Diffusion in an annihilating environment*

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March 4, 2003

Abstract. In this paper we study the following system of reaction-diffusion equations:

$$\begin{aligned} \partial\varrho/\partial t &= \Delta\varrho - V\varrho + \lambda\delta_0, \qquad && \varrho(0,x) \equiv 0, \\ \partial V/\partial t &= -\varrho V, && V(0,x) \equiv 1. \end{aligned}$$

Here $\varrho(t,x)$ and $V(t,x)$ are functions of time $t \in [0,\infty)$ and space $x \in \mathbb{R}^d$. This system describes a continuum version of a model in which particles are injected at the origin at rate λ, perform independent simple symmetric random walks on $\mathbb{Z}^d$, and are annihilated at rate 1 by traps located at the sites of $\mathbb{Z}^d$ in such a way that the trap disappears with the particle. This lattice model was studied by a number of authors, who obtained the asymptotic size and shape of the front separating the zone of particles from the zone of traps as well as the asymptotic particle density profile to leading order, in the limit of large time. The continuum model has similar behavior but allows for a more detailed study. As t increases, the particle density $\varrho(t,\cdot)$ inflates and the trap density $V(t,\cdot)$ deflates on a growing ball with radius $R_*(t)$ centered at the origin. We derive the sharp asymptotics of the front position $R_*(t)$, identify the shape of $V(t,\cdot)$ near the surface of the ball, and obtain the limiting profile of $\varrho(t,\cdot)$ inside the ball after appropriate scaling. We also identify the analogues of the total number and the age distribution of particles that are alive. It turns out that the cases $d \geq 3$, $d = 2$, and $d = 1$ exhibit different behavior.

Acknowledgment. J. G. and S. A. M. gratefully acknowledge the hospitality of EURANDOM, where a part of this research was carried out.

Mathematics Subject Classification (2000): 60J70, 60J65

* Partially supported by the DFG-Schwerpunkt 1033

1. Basic equations and elementary observations

1.1. Motivation

We begin by describing the microscopic model that motivates our investigation. Initially, let each site of the lattice $\mathbb{Z}^d$ be occupied by a 'trap'. Suppose that the origin acts as a source that produces 'particles' according to a Poisson stream with rate $\lambda > 0$. Each particle performs a simple symmetric random walk at rate 1, independently of the other particles. If a particle meets a trap, then at rate 1 both the particle and the trap are annihilated. Thus, particles interact with each other indirectly, via the annihilation of traps. In this model, it is of interest to locate the 'front' that separates the zone of particles from the zone of traps, to describe the evolution of the densities of particles and traps both near and away from this front, to derive a macroscopic scaling limit for the particle density, and to identify the total number and the age distribution of particles that are alive.

Lawler, Bramson and Griffeath [13] proved that in $d \geq 3$ the asymptotic shape of the trap free region at time t approaches a ball with radius $R(t) \sim (\lambda t/\omega_d)^{1/d}$ as $t \to \infty$, with ω_d the volume of a unit ball in $\mathbb{R}^d$. Thus, of the λt particles that are born up to time t only $o(t)$ are alive at time t. This comes from the observation that $R(t)$ is much smaller than the diffusive scale $\sqrt{t}$. Gravner and Quastel [8] extended this result to $d = 2$, showing that $R(t) \sim \kappa_* \sqrt{t}$ as $t \to \infty$, with $\kappa_* = \kappa_*(\lambda)$ being the unique solution of the equation $e^{-\kappa^2/4} = (\pi/\lambda)\kappa^2$. Thus, a fraction $0 < 1-(\pi/\lambda)\kappa_*^2 < 1$ of the particles born up to time t is alive at time t, and this leads to a hydrodynamic limit behavior of the particle density that is described by a certain Stefan problem. In $d = 1$ Gravner and Quastel [8] found that $R(t) \sim \sqrt{2t \log t}$ as $t \to \infty$, in which case all except $o(t)$ of the particles born up to time t are alive at time t and the front is pushed outwards by a small group of particles performing a large deviation of order $\sqrt{t \log t} \gg \sqrt{t}$.

Further extensions of these results were obtained for the situation where the injection rate at the origin is time-dependent: $\lambda = \lambda(t)$ (Ben Arous and Ramírez [4]; Quastel [14]; Ben Arous, Quastel and Ramírez [1]). It turns out that there are three regimes – subcritical, critical and supercritical – for which $t^{-d/2} \int_0^t \lambda(s)\, ds \to c$ as $t \to \infty$ with $c = 0$, $c \in (0, \infty)$ and $c = \infty$, respectively, exhibiting different behavior. In the subcritical regime $R(t) \sim (N(t)/\omega_d)^{1/d}$ as $t \to \infty$, with $N(t) = \int_0^t \lambda(s)\, ds$, in the critical regime there is diffusive scaling, while in the supercritical regime the growth is driven by large deviations. The results in the latter regime are still incomplete.

A further interesting question is to find the tail behavior of the survival probability of particles born at a given time. This question was addressed by Ben Arous and Ramírez [2], [3] and again depends on large deviations.

An interesting variant of the model is the one where time is discrete, particles and traps annihilate each other upon first contact, and the next particle is released from the origin only when the previous particle annihilates a trap.

This variant, which is called "internal diffusion limited aggregation", can be viewed as the $\lambda \downarrow 0$ limit of the original model and was introduced by Diaconis and Fulton [7]. Lawler, Bramson and Griffeath [13] proved that in $d \geq 1$ the asymptotic shape of the trap free region is a ball (with volume equal to the number of released particles). Lawler [12] obtained an upper bound on the fluctuations of the trap free region: in $d \geq 2$ the difference between the radius of the 'inner' and the 'outer' ball sandwiching the trap front is at most of order $n^{1/3}(\log n)^{\alpha(d)}$ for a certain $\alpha(d)$ when their radii are of order n. Blachère [5], [6] has brought this upper bound down to order $(\log n)^{\beta(d)}$ for a certain $\beta(d)$.

1.2. Continuum model

Instead of considering the particle vs. trap picture, we will study these problems in terms of a deterministic continuum model consisting of a coupled system of parabolic differential equations for the *particle density* $\varrho = \varrho(t,x)$ and the *trap density* $V = V(t,x)$ in all spatial dimensions $d \geq 1$. More precisely, we are interested in the long-time asymptotics of the following Cauchy problem:

$$\begin{aligned} \frac{\partial \varrho}{\partial t} &= \Delta \varrho - V\varrho + \lambda \delta_0, \qquad && \varrho(0,x) \equiv 0, \\ \frac{\partial V}{\partial t} &= -\varrho V, && V(0,x) \equiv 1. \end{aligned} \tag{1.1}$$

Here $\lambda > 0$ is the intensity of the δ-source at the origin.

System (1.1) has a unique weak solution in the class of functions (ϱ, V) satisfying:

(i) ϱ is continuous on $[0,\infty)\times(\mathbb{R}^d\setminus\{0\})$ and of class $C^{1,2}$ on $(0,\infty)\times(\mathbb{R}^d\setminus\{0\})$;

(ii) V is continuous on $([0,\infty)\times\mathbb{R}^d)\setminus\{(0,0)\}$ and of class $C^{1,1}$ on $(0,\infty)\times(\mathbb{R}^d\setminus\{0\})$;

(iii) $0 \leq \varrho \leq \varrho_0$ and $0 \leq V \leq 1$.

Here ϱ_0 is the free particle density, i.e. the solution of the majorizing heat equation with δ-source,

$$\frac{\partial \varrho_0}{\partial t} = \Delta \varrho_0 + \lambda \delta_0, \qquad \varrho_0(0,x) \equiv 0, \tag{1.2}$$

given by

$$\varrho_0(t,x) = \lambda \int_0^t (4\pi s)^{-d/2} \exp\left\{-|x|^2/4s\right\} ds. \tag{1.3}$$

Since $0 \leq V \leq 1$, a comparison of (1.1) with (1.2) yields

$$e^{-t}\varrho_0(t,x) \leq \varrho(t,x) \leq \varrho_0(t,x) \tag{1.4}$$

for all (t,x). In particular, $\varrho(t,x) \to 0$ as $|x| \to \infty$. Note that, in dimensions $d \geq 2$, the functions ϱ_0 and ϱ have a singularity at $x = 0$ and V is discontinuous at $(t, x) = (0, 0)$, while in dimension $d = 1$ these functions are regular. The total number of particles satisfies

$$\int \varrho(t, x)\, dx \le \int \varrho_0(t, x)\, dx = \lambda t, \qquad t \ge 0.$$

The solution of (1.1) admits the (implicit) Feynman-Kac representation

$$\begin{aligned} \varrho(t, x) &= \lambda \int_0^t ds\ \mathbb{E}_x \exp\left\{- \int_0^s V(t - u, \beta(u))\, du\right\} \delta_0(\beta(s)), \\ V(t, x) &= \exp\left\{- \int_0^t \varrho(s, x)\, ds\right\}, \end{aligned} \tag{1.5}$$

where $(\beta(t), \mathbb{P}_x)$ is Brownian motion with generator Δ starting at x and $\mathbb{E}_x$ denotes expectation with respect to the probability measure $\mathbb{P}_x$. It is obvious from this representation that

$$0 < V(t, x) < 1$$

for $(t, x) \in (0, \infty) \times (\mathbb{R}^d \setminus \{0\})$ (and also for $x = 0$ in dimension $d = 1$). In dimension $d \ge 3$,

$$0 < \varrho(t, x) < \lambda G(x) \tag{1.6}$$

for all $(t, x) \in (0, \infty) \times (\mathbb{R}^d \setminus \{0\})$, where

$$G(x) = \frac{c_d}{|x|^{d-2}} \text{ with } c_d = \frac{\Gamma(\frac{d-2}{2})}{4\pi^{d/2}} \tag{1.7}$$

denotes the Green function associated with the d-dimensional Laplace operator. (The latter function coincides with the integral on the right of the first equation in (1.5) with V replaced by 0 and t replaced by ∞.)

Since problem (1.1) remains invariant under the action of the orthogonal group, its solution is spherically symmetric. With $r = |x|$, we can and will frequently write $\varrho(t, r)$ and $V(t, r)$ instead of $\varrho(t, x)$ and $V(t, x)$, respectively. Hence, we may rewrite (1.1) in the form

$$\begin{aligned} \frac{\partial \varrho}{\partial t} &= \frac{\partial^2 \varrho}{\partial r^2} + \frac{d-1}{r} \frac{\partial \varrho}{\partial r} - V \varrho + \lambda \delta_0^{(d)}, \qquad && \varrho(0, r) \equiv 0, \\ \frac{\partial V}{\partial t} &= -\varrho V, && V(0, r) \equiv 1. \end{aligned} \tag{1.8}$$

where $\delta_0^{(d)}(r) = (d\omega_d r^{d-1})^{-1} \delta_0(r)$ and $\delta_0(r)$ are the 'd-dimensional' and the usual δ-function on $\mathbb{R}_+$, respectively. Here and in the following,

$$\omega_d = \frac{\pi^{d/2}}{\Gamma(\frac{d+2}{2})} \tag{1.9}$$

denotes the volume of a unit ball in $\mathbb{R}^d$. Note that

$$(d - 2) d c_d \omega_d = 1, \qquad d \ge 3. \tag{1.10}$$

1.3. Two key lemmas

We next state some simple monotonicity properties of ϱ and V. These will be used frequently throughout the paper.

Lemma 1.1. *(Monotonicity properties)*

a) The function $\varrho(t,r)$ is strictly increasing in t and strictly decreasing in r $(t,r>0)$.

b) The function $V(t,r)$ is strictly decreasing in t and strictly increasing in r $(t,r>0)$.

Proof. It is obvious from (1.5) that $V(t,r)$ is strictly decreasing in t and, consequently, $\varrho(t,r)$ is strictly increasing in t. As mentioned earlier, $\varrho(t,r)\to 0$ as $r\to\infty$ for all $t>0$. Suppose that $\varrho(t,r)$ is not strictly decreasing in r for some $t>0$. Then there exists an $r_0>0$ such that $(\partial/\partial r)\varrho(t,r_0)=0$ and $(\partial^2/\partial r^2)\varrho(t,r_0)\le 0$. We already know that $(\partial/\partial t)\varrho(t,r_0)\ge 0$. But this contradicts the first equation in (1.8) at the point (t,r_0). Hence, $\varrho(t,r)$ is strictly decreasing in r. Consequently, the second equation in (1.5) tells us that $V(t,r)$ is strictly increasing in r. $\square$

Let $W=1-V$ denote the density of traps annihilated up to time t. We next state a *conservation law* that turns out to be crucial for the whole paper.

Lemma 1.2. *(Conservation law)*

$$\int \varrho(t,x)\,dx+\int W(t,x)\,dx=\lambda t,\qquad t>0. \tag{1.11}$$

Proof. Rewriting the second equation in (1.1) in terms of W and adding it to the first equation, we find that

$$\frac{\partial}{\partial t}(\varrho+W)=\Delta\varrho+\lambda\delta_0. \tag{1.12}$$

Integrating over time and space, we arrive at the desired assertion. $\square$

Relation (1.11) says that the number of particles alive at time t plus the number of traps annihilated up to t equals the total number of particles born up to this time.

We also need a probabilistic representation for the total number of particles alive. After time-reversal of Brownian motion in the first equation of (1.5) and integration over x, we find that

$$\int \varrho(t,x)\,dx=\lambda\int_0^t ds\,\mathbb{E}_0\exp\left\{-\int_0^s V(t-s+u,\beta(u))\,du\right\}. \tag{1.13}$$

1.4. Outline of the paper

For the deterministic continuum model described by (1.1) we will recover essentially all the asymptotic results that were obtained for the random discrete model as summarized in Section 1.1, but we will be able to push further. In Section 2 we state the scaling limit of the particle and trap densities (Theorem 2.1), the asymptotics of the total number of particles that are alive (Theorem 2.2), and the sharp asymptotics of the trap front position (Theorem 2.3). In Section 3 we find the limiting profile of the trap density near the trap front (Theorem 3.2). In Section 4 we prove the theorems stated in Section 2. In Section 5 we identify the age distribution of particles alive (Theorem 5.1). In an Appendix we recall some properties of the Green function.

Throughout this paper we stick to the case where the source at the origin has a constant rate λ, but we believe that our arguments are flexible enough to allow for an extension to a finite number of localized sources with a time-dependent creation rate $\lambda(t)$.

There is a substantial literature on the long-time behavior of reaction-diffusion systems. However, most of the literature on equations of type (1.1) (see e.g. Hilhorst et al. [9], [10], [11] and references therein) deals with equations without the δ-source, which plays a central role in our results.

2. Scaling limit, total number of particles alive, and trap front position

This section contains part of our main results: scaling limit of the particle and trap densities (Section 2.1), asymptotics of the number of particles alive (Section 2.2), and sharp asymptotics of the trap front position (Section 2.3). Proofs of the results are deferred to Section 4.

