Random subgraphs of finite graphs: II. The lace expansion and the triangle condition

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Abstract

In a previous paper, we defined a version of the percolation triangle condition that is suitable for the analysis of bond percolation on a finite connected transitive graph, and showed that this triangle condition implies that the percolation phase transition has many features in common with the phase transition on the complete graph. In this paper, we use a new and simplified approach to the lace expansion to prove quite generally that for finite graphs that are tori the triangle condition for percolation is implied by a certain triangle condition for simple random walks on the graph.

The latter is readily verified for several graphs with vertex set $\{0, 1, \ldots, r-1\}^n$, including the n-cube, the n-dimensional torus with nearest-neighbor bonds with n large and fixed, and the n-dimensional torus with spread-out (long range) bonds with $n > 6$ fixed. The conclusions of our previous paper thus apply to the percolation phase transition for each of the above examples.

1 Introduction and results

1.1 Introduction

The percolation phase transition on the complete graph is well understood and forms a central part of modern graph theory [4, 6, 20]. In the language of mathematical physics, the phase transition is *mean-field*. It can be expected that the percolation phase transition on many other high-dimensional finite graphs will be similar to that for the complete graph. In other words, mean-field behaviour will apply much more generally.

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In a previous paper [7], we introduced the finite-graph triangle condition, and proved that it is a sufficient condition for several aspects of the phase transition on a finite connected transitive graph to be mean-field. This triangle condition is an adaptation of the well-known triangle condition of Aizenman and Newman [3] for infinite graphs. In this paper, we verify the finite-graph triangle condition for a class of graphs with the structure of high-dimensional tori. Examples include the n-cube, the Hamming cube and periodic approximations to $\mathbb{Z}^n$ for large n.

Our proof of the triangle condition is based on an adaptation of the percolation lace expansion of Hara and Slade [13] from $\mathbb{Z}^n$ to finite tori. We use the same expansion as [13] but our proof of convergence of the expansion is new and improved. This is the first time that the lace expansion has been applied in a setting where finite-size scaling plays a role. An advance in our application of the lace expansion is that we prove a general theorem that the percolation triangle condition on a finite torus is a consequence of a corresponding condition for *random walks* on the torus. Thus we are able to verify the percolation triangle condition for our examples by a relatively simple analysis of random walks on these graphs.

1.2 The triangle condition on infinite graphs

Let $\mathbb{V}$ be a finite or infinite set and let $\mathbb{B}$ be a subset of the set of all two-element subsets $\{x, y\} \subset \mathbb{V}$. Then $\mathbb{G} = (\mathbb{V}, \mathbb{B})$ is a finite or infinite graph with vertex set $\mathbb{V}$ and bond (or edge) set $\mathbb{B}$. The degree of a vertex $x \in \mathbb{V}$ is defined to be the number of edges containing x. A bijective map $\varphi : \mathbb{V} \to \mathbb{V}$ is called a graph-isomorphism if $\{\varphi(x), \varphi(y)\} \in \mathbb{B}$ whenever $\{x, y\} \in \mathbb{B}$. We say that $\mathbb{G}$ is *transitive* if for each pair $x, y \in \mathbb{V}$ there is a graph-isomorphism φ with $\varphi(x) = y$. We will always assume that $\mathbb{G}$ is connected and transitive, and denote the common degree of each vertex by Ω.

We consider percolation on $\mathbb{G}$. That is, we associate independent Bernoulli random variables to the edges, taking the value "occupied" with probability p and "vacant" with probability $1 - p$, where $p \in [0, 1]$ is a parameter. Let $x \leftrightarrow y$ denote the event that vertices x and y are connected by a path in $\mathbb{G}$ consisting of occupied bonds, and let $C(x) = \{y \in \mathbb{V} : x \leftrightarrow y\}$ denote the connected cluster of x. Let

$$\tau_p(x, y) = \mathbb{P}_p(x \leftrightarrow y) \tag{1.1}$$

denote the *two-point function* and define the *susceptibility* by

$$\chi(p) = \mathbb{E}_p|C(0)|. \tag{1.2}$$

For many infinite graphs, such as $\mathbb{Z}^n$ with $n \geq 2$, or for a regular tree with degree at least three, there is a $p_c = p_c(\mathbb{G}) \in (0, 1)$ such that

$$p_c(\mathbb{G}) = \sup\{p : \chi(p) < \infty\} = \inf\{p : \mathbb{P}_p(|C(0)| = \infty) > 0\}. \tag{1.3}$$

Thus $\chi(p) < \infty$ if and only if $p < p_c$, $\mathbb{P}_p(|C(0)| = \infty) > 0$ if $p > p_c$, and $\mathbb{P}_p(|C(0)| = \infty) = 0$ for $p < p_c$. The equality of the infimum and supremum of (1.3) is a theorem of [2, 22].

Percolation on a tree is well understood [10, Chapter 10], and infinite graphs whose percolation phase transition is analogous to the transition on a tree are said to exhibit mean-field behaviour. In 1984, Aizenman and Newman [3] introduced the triangle condition as a sufficient condition for mean-field behaviour. The triangle condition is defined in terms of the *triangle diagram*

$$\nabla_p(x, y) = \sum_{w,z \in \mathbb{V}} \tau_p(x, w)\tau_p(w, z)\tau_p(z, y), \tag{1.4}$$

and states that for all $x \in \mathbb{V}$

$$\nabla_{p_c}(x,x) < \infty. \tag{1.5}$$

It is predicted that the triangle condition on $\mathbb{Z}^n$ holds for all $n > 6$. Aizenman and Newman used a differential inequality for $\chi(p)$ to show that the triangle condition implies that

$$\chi(p) = \Theta((p_c - p)^{-\gamma}) \quad \text{uniformly in } p < p_c, \tag{1.6}$$

with $\gamma = 1$, and Nguyen [23] extended this to show that

$$\frac{\mathbb{E}_p[|C(0)|^{t+1}]}{\mathbb{E}[|C(0)|^t]} = \Theta((p_c - p)^{-\Delta_{t+1}}) \quad \text{uniformly in } p < p_c, \tag{1.7}$$

with $\Delta_{t+1} = 2$ for $t = 1, 2, 3, \ldots$. Subsequently, Barsky and Aizenman [5] showed, in particular, that the triangle condition also implies that the percolation probability obeys

$$\mathbb{P}_p(|C(0)| = \infty) = \Theta((p - p_c)^{\hat{\beta}}) \quad \text{uniformly in } p \geq p_c, \tag{1.8}$$

with $\hat{\beta} = 1$.

In 1990, Hara and Slade established the triangle condition for nearest-neighbor bond percolation on $\mathbb{Z}^n$ for large n (it is now known that $n \geq 19$ is large enough), and for a wide class of long-range models, called *spread-out* models, for $n > 6$ [13, 14]. Their proof of the triangle condition was based on the *lace expansion*, an adaptation of an expansion introduced in 1985 by D.C. Brydges and T. Spencer [9] to study the self-avoiding walk in high dimensions. Since the late 1980s, lace expansion methods have been used to derive detailed estimates on the critical behaviour of several models in high dimensions; see [14, 21, 25] for reviews. Recent extensions of the lace expansion for percolation can be found in [12, 16].

1.3 The triangle condition on finite graphs

On a finite graph, $|C(0)| \leq |\mathbb{V}| < \infty$. Thus, there cannot be a phase transition characterized by divergence to infinity of the susceptibility or existence of an infinite cluster. Instead, the phase transition takes place in a small window of p values, below which clusters are typically small in size and above which a single giant cluster coexists with many relatively small clusters. The basic example is the phase transition on the complete graph.

Let $\mathbb{G}$ be a connected transitive finite graph, let $V = |\mathbb{V}| < \infty$ denote its number of vertices, and let Ω denote the common degree of these vertices. The susceptibility $\chi(p) = \mathbb{E}_p|C(0)|$ is an increasing function of p, with $\chi(0) = 1$ and $\chi(1) = V$. In [7], we defined the *critical threshold* $p_c = p_c(\mathbb{G})$ to be the unique solution to the equation

$$\chi(p_c(\mathbb{G})) = \lambda V^{1/3}, \tag{1.9}$$

where λ is a fixed small parameter. The flexibility in the choice of λ in (1.9) is connected with the fact that the phase transition in a finite system is smeared out over a window rather than occurring at a sharply defined threshold, and any value in the window could be chosen as a threshold.

On a finite graph, the triangle diagram (1.4) is bounded above by V^2, and thus (1.5) is satisfied trivially. In [7], we defined the triangle condition for a finite graph to be the statement that

$$\nabla_{p_c(\mathbb{G})}(x,y) \leq \delta_{x,y} + a_0, \tag{1.10}$$

where a_0 is sufficiently small. In particular, (1.10) implies that $\nabla_{p_c(\mathbb{G})}(x, y)$ is uniformly bounded as $V \to \infty$. In addition, we defined the *stronger* triangle condition to be the statement that there are constants K_1, K_2 such that for $p \le p_c(\mathbb{G})$

$$\nabla_p(x, y) \le \delta_{x,y} + K_1\Omega^{-1} + K_2\frac{\chi^3(p)}{V}. \tag{1.11}$$

Note that (1.10) is a consequence of (1.11), provided Ω is sufficiently large and λ is sufficiently small. Moreover, since $\sum_y \nabla_{p_c}(x, y) = \chi^3(p_c) = \lambda^3 V$, (1.10) implies that $\lambda^3 \le V^{-1} + a_0$ and hence λ must be taken to be small for the triangle condition to hold.

As described in more detail below, we showed in [7] that the triangle condition (1.10) implies that the percolation phase transition on a finite graph shares many features with the transition on the complete graph. In this paper, we prove (1.11) and hence (1.10) for several finite graphs, assuming that λ is a sufficiently small constant. These graphs all have vertex set $\mathbb{V} = \{0, 1, \ldots, r-1\}^n$ for some $r \ge 2$ and $n \ge 1$, with periodic boundary conditions. We consider various edge sets.

