \max\{x_i, y_i\}$ for $1 \leq i \leq n$. A subset V of $\mathbb{R}^n$ is called a *lattice* if $x \wedge y \in V$ and $x \vee y \in V$ for all $x, y \in V$.

Let $V \subset \mathbb{R}^n$ and $W \subset \mathbb{R}^m$ be lattices. If $f : V \to W$ is such that $f(x \wedge y) = f(x) \wedge f(y)$ and $f(x \vee y) = f(x) \vee f(y)$ for all $x, y \in V$, then f is said to be a *lattice homomorphism*. Note that every lattice homomorphism is order-preserving, because $x \leq y$ implies $x = x \wedge y$, so that $f(x) = f(x \wedge y) = f(x) \wedge f(y)$ and hence $f(x) \leq f(y)$.

A norm $\|\cdot\|$ on $\mathbb{R}^n$ is called *monotone* if $\|x\| \leq \|y\|$ for all $0 \leq x \leq y$. It is said to be *strictly monotone* if $0 \leq x < y$ implies $\|x\| < \|y\|$. A norm $\|\cdot\|$ is called *strictly convex* if its unit sphere contains no line segment. Equivalently, $\|x + y\| = \|x\| + \|y\|$ and $y \neq 0$ implies $x = (\|x\|/\|y\|)y$. We remark that if a norm $\|\cdot\|$ on $\mathbb{R}^n$ is strictly convex and monotone, then it is also strictly monotone. Indeed $0 \le x < y$ implies

$$\|x\| \le \|1/2(x+y)\| \le 1/2(\|x\| + \|y\|) \le \|y\|. \tag{1}$$

Therefore if $\|x\| = \|y\|$, then (1) and the fact that $\|\cdot\|$ is strictly convex give $x = (\|x\|/\|y\|)y$ and hence $x = y$.

A map $f : X \to \mathbb{R}^n$, with $X \subset \mathbb{R}^n$, is called *nonexpansive* with respect to $\|\cdot\|$ if

$$\|f(x) - f(y)\| \le \|x - y\| \quad \text{for all } x, y \in X. \tag{2}$$

In particular, if equality holds in (2) for all $x, y \in X$, then f is said to be an *isometry*. A point $x \in X$ is called a *periodic point* of $f : X \to X$ if there exists an integer $p \ge 1$ such that $f^p(x) = x$. The smallest such p is said to be the *minimal period* of x.

3 Nonexpansive projections and their ranges

A (possibly nonlinear) map $r : X \to X$ is called a *projection* if $r^2(x) = r(x)$ for all $x \in X$. The following lemma, which explains the connection between nonexpansive projections and the iterates of nonexpansive maps, is a nonlinear generalisation of Lemma 3.1 in [7]. It uses the following notation. If $Y \subset \mathbb{R}^n$ is nonempty and $\mathbb{R}^n$ is equipped with a norm $\|\cdot\|$, then $d(x, Y) = \inf\{\|x - y\| : y \in Y\}$ for $x \in \mathbb{R}^n$.

Lemma 3.1. *Let $\|\cdot\|$ be a norm on $\mathbb{R}^n$, and let $X \subset \mathbb{R}^n$ be a closed subset with $0 \in X$. If $f : X \to X$ is nonexpansive (with respect to $\|\cdot\|$) and $f(0) = 0$, then there exists a sequence of integers $(k_i)_i$ with $k_i \to \infty$ such that*

$$r(x) = \lim_{i \to \infty} f^{k_i}(x) \tag{3}$$

exists for each $x \in X$, and the convergence is uniform on bounded subsets of X. Moreover $(k_i)_i$ can be chosen such that the map $r : X \to X$ is a nonexpansive projection and such that the restriction of f to $r(X)$ is a bijective isometry that maps $r(X)$ onto itself, and $d(f^k(x), r(X)) \to 0$ as $k \to \infty$, for each $x \in X$.