2.1. Scaling limit

A natural approach to study the long-time behavior of (ϱ, V) is to derive an appropriate scaling limit. To this end we introduce a small scaling parameter $\varepsilon > 0$ and consider the rescaled functions

$$\begin{aligned} \varrho_\varepsilon(t,x) &= \frac{1}{\varepsilon^{d-2}}\,\varrho\left(\frac{t}{\varepsilon^d}, \frac{x}{\varepsilon}\right), \\ V_\varepsilon(t,x) &= V\left(\frac{t}{\varepsilon^d}, \frac{x}{\varepsilon}\right). \end{aligned} \tag{2.1}$$

Then (1.1) is equivalent to the following Cauchy problem for $(\varrho_\varepsilon, V_\varepsilon)$:

$$\begin{aligned} \varepsilon^{d-2}\frac{\partial \varrho_\varepsilon}{\partial t} &= \Delta\varrho_\varepsilon - \frac{1}{\varepsilon^2} V_\varepsilon \varrho_\varepsilon + \lambda\delta_0, && \varrho_\varepsilon(0,x) \equiv 0, \\ \frac{\partial V_\varepsilon}{\partial t} &= -\frac{1}{\varepsilon^2}\varrho_\varepsilon V_\varepsilon, && V_\varepsilon(0,x) \equiv 1. \end{aligned} \tag{2.2}$$

In dimension $d = 2$ we have diffusive scaling of space and time, and system (2.2) is ‘almost’ scaling invariant (modulo the prefactor ε^{-2} in front of the annihilation term). Thus $d = 2$ is the ‘critical’ and therefore most interesting case. For dimensions $d \geq 3$, the first equation in (2.2) turns into an elliptic (i.e., time-homogeneous) equation as $\varepsilon \downarrow 0$, and the scaling limit turns out to be closely related to the Green function for the Laplace operator. As we will see later, in dimension $d = 1$ the situation is totally different, with no natural scaling for the pair (ϱ, V).

We next introduce some notation. We define the open sets

$$D_d^0 = \left\{(t, x) \in (0, \infty) \times \mathbb{R}^d \colon |x| < R_d(t)\right\},$$
$$D_d^1 = \left\{(t, x) \in (0, \infty) \times \mathbb{R}^d \colon |x| > R_d(t)\right\},$$

where

$$R_d(t) = \begin{cases} \kappa_*(\lambda)\sqrt{t}, & \text{if } d = 2, \\ (\lambda t/\omega_d)^{1/d}, & \text{if } d \geq 3, \end{cases} \tag{2.3}$$

and $\kappa_* = \kappa_*(\lambda) \in (0, (\lambda/\omega_2)^{1/2})$ is the unique positive solution of the equation

$$\lambda e^{-\kappa_*^2/4} = \omega_2 \kappa_*^2. \tag{2.4}$$

Clearly, $\omega_2 = \pi$. For $d \geq 3$, $R_d(t)$ is the radius of a ball of volume λt, whereas for $d = 2$ the volume of the ball is smaller.

If $d \geq 3$, then we consider the Green function for the Laplacian in the unit ball with zero boundary condition given by

$$G^0(x) = \begin{cases} c_d \left(\dfrac{1}{|x|^{d-2}} - 1\right), & 0 < |x| < 1, \\ 0, & |x| \geq 1. \end{cases} \tag{2.5}$$

If $d = 2$, then we denote by ϱ^* the unique weak solution to the boundary value problem

$$\begin{aligned} \frac{\partial \varrho^*}{\partial t}(t, x) &= \Delta \varrho^*(t, x) + \lambda \delta_0(x), && |x| < R_2(t), \\ \varrho^*(t, x) &= 0, && |x| = R_2(t). \end{aligned} \tag{2.6}$$

Equation (2.6) admits the explicit solution

$$\varrho^*(t, x) = \frac{\lambda}{2\pi} \int_{|x|/\sqrt{t}}^{\kappa_*} e^{-u^2/4} \frac{du}{u}, \qquad |x| \leq R_2(t). \tag{2.7}$$

We set $\varrho^*(t, x) = 0$ for $|x| > R_2(t)$.

One of our main results is the following.

Theorem 2.1. *(Scaling limit)*

a) For $d \geq 2$,

$$\lim_{\varepsilon \downarrow 0} V_\varepsilon(t, x) = \begin{cases} 0, & (t, x) \in D_d^0, \\ 1, & (t, x) \in D_d^1, \end{cases} \tag{2.8}$$

uniformly on compact subsets of $D_d^0 \cup D_d^1$.

b) For $d \geq 3$,

$$\lim_{\varepsilon\downarrow 0} \varrho_\varepsilon(t,x) = \frac{\lambda}{R_d(t)^{d-2}}\, G^0\left(\frac{x}{R_d(t)}\right), \qquad (t,x) \in (0,\infty)\times\left(\mathbb{R}^d\setminus\{0\}\right), \quad (2.9)$$

uniformly on compact subsets.

c) For $d = 2$,

$$\lim_{\varepsilon\downarrow 0} \varrho_\varepsilon(t,x) = \varrho^*(t,x), \qquad (t,x) \in (0,\infty)\times\left(\mathbb{R}^2\setminus\{0\}\right), \quad (2.10)$$

uniformly on compact subsets.

d) For $d = 1$,

$$\lim_{\varepsilon\downarrow 0} \varepsilon\varrho\left(\frac{t}{\varepsilon^2},\frac{x}{\varepsilon}\right) = \varrho_0(t,x), \qquad (t,x) \in (0,\infty)\times\mathbb{R}^1, \quad (2.11)$$

uniformly on compact sets, with ϱ_0 *being the free particle density given by (1.3).*

Note that, modulo the factor λ, the expression on the right of (2.9) coincides with the Green function for the d-dimensional Laplacian in the centered ball of radius $R_d(t)$ with zero boundary condition. Assertion d) is the diffusive scaling limit in which the particle density does not feel the annihilation by traps. Indeed, in dimension $d = 1$ annihilation occurs at a (scaled) distance of order $|x| = \varepsilon\sqrt{(t/\varepsilon^2)\log(t/\varepsilon^2)} \to \infty$ as $\varepsilon \downarrow 0$ (cf. Theorem 2.3 below).

The proof of Theorem 2.1 will be given in Section 4, separately for the cases $d \geq 3$, $d = 2$, and $d = 1$.

2.2. Total number of particles alive

The following statement is almost immediate from Theorem 2.1.

Theorem 2.2. *(Total number of particles alive) As* $t \to \infty$,

$$\int \varrho(t,x)\,dx \begin{cases} \sim \frac{\lambda}{2d}\left(\frac{\lambda t}{\omega_d}\right)^{2/d} & \text{for } d \geq 3,\\ \sim (1-e^{-\kappa_*^2/4})\,\lambda t & \text{for } d = 2,\\ = \lambda t - \sqrt{8t\log t}\,(1+o(1)) & \text{for } d = 1. \end{cases} \quad (2.12)$$

This again shows the different behavior in different dimensions. In dimensions $d \geq 3$, only an asymptotically vanishing fraction of order $O(t^{-(d-2)/d})$ of the total amount λt of particles born up to time t is alive at time t. In dimension $d = 2$, this fraction tends to $1 - \exp\{-\kappa_*^2/4\} \in (0,1)$, whereas in dimension $d = 1$ it tends to 1 as $t \to \infty$.

The proof of Theorem 2.2 will be carried out in Section 4, separately for $d \geq 3$, $d = 2$, and $d = 1$.

2.3. Sharp asymptotics of the trap front position

For each $t > 0$, we define the *position of the trap front* at time t as the unique positive radius $R_*(t)$ for which

$$\int_{|x|<R_*(t)} V(t,x)\,dx = \int_{|x|>R_*(t)} [1 - V(t,x)]\,dx. \tag{2.13}$$

I.e., the amount of traps remaining inside the centered ball of radius $R_*(t)$ is in balance with the amount of traps annihilated outside this ball.

Recall that $\kappa_* = \kappa_*(\lambda)$ is given by (2.4). One of our main results is the following.

Theorem 2.3. *(Sharp asymptotics of trap front position) As $t \to \infty$,*

$$R_*(t) \begin{cases} = \left(\frac{\lambda t}{\omega_d}\right)^{1/d} - \frac{\lambda}{2d^2\omega_d}\left(\frac{\lambda t}{\omega_d}\right)^{-(d-3)/d}(1+o(1)) & \text{for } d \geq 3, \\ \sim \kappa_* \sqrt{t} & \text{for } d = 2, \\ \sim \sqrt{2t\log t} & \text{for } d = 1. \end{cases}$$

Note that the shift by $-\lambda/(18\omega_3)$ in dimension $d = 3$ is due to the fact that the number of particles $\int \varrho(t,x)\,dx$ alive at time t has the same order as the surface of a ball with radius $R_*(t)$, whereas in dimension $d \geq 4$ it is asymptotically negligible in this respect. Due to the more subtle mechanism of front propagation, for $d = 2$ and $d = 1$ our result is not as sharp as for $d \geq 3$.

The proof of Theorem 2.3 will be given in Section 4, separately for $d \geq 3$, $d = 2$, and $d = 1$.

3. Limiting profile of the trap density near the trap front

This section contains one more main result: the limiting profile of the trap density near the trap front. After some preliminaries (Section 3.1), we formulate the precise statement and a key lemma (Section 3.2), and give proofs (Section 3.3).

3.1. Preliminaries

In (2.13) we introduced the position $R_*(t)$ of the trap front at time t. We begin this section with an alternative definition of the location of this front. To this end we fix $h \in (0,1)$ arbitrarily. We will see below that the equation

$$V\left(t, R_h(t)\right) = h \tag{3.1}$$

admits a unique solution $R_h(t) > 0$ for all $t > t_0(h)$, with $t_0(h) = 0$ for $d \geq 2$ and $t_0(h)$ the solution of $V(t_0(h), 0) = h$ for $d = 1$ (recall that ϱ has a singularity at the origin for $d \geq 2$). Clearly, R_h is the front separating the domain $\{V <$

$h\}$ from the domain $\{V > h\}$. We therefore call R_h the *h-front*. Later, in Theorem 3.2 c), we will see how R_h and R_* are related to each other.

Existence and uniqueness of $R_h(t)$ follow from the observation that $V(t,r)$ is continuous and strictly increasing in r (Lemma 1.1), $V(t,r) \to 1$ as $r \to \infty$, and $V(t,0) = 0$ for $d \geq 2$, and $V(t,0) \in (0,1)$, $V(t,0) \to 0$ as $t \to \infty$ for $d = 1$. Since $\varrho(t,r) \to 0$ as $r \to \infty$ and $\varrho(t,r) \to c(t)$ as $r \to 0$, with $c(t) = \infty$ for $d \geq 2$ and increasing $c(t) \in (0,\infty)$ for $d = 1$ (see (1.4)), the last properties follow from the second formula in (1.5).

Note also that $R_h(t)$ is strictly increasing and continuous in t and $R_h(t) \to \infty$ as $t \to \infty$. This is a consequence of the monotonicity and continuity of $V(t,r)$ in t and r and the fact that $V(t,r) \to 0$ as $t \to \infty$.

In Section 3.2 we will show that the profile of the trap density $V(t,\cdot\,)$ in polar coordinates around the trap front position $R_*(t)$ approaches a non-degenerate limiting profile v_* as $t \to \infty$. It will turn out that this profile is of the form

$$v_*(r) = e^{-\eta_*(r)}, \qquad r \in \mathbb{R}, \tag{3.2}$$

where η_* is the unique strictly decreasing positive solution η of the equation

$$\eta'' = 1 - e^{-\eta} \qquad \text{on } \mathbb{R} \tag{3.3}$$

for which

$$\int_{-\infty}^{0} v_*(r)\,dr = \int_0^\infty \left[1 - v_*(r)\right]\,dr. \tag{3.4}$$

Note that this balance condition corresponds to the balance condition (2.13) for the original trap density. Existence and uniqueness of v_*, respectively, η_* as well as some additional properties of these functions are established in the next lemma.

Lemma 3.1. *(Profile properties)*

a) Equation (3.3) admits a strictly decreasing positive solution η, which is unique modulo shifts and strictly convex. Each such solution η satisfies

$$\eta(r) \sim \frac{r^2}{2} \qquad \textit{as } r \to -\infty, \tag{3.5}$$

$$\log \eta(r) \sim -r \qquad \textit{as } r \to +\infty. \tag{3.6}$$

b) Among the above solutions there is exactly one solution η_* for which the function v_* defined by (3.2) satisfies the balance condition (3.4). Moreover,

$$\eta_*(r) = \frac{r^2}{2} + 1 + O\left(e^{-r^2/2}\right) \qquad \textit{as } r \to -\infty, \tag{3.7}$$

$$\eta_*(r) = C_* e^{-r} + O\left(e^{-2r}\right) \qquad \textit{as } r \to +\infty, \tag{3.8}$$

where C_* denotes a positive constant. Consequently, $v_* \colon \mathbb{R} \to (0,1)$ is strictly increasing and satisfies

$$v_*(r) = e^{-r^2/2-1} + O\left(e^{-r^2}\right) \qquad \textit{as } r \to -\infty, \tag{3.9}$$

$$v_*(r) = 1 - C_* e^{-r} + O\left(e^{-2r}\right) \qquad \textit{as } r \to +\infty. \tag{3.10}$$

The proof of this lemma will be given in Section 3.3.

3.2. Limiting profile

For each $h \in (0,1)$, there exists a unique $r_h \in \mathbb{R}$ such that

$$\eta_*(r_h) = \log \frac{1}{h}, \quad \text{i.e.,} \quad v_*(r_h) = h.$$

Let

$$\eta_h(r) = \eta_*(r_h + r) \quad \text{and} \quad v_h(r) = v_*(r_h + r), \qquad r \in \mathbb{R}, \tag{3.11}$$

denote the correspondingly shifted versions of η_* and v_*, respectively. Recall that the position $R_h(t)$ of the h-front is defined by (3.1). For convenience, we set $V(t,r) = 0$ for $r < 0$.

We are now in a position to formulate our main result about the asymptotic profile of the trap density.

Theorem 3.2. *(Limiting profile of trap density near trap front)*
a) For each $h \in (0,1)$ and all $t > t_0(h)$,

$$0 < V(t, R_h(t) + r) < v_h(r) \qquad \text{for } -R_h(t) < r < 0, \tag{3.12}$$

$$v_h(r) < V(t, R_h(t) + r) < 1 \qquad \text{for } r > 0. \tag{3.13}$$

b) For each $h \in (0,1)$,

$$\lim_{t\to\infty} V(t, R_h(t) + r) = v_h(r) \qquad \text{uniformly in } r \in \mathbb{R}. \tag{3.14}$$

c) For each $h \in (0,1)$,

$$\lim_{t\to\infty} [R_h(t) - R_*(t)] = r_h. \tag{3.15}$$

d) Finally,

$$\lim_{t\to\infty} V(t, R_*(t) + r) = v_*(r) \qquad \text{uniformly in } r \in \mathbb{R}.$$

Note that the limiting profile v_* is the same in all dimensions. In the proof of parts b)–d) of Theorem 3.2 we will need the following lemma.