1.4 Periodic tori

There are three levels of generality that we will use. First, we use $\mathbb{G}$ to denote a finite connected transitive graph of degree Ω. Our derivation of the lace expansion, and much of the diagrammatic estimation of the lace expansion, is valid for general $\mathbb{G}$. Second, for our analysis of the lace expansion, we restrict $\mathbb{G}$ to have the vertex set of the torus $\mathbb{T} = \mathbb{T}_{r,n} = (\mathbb{Z}_r)^n$, where $\mathbb{Z}_r$ denotes the integers modulo r, for $r = 2, 3, \ldots$. The torus $\mathbb{T}_{r,n}$ is an additive group under coordinate-wise addition modulo r, with volume $V = r^n$. We allow any edge set for the torus that respects the symmetries of translation and $x \mapsto -x$ reflections. That is, we assume that the edge set is such that $\{0, x\}$ is an edge if and only if $\{y, y \pm x\}$ is an edge for any vertex y. Third, we will verify the stronger percolation triangle condition (1.11) for the following specific edge sets:

1. The narrow torus: an edge joins vertices that differ by 1 (modulo r) in exactly one component, for $r \ge 2$ fixed and $n \to \infty$. For $r = 2$, this is the n-cube. Here $\Omega = 2n$ for $r \ge 3$ and $\Omega = n$ for $r = 2$.

2. The Hamming torus: an edge joins vertices that differ in exactly one component, again with the periodic boundary condition, for $r \ge 2$ fixed and $n \to \infty$. Here $\Omega = (r-1)n$.

3. The wide torus in high dimensions: the same edge set as the narrow torus but now n is large and fixed and we study the limit $r \to \infty$ to approximate $\mathbb{Z}^n$. Here $\Omega = 2n$.

4. The wide spread-out torus in dimensions $n > 6$: an edge joins vertices $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ if $0 < \max_{i=1,\ldots,n} |x_i - y_i| \le L$ (with periodic boundary conditions) with $n > 6$ fixed, L large and fixed, in the limit $r \to \infty$ to approximate range-L percolation on $\mathbb{Z}^n$. Here $\Omega = [(2L+1)^n - 1]$ for r large compared to L.

1.5 Fourier analysis on a torus

Our method relies heavily on Fourier analysis. Fourier analysis on the torus $\mathbb{T}_{r,n}$ is a special case of a more general theory of Fourier analysis on abelian groups. Let G be a finite abelian group, with group operation denoted +. A *character* is a homomorphism $\chi : G \to \mathbb{C}$ of G into the multiplicative group of non-zero complex numbers, i.e., for all $a, b \in G$ we have

$$\chi(a + b) = \chi(a)\chi(b). \tag{1.12}$$

Let $1 \in G$ denote the identity element. The order of any element of G divides $|G|$, and hence $\chi(a)^{|G|} = \chi(a^{|G|}) = \chi(1) = 1$ for every $a \in G$. Therefore, χ maps into the $|G|$ roots of unity. The set $\hat{G}$ of characters is again an abelian group under the operation $(\chi\psi)(a) = \chi(a)\psi(a)$. The Fourier transform of a function $f : G \to \mathbb{C}$ is the function $f : \hat{G} \to \mathbb{C}$ defined by

$$\hat{f}(\chi) = \sum_{a \in G} \chi(a) f(a), \tag{1.13}$$

and the *Fourier inversion formula* is

$$f(a) = \frac{1}{|G|} \sum_{\chi \in \hat{G}} \hat{f}(\chi)\overline{\chi(a)}. \tag{1.14}$$

The *convolution* of functions f, g on G is defined by

$$(f * g)(a) = \sum_{b \in G} f(b)g(a - b), \tag{1.15}$$

and the Fourier transform of a convolution is the product of the Fourier transforms:

$$\widehat{f * g} = \hat{f}\hat{g}. \tag{1.16}$$

We now take $G = \mathbb{T}_{r,n}$, where the group action is addition modulo r. The group $\hat{\mathbb{T}}_{r,n}$ of characters is generated by $\chi^{(j)}(x) = e^{2\pi i x_j / r}$ $(j = 1, \ldots, n)$, and hence is isomorphic to $\mathbb{T}_{r,n}$. For convenience, we may equivalently regard $\hat{\mathbb{T}}_{r,n}$ as the group $\mathbb{T}^*_{r,n} = \frac{2\pi}{r}\mathbb{T}_{r,n}$. This identification is made explicit by the isomorphism $k \mapsto \chi_k$ of $\mathbb{T}^*_{r,n}$ onto $\hat{\mathbb{T}}_{r,n}$, where $\chi_k(x) = e^{ik\cdot x}$ with the dot product defined by $k \cdot x = \sum_{j=1}^n k_j x_j$. We will *always* identify the dual torus as $\mathbb{T}^*_{r,n} = \frac{2\pi}{r}\{-\lfloor \frac{r-1}{2} \rfloor, \ldots, \lceil \frac{r-1}{2} \rceil\}^n$, so that each component of $k \in \mathbb{T}^*_{r,n}$ is between $-\pi$ and π. The reason for this identification is that the point $k = 0$ plays a special role, and we do not want to see it mirrored at the point $(2\pi, \ldots, 2\pi)$. The Fourier transform of $f : \mathbb{T}_{r,n} \to \mathbb{C}$ can be written as

$$\hat{f}(k) = \sum_{x \in \mathbb{T}_{r,n}} f(x) e^{ik\cdot x} \qquad (k \in \mathbb{T}^*_{r,n}), \tag{1.17}$$

with the inverse Fourier transform given by

$$f(x) = \frac{1}{V} \sum_{k \in \mathbb{T}^*_{r,n}} \hat{f}(k) e^{-ik\cdot x}. \tag{1.18}$$

1.6 The triangle diagram in Fourier form

It is convenient to regard the two-point function or triangle diagram as a function of a single variable, e.g., $\tau_p(x, y) = \tau_p(y - x)$. With this identification,

$$\hat{\tau}_p(k) = \sum_{x \in \mathbb{T}_{r,n}} \tau_p(0, x) e^{ik\cdot x}, \tag{1.19}$$

where 0 denotes the origin of $\mathbb{T}_{r,n}$. It is shown in [3] that $\hat{\tau}_p(k) \geq 0$ for all $k \in \mathbb{T}^*_{r,n}$. The expected cluster size and two-point function are related by

$$\chi(p) = \mathbb{E}_p|C(0)| = \sum_{x\in\mathbb{T}_{r,n}} \mathbb{E}_p I[x \in C(0)] = \sum_{x\in\mathbb{T}_{r,n}} \tau_p(0,x) = \hat{\tau}_p(0), \tag{1.20}$$

where $I[E]$ denotes the indicator function for the event E. In particular,

$$\hat{\tau}_{p_c}(0) = \chi(p_c) = \lambda V^{1/3}. \tag{1.21}$$

Recalling (1.15), the triangle diagram (1.4) can be written as

$$\nabla_p(x,y) = (\tau_p * \tau_p * \tau_p)(y-x). \tag{1.22}$$

By (1.16) and (1.18), this implies that $\hat{\nabla}_p(k) = \hat{\tau}_p(k)^3$ and

$$\nabla_p(x,y) = \frac{1}{V}\sum_{k\in\mathbb{T}^*_{r,n}} \hat{\nabla}_p(k) e^{-ik\cdot(y-x)} = \frac{1}{V}\sum_{k\in\mathbb{T}^*_{r,n}} \hat{\tau}_p(k)^3 e^{-ik\cdot(y-x)}. \tag{1.23}$$

By (1.21), the contribution to the right side of (1.23) due to the term $k=0$ is $V^{-1}\lambda^3 V = \lambda^3$.

1.7 Main results

1.7.1 The random walk triangle condition

For $x, y \in \mathbb{T}_{r,n}$, let

$$D(x,y) = D(y-x) = \frac{1}{\Omega} I[\{x,y\} \in \mathbb{B}], \tag{1.24}$$

where $\mathbb{B}$ denotes a particular choice of edge set for the torus. As in Section 1.4, we assume that $\mathbb{B}$ is symmetric under translations and under $x \mapsto -x$ reflections. Thus $D(x)$ represents the 1-step transition probability for a random walk to step from 0 to a neighbor x. We make the following assumptions on D, which can be regarded as assumptions on the edge set $\mathbb{B}$.

Assumption 1.1. *There exists $\beta > 0$ such that*

$$\max_{x\in\mathbb{T}_{r,n}} D(x) \leq \beta \tag{1.25}$$

and

$$\frac{1}{V}\sum_{k\in\mathbb{T}^*_{r,n}:k\neq 0} \frac{\hat{D}(k)^2}{[1-\hat{D}(k)]^3} \leq \beta. \tag{1.26}$$

The assumption (1.25) is straightforward. As we will discuss in more detail in Section 2, the critical two-point function for random walks is $[1-\hat{D}(k)]^{-1}$, and comparing with the right side of (1.23), the assumption (1.26) can be interpreted as a kind of generalized triangle condition for random walks. For any D defined by (1.24), (1.25) implies that $\beta \geq \Omega^{-1} \geq V^{-1}$. We will require below that β be small.

Random walks on each of the four tori listed in Section 1.4 obeys Assumption 1.1 with β proportional to Ω^{-1}, as the following proposition shows. The proof of the proposition is given in Section 2.

Proposition 1.2. *There is a constant $a > 0$, which remains fixed as the volume $V = r^n$ goes to infinity, such that random walks on each of the four tori listed in Section 1.4 obey Assumption 1.1, with $\beta = an^{-1}$ for the narrow torus, the Hamming torus and the wide torus in high dimensions, and with $\beta = aL^{-n}$ for the wide spread-out torus in dimensions $n > 6$.*

1.7.2 The triangle condition and its consequences

Our main result is that if Assumption 1.1 holds with appropriately small parameters, then the percolation triangle condition holds. By Proposition 1.2, this establishes the triangle condition for the four tori listed in Section 1.1.