The proof of this lemma relies on the following consequence of the Arzela-Ascoli theorem.

Lemma 3.2. *Let $\|\cdot\|$ be a norm on $\mathbb{R}^n$, and let X be a closed subset of $\mathbb{R}^n$ with $0 \in X$. If $f : X \to X$ is nonexpansive and $f(0) = 0$, then every subsequence of $(f^k)_k$ has a convergent subsequence that converges uniformly on bounded subsets to a continuous map $g : X \to X$.*

Proof of Lemma 3.2. For each $N \in \mathbb{N}$ let $X_N = X \cap \{x : \|x\| \le N\}$. Clearly X_N is a compact set containing the origin. As f is nonexpansive and $f(0) = 0$ we have that $(f^k_{|X_N})_k$ is a bounded equicontinuous family of maps. Thus we can apply the Arzela-Ascoli theorem to see that $(f^k_{|X_1})_k$ has a uniformly convergent subsequence $(f^{k_i}_{|X_1})_i$. Likewise $(f^{k_i}_{|X_2})_i$ has a uniformly convergent subsequence. By repeating this argument we obtain a sequence of successive subsequences of $(f^k)_k$ such that the N-th subsequence converges uniformly on X_N.

Clearly any two limit maps coincide on the set where they are both defined. By a diagonal argument we find a subsequence $(f^{m_i})_i$ and a map $g : X \to \mathbb{R}^n$ such that $(f^{m_i})_i$ converges uniformly to g on X_N for each $N \in \mathbb{N}$. As X is closed $g(x) \in X$, so that g maps X into X. $\square$

Let us now prove lemma 3.1.

Proof of Lemma 3.1. From the previous lemma we know that there exists a subsequence $(f^{m_i})_i$ that converges uniformly on bounded subsets of X. By passing to a subsequence, if necessary, we may assume that $m_{i+1} - m_i \to \infty$ as $i \to \infty$. If we apply the previous lemma again to $(f^{m_{i+1}-m_i})_i$ we find a subsequence $(f^{m_{i_j+1}-m_{i_j}})_j$ that converges uniformly on bounded subsets of X. Now define

$$r(x) = \lim_{j\to\infty} f^{m_{i_j+1}-m_{i_j}}(x) \quad \text{for } x \in X. \tag{4}$$

Obviously r is nonexpansive and $r(X) \subset X$.

Before we prove the other assertions we make some auxiliary observations. Let $N > 0$ and define $X_N = X \cap \{x : \|x\| \leq N\}$. Remark that $(f^k(X_N))_k$ is a decreasing sequence of compact sets. Define

$$Y_N = \cap_k f^k(X_N) \quad \text{and} \quad Y = \cup_{N>0} Y_N.$$

Claim 1. If $(x^i)_i$ is a sequence in X_N, $(k_i)_i$ is a sequence in $\mathbb{N}$ with $k_i \to \infty$, and z is such that $(f^{k_i}(x^i))_i \to z$ as $i \to \infty$, then $z \in Y_N$.

Indeed, for each $m \in \mathbb{N}$ we have that $f^{k_i}(x^i) \in f^m(X_N)$ for i sufficiently large, and hence $z \in f^m(X_N)$. Therefore $z \in \cap_m f^m(X_N) = Y_N$.

Claim 2. If $(k_i)_i$ is sequence in $\mathbb{N}$ such that $k_i \to \infty$ and $(f^{k_i}_{|X_N})_i$ converges uniformly on X_N to a map g, then $g(X_N) = Y_N$.