Lemma 3.3. *For any $d \geq 1$ and $h \in (0,1)$,*

$$\lim_{t\to\infty} \varrho(t, R_h(t)) = 0. \tag{3.16}$$

Proof. In dimensions $d \geq 3$, assertion (3.16) follows from the trivial bound $\varrho(t, R_h(t)) \leq \lambda G(R_h(t)) \to 0$ as $t \to \infty$. In dimensions $d = 2$ and $d = 1$ we need to know the asymptotic behavior of $R_h(t)$ up to equivalence to derive (3.16). This and the proof of (3.16) will be carried out in Sections 4.2 and 4.3 with the help of assertion a) but none of the assertions b)–d) of Theorem 3.2. $\square$

3.3. Proofs of Lemma 3.1 and Theorem 3.2

In the remainder of this section we will first prove Lemma 3.1 and afterwards present the proof of Theorem 3.2.

Proof of Lemma 3.1. a) The system of first order differential equations for (η, η') corresponding to equation (3.3) has a saddle point at $(0,0)$ with the vector $(1, -1)$ being tangential to the stable curve (manifold) through the origin. Hence, there exists a solution η of (3.3) with $\eta(r) > 0$ and $\eta'(r) < 0$ for large r and $\eta(r), \eta'(r) \to 0$ as $r \to +\infty$ in such a way that

$$\frac{\eta'(r)}{\eta(r)} \longrightarrow -1 \qquad \text{as } r \to +\infty. \tag{3.17}$$

This solution is positive everywhere. Otherwise there would exist an $r_0 \in \mathbb{R}$ such that $\eta(r_0) = 0$ and $\eta(r) > 0$ for $r > r_0$. But, since $\eta(r) \to 0$ as $r \to +\infty$, the function η could not be convex on (r_0, ∞) in contradiction with (3.3). Now, as a consequence of positivity and equation (3.3), η is strictly convex and therefore also strictly decreasing on $\mathbb{R}$. In this way we have shown the existence of a solution of (3.3) with the desired properties. Clearly, each spatial shift of such a solution is again such a solution.

It is obvious from the above convexity argument that each strictly decreasing positive solution η of (3.3) satisfies $\eta(r) \to \infty$ as $r \to -\infty$ and, moreover, $(\eta(r), \eta'(r)) \to (0,0)$ as $r \to +\infty$. Since the system (η, η') has only one stable curve entering $(0,0)$ through the quadrant $(0,\infty) \times (-\infty, 0)$, we obtain uniqueness modulo shifts.

Now let η be any strictly decreasing positive solution of (3.3). Then

$$\eta'(r) = \eta'(0) - \int_r^0 \left(1 - e^{-\eta(u)}\right) du.$$

Since $\eta(u) \to \infty$ as $u \to -\infty$, the integral on the right is asymptotically equivalent to $-r$ as $r \to -\infty$. Hence, $\eta'(r) \sim r$ as $r \to -\infty$. This implies assertion (3.5). Assertion (3.6) follows from (3.17).

b) The existence and uniqueness of η_* are obvious from assertion a). It remains to prove the asymptotic formulas (3.7) and (3.8), from which (3.9) and (3.10) are immediate.

It follows from equation (3.3), definition (3.2), and balance condition (3.4) that

$$\begin{aligned}\eta_*'(r) &= -\int_r^\infty \left(1 - e^{-\eta_*(u)}\right) du \\ &= -\int_r^0 \left(1 - e^{-\eta_*(u)}\right) du - \int_{-\infty}^0 e^{-\eta_*(u)}\, du \\ &= r - \int_{-\infty}^r e^{-\eta_*(u)}\, du. \end{aligned} \tag{3.18}$$

Hence, taking into account (3.5), we see that

$$\eta_*'(r) = r - O\left(e^{-r^2/4}\right) \qquad \text{as } r \to -\infty. \tag{3.19}$$

Multiplying both sides of (3.3) with η' and integrating over the interval (r, ∞), we get, after rearranging the individual terms,

$$\eta_*(r) = \frac{1}{2}\eta_*'(r)^2 + 1 - e^{-\eta_*(r)}. \tag{3.20}$$

After substituting for $\eta_*'(r)$ and $\eta_*(r)$ on the right the asymptotic expressions (3.19) and (3.5), respectively, we find that

$$\eta_*(r) = \frac{r^2}{2} + 1 + o(1) \qquad \text{as } r \to \ -\infty. \tag{3.21}$$

Now we may substitute (3.21) into (3.18) and repeat the above arguments, to arrive at assertion (3.7).

It follows from (3.20) by a Taylor expansion that

$$\eta_*'(r)^2 = \eta_*(r)^2 + O\left(\eta_*(r)^3\right) \qquad \text{as } r \to +\infty.$$

Thus,

$$\frac{\eta_*'(r)}{\eta_*(r)} = -1 + O\left(\eta_*(r)\right) \qquad \text{as } r \to +\infty. \tag{3.22}$$

Hence, taking into account (3.6), we get

$$(\log \eta_*(r))' = -1 + O\left(e^{-r/2}\right) \qquad \text{as } r \to +\infty. \tag{3.23}$$

By integrating over $(0, r)$, we conclude from this that

$$\eta_*(r) = C_* e^{-r - O(e^{-r/2})} \qquad \text{as } r \to +\infty \tag{3.24}$$

for some positive constant C_*. After substituting this into the expression on the right of (3.22), we see that in (3.23), and therefore also in (3.24), the term $O(e^{-r/2})$ may be replaced by $O(e^{-r})$. In this way we arrive at assertion (3.8). $\square$

Proof of Theorem 3.2. Fix $h \in (0,1)$ arbitrarily and define a function $\tilde\eta$ via

$$V(t,r) = e^{-\tilde\eta(t,r)}, \qquad (t,r) \in (0,\infty)^2. \tag{3.25}$$

Comparing this with the second equation in (1.5), we find that

$$\tilde\eta(t,r) = \int_0^t \varrho(s,r)\, ds. \tag{3.26}$$

Now rewrite (1.12) in polar coordinates,

$$\frac{\partial}{\partial t}(\varrho + W) = \frac{\partial^2 \varrho}{\partial r^2} + \frac{d-1}{r}\frac{\partial \varrho}{\partial r} + \lambda\delta_0^{(d)},$$

integrate both sides over the time interval $(0,t)$, remember that $W = 1 - V$, and use (3.25) and (3.26), to arrive, for each $t > 0$, at the differential equation

$$\frac{\partial^2 \tilde\eta}{\partial r^2} + \frac{d-1}{r}\frac{\partial \tilde\eta}{\partial r} = 1 - e^{-\tilde\eta} + \varrho, \qquad r \in (0,\infty). \tag{3.27}$$

It is remarkable that (3.27) is not a parabolic equation but, for each t, a (non-linear) *elliptic* equation given ϱ.

We compare $\tilde\eta$ with the function η_h given by

$$\eta_h(t,r) = \eta_h\left(r - R_h(t)\right), \qquad (t,r) \in (t_0(h),\infty) \times (0,\infty).$$

For each $t > t_0(h)$, the latter satisfies the equation

$$\frac{\partial^2 \eta_h}{\partial r^2} = 1 - e^{-\eta_h}. \tag{3.28}$$

Moreover,

$$\eta_h(t, R_h(t)) = \tilde\eta(t, R_h(t)) = \log\frac{1}{h}. \tag{3.29}$$

Hence, for each $t > t_0(h)$, the difference $\zeta(t,\cdot\,) = \eta_h(t,\cdot\,) - \tilde\eta(t,\cdot\,)$ satisfies

$$\begin{aligned} &\frac{\partial^2 \zeta}{\partial r^2}(t,r) + \frac{d-1}{r}\frac{\partial \zeta}{\partial r}(t,r) - \varphi(t,r)\zeta(t,r) = -\varrho(t,r) + \frac{d-1}{r}\frac{\partial \eta_h}{\partial r}(t,r), \\ &\hspace{22em} r \in (R_h(t),\infty), \\ &\zeta(t, R_h(t)) = 0, \end{aligned} \tag{3.30}$$

where the potential φ has the form

$$\varphi(t,r) = \frac{e^{-\tilde\eta(t,r)} - e^{-\eta_h(t,r)}}{\eta_h(t,r) - \tilde\eta(t,r)}$$

if $\tilde\eta(t,r) \neq \eta_h(t,r)$ and $\varphi(t,r) = e^{-\eta_h(t,r)}$ if $\tilde\eta(t,r) = \eta_h(t,r)$. This is an elliptic boundary value problem in the exterior of the centered ball of radius $R_h(t)$ with zero boundary condition. By (3.29) and monotonicity, $\eta_h(t,r), \tilde\eta(t,r) < \log(1/h)$

for $r > R_h(t)$ and, consequently, $\varphi(t,r) > h$ for $r > R_h(t)$. Moreover, $\zeta(t,r) \to 0$ as $r \to +\infty$. Note also that the expression on the right of (3.30) is negative. We may therefore apply the maximum principle to this Dirichlet problem, to find that

$$\begin{aligned} 0 < \zeta(t,r) &< \frac{1}{h} \max_{q \geq R_h(t)} \left[\varrho(t,q) - \frac{d-1}{q} \frac{\partial \eta_h}{\partial r}(t,q) \right] \\ &= \frac{1}{h} \left[\varrho(t, R_h(t)) - \frac{d-1}{R_h(t)} \eta_h'(0) \right] \end{aligned} \tag{3.31}$$

for each $t > t_0(h)$ and all $r > R_h(t)$. Here we have used that $\varrho(t,r)$ and the function $-(\partial/\partial r)\eta_h(t,r) = -\eta_h'(r - R_h(t))$ are positive and decreasing in r.

a) The left part of (3.31) proves inequality (3.13). To prove inequality (3.12), fix $h \in (0,1)$, $t > t_0(h)$, and $-R_h(t) < r_0 < 0$ arbitrarily. Define $h_0 = V(t, R_h(t) + r_0)$. Clearly $h_0 < h$, $t > t_0(h_0)$, and $R_{h_0}(t) = R_h(t) + r_0$. Now, applying inequality (3.13) for h_0 instead of h, we see that

$$v_{h_0}(-r_0) < V(t, R_{h_0}(t) - r_0) = V(t, R_h(t)) = v_h(0).$$

But, since the functions $v_{h_0}(-r_0 + \cdot\,)$ and $v_h(\cdot)$ are strictly increasing and shifts of each other, they do not intersect. Hence, the graph of the first lies below the graph of the second. In particular,

$$V(t, R_h(t) + r_0) = v_{h_0}(0) < v_h(r_0),$$

which is the desired estimate.

b) Assertion (3.16) of Lemma 3.3 together with $R_h(t) \to \infty$ implies that the expression on the right of (3.31) tends to zero as $t \to \infty$. From this we conclude that

$$\lim_{t\to\infty} \sup_{r \geq 0} |V(t, R_h(t) + r) - v_h(r)| = 0 \tag{3.32}$$

for each $h \in (0,1)$. Fix $H \in (0,1)$ arbitrarily. We are now going to prove (3.14) with h replaced by H. To this end, choose $h \in (0,H)$ arbitrarily small and rewrite (3.32) in the form

$$\begin{aligned} &\lim_{t\to\infty} \sup_{r \geq -(R_H(t) - R_h(t))} |V(t, R_H(t) + r) - v_H(r + R_H(t) - R_h(t) - (r_H - r_h))| \\ &= 0. \end{aligned} \tag{3.33}$$

In particular, since $R_H(t) > R_h(t)$ and $V(t, R_H(t)) = v_H(0)$ for all t, we have that $v_H(R_H(t) - R_h(t) - (r_H - r_h)) \to v_H(0)$ as $t \to \infty$ and, consequently,

$$\lim_{t\to\infty} [R_H(t) - R_h(t)] = r_H - r_h. \tag{3.34}$$

We therefore conclude from (3.33) that

$$\lim_{t\to\infty} \sup_{r \geq -(r_H - r_h) + \delta} |V(t, R_H(t) + r) - v_H(r)| = 0$$

for each $\delta > 0$. Since $r_h \to -\infty$ as $h \downarrow 0$ and because of the additional bound (3.12) (with h replaced by H), this yields the desired uniform convergence.

c) Recall that $v_h(\cdot) = v_*(r_h + \cdot)$. For $h_* = v_*(0)$ we have $r_{h_*} = 0$ and $v_{h_*} = v_*$. An application of the assertions a) and b) shows that

$$\lim_{t\to\infty} \frac{1}{R_h(t)^{d-1}} \int_{|x|<R_h(t)} V(t,x)\,dx = d\omega_d \int_{-\infty}^{r_h} v_*(r)\,dr \tag{3.35}$$

and

$$\lim_{t\to\infty} \frac{1}{R_h(t)^{d-1}} \int_{|x|>R_h(t)} [1 - V(t,x)]\,dx = d\omega_d \int_{r_h}^{\infty} [1 - v_*(r)]\,dr \tag{3.36}$$

for each $h \in (0,1)$. Because of the balance condition (3.4), the expression in (3.35) is less than (3.36) if $h < h_*$ and larger than (3.36) if $h > h_*$. In other words,

$$\int_{|x|<R_h(t)} V(t,x)\,dx < \int_{|x|>R_h(t)} [1 - V(t,x)]\,dx \qquad \text{for large } t$$

if $h < h_*$, and the opposite inequality holds if $h > h_*$. Comparing this with the defining equation (2.13) for $R_*(t)$, we find that

$$R_h(t) < R_*(t) < R_H(t) \qquad \text{for large } t$$

whenever $0 < h < h_* < H < 1$. Because of (3.34) and the continuity of r_h as a function of h, this implies that

$$R_*(t) = R_{h_*}(t) + o(1) \qquad \text{as } t \to \infty. \tag{3.37}$$

We may now use (3.34) once more (with $h = h_*$ or $H = h_*$) to arrive at assertion (3.15).

d) Assertion d) follows from (3.14) for $h = h_*$ and (3.37). □

Remark 3.4. In the proof of parts b)–d) of Theorem 3.2 we have applied Lemma 3.3, the proof of which has been postponed to Section 4.2 and Section 4.3 in dimension $d = 2$ and $d = 1$, respectively. *In that proof we will use assertion a), but none of the assertions b)–d) of Theorem 3.2.*

4. Proof of the theorems in Section 2

This section is long and contains the proofs of Theorems 2.1, 2.2 and 2.3. The proof proceeds differently in dimensions $d \geq 3$ (Section 4.1), $d = 2$ (Section 4.2), and $d = 1$ (Section 4.3).

As an immediate consequence of the conservation law (1.11) and the definition (2.13) of the position $R_*(t)$ of the trap front, we have

$$\omega_d R_*(t)^d = \lambda t - \int \varrho(t,x)\,dx. \tag{4.1}$$

Hence, in order to determine $R_*(t)$, we need to control $\int \varrho(t,x)\,dx$ and vice versa, which shows that Theorems 2.2 and 2.3 are closely linked together.