Theorem 1.3 (The triangle condition). *Consider the torus $\mathbb{T}_{r,n}$ with edge set such that $\{0, x\}$ is an edge if and only if $\{y, y \pm x\}$ is an edge for any vertex y. If Assumption 1.1 holds and if $\lambda^3 \vee \beta$ is sufficiently small, then there are constants K_1, K_2 such that the stronger triangle condition* (1.11) *holds in the form*

$$\nabla_{p_c}(x, y) = \delta_{x,y} + K_1\beta + K_2\frac{\chi^3(p)}{V}. \tag{1.27}$$

This establishes (1.11) for our four tori, since β is proportional to Ω^{-1} in Proposition 1.2. It follows that the various consequences of the triangle condition established in [7] hold for these four tori. We now summarize these consequences in this context. In the following discussion, the "four tori" are the four tori listed in Section 1.4, in the limits $r \to \infty$ or $n \to \infty$ as indicated there. Also, we assume throughout the discussion that λ is chosen sufficiently small and V sufficiently large that the results of [7] apply. In particular, we assume that $\lambda V^{1/3}$ is bounded below by a large positive constant, as required in [7, Theorems 1.2–1.4]. Note that if λ is a fixed positive constant then this condition merely states that V is large. We write $|C(x)|$ for the number of vertices in the cluster of x, let $\mathcal{C}_{\max}$ denote a cluster of maximal size, and let

$$|\mathcal{C}_{\max}| = \max\{|C(x)| : x \in \mathbb{V}\}. \tag{1.28}$$

The asymptotic behaviour of the critical value p_c is given in [7, Theorem 1.5] as follows.

Theorem 1.4 (Critical threshold). *For the four tori,*

$$p_c = \frac{1}{\Omega}\Big[1 + O(\Omega^{-1}) + O(\lambda^{-1}V^{-1/3})\Big]. \tag{1.29}$$

For the subcritical phase, the following results are consequences of [7, Theorems 1.2, 1.5]. A version of (1.30) valid for all $p \le p_c$ is given in [7, Theorem 1.5]; see (6.1) below.

Theorem 1.5 (Subcritical phase). *Let $p = p_c - \Omega^{-1}\epsilon$ with $\epsilon \ge 0$. For the four tori, the following hold.*
i) If $\epsilon\lambda V^{1/3} \to 0$ as $V \to \infty$, then as $V \to \infty$,

$$\chi(p) = \frac{1}{\epsilon}[1 + o(1)]. \tag{1.30}$$

ii) For all $\epsilon \ge 0$,

$$10^{-4}\chi^2(p) \le \mathbb{E}_p\Big(|\mathcal{C}_{\max}|\Big) \le 2\chi^2(p)\log(V/\chi^3(p)), \tag{1.31}$$

$$\mathbb{P}_p\Big(|\mathcal{C}_{\max}| \le 2\chi^2(p)\log(V/\chi^3(p))\Big) \ge 1 - \frac{\sqrt{e}}{[2\log(V/\chi^3(p))]^{3/2}}, \tag{1.32}$$

and, for $\omega \ge 1$,

$$\mathbb{P}_p\Big(|\mathcal{C}_{\max}| \ge \frac{\chi^2(p)}{3600\omega}\Big) \ge \Big(1 + \frac{36\chi^3(p)}{\omega V}\Big)^{-1}. \tag{1.33}$$

Inside a scaling window of width proportional to $V^{-1/3}$, the following results are consequences of [7, Theorem 1.3].

Theorem 1.6 (Scaling Window). *Fix* $\Lambda < \infty$. *Let* $p = p_c + \Omega^{-1}\epsilon$ *with* $|\epsilon| \le \Lambda V^{-1/3}$. *Then there exist constants* $b_1, \ldots, b_8$ *such that the following hold.*
i) If $k \le b_1 V^{2/3}$, *then*

$$\frac{b_2}{\sqrt{k}} \le P_{\ge k}(p) \le \frac{b_3}{\sqrt{k}}. \tag{1.34}$$

ii)

$$b_4 V^{2/3} \le \mathbb{E}_p\Big[|\mathcal{C}_{\max}|\Big] \le b_5 V^{2/3} \tag{1.35}$$

and, if $\omega \ge 1$, *then*

$$\mathbb{P}_p\Big(\omega^{-1}V^{2/3} \le |\mathcal{C}_{\max}| \le \omega V^{2/3}\Big) \ge 1 - \frac{b_6}{\omega}. \tag{1.36}$$

iii)

$$b_7 V^{1/3} \le \chi(p) \le b_8 V^{1/3}. \tag{1.37}$$

In the above statements, the constants b_2 *and* b_3 *can be chosen independent of* λ *and* Λ, *the constants* b_5 *and* b_8 *depend on* Λ *and not* λ, *and the constants* b_1, b_4, b_6 *and* b_7 *depend on both* λ *and* Λ.

For the supercritical phase, the following results are consequences of [7, Theorem 1.4].

Theorem 1.7 (Supercritical phase). *Let* $p = p_c + \epsilon\Omega^{-1}$ *with* $\epsilon \ge 0$.
i)

$$\mathbb{E}_p(|\mathcal{C}_{\max}|) \le 21\epsilon V + 7V^{2/3}, \tag{1.38}$$

and, for all $\omega > 0$,

$$\mathbb{P}_p\Big(|\mathcal{C}_{\max}| \le \omega(V^{2/3} + \epsilon V)\Big) \ge 1 - \frac{21}{\omega}. \tag{1.39}$$

ii)

$$\chi(p) \le 81(V^{1/3} + \epsilon^2 V). \tag{1.40}$$

Theorem 1.7 provides upper bounds on the size of clusters in the supercritical phase. To see that a phase transition occurs at p_c, one wants a *lower* bound. We have not proved a lower bound at the level of generality of all four tori, but we have obtained a lower bound for the case of the n-cube $\mathbb{T}_{2,n} = \{0,1\}^n$. This is the content of the following theorem, which is proved in [8, Theorem 1.5]. The statement that E_n occurs a.a.s. means that $\lim_{n\to\infty} \mathbb{P}(E_n) = 0$.

Theorem 1.8 (Supercritical phase for the n-cube). *Let* $\mathbb{G} = \mathbb{T}_{2,n}$, *let* λ *be sufficiently small, and let* $p = p_c + \epsilon n^{-1}$. *There are constants* c_1, c_2 *such that for all* $e^{-c_1 n^{1/3}} \le \epsilon \le 1$,

$$|\mathcal{C}_{\max}| \ge c_2\epsilon 2^n \qquad a.a.s.\ as\ n \to \infty, \tag{1.41}$$

$$\chi(p) \ge [1 + o(1)](c_2\epsilon)^2 2^n \qquad as\ n \to \infty. \tag{1.42}$$

1.8 Discussion

1.8.1 Restriction to high dimensional graphs

Our results show that the phase transition for percolation on general random graphs obeying the triangle condition shares several features with the phase transition for the random graph. The mean-field behaviour of the random graph is expected to apply only to graphs that are in some sense high-dimensional, and our entire approach is restricted to high dimensional graphs. As discussed in [7, Section 3.4.2], we do not expect the definition (1.9) of the critical threshold to be correct for finite approximations to low-dimensional graphs, such as $\mathbb{Z}^n$ for $n < 6$. Neither do we expect the triangle condition to be relevant in low dimensions.

1.8.2 The lace expansion

There are two steps in applying the lace expansion: derivation of the expansion, and a proof of convergence. The derivation of the lace expansion in [13] applies immediately to transitive finite graphs, as only translation invariance is needed to derive the expansion. Our proof of convergence of the expansion uses the group structure of the torus for Fourier analysis, as well as the $x \mapsto -x$ symmetry of the torus. The proof is an adaptation of the original convergence proof of [13], but is conceptually simpler and the idea of basing the proof on Assumption 1.1 is new. In addition, we benefit from working on a finite set where Fourier integrals are simply finite sums.

Since every finite abelian group is a direct product of cylic groups, our restriction to the torus actually covers all abelian groups, apart from the fact that we consider constant widths in all directions. It would be straightforward to generalize our results to tori with different widths in different directions. This leaves open the case of more general graphs and non-abelian groups, which would require a replacement for both the $x \mapsto -x$ symmetry of the torus and the commutative law.

1.8.3 Bulk versus periodic boundary conditions

A natural question for $\mathbb{Z}^n$ is the following. For $p = p_c(\mathbb{Z}^n)$, consider the restriction of percolation configurations to a large box of side r, centered at the origin. How large is the largest cluster in the box, as $r \to \infty$? The combined results of Aizenman [1] and Hara, van der Hofstad and Slade [12] show that for spread-out models with $n > 6$ the largest cluster has size of order r^4, and there are order r^{d-6} clusters of this size. For the nearest-neighbor model in dimensions $n \gg 6$, the same results follow from the combined results of [1] and Hara [11]. These results apply under the *bulk* boundary condition, in which the clusters in the box are defined to be the intersection of the box with clusters in the infinite lattice (and thus clusters in the box need not be connected within the box). In terms of the volume $V = r^n$ of the box, the largest cluster at $p_c(\mathbb{Z}^n)$ therefore has size $V^{4/n}$. Aizenman [1] raised the interesting question whether the $r^4 = V^{4/n}$ would change to $r^{2n/3} = V^{2/3}$ if the periodic boundary condition is used instead of the the bulk boundary condition.

Theorem 1.6 shows that for p within a scaling window of width proportional to $V^{-1/3}$, centered at $p_c(\mathbb{T}_{r,n})$, the largest cluster is of size $V^{2/3}$ both for the sufficiently spread-out model with $n > 6$ and the nearest-neighbor model with n sufficiently large. An affirmative answer to Aizenman's question would then follow if we knew that $p_c(\mathbb{Z}^d)$ were within this scaling window. It would be interesting to investigate this further.