To prove the claim note that $g(X_N) \subset Y_N$ by Claim 1. Now let $y \in Y_N$. Then there exists a sequence $(x^l)_l$ in X_N with $y = f^l(x^l)$ for all $l \in \mathbb{N}$. Since X_N is compact, the sequence $(x^{k_i})_i$ has a convergent subsequence given by $(x^{k_{i_j}})_j$ with limit $u \in X_N$ and

$$\|f^{k_{i_j}}(x^{k_{i_j}}) - g(u)\| \leq \|f^{k_{i_j}}(x^{k_{i_j}}) - g(x^{k_{i_j}})\| + \|g(x^{k_{i_j}}) - g(u)\| \to 0$$

as $i \to \infty$, by uniform convergence. Hence $y = g(u)$ and thus $g(X_N) \supset Y_N$.

Next we show that $f(Y) = Y$. It suffices to show that $f(Y_N) = Y_N$ for all $N > 0$, because $\cup_{N>0} f(Y_N) = f(\cup_{N>0} Y_N)$. As $(f^k(X_N))_k$ is decreasing, we find that $f(Y_N) = f(\cap_k f^k(X_N)) \subset \cap_k f^{k+1}(X_N) = Y_N$. Conversely let $y \in Y_N$. Then there exists a sequence $(x^k)_k$ in X_N with $y = f^k(x^k) = f(f^{k-1}(x^k))_k$ for all $k \geq 1$. The sequence $(f^{k-1}(x^k))_k$ has a convergent subsequence in X_N and according to Claim 1 its limit z is in Y_N. As f is continuous this shows that $y = f(z)$, and hence $Y_N \subset f(Y_N)$.

In order prove that r is a projection onto Y we first note that Claim 1 implies that $r(x) \in Y_N$ for all $x \in X_N$, so that $r(X) \subset Y$. To show that $r(y) = y$ for all $y \in Y$ we let $y \in Y$. Take $N > 0$ such that $y \in Y_N$. Recall that we have chosen $(m_i)_i$ such that $(f^{m_i})_i$ is uniformly convergent on X_N. Let g be its limit. By Claim 2 there exists $x \in X_N$ with $g(x) = y$. Consider the equality

$$f^{m_{i_j+1}}(x) = f^{m_{i_j+1}-m_{i_j}}(f^{m_{i_j}}(x)) \quad \text{for } j \geq 1.$$

Clearly the left hand side converges to $g(x)$. As $f^{m_i}(x) \in X_N$ for each i, and $(f^{m_{i_j+1}-m_{i_j}})_j$ converges uniformly to r on X_N, the right hand side converges to $r(g(x))$. Therefore $r(y) = y$ and hence r is a projection of X onto Y.

To see that f is an isometry on Y we observe that by the definition of r and the nonexpansiveness of f the following inequalities are true for all $x, y \in Y$:

$$\begin{aligned} \|x - y\| &= \|r(x) - r(y)\| = \lim_{j\to\infty} \|f^{m_{i_j+1}-m_{i_j}}(x) - f^{m_{i_j+1}-m_{i_j}}(y)\| \\ &\leq \|f(x) - f(y)\| \leq \|x - y\|. \end{aligned}$$

Finally we show that $d(f^k(x), Y) \to 0$ as $k \to \infty$, for each $x \in X$. Let $x \in X_N$ and $\varepsilon > 0$. Take $M \in \mathbb{N}$ such that

$$\|f^{m_{i_M+1}-m_{i_M}}(x) - r(x)\| < \varepsilon.$$

This implies that for each $k \geq m_{i_M+1} - m_{i_M}$,

$$\begin{aligned} d(f^k(x), Y) &\leq \|f^k(x) - f^{k-(m_{i_M+1}-m_{i_M})}(r(x))\| \\ &\leq \|f^{m_{i_M+1}-m_{i_M}}(x) - r(x)\| < \varepsilon, \end{aligned}$$

and hence the proof of the lemma is complete. □

Remark that if $f : X \to X$ is order-preserving, then the projection r in Lemma 3.1 is by (3) order-preserving. If in addition the domain $X \subset \mathbb{R}^n$ is a lattice and the norm is strictly monotone, then the range of r is a lattice. This is an immediate consequence of the following observation.