4.1. The case $d \geq 3$

As a preliminary step, we derive a rough upper bound on the total amount of particles alive at time t. Afterwards this bound will be sharpened to the precise asymptotics of Theorem 2.2.

Lemma 4.1. *(Rough bound on number of particles alive) Let $d \geq 3$. Then for $t > 2$,*

$$\int \varrho(t,x)\,dx \leq C(d,\lambda)\,t^{2/d}\log t, \tag{4.2}$$

where $C(d,\lambda)$ is a positive constant depending on d and λ only.

The proof of this lemma is based on the following statement.

Lemma 4.2. *(Isoperimetric inequality) For arbitrary positive numbers r, s, t and all measurable functions $w\colon \mathbb{R}^d \to [0,1]$ with*

$$\int w(x)\,dx \leq \omega_d r^d, \tag{4.3}$$

the following inequality holds:

$$\mathbb{E}_0 e^{sw(\beta(t))} \leq 1 + (e^s - 1)\mathbb{P}_0\left(|\beta(t)| \leq r\right). \tag{4.4}$$

Note that both in (4.3) and in (4.4) equality holds if w is the indicator function of the centered ball of radius r.

Proof. Note that $e^{sx} \leq 1 + (e^s - 1)x$ for all $s \geq 0$ and $x \in [0,1]$. Consequently, under our restrictions on w,

$$\mathbb{E}_0 e^{sw(\beta(t))} \leq 1 + (e^s - 1)\mathbb{E}_0 w(\beta(t)).$$

Expressing the expectation on the right with the help of the Gaussian kernel, we easily see that the expectation $\mathbb{E}_0 w(\beta(t))$ is maximal when w is the indicator function of the centered ball of radius r. $\square$

Proof of Lemma 4.1. Using the monotonicity of $V(t,x)$ in t, we conclude from the Feynman-Kac formula (1.13) that

$$\int \varrho(t,x)\,dx \leq \lambda \int_0^t ds\ \mathbb{E}_0 \exp\left\{-\int_0^s V(t,\beta(u))\,du\right\}.$$

Using Jensen's inequality, and recalling that $V = 1 - W$, we find that

$$\begin{aligned}\int \varrho(t,x)\,dx &\leq \lambda \int_0^t ds\,\frac{1}{s}\int_0^s du\ \mathbb{E}_0 e^{-sV(t,\beta(u))} \\ &= \lambda \int_0^t ds\, e^{-s}\,\frac{1}{s}\int_0^s du\ \mathbb{E}_0 e^{sW(t,\beta(u))}.\end{aligned} \tag{4.5}$$

We know that $0 \leq W(t,x) \leq 1$ and, because of the conservation law (1.11),

$$\int W(t,x)\, dx \leq \lambda t.$$

Hence we may apply the isoperimetric inequality of Lemma 4.2 to obtain

$$\mathbb{E}_0 e^{sW(t,\beta(u))} \leq 1 + (e^s - 1)\mathbb{P}_0\left(|\beta(u)| \leq R_d(t)\right),$$

where, as in (2.3), $R_d(t)$ is the radius of a ball of volume λt. After inserting this into (4.5), interchanging the order of integration, and performing simple estimations, we find that

$$\int \varrho(t,x)\, dx \leq \lambda + \lambda \int_0^t \frac{1-e^{-s}}{s}\, ds \int_0^\infty \mathbb{P}_0\left(|\beta(u)| \leq R_d(t)\right)\, du.$$

The first integral on the right is asymptotically equivalent to $\log t$. The second integral on the right is equal to the integral of the Green function (1.7) over the centered ball of radius $R_d(t)$, which is a constant multiple of $R_d(t)^2 = (\lambda t/\omega_d)^{2/d}$. This finally yields (4.2). $\square$

Lemma 4.3. *(Rough asymptotics of trap front positions) Let $d \geq 3$.*

a)

$$R_*(t) \sim R_d(t) \qquad \text{as } t \to \infty.$$

b) For each $h \in (0,1)$,

$$R_h(t) \sim R_d(t) \qquad \text{as } t \to \infty.$$

Proof. Assertion a) is an immediate consequence of (4.1) and (4.2). Assertion b) follows from a) and Theorem 3.2 c). $\square$

We next turn to the proof of the scaling limit.

Proof of Theorem 2.1 a) and b) for $d \geq 3$.

1^0 To prove assertion a) for $d \geq 3$, we fix $h \in (0,1)$ and compact sets $K_0 \subset D_d^0$ and $K_1 \subset D_d^1$ arbitrarily. It will be enough to show that

$$\limsup_{\varepsilon \downarrow 0} \sup_{(t,x)\in K_0} V_\varepsilon(t,x) \leq h \tag{4.6}$$

and

$$\liminf_{\varepsilon \downarrow 0} \inf_{(t,x)\in K_1} V_\varepsilon(t,x) \geq h. \tag{4.7}$$

To this end, let

$$R_h^\varepsilon(t) = \varepsilon R_h\left(\frac{t}{\varepsilon^d}\right) \tag{4.8}$$

denote the rescaled h-front. Then, by Lemma 4.3 b),

$$\lim_{\varepsilon \downarrow 0} R_h^\varepsilon(t) = R_d(t), \tag{4.9}$$

uniformly in t on compact subsets of $(0, \infty)$. This implies that

$$|x| < R_h^\varepsilon(t) \qquad \text{for } (t, x) \in K_0 \tag{4.10}$$

and all sufficiently small ε. From this it follows that

$$V_\varepsilon(t, x) < V_\varepsilon\left(t, R_h^\varepsilon(t)\right) = h \qquad \text{for } (t, x) \in K_0 \tag{4.11}$$

and all sufficiently small ε. This yields (4.6). Similarly, with K_0 replaced by K_1 and the inequalities opposite to (4.10) and (4.11), we obtain (4.7).

2^0 Introducing the potential

$$V_{R_h(t),h}(x) = \begin{cases} 0, & |x| \le R_h(t), \\ h, & |x| > R_h(t), \end{cases}$$

we find that

$$V(s, x) \ge V(t, x) \ge V_{R_h(t),h}(x), \qquad (s, x) \in [0, t] \times \mathbb{R}^d.$$

This leads to the simple, but fundamental, observation that

$$\varrho(t, x) \le \lambda G_{R_h(t),h}(x), \qquad (t, x) \in (0, \infty) \times (\mathbb{R}^d \setminus \{0\}), \tag{4.12}$$

where $G_{R_h(t),h}$ denotes the Green function associated with the potential $V_{R_h(t),h}$ considered in the Appendix. Inequality (4.12) may be derived by comparing the Feynman-Kac representation (1.5) of ϱ with the corresponding formula for $\lambda G_{R_h(t),h}$. The latter is obtained from (1.5) by first replacing the potential V by $V_{R_h(t),h}$ and then integrating up to ∞ instead of t.

3^0 We now turn to the proof of assertion b) of Theorem 2.1. Since $\varrho_\varepsilon(t, r)$ is monotone in t and r, it will be enough to prove pointwise convergence of the scaled particle density ϱ_ε as $\varepsilon \downarrow 0$. We first derive the corresponding *upper bound* for ϱ_ε. To this end we fix $h \in (0, 1)$ arbitrarily. Combining the estimate (4.12) with the bound of Lemma A.1 a) and taking into account the monotonicity of $G_{R,h}$, we obtain

$$\lambda^{-1}\varrho(t, r) \le G(r \wedge R_h(t)) - G(R_h(t)) + \frac{(d-2)c_d}{\sqrt{h} R_h(t)^{d-1}}$$

for all $t, r > 0$. Here, as in (1.7), $G(r) = c_d / r^{d-2}$ denotes the Green function for the d-dimensional Laplacian, but now written in polar coordinates. After rescaling with the parameter $\varepsilon > 0$, we may rewrite this as

$$\lambda^{-1}\varrho_\varepsilon(t, r) \le G\left(r \wedge R_h^\varepsilon(t)\right) - G\left(R_h^\varepsilon(t)\right) + \varepsilon \, \frac{(d-2)c_d}{\sqrt{h} R_h^\varepsilon(t)^{d-1}}.$$

Together with (4.9), this yields

$$\lambda^{-1} \limsup_{\varepsilon \downarrow 0} \varrho_\varepsilon(t,r) \leq G(r \wedge R_d(t)) - G(R_d(t)) \tag{4.13}$$

for $(t,r) \in (0,\infty)^2$. The expression on the right is the Green function in polar coordinates for the Laplacian in the centered ball of radius $R_d(t)$ with zero boundary condition. Hence, (4.13) is the desired upper bound.

4^0 It remains to derive the corresponding *lower bound* for ϱ_ε. Recalling the definition (2.1) of ϱ_ε and V_ε and using the scaling invariance of Brownian motion, we derive from the Feynman-Kac representation (1.5) the formula

$$\lambda^{-1} \varrho_\varepsilon(t,x) = \int_0^{t/\varepsilon^{d-2}} ds\, \mathbb{E}_x \exp\left\{ -\frac{1}{\varepsilon^2} \int_0^s V_\varepsilon\left(t - \varepsilon^{d-2}u, \beta(u)\right) du \right\} \delta_0(\beta(s)). \tag{4.14}$$

Now fix $(t,x) \in (0,\infty) \times (\mathbb{R}^d \setminus \{0\})$ arbitrarily. Pick $\delta > 0$ so small that $t > 2\delta$ and $R_d(t-\delta) > \delta$, and introduce the stopping time

$$\tau_\delta(t) = \inf\left\{u \geq 0 \colon |\beta(u)| \geq R_d(t-\delta) - \delta\right\}.$$

Taking into account the monotonicity of V_ε, we derive from (4.14) the bound

$$\lambda^{-1} \varrho_\varepsilon(t,x) \geq \int_0^{\delta/\varepsilon^{d-2}} ds\, \mathbb{E}_x \exp\left\{ -\frac{1}{\varepsilon^2} \int_0^s V_\varepsilon(t-\delta, \beta(u))\, du \right\} 1\!\!1\,(\tau_\delta(t) > s)\, \delta_0(\beta(s)). \tag{4.15}$$

Pick $h \in (0,1)$. It follows from (3.9) and (3.11) that

$$r^d v_h(-r) \longrightarrow 0 \qquad \text{as } r \to +\infty.$$

Hence, using (3.12) and (4.9), we obtain

$$\frac{1}{\varepsilon^d} V_\varepsilon(t-\delta, y) \longrightarrow 0 \tag{4.16}$$

as $\varepsilon \downarrow 0$ if $|y| < R_d(t-\delta) - \delta$. Moreover,

$$\int_{\delta/\varepsilon^{d-2}}^\infty ds\, \mathbb{E}_x \delta_0(\beta(s)) = \int_{\delta/\varepsilon^{d-2}}^\infty ds\, (4\pi s)^{-d/2} \exp\{-|x|^2/(4s)\} \longrightarrow 0 \tag{4.17}$$

as $\varepsilon \downarrow 0$. Combining (4.15) with (4.16) and (4.17), we find that

$$\liminf_{\varepsilon \downarrow 0} \lambda^{-1} \varrho_\varepsilon(t,x) \geq \int_0^\infty ds\, \mathbb{E}_x 1\!\!1\,(\tau_\delta(t) > s)\, \delta_0(\beta(s)).$$

But the expression on the right coincides with the Green function for the Laplace operator in the centered ball of radius $R_d(t-\delta)-\delta$ with zero boundary condition. As $\delta \downarrow 0$, the latter function converges to the corresponding Green function for $\delta = 0$. In this way we arrive at the desired lower bound. $\square$

We are now in a position to derive the asymptotics of the total number of particles alive.

Proof of Theorem 2.2 for $d \geq 3$. Using the rescaled particle density $\varrho_\varepsilon(t,x)$ with t replaced by 1, ε replaced by $t^{-1/d}$, and x replaced by $t^{-1/d}x$, we find that

$$\varrho(t,x) = t^{(2-d)/d} \varrho_{t^{-1/d}}(1, t^{-1/d}x).$$

Therefore, picking any $h \in (0,1)$, we have

$$\int_{|x| \leq R_h(t)} \varrho(t,x)\,dx = t^{2/d} \int_{|y| \leq \frac{R_h(t)}{t^{1/d}}} \varrho_{t^{-1/d}}(1,y)\,dy. \tag{4.18}$$

By Lemma 4.3 b) and definition (2.3), $R_h(t)/t^{1/d} \to R_d(1)$ as $t \to \infty$, where $R_d(1) = (\lambda/\omega_d)^{1/d}$. From Theorem 2.1 b) we know that

$$\varrho_{t^{-1/d}}(1,y) \longrightarrow \frac{\lambda}{R_d(1)^{d-2}}\, G^0\left(\frac{y}{R_d(1)}\right)$$

pointwise as $t \to \infty$, where the Green function G^0 is given by the explicit formula (2.5). Moreover, according to (1.6), $\varrho(t,\cdot\,)$, and therefore also $\varrho_{t^{-1/d}}(1,\cdot\,)$, is dominated by λ times the locally integrable Green function G. Hence, an application of Lebesgue's dominated convergence theorem shows that the integral on the right of (4.18) converges to

$$\frac{\lambda}{R_d(1)^{d-2}} \int_{|y| \leq R_d(1)} G^0\left(\frac{y}{R_d(1)}\right) dy = \frac{\lambda}{2d} R_d(1)^2 (d-2) d c_d \omega_d.$$

Hence, taking into account (1.10), we obtain

$$\int_{|x| \leq R_h(t)} \varrho(t,x)\,dx \sim \frac{\lambda}{2d}\left(\frac{\lambda t}{\omega_d}\right)^{2/d} \qquad \text{as } t \to \infty.$$

It remains to show that

$$\int_{|x| > R_h(t)} \varrho(t,x)\,dx = o(t^{2/d}) \qquad \text{as } t \to \infty. \tag{4.19}$$

A combination of the bound (4.12) with the bound for the Green function $G_{R_h(t),h}$ in Lemma A.1 c) of the Appendix yields

$$\varrho(t,r) \leq \lambda \frac{C(d)}{\sqrt{h} R_h(t)^{\frac{d-1}{2}} r^{\frac{d-1}{2}}} \exp\left\{-\sqrt{h}\,(r - R_h(t))\right\}$$

for all sufficiently large t and all $r > R_h(t)$. But, as one easily checks, this implies that the integral on the left of (4.19) in fact stays bounded as $t \to \infty$. $\square$

Proof of Theorem 2.3 for $d \geq 3$. The sharp asymptotics of $R_*(t)$ is now immediate from (4.1) and Theorem 2.2 for $d \geq 3$. $\square$

4.2. The case $d = 2$

Given $\alpha > 0$, consider the two-dimensional boundary value problem

$$\begin{aligned} \frac{\partial \varrho_\alpha^*}{\partial t}(t,x) &= \Delta \varrho_\alpha^*(t,x) + \lambda \delta_0(x) && \text{for } |x| < \alpha\sqrt{t}, \\ \varrho_\alpha^*(t,x) &= 0 && \text{for } |x| = \alpha\sqrt{t}. \end{aligned} \tag{4.20}$$

One easily checks that the unique weak solution ϱ_α^* to this problem with $0 \leq \varrho_\alpha^* \leq \varrho_0$ (and ϱ_0 taken from (1.3)) is given by

$$\varrho_\alpha^*(t,x) = \frac{\lambda}{2\pi} \int_{|x|/\sqrt{t}}^{\alpha} e^{-u^2/4}\, \frac{du}{u}, \qquad 0 < |x| \leq \alpha\sqrt{t}.$$

In particular, $\varrho_{\kappa_*}^*$ coincides with the function ϱ_* given by (2.6) and (2.7). Define

$$\Psi(\alpha) = \lambda \left(1 - e^{-\alpha^2/4}\right)$$

and note that

$$\frac{1}{t} \int_{|x|<\alpha\sqrt{t}} \varrho_\alpha^*(t,x)\, dx = \Psi(\alpha), \qquad t > 0. \tag{4.21}$$

Set $\Psi(\infty) = \lambda$. We need the following lemma.