1.9 Organization

The remainder of this paper is organized as follows. In Section 2, we analyse random walks on a torus and verify Assumption 1.1 for the four tori listed in Section 1.4. In Section 3, we give a self-contained derivation of the lace expansion. In Section 4, we estimate the Feynman diagrams that arise in the lace expansion. The results of Sections 3 apply on an arbitrary transitive graph $\mathbb{G}$ (finite or infinite). Parts of Section 4 also apply in this general context, but in Section 4.2 we will specialize to $\mathbb{T}_{r,n}$. In Section 5, we analyse the lace expansion on an arbitrary torus that obeys Assumption 1.1, thereby proving Theorem 1.3. Finally, in Section 6, we establish a detailed relation between the Fourier transforms of the two-point functions for percolation and random walks.

2 Proof of Proposition 1.2

2.1 The random walk two-point function

Consider a random walk on $\mathbb{T}_{r,n}$ where the transition probability for a step from x to y is equal to $D(x,y)$, with D given by (1.24). We assume that the edge set of the torus is invariant under translations and $x \mapsto -x$ reflections. The *two-point function* for the random walk is defined by

$$C_\mu(0,x) = \sum_{\omega:0\to x} \mu^{|\omega|}, \tag{2.1}$$

where $0 \le \mu < \Omega^{-1}$, the sum is over all random walks ω from 0 to x that take any number of steps $|\omega|$, and the "zero-step" walk contributes $\delta_{0,x}$. This is well-defined, because the fact that there are Ω^m nearest-neighbor random walks of length m starting from the origin implies that

$$C_\mu(0,x) \le \sum_{x\in\mathbb{T}_{r,n}} C_\mu(0,x) = \sum_{m=0}^{\infty} \Omega^m \mu^m = \frac{1}{1-\mu\Omega} \qquad (\mu < \Omega^{-1}), \tag{2.2}$$

i.e., the corresponding susceptibility $\sum_{x\in\mathbb{T}_{r,n}} C_\mu(0,x)$ is finite. Probabilistically, $C_{1/\Omega}(0,x)$ represents the expected number of visits to x for an infinite random walk starting at 0. Since the torus is finite, the random walk is recurrent, and hence $C_{1/\Omega}(0,x)$ is infinite for all x. We therefore must keep $\mu < \Omega^{-1}$ when dealing with $C_\mu(0,x)$. The value $\mu = \Omega^{-1}$ plays the role of the critical point for random walks.

Using translation invariance, we can write $C_\mu(x,y) = C_\mu(y-x)$. By conditioning on the first step, we see that the two-point function obeys the convolution equation

$$C_\mu(x) = \delta_{0,x} + \mu\Omega(D * C_\mu)(x). \tag{2.3}$$

Taking the Fourier transform of (2.3) gives $\hat{C}_\mu(k) = 1 + \mu\Omega\hat{D}(k)\hat{C}_\mu(k)$ and hence

$$\hat{C}_\mu(k) = \frac{1}{1-\mu\Omega\hat{D}(k)}. \tag{2.4}$$

Note that $\hat{C}_\mu(0) < \infty$ for $\mu < \Omega^{-1}$ but $\hat{C}_{1/\Omega}(0) = \infty$. Although $C_{1/\Omega}(x)$ is infinite, the formula (2.4) does not diverge for $\mu = \Omega^{-1}$ for all k for which $\hat{D}(k) \neq 1$. Apart from any such singular points (usually arising only for $k = 0$), the expression $\hat{C}_1(k) = [1-\hat{D}(k)]^{-1}$ is finite. The factor $[1-\hat{D}(k)]^{-3}$ that appears in (1.26) is thus the same as $\hat{C}_1(k)^3$. Comparing with (1.23), we see that (1.26) is closely related to a triangle condition for random walks.

2.2 Random walk estimates

In this section we prove Proposition 1.2, which for convenience we restate as Proposition 2.1.

Proposition 2.1. *There is a constant $a > 0$, which remains fixed as the volume $V = r^n$ goes to infinity, such that random walks on each of the four tori listed in Section 1.4 obey Assumption 1.1, with $\beta = an^{-1}$ for the narrow torus, the Hamming torus and the wide torus in high dimensions, and with $\beta = aL^{-n}$ for the wide spread-out torus in dimensions $n > 6$.*

The proof is given throughout the remainder of Section 2.2. We first note that (1.25) is trivial since the maximal value of $D(x)$ is Ω^{-1} and this is less than $\beta = a\Omega^{-1}$ provided $a \geq 1$. We verify the substantial assumption (1.26) below.

2.2.1 The random walk triangle condition (1.26)

For $k \in \mathbb{T}^*_{r,n} = \frac{2\pi}{r}\{-\lfloor \frac{r-1}{2} \rfloor, \ldots, \lceil \frac{r-1}{2} \rceil\}^n$, we define

$$|k|^2 = \sum_{i=1}^n k_i^2. \tag{2.5}$$

The narrow and Hamming tori. Here r is fixed and $n \to \infty$. We first claim that

$$\frac{1}{V} \sum_{k \in \mathbb{T}^*_{r,n}} \hat{D}(k)^{2i} \leq \frac{a_i}{n^i} \qquad (i = 1, 2, 3, \ldots), \tag{2.6}$$

where a_i depends on r for the Hamming torus. To prove (2.6), we observe that the left side is equal to the probability that a random walk on $\mathbb{T}_{r,n}$ that starts at the origin returns to the origin after $2i$ steps. This probability is equal to Ω^{-2i} times the number of walks that make the transition from 0 to 0 in $2i$ steps. The number of such walks is bounded above by Ω^i (counting all possibilities for the first i steps) times a factor b_i which counts the maximal number of ways that a random walk can return to the origin from a vertex reachable in i steps. The latter depends on i, and depends on r for the Hamming torus, but it does not depend on n (the return walk must remain in an i-dimensional subgraph). This proves (2.6).

We will also use elementary *infrared bounds*. By the symmetry of D,

$$\hat{D}(k) = \sum_{x \in \mathbb{T}_{r,n}} D(x) \cos(k \cdot x). \tag{2.7}$$

For the narrow or wide torus, this gives

$$\hat{D}(k) = \frac{1}{n} \sum_{j=1}^n \cos k_j. \tag{2.8}$$

Since $1 - \cos t \geq 2\pi^{-2} t^2$ for $|t| \leq \pi$, this implies that

$$1 - \hat{D}(k) = \frac{1}{n} \sum_{j=1}^n (1 - \cos k_j) \geq \frac{2}{\pi^2} \frac{|k|^2}{n}. \tag{2.9}$$

For the Hamming torus, we have

$$1-\hat{D}(k) = \sum_{x\in\mathbb{T}_{r,n}} D(x)[1-\cos(k\cdot x)]$$
$$\geq \frac{1}{(r-1)n}\sum_{x:|x|=1}[1-\cos(k\cdot x)] \geq \frac{1}{r-1}\frac{2}{\pi^2}\frac{|k|^2}{n}, \tag{2.10}$$

applying (2.9) in the last step. We combine (2.9)–(2.10) by setting $\eta = \frac{2}{\pi^2}$ for the narrow (or wide) torus and $\eta = \frac{1}{r-1}\frac{2}{\pi^2}$ for the Hamming torus, obtaining the infrared bound

$$1-\hat{D}(k) \geq \eta\frac{|k|^2}{n}. \tag{2.11}$$

in either case.

By (2.6) and the Cauchy–Schwarz inequality,

$$\frac{1}{V}\sum_{k\in\mathbb{T}^*_{r,n}:k\neq 0}\frac{\hat{D}(k)^{2i}}{[1-\hat{D}(k)]^3} \leq \frac{a_{2i}^{1/2}}{n^i}\left(\sum_{k\in\mathbb{T}^*_{r,n}:k\neq 0}\frac{1}{[1-\hat{D}(k)]^6}\right)^{1/2}. \tag{2.12}$$

It suffices to show that the sum on the right side of (2.12) is bounded uniformly in n. For this, we fix an $\epsilon > 0$ and divide the sum according to whether $|k|^2 \leq \epsilon n$ or $|k|^2 > \epsilon n$. It follows from (2.11) that the contribution to the sum due to $|k|^2 > \epsilon n$ is bounded by a constant depending on ϵ.

On the other hand, by (2.11),

$$\frac{1}{V}\sum_{k\in\mathbb{T}^*_{r,n}:k\neq 0,\,|k|^2\leq\epsilon n}\frac{1}{[1-\hat{D}(k)]^6} \leq \frac{n^6}{\eta^6}\frac{1}{V}\sum_{k\in\mathbb{T}^*_{r,n}:k\neq 0,\,|k|^2\leq\epsilon n}\frac{1}{|k|^{12}}. \tag{2.13}$$

By the Cauchy–Schwarz inequality, $|k|^2 \geq n^{-1}\|k\|_1^2 \geq n^{-1}m(k)^2$, where $m(k)$ denotes the number of nonzero components of k. Therefore,

$$\frac{1}{V}\sum_{k\in\mathbb{T}^*_{r,n}:k\neq 0,\,|k|^2\leq\epsilon n}\frac{1}{[1-\hat{D}(k)]^6} \leq \frac{n^{12}}{\eta^6}\frac{1}{r^n}\sum_{m=1}^{r^2\epsilon n}\binom{n}{m}(r-1)^m\frac{1}{m^{12}}. \tag{2.14}$$

In (2.14), the binomial coefficient counts the number of ways to choose m nonzero components from n, the factor $(r-1)^m$ counts the number of values that each nonzero component can assume, and the upper limit of summation reflects the fact that there cannot be more than $r^2(2\pi)^{-2}\epsilon n \leq r^2\epsilon n$ nonzero components when $|k|^2 \leq \epsilon n$. The right side of (2.14) is at most

$$\frac{n^{12}}{\eta^6}\sum_{m=1}^{r^2\epsilon n}\binom{n}{m}\Big(1-\frac{1}{r}\Big)^m\Big(\frac{1}{r}\Big)^{n-m} \leq \frac{n^{12}}{\eta^6}\mathbb{P}(X\leq r^2\epsilon n), \tag{2.15}$$

where X is a binomial random variable with parameters $(n, 1-r^{-1})$. Since $\mathbb{E}[X] = n(1-r^{-1})$, the right side of (2.15) is exponentially small in n if we choose $\epsilon < r^{-2}(1-r^{-1})$, by standard large deviation bounds.