Lemma 3.3. *If $X \subset \mathbb{R}^n$ is a lattice, and $f : X \to \mathbb{R}^n$ is order-preserving and nonexpansive with respect to a strictly monotone norm, then $\{z \in X : f(z) = z\}$ is a lattice.*

Proof. Suppose that $x, y \in \{z \in X : f(z) = z\}$. As f is order-preserving we have that $f(x \wedge y) \leq f(x)$ and $f(x \wedge y) \leq f(y)$, and hence $f(x \wedge y) \leq f(x) \wedge f(y) = x \wedge y$. Seeking a contradiction we suppose that $f(x \wedge y) < f(x) \wedge f(y)$. As $\|\cdot\|$ is strictly monotone this yields

$$\|y - x \wedge y\| \geq \|f(y) - f(x \wedge y)\| > \|f(y) - f(x) \wedge f(y)\| = \|y - x \wedge y\|,$$

which is a contradiction.

Similarly, $f(x) \leq f(x \vee y)$ and $f(y) \leq f(x \vee y)$, so that $x \vee y = f(x) \vee f(y) \leq f(x \vee y)$. If equality does not hold, then

$$\|x \vee y - y\| \geq \|f(x \vee y) - f(y)\| > \|f(x) \vee f(y) - f(y)\| = \|x \vee y - y\|,$$

which is again a contradiction. □

Consequently we have the following corollary.

Corollary 3.4. *Let X be a lattice in $\mathbb{R}^n$. If $r : X \to X$ is an order-preserving projection and r is nonexpansive with respect to a strictly monotone norm, then $r(X)$ is a lattice.*

We remark that the range of an order-preserving nonexpansive projection on a lattice need not be a lattice if the norm is not strictly monotone.

It is not difficult to see, as the following lemma shows, that the range of a nonexpansive projection is convex if the norm is strictly convex.

Lemma 3.5. *Let $X \subset \mathbb{R}^n$ be a convex set. If $r : X \to X$ is a projection and r is nonexpansive with respect to a strictly convex norm, then $r(X)$ is convex.*

Proof. Let $x, y \in r(X)$ and $0 \leq \lambda \leq 1$. Observe that $z = \lambda x + (1-\lambda)y$ is the unique element that satisfies $\|x-z\| = (1-\lambda)\|x-y\|$ and $\|y-z\| = \lambda\|x-y\|$, as $\|\cdot\|$ is strictly convex. Since $r(x) = x$, $r(y) = y$, and r is nonexpansive we have that $\|x - r(z)\| \leq (1-\lambda)\|x-y\|$ and $\|y - r(z)\| \leq \lambda\|x-y\|$. Now the triangle inequality yields that these inequalties are equalities, and hence $r(z) = z$. $\square$

The above observations concerning the ranges of the nonexpansive projections motivate the following two lemmas.

Lemma 3.6. *If $Y \subset \mathbb{R}^n$ is a convex lattice and $0 \in Y$, then the linear span of Y is a lattice.*

Proof. Let $R = \{\alpha x - \beta y : \alpha, \beta \geq 0 \text{ and } x, y \in Y\}$. We claim that R is the linear span of Y. To prove the claim it suffices to show that R is a linear subspace, since $Y \subset R \subset \operatorname{span} Y$. Clearly $x \in R$ implies that $\lambda x \in R$ for all $\lambda \in \mathbb{R}$. Further if $x_1, x_2 \in R$, then there exist $\alpha_1, \alpha_2, \beta_1, \beta_2 \geq 0$ and $u_1, u_2, v_1, v_2 \in Y$ such that $x_1 = \alpha_1 u_1 - \beta_1 v_1$ and $x_2 = \alpha_2 u_2 - \beta_2 v_2$. Now put $y = 0$ if $\alpha_1 = 0$ and $\alpha_2 = 0$, and