Lemma 4.4. *(Rough bounds on number of particles alive) Let $d = 2$. Then, for each $h \in (0,1)$,*

$$\limsup_{t\to\infty} \frac{1}{t} \int \varrho(t,x)\, dx \leq \Psi\left(\limsup_{t\to\infty} \frac{R_h(t)}{\sqrt{t}}\right), \tag{4.22}$$

$$\liminf_{t\to\infty} \frac{1}{t} \int \varrho(t,x)\, dx \geq \Psi\left(\liminf_{t\to\infty} \frac{R_h(t)}{\sqrt{t}}\right). \tag{4.23}$$

Proof. 1^0 We first remark that the solution of (4.20) admits the Feynman-Kac representation

$$\varrho_\alpha^*(t,x) = \lambda \int_0^t ds\, \mathbb{E}_x \mathbb{1}\Big(|\beta(u)| < \alpha\sqrt{t-u} \text{ for } u \in [0,s]\Big) \delta_0(\beta(s)).$$

After time-reversal and integration over x, we obtain

$$\int_{|x|<\alpha\sqrt{t}} \varrho_\alpha^*(t,x)\, dx = \lambda \int_0^t ds\, \mathbb{P}_0\Big(|\beta(u)| < \alpha\sqrt{t-s+u} \text{ for } u \in [0,s]\Big).$$

After Brownian scaling this reads as

$$\frac{1}{t}\int_{|x|<\alpha\sqrt{t}} \varrho_\alpha^*(t,x)\,dx = \lambda \int_0^1 ds\,\mathbb{P}_0\Big(|\beta(u)| < \alpha\sqrt{1-s+u} \text{ for } u\in[0,s]\Big).$$

Comparing this with (4.21), we find that

$$\lambda \int_0^1 ds\,\mathbb{P}_0\Big(|\beta(u)| < \alpha\sqrt{1-s+u} \text{ for } u\in[0,s]\Big) = \Psi(\alpha). \tag{4.24}$$

2^0 We now turn to the proof of (4.22). Because of the conservation law (1.11), inequality (4.22) is trivial if the upper limit on the right is infinite. Let us therefore suppose that it is finite and fix

$$\alpha > \limsup_{t\to\infty} \frac{R_h(t)}{\sqrt{t}}$$

arbitrarily. It follows from the definition of $R_h(t)$ in (3.1) that

$$V(t,r) \geq h 1\!\!1_{(R_h(t),\infty)}(r). \tag{4.25}$$

Substituting this into the Feynman-Kac formula (1.13) and performing Brownian scaling, we obtain

$$\begin{aligned}
&\frac{1}{t}\int \varrho(t,x)\,dx \\
&\quad\leq \lambda \int_0^1 ds\,\mathbb{E}_0 \exp\left\{-ht\int_0^s 1\!\!1\left(|\beta(u)| > \frac{R_h(t(1-s+u))}{\sqrt{t}}\right) du\right\} \\
&\quad\leq \lambda \int_0^1 ds\,\mathbb{P}_0\Big(|\beta(u)| < \alpha\sqrt{1-s+u} \text{ for } u\in[0,s]\Big) \\
&\qquad + \lambda \int_0^1 ds\,\mathbb{E}_0 \exp\left\{-ht\int_0^s 1\!\!1\left(|\beta(u)| > \frac{R_h(t(1-s+u))}{\sqrt{t}}\right) du\right\} \\
&\qquad\qquad \times 1\!\!1\Big(|\beta(u)| > \alpha\sqrt{1-s+u} \text{ for some } u\in[0,s]\Big).
\end{aligned} \tag{4.26}$$

Suppose that $|\beta(u_0)| > \alpha\sqrt{1-s+u_0}$ for some $u_0 \in [0,s]$ and $s\in[0,1)$. Then, by the continuity of Brownian paths and our choice of α,

$$|\beta(u)| > \alpha\sqrt{1-s+u} > \frac{R_h(t(1-s+u))}{\sqrt{t}}$$

for all sufficiently large t and all u in a certain neighborhood of u_0. By Fatou's lemma, this implies that

$$\liminf_{t\to\infty} \int_0^s 1\!\!1\left(|\beta(u)| > \frac{R_h(t(1-s+u))}{\sqrt{t}}\right) du > 0.$$

Therefore, an application of Lebesgue's dominated convergence theorem shows that the second integral on the right of (4.26) tends to zero as $t \to \infty$. But, according to (4.24), the first term on the right equals $\Psi(\alpha)$. Hence,

$$\limsup_{t\to\infty} \frac{1}{t} \int \varrho(t,x)\,dx \le \Psi(\alpha).$$

Since α can be chosen arbitrarily close to $\limsup_{t\to\infty} R_h(t)/\sqrt{t}$, this proves assertion (4.22).

3^0 In order to prove assertion (4.23), because $\Psi(0) = 0$, we may assume without loss of generality that $\liminf_{t\to\infty} R_h(t)/\sqrt{t} > 0$ and fix α arbitrarily with

$$0 < \alpha < \liminf_{t\to\infty} \frac{R_h(t)}{\sqrt{t}}.$$

From the upper bound in (3.12) of Theorem 3.2 a) (recall Remark 3.4), (3.11) and (3.9) we conclude that

$$V(s,r) \le \frac{C_h}{t^2}, \qquad r \le R_h(s) - r_h - 2\sqrt{\log t},$$

for $s \in [0,t]$ and $t > 1$, where C_h denotes a positive constant depending on h only. Taking this into account, we derive from (1.13) that for such t,

$$\begin{aligned}
&\frac{1}{t}\int \varrho(t,x)\,dx \\
&\ge \frac{\lambda e^{-C_h/t}}{t} \int_0^t ds\, \mathbb{P}_0\left(|\beta(u)| \le R_h(t-s+u) - r_h - 2\sqrt{\log t} \text{ for } u \in [0,s]\right) \\
&= \lambda e^{-C_h/t} \int_0^1 ds\, \mathbb{P}_0\left(|\beta(u)| \le \frac{R_h(t(1-s+u)) - r_h - 2\sqrt{\log t}}{\sqrt{t}} \text{ for } u \in [0,s]\right).
\end{aligned}$$

By our choice of α and Fatou's lemma,

$$\begin{aligned}
\liminf_{t\to\infty} \mathbb{P}_0 &\left(|\beta(u)| \le \frac{R_h(t(1-s+u)) - r_h - 2\sqrt{\log t}}{\sqrt{t}} \text{ for } u \in [0,s]\right) \\
&\ge \mathbb{P}_0\left(|\beta(u)| < \alpha\sqrt{1-s+u} \text{ for } u \in [0,s]\right)
\end{aligned}$$

for all $s \in [0,1]$. Hence,

$$\liminf_{t\to\infty} \frac{1}{t}\int \varrho(t,x)\,dx \ge \lambda \int_0^1 ds\, \mathbb{P}_0\left(|\beta(u)| < \alpha\sqrt{1-s+u} \text{ for } u \in [0,s]\right).$$

According to (4.24), the expression on the right equals $\Psi(\alpha)$, and this proves assertion (4.23). $\square$

With the help of Lemma 4.4 we now derive rough bounds for the h-fronts.

Lemma 4.5. *(Rough bounds for h-fronts) Fix $h \in (0,1)$ arbitrarily and define*

$$\alpha_{0,h} = \sqrt{\frac{\lambda}{(1-h)\pi}} \qquad \textit{and} \qquad \alpha_{1,h} = \sqrt{\frac{\lambda - \Psi(\alpha_{0,h})}{\pi}}. \tag{4.27}$$

Then

$$\alpha_{1,h} \le \liminf_{t\to\infty} \frac{R_h(t)}{\sqrt{t}} \le \limsup_{t\to\infty} \frac{R_h(t)}{\sqrt{t}} \le \alpha_{0,h}. \tag{4.28}$$

Proof. Since $W(t,x) = 1 - V(t,x) \ge 1-h$ for $|x| \le R_h(t)$, we find that

$$\int W(t,x)\,dx \ge (1-h)\pi R_h^2(t).$$

Hence, using the conservation law (1.11), we conclude that

$$(1-h)\pi R_h^2(t) \le \lambda t.$$

This yields the upper bound in (4.28). By the lower bound in (3.13) of Theorem 3.2 a) (recall Remark 3.4), $W(t,r) < 1 - v_h(r - R_h(t))$ for $r > R_h(t)$. Therefore,

$$\int W(t,x)\,dx \le \pi R_h^2(t) + 2\pi R_h(t) \int_0^\infty (1 - v_h(r))\; dr + 2\pi \int_0^\infty r\,(1 - v_h(r))\; dr.$$

It follows from (3.11) and (3.10) that both integrals on the right are finite. Hence, there exist constants $C_{1,h}$ and $C_{2,h}$ such that

$$\int W(t,x)dx \le \pi\left(R_h(t) + C_{1,h}\right)^2 + C_{2,h}.$$

Substituting this into the conservation law (1.11), we obtain

$$\lambda - \pi\left(\frac{R_h(t) + C_{1,h}}{\sqrt{t}}\right)^2 - \frac{C_{2,h}}{t} \le \frac{1}{t}\int \varrho(t,x)\,dx.$$

Combining this with the upper bound (4.22) of Lemma 4.4, we conclude that

$$\limsup_{t\to\infty}\left(\lambda - \pi\left(\frac{R_h(t)+C_{1,h}}{\sqrt{t}}\right)^2\right) \le \Psi\left(\limsup_{t\to\infty}\frac{R_h(t)}{\sqrt{t}}\right) \le \Psi(\alpha_{0,h}),$$

where the last inequality comes from the upper bound in (4.27) just proved. This implies the lower bound in (4.28). □

We now turn to the identification of the h-fronts. Recall that $\kappa_* = \kappa_*(\lambda)$ is defined by (2.4).

Lemma 4.6. *(Asymptotics of h-fronts) Let $d = 2$. Then, for each $h \in (0,1)$,*

$$R_h(t) \sim \kappa_* \sqrt{t} \qquad \text{as } t \to \infty.$$

Proof. Fix $h \in (0,1)$ arbitrarily and recall (4.8). It will be enough to show that

$$\lim_{\varepsilon \downarrow 0} R_h^\varepsilon(t) = \kappa_* \sqrt{t}, \qquad t > 0. \tag{4.29}$$

1^0 It follows from Lemma 4.5 that

$$0 < \alpha_{1,h} \sqrt{t} \le \liminf_{\varepsilon \downarrow 0} R_h^\varepsilon(t) \le \limsup_{\varepsilon \downarrow 0} R_h^\varepsilon(t) \le \alpha_{0,h} \sqrt{t}$$

for all $t > 0$. Moreover, as a consequence of (4.1) and the monotonicity of $\varrho(t,x)$ in t,

$$0 \le R_h^\varepsilon(t)^2 - R_h^\varepsilon(s)^2 \le \frac{\lambda}{\pi}(t-s) \qquad \text{for } 0 < s < t.$$

Hence, the monotone functions R_h^ε are vaguely compact as $\varepsilon \downarrow 0$, and each limiting function $\tilde{R}_h$ is continuous and satisfies

$$\alpha_{1,h} \sqrt{t} \le \tilde{R}_h(t) \le \alpha_{0,h} \sqrt{t}. \tag{4.30}$$

Let therefore $\varepsilon_n \downarrow 0$ be chosen in such a way that the limit

$$\lim_{n \to \infty} R_h^{\varepsilon_n}(t) = \tilde{R}_h(t) \tag{4.31}$$

exists for all $t > 0$. Then this convergence is uniform in t on compact subsets of $(0, \infty)$. To prove (4.29) it will therefore be enough to show that

$$\tilde{R}_h(t) = \kappa_* \sqrt{t}, \qquad t > 0. \tag{4.32}$$

2^0 We next proceed as in step 1^0 of the proof of the scaling limit for $d \ge 3$ carried out after Lemma 4.3, to obtain

$$\lim_{n \to \infty} V_{\varepsilon_n}(t,x) = \begin{cases} 0, & (t,x) \in \tilde{D}_h^0, \\ 1, & (t,x) \in \tilde{D}_h^1, \end{cases} \tag{4.33}$$

where $\tilde{D}_h^0 = \{(t,x)\colon |x| < \tilde{R}_h(t)\}$ and $\tilde{D}_h^1 = \{(t,x)\colon |x| > \tilde{R}_h(t)\}$. This convergence is uniform on compact subsets of $\tilde{D}_h^0 \cup \tilde{D}_h^1$.