The wide torus. Here n is large but fixed and $r \to \infty$. The proof of (2.6) applies without change in this case, with a_i independent of r. Therefore (2.12) also applies, and it suffices to show that

$$\frac{1}{V} \sum_{k \in \mathbb{T}^*_{r,n} : k \neq 0} \frac{1}{[1 - \hat{D}(k)]^6} \tag{2.16}$$

is bounded uniformly in large n and large r. Apart from the missing term $k = 0$, (2.16) is a Riemann sum approximation to the integral

$$\int_{[-\pi,\pi]^n} \frac{1}{[1 - \hat{D}(k)]^6} \frac{d^n k}{(2\pi)^n}, \tag{2.17}$$

which is finite for $n > 12$ by (2.11). We prove that (2.16) converges to the improper integral (2.17), using the dominated convergence theorem as follows.

Let $B(0, r) = (-\frac{\pi}{r}, \frac{\pi}{r}]^n \subset \mathbb{R}^n$. For each $k \in (-\pi, \pi]^n$ there is a unique $k_r \in \mathbb{T}^*_{r,n}$ such that $k \in k_r + B(0, r)$ and we define

$$\hat{D}_r(k) = \begin{cases} \hat{D}(k_r) & (k_r \neq 0) \\ \infty & (k_r = 0). \end{cases} \tag{2.18}$$

(The term $k_r = 0$ has been singled out here because it is not included in the sum (2.16).) Thus $\hat{D}_r(k)$ is constant on the cubes $k_r + B(0, r)$ for $k_r \in \mathbb{T}^*_{r,n}$, the identity $V|B(0, r)| = (2\pi)^n$ holds, and

$$\frac{1}{V} \sum_{k \in \mathbb{T}^*_{r,n} : k \neq 0} \frac{1}{[1 - \hat{D}(k)]^6} = \int_{[-\pi,\pi]^n} \frac{1}{[1 - \hat{D}_r(k)]^6} \frac{d^n k}{(2\pi)^n}. \tag{2.19}$$

The function $[1 - \hat{D}_r(k)]^{-6}$ converges pointwise to $[1 - \hat{D}(k)]^{-6}$ for $k \neq 0$ and to 0 for $k \neq 0$. Also, by the infrared bound (2.11),

$$\frac{1}{[1 - \hat{D}(k_r)]^6} \leq \frac{n^6}{\eta^6 |k_r|^{12}} \tag{2.20}$$

for every nonzero $k_r \in \mathbb{T}^*_{r,n}$. For each nonzero $k_r \in \mathbb{T}^*_{r,n}$ and $k \in k_r + B(0, r)$, we have $\|k_r\|_\infty \geq \frac{2\pi}{r}$ and $\|k - k_r\|_\infty \leq \frac{\pi}{r}$, so that $\|k_r\|_\infty \geq \frac{2}{3}\|k\|_\infty$. This implies that $|k_r|^2 \geq \|k_r\|_\infty^2 \geq \frac{4}{9}\|k\|_\infty^2 \geq \frac{4}{9n}|k|^2$. Therefore, for every $k \in (-\pi, \pi]^n$,

$$\frac{1}{[1 - \hat{D}_r(k)]^6} \leq \Big(\frac{9}{4}\Big)^6 \frac{n^{12}}{\eta^6 |k|^{12}}, \tag{2.21}$$

which is integrable when $n > 12$. Therefore, by dominated convergence, (2.16) converges to (2.17) as $r \to \infty$, and thus is bounded by twice the integral (2.17) for r sufficiently large depending on n.

Moreover, the integral (2.17) is bounded uniformly in $n \geq 13$, by the following argument. For $A > 0$ and $m \geq 1$,

$$\frac{1}{A^m} = \frac{1}{(m-1)!} \int_0^\infty t^{m-1} e^{-tA} dt. \tag{2.22}$$

Hence, by (2.8),

$$\int_{[-\pi,\pi]^n} \frac{1}{[1 - \hat{D}(k)]^6} \frac{d^n k}{(2\pi)^n} = \frac{1}{5!} \int_0^\infty dt\, t^5 \Big(\int_{-\pi}^{\pi} e^{-tn^{-1}(1-\cos\theta)} \frac{d\theta}{2\pi} \Big)^n. \tag{2.23}$$

The right side is non-increasing in n, since $\|f\|_p \leq \|f\|_q$ for $0 < p \leq q \leq \infty$ on a probability space.

The wide spread-out torus. Now $n > 6$ is fixed, L is fixed and large, and $r \to \infty$. We first note that $\hat{D}(k)$ does not depend on r if r is large compared to L. Thus, we can apply bounds on $\hat{D}(k)$ with $D(x)$ regarded as the step distribution of a random walk on $\mathbb{Z}^n$. The latter is analysed in [18, Appendix A], where it is shown that there is an η depending only on n such that the infrared bound

$$1 - \hat{D}(k) \geq \eta\Big(1 \wedge \frac{L^2|k|^2}{n}\Big) \tag{2.24}$$

holds for all $k \in \mathbb{T}^*_{r,n}$,

By a Riemann sum argument similar to that used above,

$$\frac{1}{V} \sum_{k \in \mathbb{T}^*_{r,n}: k \neq 0} \frac{|\hat{D}(k)|^2}{[1 - \hat{D}(k)]^3} \leq 2 \int_{[-\pi,\pi]^n} \frac{|\hat{D}(k)|^2}{[1 - \hat{D}(k)]^3} \frac{d^n k}{(2\pi)^n}, \tag{2.25}$$

for r sufficiently large depending on L, n. We bound this integral by considering separately the regions where $|k|^2 \geq nL^{-2}$ and $|k|^2 \leq nL^{-2}$.

For the contribution to the integral on the right side of (2.25) due to $|k|^2 \geq nL^{-2}$, we use (2.24) to obtain

$$\int_{k \in [-\pi,\pi]^n: |k|^2 \geq \frac{n}{L^2}} \frac{|\hat{D}(k)|^2}{[1 - \hat{D}(k)]^3} \frac{d^n k}{(2\pi)^n} \leq \eta^{-3} \int_{[-\pi,\pi]^n} |\hat{D}(k)|^2 \frac{d^n k}{(2\pi)^n} = \eta^{-3}\Omega^{-1}. \tag{2.26}$$

Here, we used the fact that the middle integral is the probability that the spread-out random walk returns to its starting vertex after two steps, which equals Ω^{-1}. For the contribution to the integral in (2.25) due to $|k|^2 \leq nL^{-2}$, we use (2.24) and $|\hat{D}(k)|^2 \leq 1$ to obtain

$$\int_{|k|^2 \leq \frac{n}{L^2}} \frac{|\hat{D}(k)|^2}{[1 - \hat{D}(k)]^3} \frac{d^n k}{(2\pi)^n} \leq \Big(\frac{n}{\eta L^2}\Big)^3 \int_{|k|^2 \leq \frac{n}{L^2}} \frac{1}{|k|^6} \frac{d^n k}{(2\pi)^n} = C_{n,\eta} L^{-n}. \tag{2.27}$$

Summing the two contributions yields (1.26).

2.2.2 A consequence of Assumption 1.1

Finally, we note for future reference that (1.26) implies that

$$\frac{1}{V} \sum_{k \in \mathbb{T}^*_{r,n}: k \neq 0} \frac{1}{[1 - \hat{D}(k)]^3} \leq 1 + 6\beta. \tag{2.28}$$

To see this, we use the identity

$$\frac{1}{[1 - \hat{D}]^3} = 1 + 3\hat{D} + \frac{3\hat{D}^2}{1 - \hat{D}} + \frac{2\hat{D}^2}{[1 - \hat{D}]^2} + \frac{\hat{D}^2}{[1 - \hat{D}]^3}, \tag{2.29}$$

and note that the sum of the last three terms on the right side is at most $6\hat{D}^2[1 - \hat{D}]^{-3}$, and their normalized sum over k is thus at most 6β, by (1.26). Since the normalized sum over *all* $k \in \mathbb{T}_{r,n}$ of $3\hat{D}(k)$ is $3D(0) = 0$, its sum over nonzero k is $-3V^{-1} < 0$. This proves (2.28).

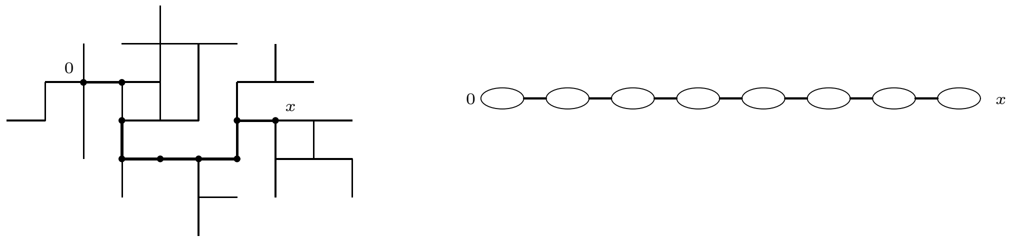

Figure 1: A percolation cluster with a string of 8 sausages joining 0 to x, and a schematic representation of the string. The 7 pivotal bonds are shown in bold.

3 The lace expansion

We begin in Section 3.1 with a brief overview of the lace expansion, and then give a self-contained and detailed derivation of the expansion in Section 3.2.

The term "lace" was used by Brydges and Spencer [9] for a certain graphical construction that arose in the expansion they invented to study the self-avoiding walk. Although the lace expansion for percolation evolved from the expansion for the self-avoiding walk, this graphical construction does not occur for percolation, and so the term "lace" expansion is a misnomer in the percolation context. However, the name has stuck for historical reasons.