$$y = \left(\frac{\alpha_1}{\alpha_1+\alpha_2}\right)u_1 + \left(\frac{\alpha_2}{\alpha_1+\alpha_2}\right)u_2$$

otherwise. Likewise let $z = 0$ if $\beta_1 = 0$ and $\beta_2 = 0$, and

$$z = \left(\frac{\beta_1}{\beta_1+\beta_2}\right)v_1 + \left(\frac{\beta_2}{\beta_1+\beta_2}\right)v_2$$

otherwise. As Y is convex and $0 \in Y$ we have that $y, z \in Y$. Therefore

$$x_1 + x_2 = \alpha_1 u_1 + \alpha_2 u_2 - \beta_1 v_1 - \beta_2 v_2 = (\alpha_1+\alpha_2)y - (\beta_1+\beta_2)z,$$

is a member of R.

To prove that R is a lattice we show that $x \vee 0 \in R$ for all $x \in R$. This is sufficient as $y \vee z = y + (z-y) \vee 0$ and $y \wedge z = -(-y \vee -z)$. So let $x \in R$. Then $x = \alpha u - \beta v$ for some $\alpha, \beta \geq 0$ and $u, v \in Y$. Take $M > \alpha, \beta$ and put $a = \alpha u/M$ and $b = \beta v/M$. As $0 \in Y$ we have that a and b are both in Y. This implies that $a \vee b \in Y$, because Y is a lattice. Therefore

$$x \vee 0 = M(a-b) \vee 0 = M((a-b) \vee 0) = M(a \vee b - b) = M(a \vee b) - Mb$$

is in R, and this proves the lemma. $\square$

Subspaces of $\mathbb{R}^n$ that are lattices have a so-called positive block basis. A basis $\{v^1, \ldots, v^k\}$ for a subspace V of $\mathbb{R}^n$ is said to be a *block basis* if $|v^i| \wedge |v^j| = 0$ for all $i \neq j$. If moreover $v^1 \geq 0, \ldots, v^k \geq 0$ it is called a *positive block basis.*

Lemma 3.7 (Section 2.a [8]). *A subspace of $\mathbb{R}^n$ is a lattice if and only if it has a positive block basis.*

4 Isometries and lattice homomorphisms

There exists a relation between order-preserving isometries and lattice homomorphisms as the following proposition shows.

Proposition 4.1. *Let $Y \subset \mathbb{R}^n$ be a lattice with $0 \in Y$. If $f : Y \to Y$, with $f(0) = 0$, is order-preserving and f is an isometry with respect to a monotone norm, then f is a lattice homomorphism.*

Proof. The proof is based on two claims.

Claim 1. For each $x, y \in Y$ there exists a sequence of integers $(k_i)_i$ with $k_i \to \infty$ such that

$$\|f^{k_i}(x) - x\| \to 0, \quad \|f^{k_i}(y) - y\| \to 0,$$
$$\|f^{k_i}(x \wedge y) - x \wedge y\| \to 0, \quad \text{and} \quad \|f^{k_i}(x \vee y) - x \vee y\| \to 0.$$

Indeed since f is an isometry and $f(0) = 0$, the set $\{f^k(z) : k \geq 0\}$ is bounded for each $z \in Y$, so that its closure is compact. Therefore we can find x', y', u, and v in the closure of Y, and a sequence $(m_i)_i$ with $m_{i+1} - m_i \to \infty$ such that

$$\|f^{m_i}(x) - x'\| \to 0, \quad \|f^{m_i}(y) - y'\| \to 0,$$
$$\|f^{m_i}(x \wedge y) - u\| \to 0, \text{ and } \|f^{m_i}(x \vee y) - v\| \to 0.$$

Now put $k_i = m_{i+1} - m_i$ for $i \geq 1$, and observe that as f is an isometry:

$$\|f^{k_i}(x) - x\| = \|f^{m_{i+1}}(x) - f^{m_i}(x)\| \leq \|f^{m_{i+1}}(x) - x'\| + \|f^{m_i}(x) - x'\|$$

for all $i \geq 1$. Therefore $\|f^{k_i}(x) - x\| \to 0$ as $i \to \infty$. The same argument can be used for y, $x \wedge y$ and $x \vee y$, and this proves the claim.