3^0 We next show that

$$\lim_{n \to \infty} \varrho_{\varepsilon_n}(t,x) = \tilde{\varrho}_h(t,x), \tag{4.34}$$

uniformly in (t,x) on compact subsets of $(0,\infty) \times (\mathbb{R}^2 \setminus \{0\})$, where

$$\tilde{\varrho}_h(t,x) = \lambda \int_0^t ds \, \mathbb{E}_x \mathbb{1}\Big(|\beta(u)| < \tilde{R}_h(t-u) \text{ for } u \in [0,s]\Big)\, \delta_0(\beta(s)). \tag{4.35}$$

This goes as follows. Because of the monotonicity of ϱ_{ε_n} and the continuity of $\tilde{\varrho}_h$, it will be enough to prove pointwise convergence. To this end, we fix $(t,x) \in (0,\infty) \times (\mathbb{R}^2 \setminus \{0\})$ arbitrarily. The Feynman-Kac representation (4.14) for $d=2$ reads

$$\varrho_{\varepsilon_n}(t,x) = \lambda \int_0^t ds\, \mathbb{E}_x \exp\left\{-\frac{1}{\varepsilon_n^2}\int_0^s V_{\varepsilon_n}(t-u,\beta(u))\,du\right\}\delta_0(\beta(s)).$$

Therefore

$$\begin{aligned}\varrho_{\varepsilon_n}(t,x) \le \tilde{\varrho}_h(t,x) + \lambda \int_0^t ds\, \mathbb{E}_x \exp&\left\{-\frac{1}{\varepsilon_n^2}\int_0^s V_{\varepsilon_n}(t-u,\beta(u))\,du\right\} \\ &\times \mathbb{1}\Big(|\beta(u)| \ge \tilde{R}_h(t-u) \text{ for some } u \in [0,s]\Big)\,\delta_0(\beta(s)).\end{aligned}$$

Because of (4.33),

$$\frac{1}{\varepsilon_n^2} V_{\varepsilon_n}(t-u,y) \longrightarrow \infty \qquad \text{as } n \to \infty$$

if $|y| > \tilde{R}_h(t-u)$. Hence, the integral on the right of the last bound tends to zero as $n \to \infty$ by Fatou's lemma and Lebesgue's dominated convergence theorem, and so we arrive at

$$\limsup_{n\to\infty} \varrho_{\varepsilon_n}(t,x) \le \tilde{\varrho}_h(t,x).$$

To derive the opposite bound we choose $\delta > 0$ arbitrarily and estimate

$$\begin{aligned}\varrho_{\varepsilon_n}(t,x) \ge \lambda \int_0^t ds\, \mathbb{E}_x \exp&\left\{-\frac{1}{\varepsilon_n^2}\int_0^s V_{\varepsilon_n}(t-u,\beta(u))\,du\right\} \\ &\times \mathbb{1}\Big(|\beta(u)| < \tilde{R}_h(t-u) - \delta \text{ for } u \in [0,s]\Big)\,\delta_0(\beta(s)).\end{aligned}$$

Similarly to (4.16), using (4.31) instead of (4.9), we find that

$$\frac{1}{\varepsilon_n^2} V_{\varepsilon_n}(t-u,y) \longrightarrow 0 \qquad \text{as } n \to \infty$$

if $|y| \le \tilde{R}_h(t-u) - \delta$. Therefore, as before,

$$\liminf_{n\to\infty} \varrho_{\varepsilon_n}(t,x) \ge \lambda \int_0^t ds\, \mathbb{E}_x \mathbb{1}\Big(|\beta(u)| < \tilde{R}_h(t-u) - \delta \text{ for } u \in [0,s]\Big)\,\delta_0(\beta(s)).$$

Letting $\delta \downarrow 0$, we arrive at

$$\liminf_{n\to\infty} \varrho_{\varepsilon_n}(t,x) \ge \tilde{\varrho}_h(t,x).$$

4^0 We next rescale equation (1.12) for $d=2$ to obtain

$$\frac{\partial}{\partial t}\left(\varrho_\varepsilon + W_\varepsilon\right) = \Delta \varrho_\varepsilon + \lambda \delta_0,$$

where $W_\varepsilon = 1 - V_\varepsilon$. Rewriting this equation in its weak form, we get

$$\int_0^\infty dt \int_{\mathbb{R}^2} dx \left[\frac{\partial\varphi}{\partial t}(t,x)\Big(\varrho_{\varepsilon_n}(t,x) + W_{\varepsilon_n}(t,x)\Big) + \Delta\varphi(t,x)\varrho_{\varepsilon_n}(t,x)\right] + \lambda \int_0^\infty dt\, \varphi(t,0) = 0$$

for any C^∞-function φ on $(0,\infty)\times\mathbb{R}^2$ with compact support. Because of (4.34), (4.33), and $0 \le \varrho_{\varepsilon_n} \le \varrho_0$ with ϱ_0 given by (1.3), we may pass to the limit for $n \to \infty$ to arrive at

$$\int_0^\infty dt \int_{\mathbb{R}^2} dx \left[\frac{\partial\varphi}{\partial t}(t,x)\Big(\tilde{\varrho}_h(t,x) + 1\!\!1_{\tilde{D}_h^0}(t,x)\Big) + \Delta\varphi(t,x)\tilde{\varrho}_h(t,x)\right] + \lambda \int_0^\infty dt\, \varphi(t,0) = 0. \tag{4.36}$$

Since, by (4.35),

$$\tilde{\varrho}_h > 0 \text{ on } \tilde{D}_h^0 \quad \text{and} \quad \tilde{\varrho}_h = 0 \text{ on } \tilde{D}_h^1, \tag{4.37}$$

this means that $\tilde{\varrho}_h$ is a weak solution of the Stefan problem

$$\begin{aligned} &\frac{\partial}{\partial t}\left[(\tilde{\varrho}_h + 1)\, 1\!\!1_{\tilde{\varrho}_h>0}\right] = \Delta\tilde{\varrho}_h + \lambda\delta_0, \\ &\tilde{\varrho}_h(0,\cdot\,) \equiv 0. \end{aligned} \tag{4.38}$$

Moreover, $0 \le \tilde{\varrho}_h \le \varrho_0$ and $\int_0^T dt \int_{\mathbb{R}^2} dx\, \varrho_0^2(t,x) < \infty$ for all $T > 0$. According to Gravner and Quastel [8], Lemma 2.7, such a solution is unique. But it is not difficult to check that the function ϱ^* given by (2.7) also fulfills (4.36) and (4.37). Hence,

$$\tilde{\varrho}_h = \varrho^*. \tag{4.39}$$

This together with (4.37) implies (4.32) (recall (2.3) and (2.6)) and therefore proves Lemma 4.6. □

Proof of Theorem 2.1 a) and c) for $d = 2$. Assertion a) of Theorem 2.1 for $d = 2$ now follows from (4.32) and (4.33). Assertion c) of Theorem 2.1 is immediate from (4.34) and (4.39). □

Proof of Theorem 2.2 for $d = 2$. By an application of Lebesgue's dominated convergence theorem, it follows from Theorem 2.1 c), $0 \le \varrho_\varepsilon \le \varrho_0$, and formula (2.7) that

$$\lim_{\varepsilon\downarrow 0} \int \varrho_\varepsilon(t,x)\,dx = \int \varrho^*(t,x)\,dx = \left(1 - e^{-\kappa_*^2/4}\right)\lambda t$$

for all $t > 0$. This obviously implies the desired assertion. □

Proof of Lemma 3.3 for $d = 2$. It immediately follows from Theorem 2.1 c) and Lemma 4.6 that

$$\varrho_\varepsilon(t, R_h^\varepsilon(t)) \longrightarrow \varrho^*(t, \kappa_* \sqrt{t}) = 0$$

as $\varepsilon \downarrow 0$. This implies (3.16) after we pick $t = 1$ and use (2.1). $\square$

Proof of Theorem 2.3 for $d = 2$. Since Lemma 3.3 is proven for $d = 2$, we may now apply Theorem 3.2 c) (recall Remark 3.4) to see that the assertion follows from Lemma 4.6. $\square$

4.3. The case $d = 1$

The key result in this section is the following lemma about the asymptotic behavior of the h-fronts in dimension $d = 1$.

Lemma 4.7. *(Asymptotics of h-fronts) Let $d = 1$. Then, for each $h \in (0, 1)$,*

$$R_h(t) \sim \sqrt{2t \log t} \qquad \text{as } t \to \infty.$$

Proof. Fix $h \in (0, 1)$ arbitrarily. A combination of the conservation law (1.11) with the Feynman-Kac formula (1.13) yields

$$\int W(t,x)\,dx = \lambda \int_0^t ds\, \left(1 - \mathbb{E}_0 \exp\left\{-\int_0^{t-s} V(s+u, \beta(u))\,du\right\}\right). \tag{4.40}$$

Using the upper bound (3.12) in Theorem 3.2 a), (3.11), and (3.9), we find that

$$\begin{aligned}\int W(t,x)\,dx &\geq 2\int_0^{R_h(t)} W(t,r)\,dr \geq 2R_h(t) - 2\int_{-\infty}^0 v_h(r)\,dr \\ &= 2R_h(t) - C_h \end{aligned} \tag{4.41}$$

for some positive constant C_h and all $t > 0$. Similarly, using the lower bound (3.13) in Theorem 3.2 a), (3.11), and (3.10), we obtain

$$\int W(t,x)\,dx \leq 2R_h(t) + C_h' \tag{4.42}$$

for some positive constant C_h' and all $t > 0$. Let us further note that

$$\mathbb{P}_0(\beta(u) > r) \leq \frac{1}{\sqrt{\pi}(r/\sqrt{u})}\; e^{-(r/\sqrt{u})^2/4} \qquad \text{for } r, u > 0, \tag{4.43}$$

$$\mathbb{P}_0(\beta(u) > r) \sim \frac{1}{\sqrt{\pi}(r/\sqrt{u})}\; e^{-(r/\sqrt{u})^2/4} \qquad \text{as } r/\sqrt{u} \to \infty. \tag{4.44}$$

1^0 We first show that

$$\liminf_{t\to\infty} \frac{R_h(t)}{\sqrt{2t\log t}} \geq 1. \tag{4.45}$$

To this end we proceed as in (4.5) and use the bound (4.25), to find that

$$\int \varrho(t,x)\,dx \leq \lambda \int_0^t ds\, \frac{1}{s} \int_0^s du\, \mathbb{E}_0 \exp\left\{-hs 1\!\!1(\beta(u) > R_h(t))\right\}.$$

Combining this with the conservation law (1.11), we see that

$$\begin{aligned} \int W(t,x)\,dx &\geq \lambda \int_0^t ds\, \frac{1}{s} \int_0^s du\, \mathbb{E}_0 \Big[1 - \exp\left\{-hs 1\!\!1(\beta(u) > R_h(t))\right\}\Big] \\ &= \lambda \int_0^t du\, \left(\int_u^t ds\, \frac{1-e^{-hs}}{s}\right) \mathbb{P}_0\left(|\beta(u)| > R_h(t)\right). \end{aligned}$$

After Brownian scaling, this reads as

$$\int W(t,x)\,dx \geq \lambda t \int_0^1 du\, \left(\int_u^1 ds\, \frac{1-e^{-hts}}{s}\right) \mathbb{P}_0\left(|\beta(u)| > \frac{R_h(t)}{\sqrt{t}}\right). \tag{4.46}$$

Now choose $\varepsilon \in (0,1)$ arbitrarily. Then, after substituting (4.42) into (4.46) and integrating on the right of (4.46) over $(1-\varepsilon, 1)$ only, we find that

$$\frac{R_h(t)}{\sqrt{t}} \geq C_{\varepsilon,\lambda} \sqrt{t}\, \mathbb{P}_0\left(|\beta(1-\varepsilon)| > \frac{R_h(t)}{\sqrt{t}}\right)$$

for all sufficiently large t, where $C_{\varepsilon,\lambda}$ is a positive constant that depends on ε and λ but not on t. It is obvious from this that $R_h(t)/\sqrt{t} \to \infty$, and because of (4.44) we get the asymptotics

$$\mathbb{P}_0\left(|\beta(1-\varepsilon)| > \frac{R_h(t)}{\sqrt{t}}\right) \sim \sqrt{\frac{1-\varepsilon}{\pi}} \left(\frac{R_h(t)}{\sqrt{t}}\right)^{-1} \exp\left\{-\frac{\left(R_h(t)/\sqrt{t}\right)^2}{4(1-\varepsilon)}\right\}$$

as $t \to \infty$. Hence, there exists a positive constant $C'_{\varepsilon,\lambda}$ such that

$$\left(\frac{R_h(t)}{\sqrt{t}}\right)^2 \geq C'_{\varepsilon,\lambda} \sqrt{t}\, \exp\left\{-\frac{\left(R_h(t)/\sqrt{t}\right)^2}{4(1-\varepsilon)}\right\}$$

for large t. From this we conclude that

$$\frac{R_h(t)}{\sqrt{t}} > (1-\varepsilon)\sqrt{2\log t}$$

for large t. Since ε can be chosen arbitrarily small, this proves assertion (4.45).

2^0 We next show that

$$\liminf_{t\to\infty} \frac{R_h(t)}{\sqrt{2t\log t}} \le 1. \tag{4.47}$$

Assume that this statement is false. Then there exist $\varepsilon > 0$ and $t_0 = t_0(\varepsilon) > e$ such that

$$R_h(t) \ge R^{(\varepsilon)}(t) \qquad \text{for } t > t_0, \tag{4.48}$$

where

$$R^{(\varepsilon)}(t) = (1+\varepsilon)\sqrt{2t\log t}. \tag{4.49}$$

Choose $\delta > 0$ arbitrarily, and define

$$t_n = e^{\delta n} \tag{4.50}$$

and $n_0 = n_0(\varepsilon)$ such that $t_{n_0} \ge t_0$. After combining (4.48), (4.41), and (4.40), we get for $n > n_0$ and any $\delta' \in (0,1)$,

$$\begin{aligned}
2R^{(\varepsilon)}(t_n) - C_h &\le 2R_h(t_n) - C_h \\
&\le \lambda \int_0^{t_n} ds \left(1 - \mathbb{E}_0 \exp\left\{-\int_0^{t_n-s} V(s+u,\beta(u))\,du\right\}\right) \\
&\le \lambda \int_0^{t_n} ds \left(1 - \mathbb{E}_0 \exp\left\{-\int_0^{t_n-s} v_h(|\beta(u)| - R_h(s+u))\,du\right\}\right. \\
&\qquad\qquad \left. \times \mathbb{1}\Big(|\beta(u)| < (1-\delta')R_h(s+u) \text{ for } u \in [0, t_n - s]\Big)\right),
\end{aligned}$$

where in the last line we have again used the upper bound in Theorem 3.2 a) (recall Remark 3.4). But the expression on the right equals

$$\begin{aligned}
&\lambda \int_0^{t_n} ds\, \mathbb{P}_0\Big(|\beta(u)| \ge (1-\delta')R_h(s+u) \text{ for some } u \in [0, t_n - s]\Big) \\
&+ \lambda \int_0^{t_n} ds\, \mathbb{E}_0 \left(1 - \exp\left\{-\int_0^{t_n-s} v_h(|\beta(u)| - R_h(s+u))\,du\right\}\right) \\
&\qquad\qquad \times \mathbb{1}\Big(|\beta(u)| < (1-\delta')R_h(s+u) \text{ for } u \in [0, t_n - s]\Big).
\end{aligned}$$

Using the inequality $1 - e^{-\eta} \le \eta$, we may estimate the second term on the right from above by

$$\lambda \int_0^{t_n} ds \int_0^{t_n-s} du\, v_h\left(-\delta' R_h(s+u)\right) \le \lambda \int_0^{\infty} du\; u\, v_h\left(-\delta' R_h(u)\right).$$

The integral on the right is finite by (3.11), (3.9), and (4.48). Combining the above estimates, taking again into account (4.48), and applying the reflection principle for Brownian motion, we see that there exist positive constants C_1 and