3.1 Overview of the lace expansion

In this section, we give a brief introduction to the lace expansion, with an indication of how it is used to prove the triangle condition of Theorem 1.3. Since the analysis will involve the Fourier transform, we restrict attention here to percolation on the narrow torus $\mathbb{T}_{r,n}$, with $r \geq 2$ and n large. Each vertex has degree $\Omega = 2n$ for $r \geq 3$ and $\Omega = n$ for $r = 2$. However, in Section 3.2 the expansion will be derived on an arbitrary transitive graph $\mathbb{G}$.

For $p = \Omega^{-1}$, the probability that the origin is in a cycle of length 4 is bounded above by $\Omega^2\Omega^{-4} = \Omega^{-2}$. Larger cycles are more unlikely. This suggests that in a typical percolation configuration connecting 0 and x, the backbone for the connection is a random walk path with few cycles. Thus it makes sense to attempt to relate $\tau_p(x)$ to the two-point function for random walks.

Although unlikely, cycles do exist in percolation clusters, and due to cycles the percolation critical threshold is shifted from the tree critical value $(\Omega-1)^{-1}$ to $p_c(\mathbb{G})$. It is therefore necessary to take the cycles seriously into account. Given a percolation cluster containing 0 and x, we call any bond whose removal would disconnect 0 from x a *pivotal* bond. The connected components that remain after removing all pivotal bonds are called *sausages*. A sausage may contain a cycle that gives alternate paths from 0 to x, as in the second sausage of Figure 1, but this is the unlikely case.

Since they are separated by at least one pivotal bond by definition, no two sausages can have a common vertex. Thus the sausages are constrained to be mutually avoiding. However, this is a weak constraint, since sausage intersections require a cycle, and cycles are unlikely. This makes it reasonable to attempt to apply an inclusion-exclusion analysis, where the connection from 0 to x

is treated as a random walk taking independent steps, with correction terms taking into account the avoidance constraint.

The lace expansion of Hara and Slade [13] makes this procedure precise. It produces a convolution equation

$$\tau_p(0,x) = \delta_{0,x} + p\Omega(D * \tau_p)(0,x) + p\Omega(\Pi_p * D * \tau_p)(0,x) + \Pi_p(0,x) \tag{3.1}$$

for the two-point function, valid for $p \leq p_c(\mathbb{T}_{r,n})$. The expansion gives explicit but complicated formulas for the function $\Pi_p : \mathbb{T}_{r,n} \times \mathbb{T}_{r,n} \to \mathbb{R}$, which will turn out to be small under Assumption 1.1. In particular, $\hat{\Pi}_p(k) = O(\Omega^{-1})$ uniformly in $p \leq p_c(\mathbb{G})$. Putting $\Pi_p = 0$ in (3.1) gives (2.3), and in this sense the percolation two-point function can be regarded as a small perturbation of the random walk two-point function.

Since the Fourier transform of a convolution is the product of Fourier transforms, (3.1) can be rewritten as

$$\hat{\tau}_p(k) = \frac{1 + \hat{\Pi}_p(k)}{1 - p\Omega\hat{D}(k)[1 + \hat{\Pi}_p(k)]}. \tag{3.2}$$

We will prove estimates to show that $\hat{\Pi}_p(k)$ can be well approximated by $\hat{\Pi}_p(0)$. Since $\hat{\Pi}_p(k)$ is also small compared to 1, (3.2) suggests that the approximation

$$\hat{\tau}_p(k) \approx \frac{1}{1 - p\Omega[1 + \hat{\Pi}_p(0)]\hat{D}(k)} \tag{3.3}$$

is reasonable (where $\approx$ denotes an uncontrolled approximation). Comparing with (2.4), this suggests that

$$\hat{\tau}_p(k) \approx \hat{C}_{\mu_p}(k) \qquad \text{with} \qquad \mu_p\Omega = p\Omega[1 + \hat{\Pi}_p(0)]. \tag{3.4}$$

We will make this approximation precise in (5.4) and (6.6). Since $\hat{D}(0) = 1$, if we set $k = 0$ in (3.2) and solve for $p\Omega$ then we obtain

$$p\Omega = \frac{1}{1 + \hat{\Pi}_p(0)} - \hat{\tau}_p(0)^{-1}. \tag{3.5}$$

For $p = p_c = p_c(\mathbb{T}_{r,n})$, (3.5) states that

$$p_c\Omega = \frac{1}{1 + \hat{\Pi}_{p_c}(0)} - \lambda^{-1}V^{-1/3}, \tag{3.6}$$

and hence $\mu_{p_c}\Omega \approx 1 - \lambda^{-1}V^{-1/3}$. This should be compared with the critical value $\mu\Omega = 1$ for the random walk.

For the triangle condition, we analyse the Fourier representation of $\nabla_p(x)$ given in (1.23). Extraction of the $k = 0$ term in (1.23) gives

$$\nabla_p(x) = \frac{\chi^3(p)}{V} + \frac{1}{V} \sum_{k \in \mathbb{T}^*_{r,n} : k \neq 0} \hat{\tau}_p(k)^3 e^{-ik\cdot x}. \tag{3.7}$$

The second term can be estimated using (3.4) and Assumption 1.1, leading to a proof of Theorem 1.3. Details are given in Section 5.

Using $\hat{\Pi}_p(k) = O(\Omega^{-1})$, (3.6) gives $p_c = \Omega^{-1} + O(\Omega^{-2})$ if $\lambda^{-1}V^{-1/3} \leq O(\Omega^{-2})$. This is the first term in an asymptotic expansion. Further terms will follow from an asymptotic expansion of $\hat{\Pi}_{p_c}(0)$ in powers of Ω^{-1}. This was done in [15] for percolation on $\mathbb{Z}^n$ and is developed more generally in [17].

3.2 Derivation of the lace expansion

In this section, we derive a version of the lace expansion (3.1) that contains a remainder term. We use the method of [13], which requires no change for a transitive graph $\mathbb{G}$, and we closely follow the presentation of [19]. We assume for simplicity that $\mathbb{G}$ is finite, but with minor modifications the analysis also applies when $\mathbb{G}$ is infinite provided there is almost surely no infinite cluster.

Fix $p \in [0,1]$. We write $\tau(x) = \tau_p(x)$ for brevity, and generally drop subscripts indicating dependence on p. For each $M = 0, 1, 2, \ldots$, the expansion takes the form

$$\tau(0,x) = \delta_{0,x} + p\Omega(D * \tau)(0,x) + p\Omega(\Pi_M * D * \tau)(0,x) + \Pi_M(0,x) + R_M(0,x). \tag{3.8}$$

Here $D(0,x)$ is given by (1.24), the function $\Pi_M : \mathbb{V} \times \mathbb{V} \to \mathbb{R}$ is the key quantity in the expansion, and $R_M(0,x)$ is a remainder term. The dependence of Π_M on M is given by

$$\Pi_M(0,x) = \sum_{N=0}^{M} (-1)^N \Pi^{(N)}(0,x), \tag{3.9}$$

with $\Pi^{(N)}(0,x)$ independent of M. The alternating sign in (3.9) arises via repeated inclusion-exclusion. In Section 5, we will prove that for $\mathbb{G} = \mathbb{T}_{r,n}$ with Assumption 1.1, and for $p \le p_c(\mathbb{T}_{r,n})$

$$\lim_{M\to\infty} \sum_x |R_M(0,x)| = 0, \tag{3.10}$$

which leads to (3.1) with $\Pi = \Pi_\infty$. Convergence properties of (3.9) when $M = \infty$ will also be established in Section 5. The remainder of this section gives the proof of (3.8).

Given increasing events E, F, we use the notation $E \circ F$ to denote the event that E and F occur disjointly. Roughly speaking, $E \circ F$ is the set of bond configurations for which there exist two disjoint sets of occupied bonds such that the first set guarantees the occurrence of E and the second guarantees the occurrence of F. The BK inequality asserts that $\mathbb{P}(E \circ F) \le \mathbb{P}(E)\mathbb{P}(F)$, for increasing events E and F. (See [10, Section 2.3] for a proof, and for a precise definition of $E \circ F$.) Given a bond configuration, we say that a bond is *pivotal* for $x \leftrightarrow y$ if $x \leftrightarrow y$ in the possibly modified configuration in which the bond is made occupied, whereas x is not connected to y in the possibly modified configuration in which the bond is made vacant.

Bonds are not directed. However, it will be convenient at times to regard a bond $\{u,v\}$ as directed from u to v, and we will emphasize this point of view with the notation (u,v).

We need the following definitions and lemma.

Definition 3.1. (a) Given a bond configuration, and $A \subset \mathbb{V}$, we say x and y are *connected in* A if there is an occupied path from x to y having all its endpoints in A, or if $x = y \in A$. We define a restricted two-point function by

$$\tau^A(x,y) = \mathbb{P}(x \text{ and } y \text{ are connected in } \mathbb{V}\backslash A). \tag{3.11}$$

(b) Given a bond configuration, and $A \subset \mathbb{V}$, we say x and y are *connected through* A, if $x \leftrightarrow y$ and every occupied path connecting x to y has at least one bond with an endpoint in A. This event is written as $x \overset{A}{\leftrightarrow} y$.

(c) Given a bond configuration, and a bond b, we define $\tilde{C}^b(x)$ to be the set of sites connected to x in the new configuration obtained by setting b to be vacant.

(d) Given an event E, we define the event $\{E$ occurs on $\tilde{C}^{(u,v)}(x)\}$ to be the set of configurations such that E occurs on the modified configuration in which every bond that does not have an endpoint in $\tilde{C}^{(u,v)}(x)$ is made vacant. We say that $\{E$ occurs in $\mathbb{V}\backslash\tilde{C}^{(u,v)}(x)\}$ if E occurs on the modified configuration in which every bond that does not have both endpoints in $\mathbb{V}\backslash\tilde{C}^{(u,v)}(x)$ is made vacant.