The second claim asserts the following.

Claim 2. $(\|f^k(x) \wedge f^k(y) - f^k(x \wedge y)\|)_k$ and $(\|f^k(x \vee y) - f^k(x) \vee f^k(y)\|)_k$ are increasing sequences for each $x, y \in Y$.

Because f is order-preserving

$$\begin{aligned} f^k(x) \wedge f^k(y) - f^k(x \wedge y) &= f(f^{k-1}(x)) \wedge f(f^{k-1}(y)) - f(f^{k-1}(x \wedge y)) \\ &\geq f(f^{k-1}(x) \wedge f^{k-1}(y)) - f(f^{k-1}(x \wedge y)) \quad \geq 0 \end{aligned}$$

and likewise

$$\begin{aligned} f^k(x \vee y) - f^k(x) \vee f^k(y) &= f(f^{k-1}(x \vee y)) - f(f^{k-1}(x)) \vee f(f^{k-1}(y)) \\ &\geq f(f^{k-1}(x \vee y)) - f(f^{k-1}(x) \vee f^{k-1}(y)) \quad \geq 0, \end{aligned}$$

for all $k \geq 1$. Since f is an isometry and $\|\cdot\|$ is monotone we find that

$$\begin{aligned}\|f^k(x) \wedge f^k(y) - f^k(x \wedge y)\| &\geq \|f(f^{k-1}(x) \wedge f^{k-1}(y)) - f(f^{k-1}(x \wedge y))\| \\ &= \|f^{k-1}(x) \wedge f^{k-1}(y) - f^{k-1}(x \wedge y)\|\end{aligned}$$

and

$$\begin{aligned}\|f^k(x \vee y) - f^k(x) \vee f^k(y)\| &\geq \|f(f^{k-1}(x \vee y)) - f(f^{k-1}(x) \vee f^{k-1}(y))\| \\ &= \|f^{k-1}(x \vee y) - f^{k-1}(x) \vee f^{k-1}(y)\|,\end{aligned}$$

for all $k \geq 1$. This proves the second claim.

Now to prove the proposition let $(k_i)_i$ be the sequence of Claim 1. Then clearly

$$\|f^{k_i}(x) \wedge f^{k_i}(y) - f^{k_i}(x \wedge y)\| \to 0 \quad \text{and} \quad \|f^{k_i}(x \vee y) - f^{k_i}(x) \vee f^{k_i}(y)\| \to 0$$

as $i \to \infty$. Combining this with Claim 2 yields $f(x) \wedge f(y) = f(x \wedge y)$ and $f(x) \vee f(y) = f(x \vee y)$. $\square$

The following lemma shows that an isometry on a convex set with a strictly convex norm always preserves convex combinations. This fact has been observed by Edelstein in [3, Proposition 3].

Lemma 4.2. *If $Y \subset \mathbb{R}^n$ is convex and $f : Y \to Y$ is isometric with respect to a strictly convex norm, then $f(\lambda x + (1-\lambda)y) = \lambda f(x) + (1-\lambda)f(y)$ for each $x, y \in Y$ and $0 \leq \lambda \leq 1$.*

Proof. Let $x, y \in Y$ and $0 \leq \lambda \leq 1$. As $\|\cdot\|$ is strictly convex $z = \lambda x + (1-\lambda)y$ is the unique element that satisfies $\|x-z\| = (1-\lambda)\|x-y\|$ and $\|y-z\| = \lambda\|x-y\|$. Since f is an isometry on Y this implies that $\|f(x)-f(z)\| = (1-\lambda)\|f(x)-f(y)\|$ and $\|f(y) - f(z)\| = \lambda\|f(x) - f(y)\|$. As the norm is strictly convex there is only one element $f(z)$ that satisfies the two equalities simultaneously, namely $f(z) = \lambda f(x) + (1-\lambda)f(y)$. $\square$

This lemma motivates the following result.