C_2 depending on λ, h, δ, δ' and ε but not on n such that, for $n > n_0$ and any $\delta' \in (0,1)$,

$$\begin{aligned}
&2R^{(\varepsilon)}(t_n) - C_{h,\delta',\varepsilon}\\
&\le \lambda \int_{t_{n_0}}^{t_n} ds\, \mathbb{P}_0\Big(|\beta(u)| \ge (1-\delta')R_h(s+u) \text{ for some } u \in [0, t_n - s]\Big)\\
&\le \lambda \sum_{k=n_0+1}^{n} \int_{t_{k-1}}^{t_k} ds\\
&\quad \times \sum_{j=k}^{n} \mathbb{P}_0\Big(|\beta(u)| \ge (1-\delta')R_h(s+u) \text{ for some } u \in [t_{j-1} - t_{k-1}, t_j - t_{k-1}]\Big)\\
&\le \lambda \sum_{k=n_0+1}^{n} (t_k - t_{k-1}) \sum_{j=k}^{n} \mathbb{P}_0\Big(|\beta(u)| \ge (1-\delta')R_h(t_{j-1}) \text{ for some } u \in [0, t_j]\Big)\\
&\le \lambda \sum_{j=n_0+1}^{n} t_j\, \mathbb{P}_0\Big(|\beta(u)| \ge (1-\delta')R_h(t_{j-1}) \text{ for some } u \in [0, t_j]\Big)\\
&\le 4\lambda \sum_{j=n_0+1}^{n} t_j\, \mathbb{P}_0\Big(\beta(t_j) \ge (1-\delta')R^{(\varepsilon)}(t_{j-1})\Big)\\
&\le C_2 \sum_{j=n_0+1}^{n} t_j \,\exp\left\{-\frac{(1-\delta')^2}{4}\left(\frac{R^{(\varepsilon)}(t_{j-1})}{\sqrt{t_j}}\right)^2\right\},
\end{aligned} \tag{4.51}$$

where in the last line we have also used (4.43) and

$$\frac{R^{(\varepsilon)}(t_{j-1})}{\sqrt{t_j}} \ge (1+\varepsilon)\sqrt{2\delta e^{-\delta}} \qquad \text{for } j \ge 2$$

(recall (4.49) and (4.50)). Again by (4.49) and (4.50), we find that, on the one hand,

$$R^{(\varepsilon)}(t_n) = (1+\varepsilon)\sqrt{2\delta n}\, e^{(1/2)\delta n}$$

while, on the other hand, the sum on the right of (4.51) behaves like

$$e^{\delta} \sum_{j=n_0+1}^{n} e^{\nu\delta(j-1)} \sim c\, e^{\nu\delta n} \qquad \text{as } n \to \infty,$$

where c denotes a positive constant and

$$\nu = 1 - \frac{1}{2}(1-\delta')^2 e^{-\delta}(1+\varepsilon)^2.$$

But we may adjust $\delta, \delta' \in (0,1)$ so that $\nu < 1/2$. Then this leads to the contradiction that, as $n \to \infty$, the expression on the left of inequality (4.51) grows faster than the expression on its right, so assertion (4.47) is proven.

3^0 We finally show that

$$\limsup_{t\to\infty} \frac{R_h(t)}{\sqrt{2t\log t}} \le 1. \tag{4.52}$$

Assume that this statement is false. Then there exists $\varepsilon > 0$ such that

$$\limsup_{t\to\infty} \frac{R_h(t)}{\sqrt{2t\log t}} > (1+\varepsilon)^2, \tag{4.53}$$

and, taking into account (4.47), we find $e < t_n < T_n \to \infty$ such that, for each n,

$$\begin{aligned} R_h(t_n) &= (1+\varepsilon)\sqrt{2t_n \log t_n}, \\ R_h(t) &> (1+\varepsilon)\sqrt{2t\log t} \qquad \text{for } t \in (t_n, T_n], \\ R_h(T_n) &= (1+\varepsilon)^2\sqrt{2T_n \log T_n}. \end{aligned} \tag{4.54}$$

Because of the last line in (4.54) and the monotonicity of R_h,

$$R_h(T_n + t) \ge (1+\varepsilon)^2\sqrt{2T_n\log T_n} > (1+\varepsilon)\sqrt{2(T_n+t)\log(T_n+t)}$$

for $0 \le t < \varepsilon T_n$ and all n. Together with the second line in (4.54) this implies that

$$R_h(t) > (1+\varepsilon)\sqrt{2t\log t} \qquad \text{for } t \in (t_n, (1+\varepsilon)t_n) \tag{4.55}$$

for all n. We want to prove that this is impossible. To this end we fix $\delta \in (0,\varepsilon)$ arbitrarily. Combining (4.40) with (4.41) and (4.42), we find a positive constant C_h'' such that, for each n,

$$\begin{aligned} &2\Big[R_h((1+\delta)t_n) - R_h(t_n)\Big] - C_h'' \\ &\quad \le \lambda \int_0^{(1+\delta)t_n} ds\, \mathbb{E}_0 \left(1 - \exp\left\{-\int_0^{(1+\delta)t_n - s} V(s+u, \beta(u))\, du\right\}\right) \\ &\quad\quad - \lambda \int_0^{t_n} ds\, \mathbb{E}_0 \left(1 - \exp\left\{-\int_0^{t_n - s} V(s+u, \beta(u))\, du\right\}\right) \\ &\quad = \lambda \int_{t_n}^{(1+\delta)t_n} ds\, \mathbb{E}_0 \left(1 - \exp\left\{-\int_0^{(1+\delta)t_n - s} V(s+u, \beta(u))\, du\right\}\right) \\ &\quad\quad + \lambda \int_0^{t_n} ds\, \mathbb{E}_0 \exp\left\{-\int_0^{t_n - s} V(s+u, \beta(u))\, du\right\} \\ &\qquad\qquad\qquad \times \left[1 - \exp\left\{-\int_{t_n - s}^{(1+\delta)t_n - s} V(s+u, \beta(u))\, du\right\}\right] \\ &\quad \le \lambda \int_0^{(1+\delta)t_n} ds\, \mathbb{E}_0 \left(1 - \exp\left\{-\int_0^{(1+\delta)t_n - s} V(t_n, \beta(u))\, du\right\}\right), \end{aligned} \tag{4.56}$$

where in the last line we have also used the monotonicity of $V(t,x)$ in t. Because of the upper bound (3.12) in Theorem 3.2 a) (recall Remark 3.4), we get for any $\delta' \in (0,1)$ that

$$
\begin{aligned}
&\mathbb{E}_0 \exp\left\{-\int_0^{(1+\delta)t_n - s} V(t_n, \beta(u))\, du\right\} \\
&\quad \geq \mathbb{E}_0 \exp\left\{-\int_0^{(1+\delta)t_n - s} v_h(|\beta(u)| - R_h(t_n))\, du\right\} \\
&\qquad \times 1\!\!1\Big(|\beta(u)| < (1-\delta')R_h(t_n) \text{ for } u \in [0, (1+\delta)t_n - s]\Big) \\
&\quad \geq \exp\{-(1+\delta)t_n v_h(-\delta' R_h(t_n))\} \\
&\qquad \times \mathbb{P}_0\Big(|\beta(u)| < (1-\delta')R_h(t_n) \text{ for } u \in [0, (1+\delta)t_n - s]\Big),
\end{aligned}
$$

where the first factor on the right may be further estimated from below by

$$
1 - (1+\delta)t_n v_h(-\delta' R_h(t_n)).
$$

Substituting this into (4.56), we obtain, for each n,

$$
\begin{aligned}
&2\Big[R_h((1+\delta)t_n) - R_h(t_n)\Big] - C_h'' \\
&\quad \leq \lambda(1+\delta)^2 t_n^2 v_h(-\delta' R_h(t_n)) \\
&\qquad + \lambda \int_0^{(1+\delta)t_n} ds\, \mathbb{P}_0\Big(|\beta(u)| \geq (1-\delta')R_h(t_n) \text{ for some } u \in [0,s]\Big)
\end{aligned}
\tag{4.57}
$$

It follows from (3.11), (3.9) and the first line in (4.54) that

$$
\lim_{n\to\infty} t_n^2 v_h(-\delta' R_h(t_n)) = 0. \tag{4.58}
$$

We next show that, for all n,

$$
\int_0^{(1+\delta)t_n} ds\, \mathbb{P}_0\Big(|\beta(u)| \geq (1-\delta')R_h(t_n) \text{ for some } u \in [0,s]\Big) \leq \frac{C t_n^\theta}{\sqrt{\log t_n}}, \tag{4.59}
$$

where

$$
\theta = 1 - \frac{1}{2}\frac{(1-\delta')^2(1+\varepsilon)^2}{1+\delta}
$$

and C denotes a positive constant that depends on δ, δ', ε but not on n. Applying the reflection principle for Brownian motion, using (4.43), and remembering the first line in (4.54), we obtain

$$\begin{aligned}
&\int_0^{(1+\delta)t_n} ds\,\mathbb{P}_0\Big(|\beta(u)| \geq (1-\delta')R_h(t_n) \text{ for some } u\in[0,s]\Big)\\
&\qquad \leq (1+\delta)t_n\mathbb{P}_0\Big(|\beta(u)| \geq (1-\delta')R_h(t_n) \text{ for some } u\in[0,(1+\delta)t_n]\Big)\\
&\qquad \leq 4(1+\delta)t_n\mathbb{P}_0\Big(\beta((1+\delta)t_n) \geq (1-\delta')R_h(t_n)\Big)\\
&\qquad \leq \frac{4(1+\delta)^{3/2}t_n}{\sqrt{\pi}(1-\delta')R_h(t_n)/\sqrt{t_n}}\,\exp\left\{-\frac{1}{4}\frac{(1-\delta')^2}{1+\delta}\left(\frac{R_h(t_n)}{\sqrt{t_n}}\right)^2\right\}\\
&\qquad = \frac{Ct_n^\theta}{\sqrt{\log t_n}},
\end{aligned}$$

which is the desired bound (4.59).

Choosing $\delta'\in(0,1)$ small as a function of $\delta\in(0,\varepsilon)$ and ε, we achieve that $\theta<1/2$. Then, combining (4.57) with (4.58) and (4.59) and remembering the first line in (4.54), we see that, for large n,

$$\begin{aligned}
R_h((1+\delta)t_n) &\leq R_h(t_n) + t_n^\theta\\
&\leq \left(1+t_n^{\theta-1/2}\right)(1+\varepsilon)\sqrt{2t_n\log t_n}\\
&\leq (1+\varepsilon)\sqrt{2(1+\delta)t_n\log\left((1+\delta)t_n\right)}\,.
\end{aligned}$$

But this contradicts (4.55), and so assertion (4.52) is proven.

4^0 The bounds (4.45), (4.47), and (4.52) together imply the assertion of our lemma. □

Proof of Lemma 3.3 for $d=1$. 1^0 Fix $h,r>0$ arbitrarily and let

$$\tau_x = \inf\{t\geq 0\colon \beta(t)=x\}$$

denote the first hitting time of $x\in\mathbb{R}$ by Brownian motion. We first show that

$$\int_0^t ds\,\mathbb{E}_r\exp\left\{-h\int_0^s \mathbb{1}(\beta(u)>r)\,du\right\}\delta_0(\beta(s)) \leq h^{-1/2}\mathbb{P}_0(\tau_r\leq t) \tag{4.60}$$

for all $t>0$. Indeed, performing a time-reversal of Brownian motion and applying the strong Markov property at time τ_r, we obtain

$$\begin{aligned}
&\int_0^t ds\,\mathbb{E}_r\exp\left\{-h\int_0^s \mathbb{1}(\beta(u)>r)\,du\right\}\delta_0(\beta(s))\\
&\qquad = \int_0^t ds\,\mathbb{E}_0\mathbb{1}(\tau_r\leq s)\exp\left\{-h\int_{\tau_r}^s \mathbb{1}(\beta(u)>r)\,du\right\}\delta_r(\beta(s))\\
&\qquad \leq \mathbb{E}_0\mathbb{1}(\tau_r\leq t)\int_{\tau_r}^\infty ds\,\exp\left\{-h\int_{\tau_r}^s \mathbb{1}(\beta(u)>r)\,du\right\}\delta_r(\beta(s))\\
&\qquad = \mathbb{P}_0(\tau_r\leq t)\,\mathbb{E}_r\int_0^\infty ds\,\exp\left\{-h\int_0^s \mathbb{1}(\beta(u)>r)\,du\right\}\delta_r(\beta(s)).
\end{aligned}$$

But the expectation on the right equals $1/\sqrt{h}$. This follows from the observation that the function

$$w(x) = \mathbb{E}_x \int_0^\infty ds \exp\left\{-h \int_0^s 1\!\!1(\beta(u) > r)\, du\right\} \delta_r(\beta(s))$$

is the minimal nonnegative solution of

$$w'' - h 1\!\!1_{(r,\infty)} w = -\delta_r$$

given by $w(x) = 1/\sqrt{h}$ for $x \le r$ and $w(x) = e^{-\sqrt{h}(x-r)}/\sqrt{h}$ for $x > r$. In this way we arrive at (4.60).

2^0 Fix $h \in (0,1)$ arbitrarily. Since

$$V(t-u, r) \ge V(t,r) \ge h 1\!\!1_{(R_h(t),\infty)}(r)$$

for $u \in [0,t]$ and $r \in \mathbb{R}$, we conclude from the Feynman-Kac representation (1.5) that

$$\varrho(t, R_h(t)) \le \lambda \int_0^t ds\, \mathbb{E}_{R_h(t)} \exp\left\{-h \int_0^s 1\!\!1\left(\beta(u) > R_h(t)\right)\, du\right\} \delta_0(\beta(s)).$$

Combining this with (4.60) and applying Brownian scaling, we find that

$$\varrho(t, R_h(t)) \le \frac{\lambda}{\sqrt{h}} \mathbb{P}_0\left(\tau_{R_h(t)} \le t\right) = \frac{\lambda}{\sqrt{h}} \mathbb{P}_0\left(\tau_1 \le \frac{t}{R_h^2(t)}\right).$$

As a consequence of Lemma 4.7, the probability on the right tends to zero as $t \to \infty$. $\square$

Proof of Theorem 2.3 for $d = 1$. Since Lemma 3.3 is proven, we may now apply Theorem 3.2 c) (recall Remark 3.4) to see that the assertion follows from Lemma 4.7. $\square$

Proof of Theorem 2.2 for $d = 1$. This is immediate from formula (4.1) and Theorem 2.3. $\square$

Proof of Theorem 2.1 d). Abbreviate

$$\varrho_\varepsilon(t,x) = \varepsilon \varrho\left(\frac{t}{\varepsilon^2}, \frac{x}{\varepsilon}\right), \quad V_\varepsilon(t,x) = V\left(\frac{t}{\varepsilon^2}, \frac{x}{\varepsilon}\right),$$

fix $h \in (0,1)$ arbitrarily, and introduce the rescaled h-front

$$R_h^\varepsilon(t) = \varepsilon R_h\left(\frac{t}{\varepsilon^2}\right).$$

It follows from (1.5) by scaling that ϱ_ε admits the Feynman-Kac representation

$$\varrho_\varepsilon(t,x) = \lambda \int_0^t ds\, \mathbb{E}_x \exp\left\{-\int_0^s \frac{1}{\varepsilon^2} V_\varepsilon(t-u, \beta(u))\, du\right\} \delta_0(\beta(s)). \qquad (4.61)$$

By Lemma 4.7, $R_h^\varepsilon(s) \to \infty$ as $\varepsilon \downarrow 0$ for all $s > 0$. Because of this, a combination of the upper bound (3.12) in Theorem 3.2 a) with (3.11) and (3.9) yields

$$\frac{1}{\varepsilon^2} V_\varepsilon(s,y) \le \frac{1}{\varepsilon^2} v_h \left(-\frac{1}{\varepsilon} \left(R_h^\varepsilon(s) - |y| \right) \right) \longrightarrow 0$$

as $\varepsilon \downarrow 0$ for all $(s,y) \in (0,\infty) \times \mathbb{R}$. Hence, applying Lebesgue's dominated convergence theorem, we conclude from (4.61) that

$$\varrho_\varepsilon(t,x) \longrightarrow \lambda \int_0^t ds\, \mathbb{E}_x \delta_0(\beta(s)) = \varrho_0(t,x)$$

pointwise as $\varepsilon \downarrow 0$. Since $\varrho_\varepsilon(t,r)$ is monotone in t and r, this implies the desired uniform convergence. $\square$

5. Age distribution

This section contains one more and final result on the distribution of age and space of the particles that are alive.