Lemma 3.2. *Fix $p \in [0,1]$. Given a bond (u,v), a site w and events E, F,*

$$\begin{aligned} &\mathbb{E}\Big(I[E \text{ occurs on } \tilde{C}^{(u,v)}(w) \text{ \& } (u,v) \text{ is occupied } \text{ \& } F \text{ occurs in } \mathbb{V}\backslash\tilde{C}^{(u,v)}(w)]\Big) \\ &\qquad = p\Omega D(u,v)\mathbb{E}\Big(I[E \text{ occurs on } \tilde{C}^{(u,v)}(w)]\,\mathbb{E}\Big(I[F \text{ occurs in } \mathbb{V}\backslash\tilde{C}^{(u,v)}(w)]\Big)\Big). \end{aligned} \tag{3.12}$$

The identity (3.12) *is also valid if the event* $\{(u,v)$ *is occupied*$\}$ *is removed from the left side and* $\Omega D(u,v)$ *is removed from the right side.*

Proof. The proof is by conditioning on the *bond* cluster of w which remains after setting (u,v) to be vacant, which we denote $\tilde{C}^{(u,v)}(w)_b$. Let $\mathcal{B}$ denote the set of all finite bond clusters of A. Given $B \in \mathcal{B}$, we denote the set of *vertices* in B by B_s. Conditioning on $\tilde{C}^{(u,v)}(w)_b$, we have

$$\begin{aligned} &\mathbb{E}\Big[I\Big[E \text{ occurs on } \tilde{C}^{(u,v)}(w) \text{ \& } F \text{ occurs in } \mathbb{V}\backslash\tilde{C}^{(u,v)}(w) \text{ \& } (u,v) \text{ occupied}\Big]\Big] \\ &\quad = \sum_{B\in\mathcal{B}} \mathbb{E}\Big[I[\tilde{C}^{(u,v)}(w)_b = B \text{ \& } E \text{ occurs on } \tilde{B}_s \text{ \& } F \text{ occurs in } \mathbb{V}\backslash\tilde{B}_s \text{ \& } (u,v) \text{ occupied}]\Big], \end{aligned} \tag{3.13}$$

where $\tilde{B}_s$ emphasizes that E occurs on B_s *after* setting (u,v) to be vacant. Since the first two of the four events on the right side of (3.13) depend only on bonds with an endpoint in B_s (excluding (u,v)), while the third event depends only on bonds which do not have an endpoint in B_s (again, excluding (u,v)), and the fourth event depends only on (u,v), this independence allows us to write (3.13) as

$$\begin{aligned} &p\sum_{B\in\mathcal{B}} \mathbb{E}\Big[I[\tilde{C}^{(u,v)}(w)_b = B \text{ \& } E \text{ occurs on } \tilde{B}_s]\Big]\,\mathbb{E}\Big[I[F \text{ occurs in } \mathbb{V}\backslash\tilde{B}_s]\Big] \\ &\qquad = p\mathbb{E}\Big[I[E \text{ occurs on } \tilde{C}^{(u,v)}(w)]\mathbb{E}\Big[I[F \text{ occurs in } \mathbb{V}\backslash\tilde{C}^{(u,v)}(w)]\Big]\Big]. \end{aligned} \tag{3.14}$$

This completes the proof of (3.12), since $\Omega D(u,v) = 1$ if (u,v) is a bond. The statement under (3.12) holds by the same proof. □

In the nested expectation on the right side of (3.12), the set $\tilde{C}^{(u,v)}(w)$ is a random set with respect to the outer expectation, but it is deterministic with respect to the inner expectation. The inner expectation on the right side effectively introduces a second percolation model on a second graph, which is coupled to the original percolation model via the set $\tilde{C}^{(u,v)}(w)$.

Given a configuration, we say that x is *doubly connected to* y, and we write $x \Leftrightarrow y$, if there are at least two bond-disjoint paths from x to y consisting of occupied bonds. By convention, $x \Leftrightarrow x$ for all x. To begin the expansion, we define

$$\Pi^{(0)}(0,x) = \mathbb{P}(0 \Leftrightarrow x) - \delta_{0,x} \tag{3.15}$$

and distinguish configurations with $0 \leftrightarrow x$ according to whether or not there is a double connection, to obtain

$$\tau(0,x) = \delta_{0,x} + \Pi^{(0)}(0,x) + \mathbb{P}(0 \leftrightarrow x \text{ \& } 0 \not\Leftrightarrow x). \tag{3.16}$$

If 0 is connected to x, but not doubly, then there is at least one pivotal bond for the connection, and hence a first such pivotal bond. Denoting this pivotal bond by (u, v), we can write

$$\mathbb{P}(0 \leftrightarrow x \;\&\; 0 \not\Leftrightarrow x) = \sum_{(u,v)} \mathbb{P}(0 \Leftrightarrow u \text{ and } (u, v) \text{ is occupied and pivotal for } 0 \leftrightarrow x). \tag{3.17}$$

Now comes the essential part of the expansion. Ideally, we would like to factor the probability on the right side of (3.17) as

$$\mathbb{P}(0 \Leftrightarrow u)\, \mathbb{P}((u, v) \text{ is occupied})\, \mathbb{P}(v \leftrightarrow x) = \Big(\delta_{0,u} + \Pi^{(0)}(0, u)\Big) p\Omega D(u, v)\tau(v, x). \tag{3.18}$$

This would give (3.8) with $\Pi_M = \Pi^{(0)}$ and $R_M = 0$. However, (3.17) does not factor in this way because the cluster $\tilde{C}^{(u,v)}(u)$ is constrained not to intersect the cluster $\tilde{C}^{(u,v)}(v)$, since (u, v) is pivotal. What we can do is approximate the probability on the right side of (3.17) by (3.18), and then attempt to deal with the error term.

For this purpose, we observe that

$$\begin{aligned} &\mathbb{P}(0 \Leftrightarrow u \text{ and } (u, v) \text{ is occupied and pivotal for } 0 \leftrightarrow x) \\ &= \mathbb{E}\Big(I[0 \Leftrightarrow u \text{ occurs on } \tilde{C}^{(u,v)}(0) \;\&\; (u, v) \text{ is occupied } \;\&\; v \leftrightarrow x \;\text{ occurs in } \mathbb{V}\backslash\tilde{C}^{(u,v)}(0)]\Big). \end{aligned} \tag{3.19}$$

Therefore, by Lemma 3.2,

$$\begin{aligned} &\mathbb{P}(0 \Leftrightarrow u \text{ and } (u, v) \text{ is occupied and pivotal for } 0 \leftrightarrow x) \\ &= p\Omega D(u, v)\mathbb{E}\left(I[0 \Leftrightarrow u \text{ occurs on } \tilde{C}^{(u,v)}(0)]\tau^{\tilde{C}^{(u,v)}(0)}(v, x)\right). \end{aligned} \tag{3.20}$$

On the right side, $\tau^{\tilde{C}^{(u,v)}(0)}(v, x)$ is the restricted two-point function *given* the cluster $\tilde{C}^{(u,v)}(0)$ of the outer expectation, so that in the (inner) expectation defining $\tau^{\tilde{C}^{(u,v)}(0)}(v, x)$, $\tilde{C}^{(u,v)}(0)$ should be regarded as a *fixed* set. We stress this delicate point here, as it is crucial also in the rest of the expansion. As mentioned above, the expectation defining $\tau^{\tilde{C}^{(u,v)}(0)}(v, x)$ effectively introduces a second percolation model.

It follows from (3.17) and (3.20) that

$$\begin{aligned} \mathbb{P}(0 \leftrightarrow x \;\&\; 0 \not\Leftrightarrow x) &= \sum_{(u,v)} p\Omega D(u, v)\mathbb{E}\left(I[0 \Leftrightarrow u \text{ occurs on } \tilde{C}^{(u,v)}(0)]\, \tau^{\tilde{C}^{(u,v)}(0)}(v, x)\right) \\ &= \sum_{(u,v)} p\Omega D(u, v)\mathbb{E}\left(I[0 \Leftrightarrow u]\, \tau^{\tilde{C}^{(u,v)}(0)}(v, x)\right). \end{aligned} \tag{3.21}$$

In the second equality of (3.21), we dropped the condition "occurs on $\tilde{C}^{(u,v)}(0)$," because of the fact that $\tau^{\tilde{C}^{(u,v)}(0)}(v, x) = 0$ on the event $\{0 \Leftrightarrow u\} \setminus \{0 \Leftrightarrow u \text{ occurs on } \tilde{C}^{(u,v)}(0)\}$. We write

$$\tau^{\tilde{C}^{(u,v)}(0)}(v, x) = \tau(v, x) - \Big(\tau(v, x) - \tau^{\tilde{C}^{(u,v)}(0)}(v, x)\Big) = \tau(v, x) - \mathbb{P}\Big(v \xleftrightarrow{\tilde{C}^{(u,v)}(0)} x\Big), \tag{3.22}$$

insert this into (3.21), and use (3.16) and (3.15) to obtain

$$\begin{aligned} \tau(0, x) = \delta_{0,x} + \Pi^{(0)}(0, x) &+ \sum_{(u,v)} \Big(\delta_{0,u} + \Pi^{(0)}(0, u)\Big) p\Omega D(u, v)\tau(v, x) \\ &- \sum_{(u,v)} p\Omega D(u, v)\mathbb{E}\left(I[0 \Leftrightarrow u]\, \mathbb{P}(v \xleftrightarrow{\tilde{C}^{(u,v)}(0)} x)\right). \end{aligned} \tag{3.23}$$

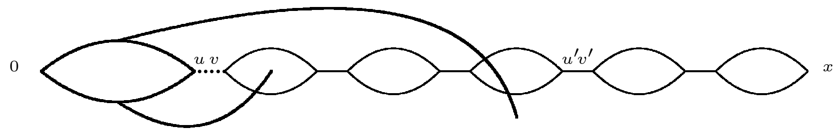

Figure 2: A possible configuration appearing in the second stage of the expansion.