Lemma 4.3. *Let $\|\cdot\|$ be a norm on $\mathbb{R}^n$, and let $Y \subset \mathbb{R}^n$ be a convex set with $0 \in Y$. If $f : Y \to Y$ is an isometry and $f(\lambda x+(1-\lambda)y) = \lambda f(x)+(1-\lambda)f(y)$ for all $x, y \in Y$ and $0 \leq \lambda \leq 1$, then there exists a unique linear isometry $F : \operatorname{span} Y \to \operatorname{span} Y$ that extends f. If in addition Y is a lattice and f is a lattice homomorphism, then F is a lattice homomorphism.*

Proof. It is straightforward to show that $f : Y \to Y$ has a unique linear extension $F : \operatorname{span} Y \to \operatorname{span} Y$. To see that F is an isometry let $z \in \operatorname{span} Y$. We know from the proof of Lemma 3.6 that $R = \{\alpha x - \beta y : \alpha, \beta \geq 0 \text{ and } x, y \in Y\}$ equals $\operatorname{span} Y$. Hence there exist $\alpha, \beta \geq 0$ and $x, y \in Y$ such that $z = \alpha x - \beta y$. Let $M > \alpha, \beta$ and put $a = \alpha x/M$ and $b = \beta y/M$. As Y is convex and $0 \in Y$ we know that a and b are both in Y, so that

$$\|F(z)\| = M\|f(a) - f(b)\| = M\|a - b\| = \|z\|.$$

To prove the second assertion we assume in addition that Y is a lattice and $f : Y \to Y$ is a lattice homomorphism. Let $z \in \operatorname{span} Y$. Then in the above notation we have that

$$z \vee 0 = M((a-b) \vee 0) = M(a \vee b - b) = M(a \vee b) - Mb.$$

As F is a linear extension of f, and f is a lattice homomorphism this implies that

$$\begin{aligned} F(z \vee 0) &= Mf(a \vee b) - Mf(b) &&= M(f(a) \vee f(b) - f(b)) \\ &= M(f(a) - f(b)) \vee 0 &&= F(z) \vee 0. \end{aligned}$$

Since F is linear, and $u \vee v = u + (v-u) \vee 0$ and $u \wedge v = -(-u \vee -v)$ for all $u, v \in \mathbb{R}^n$, the proof of the lemma is complete. □

The iterative behaviour of linear isometric lattice homomorphisms is predicted by the following lemma.

Lemma 4.4. *Let $\|\cdot\|$ be a norm on $\mathbb{R}^n$ and let V be a subspace of $\mathbb{R}^n$ that is a lattice. If $F : V \to V$ is a linear isometry and F is a lattice homomorphism, then there exists a basis for V such that the matrix representation of F with respect to this basis is a permutation matrix.*

Proof. According to Lemma 3.7 there exists a basis $\mathcal{V} = \{v^1, \ldots, v^k\}$ for V with $v^i \geq 0$ for all i, and $v^i \wedge v^j = 0$ for all $i \neq j$. Without loss of generality we may assume that $\|v^i\| = 1$ for all i. Observe that if $x = \sum_i \alpha_i v^i$, then $x \geq 0$ if and only if $\alpha_i \geq 0$ for all i. Furthermore if $y = \sum_i \beta_i v^i$, then $x \wedge y = 0$ if and only if $\alpha_i \beta_i = 0$ for all i.