As before, let (ϱ, V) denote the solution of (1.1). Given $T \ge 0$, denote by ϱ_T the unique weak solution of the initial value problem

$$\begin{aligned} \frac{\partial \varrho_T}{\partial t} &= \Delta \varrho_T - V \varrho_T + \lambda \delta_0 \qquad \text{on } (T,\infty) \times \mathbb{R}^d, \\ \varrho_T(T,\cdot\,) &\equiv 0. \end{aligned} \tag{5.1}$$

For each $t > T$, $\varrho_T(t,\cdot\,)$ may be regarded as the spatial density of particles alive at time t that were born after time T.

Let $d \ge 2$. Then, for each $t > 0$,

$$N_t([0,s] \times B) = \frac{\int_{t^{1/d}B} \varrho_{t-st^{2/d}}(t,x)\, dx}{\int_{\mathbb{R}^d} \varrho(t,x)\, dx}$$

is the relative amount of particles alive at time t that are not older than $st^{2/d}$ and are located in the Borel set $t^{1/d}B$. Hence, N_t is a probability measure on $[0, t^{(d-2)/d}] \times \mathbb{R}^d$ that describes the rescaled age distribution of the particles alive at time t as a function of their location.

The aim of this section is to study the asymptotic behavior of N_t as $t \to \infty$. To this end we denote by p_λ the heat kernel for the Laplacian in the centered ball of volume λ with zero boundary condition. In dimension $d = 2$, given $T \ge 0$, we denote by ϱ_T^* the unique weak solution to the initial boundary value problem

$$\begin{aligned} \frac{\partial \varrho_T^*}{\partial t}(t,x) &= \Delta \varrho_T^*(t,x) + \lambda \delta_0(x), && |x| < \kappa_*(\lambda)\sqrt{T+t}, \\ \varrho_T^*(t,x) &= 0, && |x| = \kappa_*(\lambda)\sqrt{T+t}, \\ \varrho_T^*(0,x) &= 0, && |x| < \kappa_*(\lambda)\sqrt{T}. \end{aligned}$$

Note that ϱ_0^* coincides with ϱ^* given by (2.6).

Our main result in this section reads:

Theorem 5.1. *a) If $d \geq 3$, then $N_t \Rightarrow N$ as $t \to \infty$ in the sense of weak convergence of probability measures on $\mathbb{R}_+ \times \mathbb{R}^d$, and the limiting measure N has density*

$$n(s,x) = \begin{cases} 2d\left(\frac{\omega_d}{\lambda}\right)^{2/d} p_\lambda(s,x), & \text{if } |x| < \lambda, \\ 0, & \text{otherwise.} \end{cases}$$

b) If $d = 2$, then $N_t \Rightarrow N$ as $t \to \infty$ in the sense of weak convergence of probability measures on $[0,1] \times \mathbb{R}^2$, and the limiting measure N has density

$$n(s,x) = \begin{cases} \left(\left(1 - e^{-\kappa_*^2/4}\right)\lambda\right)^{-1} \frac{\partial}{\partial s}\varrho_{1-s}^*(s,x), & \text{if } |x| < \kappa_*(\lambda), \\ 0, & \text{otherwise.} \end{cases}$$

The results in Theorem 5.1 should be read as follows. For $d \geq 3$, $p_\lambda(s,x)$ is the rescaled limit of the number of particles at site x alive at time 1 $(= (1-s)+s)$ that were born *at* time $1-s$. For $d = 2$, $\varrho_{1-s}^*(s,x)$ is the rescaled limit of the number of particles at site x alive at time 1 $(= (1-s)+s)$ that were born *after* time $1-s$.

Proof. The proof is sketchy, because it uses tools that have been applied extensively before.

a) Assume that $d \geq 3$. Let $\varphi\colon \mathbb{R}^d \to \mathbb{R}$ be an arbitrary continuous test function with compact support. To study the weak convergence of N_t, we consider

$$\int \varphi(x)\, N_t([0,s] \times dx) = \frac{\int \varrho_{t-st^{2/d}}(t,x)\varphi\left(\frac{x}{t^{1/d}}\right)\, dx}{\int_{\mathbb{R}^d} \varrho(t,x)\, dx}. \tag{5.2}$$

An application of the (time-reversed and rescaled) Feynman-Kac formula for ϱ_T yields

$$\begin{aligned} t^{-2/d} &\int \varrho_{t-st^{2/d}}(t,x)\varphi\left(\frac{x}{t^{1/d}}\right)\, dx \\ &= \lambda \int_0^s du\, \mathbb{E}_0 \exp\left\{-t^{2/d}\int_0^u V\left(t - t^{2/d}(u-v), t^{1/d}\beta(v)\right)\, dv\right\} \varphi(\beta(u)). \end{aligned}$$

From Lemma 4.3 b), Theorem 3.2 a), (3.11), and (3.9) we conclude that

$$\lim_{t\to\infty} t^{2/d} V\left(t - t^{2/d}(u-v), t^{1/d}\beta(v)\right) = \begin{cases} 0, & \text{if } |\beta(v)| < (\lambda/\omega_d)^{1/d}, \\ \infty, & \text{if } |\beta(v)| > (\lambda/\omega_d)^{1/d} \end{cases}$$

(cf. also Theorem 2.1 a)). Using this, we find that

$$\begin{aligned}
&\lim_{t\to\infty} t^{-2/d} \int \varrho_{t-st^{2/d}}(t,x)\varphi\left(\frac{x}{t^{1/d}}\right)\,dx \\
&\qquad = \lambda \int_0^s du\, \mathbb{E}_0 1\!\!1\left(|\beta(v)| < (\lambda/\omega_d)^{1/d} \text{ for } v\in[0,u]\right)\varphi(\beta(u)) \\
&\qquad = \lambda \int_0^s du \int dx\, p_\lambda(u,x)\varphi(x).
\end{aligned}$$

According to Theorem 2.2,

$$\lim_{t\to\infty} t^{-2/d} \int \varrho(t,x)\,dx = \frac{\lambda}{2d}\left(\frac{\lambda}{\omega_d}\right)^{2/d}.$$

Combining (5.2) with the last two limits, we arrive at

$$\lim_{t\to\infty} \int \varphi(x)\, N_t([0,s]\times dx) = 2d\left(\frac{\omega_d}{\lambda}\right)^{2/d} \int_0^s du \int dx\, p_\lambda(u,x)\varphi(x),$$

and we are done.

b) Let us now turn to the case $d=2$. In analogy with a), for $s\in[0,1]$,

$$\begin{aligned}
&t^{-1} \int \varrho_{t-st}(t,x)\varphi\left(\frac{x}{t^{1/2}}\right)\,dx \\
&\qquad = \lambda \int_0^s du\, \mathbb{E}_0 \exp\left\{-t\int_0^u V\left(t(1-(u-v)), t^{1/2}\beta(v)\right)\,dv\right\}\varphi(\beta(u)).
\end{aligned}$$

Using Theorem 4.6 a), Theorem 3.2 a), (3.11), and (3.9), we find that

$$\lim_{t\to\infty} tV\left(t(1-(u-v)), t^{1/2}\beta(v)\right) = \begin{cases} 0, & \text{if } |\beta(v)| < \kappa_*\sqrt{1-(u-v)}, \\ \infty, & \text{if } |\beta(v)| > \kappa_*\sqrt{1-(u-v)}. \end{cases}$$

Using this, we conclude that

$$\begin{aligned}
&\lim_{t\to\infty} t^{-1} \int \varrho_{t-st}(t,x)\varphi\left(\frac{x}{t^{1/2}}\right)\,dx \\
&\qquad = \lambda \int_0^s du\, \mathbb{E}_0 1\!\!1\Big(|\beta(v)| < \kappa_*\sqrt{1-(u-v)} \text{ for } v\in[0,u]\Big)\varphi(\beta(u))
\end{aligned}$$

The expression on the right is nothing but the Feynman-Kac representation for

$$\int_{|x|<\kappa_*} \varrho^*_{1-s}(s,x)\varphi(x)\,dx.$$

According to Theorem 2.2,

$$\lim_{t\to\infty} t^{-1}\int \varrho(t,x)\,dx = \left(1-e^{-\kappa_*^2/4}\right)\lambda.$$

Together with formula (5.2), the last two limits yield

$$\lim_{t\to\infty}\int \varphi(x)N_t([0,s]\times dx) = \left(\left(1-e^{-\kappa_*^2/4}\right)\lambda\right)^{-1}\int_{|x|<\kappa_*} \varrho_{1-s}^*(s,x)\varphi(x)\,dx,$$

and we are done. $\square$

A corresponding, but trivial, scaling limit for the age distribution of the particles also exists in dimension $d = 1$. Because of an obvious analogue of Theorem 2.2, almost all particles that were born after time T are still alive at time t, and, by Theorem 2.1 d), their rescaled density is asymptotically close to the free particle density ϱ_0.

A. Appendix on the Green function $G_{R,h}$ ($d \geq 3$)

Given $R > 0$ and $h > 0$, we consider the Green function $G_{R,h}$ in polar coordinates associated with the potential

$$V_{R,h}(r) = \begin{cases} 0, & 0 \leq r \leq R, \\ h, & r > R. \end{cases}$$

In other words, $G_{R,h}$ solves

$$G''_{R,h}(r) + \frac{d-1}{r}G'_{R,h}(r) - V_{R,h}(r)G_{R,h}(r) + \delta_0^{(d)}(r) = 0 \tag{A.1}$$

for r in $(0,R)\cup(R,\infty)$ subject to the gluing conditions

$$G_{R,h}(R-0) = G_{R,h}(R+0) \quad \text{and} \quad G'_{R,h}(R-0) = G'_{R,h}(R+0). \tag{A.2}$$

As before, let G denote the Green function in polar coordinates associated with the potential $V \equiv 0$. Let G_1 denote the Green function associated with the potential $V \equiv 1$. For G we have the explicit formula

$$G(r) = \frac{c_d}{r^{d-2}}, \qquad r > 0, \tag{A.3}$$

whereas for G_1 we have the following integral representation in terms of the heat kernel:

$$G_1(r) = \int_0^\infty e^{-t}(4\pi t)^{-d/2}e^{-r^2/4t}\,dt, \qquad r > 0. \tag{A.4}$$

Lemma A.1. *a) For arbitrary $R, h > 0$ and all $r \in (0,R)$,*

$$G(r) - G(R) \leq G_{R,h}(r) \leq G(r) - G(R) + \frac{|G'(R)|}{\sqrt{h}}. \tag{A.5}$$

b) For arbitrary $R, h > 0$ and all $r \in (R, \infty)$,

$$G_{R,h}(r) \leq \frac{|G'(R)|}{\sqrt{h}} \frac{G_1(\sqrt{h}r)}{G_1(\sqrt{h}R)}.$$

c) For arbitrary $R, h > 0$ with $\sqrt{h}R > 1$ and all $r \in (R, \infty)$,

$$G_{R,h}(r) \leq \frac{C(d)}{\sqrt{h}R^{\frac{d-1}{2}} r^{\frac{d-1}{2}}} e^{-\sqrt{h}(r-R)},$$

where $C(d)$ is a positive constant depending on d only.

Proof. It may be seen from equation (A.1) and the gluing conditions (A.2) that the Green function $G_{R,h}$ satisfies the scaling relation

$$G_{R,h}(r) = h^{\frac{d-2}{2}} G_{\sqrt{h}R,1}(\sqrt{h}r).$$

Because of this and the explicit formula (A.3), it will be sufficient to prove Lemma A.1 for $h = 1$. The function $G_{R,1}$ has the form

$$G_{R,1}(r) = \begin{cases} G(r) - c_0, & 0 < r < R, \\ c_1 G_1(r), & r > R, \end{cases}$$

where the constants c_0 and c_1 are determined by the gluing conditions (A.2). We obtain

$$G_{R,1}(r) = \begin{cases} G(r) - G(R) + \frac{G'(R)}{G_1'(R)} G_1(R), & 0 < r < R, \\ \frac{G'(R)}{G_1'(R)} G_1(r), & r > R. \end{cases} \tag{A.6}$$

Clearly, the derivatives G' and G_1' are negative. After making the substitution $t = r/2s$ in (A.4), we may rewrite it in the form

$$G_1(r) = \frac{1}{2(2\pi)^{d/2}} \frac{e^{-r}}{r^{(d-2)/2}} \psi(r), \tag{A.7}$$

where

$$\psi(r) = \int_0^\infty s^{\frac{d-4}{2}} \exp\left\{-r \frac{s + (1/s) - 2}{2}\right\} ds. \tag{A.8}$$

Since $s + (1/s) \geq 2$, we have $\psi'(r) < 0$ for all r. Hence, differentiating (A.7), we find that

$$|G_1'(R)| > G_1(R).$$

Inserting this in (A.6), we arrive at the assertions a) and b) of our lemma for $h = 1$. Assertion c) follows from assertion b). Indeed, by (A.3) and (A.7),

$$|G'(R)| \frac{G_1(r)}{G_1(R)} = (d-2)c_d \frac{e^{-(r-R)}}{R^{\frac{d-1}{2}} r^{\frac{d-1}{2}}} \frac{\sqrt{r}\psi(r)}{\sqrt{R}\psi(R)}.$$

But an application of the Laplace method to (A.8) yields $\sqrt{r}\psi(r) \to \sqrt{2\pi}$ as $r \to \infty$, and we are done. $\square$

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