Let A be the matrix representation of F with respect to the basis $\mathcal{V}$. Then the j-th column of A consists of the coordinates of $F(v^j)$ with respect to the basis $\mathcal{V}$. Since F is positive, each column of A is nonnegative, and hence A is a nonnegative matrix. Further observe that for each $i \neq j$ we have that $F(v^i) \wedge F(v^j) = F(v^i \wedge v^j) = F(0) = 0$, as F is a lattice homomorphism. Therefore the columns of A are disjoint, that is to say, each row of A has at most one nonzero entry. As F is a linear isometry, we know that F and A are invertible, so that each row contains exactly one nonzero entry and the nonzero entries of two different rows cannot appear in the same column.

Now let a_{ij} be a nonzero entry of A. Then $F(v^j) = a_{ij} v^i$. The map F is an isometry and $\|v^j\| = \|v^i\| = 1$, so that $|a_{ij}| = 1$. As A is nonnegative we conclude that A is a permutation matrix. □

5 The proof of Theorem 1.1

By combining the results from the previous sections we now prove the main theorem.

Proof of Theorem 1.1. Let r be the nonexpansive projection in Lemma 3.1 and let $r(X)$ be its range. The proof is based on the following claim.

Claim. There exists an integer $p \geq 1$ such that $f^p(\zeta) = \zeta$ for each $\zeta \in r(X)$, and moreover p is the order of a permutation on n letters.

If we assume the claim for a moment we can complete the proof of the theorem as follows. Let p be the integer of the claim and let $x \in X$. It suffices to show that $(f^{kp}(x))_k$ is convergent. As f is nonexpansive and $f(0) = 0$ the sequence $(f^k(x))_k$ is bounded. Therefore it has a convergent subsequence $(f^{k_i}(x))_i$, say with limit $\eta \in X$. Remark that $\eta \in r(X)$, as $d(f^k(x), r(X)) \to 0$ for $k \to \infty$ by Lemma 3.1 and $r(X)$ is closed. Thus, the claim yields that $f^p(\eta) = \eta$.

Now note that there exists an integer j with $0 \leq j < p$ such that $k_i \equiv j \bmod p$ for infinitely many i. Hence there exists a sequence $(m_l)_l$, with $m_l \to \infty$, such that $(f^{j+m_l p}(x))_l$ converges to η. As f is continuous we find that

$$\lim_{l\to\infty} f^{p+m_l p}(x) = f^{p-j}(\eta). \tag{5}$$

Put $\xi = f^{p-j}(\eta)$ and observe that $\xi \in r(X)$, as f maps $r(X)$ onto itself. Then the claim implies that

$$\|f^{kp}(x) - \xi\| = \|f^{kp}(x) - f^p(\xi)\| \leq \|f^{(k-1)p}(x) - \xi\|$$

for all $k \geq 1$, and hence $(\|f^{kp}(x) - \xi\|)_k$ is a decreasing sequence. Using (5) yields that $(f^{kp}(x))_k$ converges to ξ.

To complete the proof of the theorem we prove the claim. It follows from Corollary 3.4 and Lemma 3.5 that $r(X)$ is a convex lattice containing 0. Therefore we can combine Lemma 3.1 and Proposition 4.1 to find that the restriction of f to $r(X)$ is an isometric lattice homomorphism. Moreover it follows from Lemma 4.2 that the restriction of f to $r(X)$ preserves convex combinations.

Now let V be the linear span of $r(X)$ and let $k \leq n$ be the dimension of V. It follows from Lemma 3.6 that V is a lattice. Furthermore by Lemma 4.3 f can be extended to a linear isometric lattice homomorphism $F : V \to V$. By applying Lemma 4.4 we can find a basis $\{v^1, \ldots, v^k\}$ for V and a permutation π on k letters such that

$$F(x) = \sum_{i=1}^{k} \alpha_i v^{\pi(i)}, \quad \text{if } x = \sum_{i=1}^{k} \alpha_i v^i. \tag{6}$$

Now let p be the order of π and observe that (6) implies that $F^p(x) = x$ for all $x \in V$. Since F is an extension of f the proof of the claim is complete. $\square$

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