With $R_0(0,x)$ equal to the last term on the right side of (3.23) (including the minus sign), this proves (3.8) for $M=0$.

To continue the expansion, we would like to rewrite the final term of (3.23) in terms of a convolution with the two-point function. A configuration contributing to the expectation in the final term of (3.23) is illustrated schematically in Figure 2, in which the bonds drawn with heavy lines should be regarded as living on a different graph than the bonds drawn with lighter lines, as explained previously. Our goal is to extract a factor $\tau(x-v')$.

Given a configuration in which $v \xleftrightarrow{A} x$, the *cutting bond* (u',v') is defined to be the first pivotal bond for $v \leftrightarrow x$ such that $v \xleftrightarrow{A} u'$. It is possible that no such bond exists, as for example would be the case in Figure 2 if only the leftmost four sausages were included in the figure (using the terminology of Section 3.1), with x in the location currently occupied by u'. Let

$$E'(v,x;A) = \{v \overset{A}{\leftrightarrow} x\} \cap \{\nexists \text{ pivotal } (u',v') \text{ for } v \leftrightarrow x \text{ such that } v \overset{A}{\leftrightarrow} u'\}, \tag{3.24}$$

$$E(v,u',v',x;A) = E'(v,u';A) \cap \{(u',v') \text{ is occupied and pivotal for } v \leftrightarrow x\}. \tag{3.25}$$

By partitioning $\{v \overset{A}{\leftrightarrow} x\}$ according to the location of the cutting bond (or the lack of a cutting bond), we obtain the partition

$$\{v \overset{A}{\leftrightarrow} x\} = E'(v,x;A) \ \dot{\bigcup_{(u',v')}} \ E(v,u',v',x;A), \tag{3.26}$$

which implies that

$$\mathbb{P}(v \overset{A}{\leftrightarrow} x) = \mathbb{P}(E'(v,x;A)) + \sum_{(u',v')} \mathbb{P}(E(v,u',v',x;A)). \tag{3.27}$$

Defining

$$E''(v,u',v';A) = \{E'(v,u';A) \text{ occurs on } \tilde{C}^{(u',v')}(v)\}, \tag{3.28}$$

the event $E(v,u',v',x;A)$ can be rewritten as

$$E(v,u',v',x;A) = E''(v,u',v';A) \cap \{(u',v') \text{ occupied}\} \cap \{v' \leftrightarrow x \text{ occurs in } \mathbb{V}\backslash\tilde{C}^{(u',v')}(v)\}. \tag{3.29}$$

Using Lemma 3.2, this gives

$$\mathbb{P}(v \overset{A}{\leftrightarrow} x) = \mathbb{P}(E'(v,x;A)) + p\Omega \sum_{(u',v')} D(u',v')\, \mathbb{E}\left(I[E''(v,u',v';A)]\, \tau^{\tilde{C}^{(u',v')}(v)}(v',x)\right). \tag{3.30}$$

The events $E''(v,u',v';A)$ and $E'(v,u';A)$ differ only on configurations for which $v' \in \tilde{C}^{(u',v')}(v)$. Since $\tau^{\tilde{C}^{(u',v')}(v)}(v',x) = 0$ on such configurations, we may replace $E''(v,u',v';A)$ in (3.30) by

$E'(v,u';A)$. Using this observation, and inserting the identity (3.22) into (3.30), we obtain

$$\mathbb{P}(v \overset{A}{\leftrightarrow} x) = \mathbb{P}(E'(v,x;A)) + p\Omega \sum_{(u',v')} D(u',v')\, \mathbb{P}(E'(v,u';A))\, \tau(v',x)$$
$$- p\Omega \sum_{(u',v')} D(u',v')\, \mathbb{E}_1\Big(I[E'(v,u';A)]\, \mathbb{P}_2(v' \xleftrightarrow{\tilde{C}_1^{(u',v')}(v)} x)\Big). \tag{3.31}$$

In the last term on the right side, we have introduced subscripts for $\tilde{C}$ and the expectations, to indicate to which expectation $\tilde{C}$ belongs.

Let

$$\Pi^{(1)}(0,x) = \sum_{(u,v)} p\Omega D(u,v)\, \mathbb{E}_0\left(I[0 \Leftrightarrow u]\mathbb{P}_1(E'(v,x;\tilde{C}_0^{(u,v)}(0)))\right). \tag{3.32}$$

Inserting (3.31) into (3.23), and using (3.32), we have

$$\tau(0,x) = \delta_{0,x} + \Pi^{(0)}(0,x) - \Pi^{(1)}(0,x) + \sum_{(u,v)} \Big(\delta_{0,u} + \Pi^{(0)}(0,u) - \Pi^{(1)}(0,u)\Big) p\Omega D(u,v)\, \tau(v,x)$$
$$+ \sum_{(u,v)} p\Omega D(u,v) \sum_{(u',v')} p\Omega D(u',v')$$
$$\times \mathbb{E}_0\Big(I[0 \Leftrightarrow u]\mathbb{E}_1\Big(I[E'(v,u';\tilde{C}_0{}^{(u,v)}(0))]\mathbb{P}_2(v' \xleftrightarrow{\tilde{C}_1^{(u',v')}(v)} x)\Big)\Big). \tag{3.33}$$

This proves (3.8) for $M=2$, with $R_2(x)$ given by the last two lines of (3.33).

We now repeat this procedure recursively, rewriting $\mathbb{P}_2(v' \xleftrightarrow{\tilde{C}_1^{(u',v')}(v)} x)$ using (3.31), and so on. This leads to (3.8), with $\Pi^{(0)}$ and $\Pi^{(1)}$ given by (3.15) and (3.32), and, for $N \geq 2$,

$$\Pi^{(N)}(0,x) = \sum_{(u_0,v_0)} \cdots \sum_{(u_{N-1},v_{N-1})} \Big[\prod_{i=0}^{N-1} p\Omega D(u_i,v_i)\Big]\mathbb{E}_0 I[0 \Leftrightarrow u_0] \tag{3.34}$$
$$\times \mathbb{E}_1 I[E'(v_0,u_1;\tilde{C}_0)] \cdots \mathbb{E}_{N-1} I[E'(v_{N-2},u_{N-1};\tilde{C}_{N-2})]\mathbb{E}_N I[E'(v_{N-1},x;\tilde{C}_{N-1})],$$

$$R_M(0,x) = (-1)^{M+1} \sum_{(u_0,v_0)} \cdots \sum_{(u_M,v_M)} \Big[\prod_{i=0}^{M} p\Omega D(u_i,v_i)\Big]\mathbb{E}_0 I[0 \Leftrightarrow u_0]$$
$$\times \mathbb{E}_1 I[E'(v_0,u_1;\tilde{C}_0)] \cdots \mathbb{E}_{M-1} I[E'(v_{M-2},u_{M-1};\tilde{C}_{M-2})]$$
$$\times \mathbb{E}_M\Big[I[E'(v_{M-1},u_M;\tilde{C}_{M-1})]\mathbb{P}_{M+1}(v_M \xleftrightarrow{\tilde{C}_M} x)\Big], \tag{3.35}$$

where we have used the abbreviation $\tilde{C}_j = \tilde{C}_j^{(u_j,v_j)}(v_{j-1})$.

Since

$$\mathbb{P}_{M+1}(v_M \xleftrightarrow{\tilde{C}_M} x) \leq \tau_p(v_M,x), \tag{3.36}$$

it follows from (3.34)–(3.35) that

$$|R_M(0,x)| \leq \sum_{u_M,v_M \in \mathbb{V}} \Pi^{(M)}(0,u_M) p\Omega D(u_M,v_M)\tau_p(v_M,x). \tag{3.37}$$

4 Diagrammatic estimates for the lace expansion

In this section, we prove bounds on $\Pi^{(N)}$. These bounds are summarized in Lemma 4.1. We refer to the methods of this section as diagrammatic estimates, as we use Feynman diagrams to provide a convenient representation for upper bounds on $\Pi^{(N)}$.

4.1 The diagrams

In this section, we show how $\Pi^{(N)}$ of (3.34) can be bounded in terms of Feynman diagrams. Our approach here is essentially identical to what is done in [13, Section 2.2], apart from some notational differences, and we omit some details in the following. The results of this section apply to any transitive graph $\mathbb{G} = (\mathbb{V}, \mathbb{B})$.

Let $\mathbb{P}^{(N)}$ denote the product measure on $N+1$ copies of percolation on $\mathbb{G}$. By Fubini's Theorem and (3.34),

$$\Pi^{(N)}(0,x) = \sum_{(u_0,v_0)} \cdots \sum_{(u_{N-1},v_{N-1})} \Big[\prod_{i=0}^{N-1} pD(u_i,v_i) \Big] \\ \times \mathbb{P}^{(N)}\Big(\{0 \Leftrightarrow u_0\}_0 \cap \Big(\bigcap_{i=1}^{N-1} E'(v_{i-1},u_i;\tilde{C}_{i-1})_i \Big) \cap E'(v_{N-1},x;\tilde{C}_{N-1})_N \Big), \tag{4.1}$$

where $E'(v_{i-1},u_i;\tilde{C}_{i-1})_i$ denotes the event that $E'(v_{i-1},u_i;\tilde{C}_{i-1})$ occurs on graph i, and $\{0 \Leftrightarrow u_0\}_0$ denotes the event that $0 \Leftrightarrow u_0$ occurs on graph 0. To estimate $\Pi^{(N)}(0,x)$, it is convenient to define the events

$$F_0(v,t,z,u) = \{v \leftrightarrow t\} \circ \{t \leftrightarrow z\} \circ \{t \leftrightarrow u\} \circ \{z \leftrightarrow u\}, \tag{4.2}$$

$$F_1(v_{i-1},t_i,z_i,u_i,w_i,z_{i+1}) = \{v_{i-1} \leftrightarrow t_i\} \circ \{t_i \leftrightarrow z_i\} \circ \{t_i \leftrightarrow w_i\} \\ \circ \{z_i \leftrightarrow u_i\} \circ \{w_i \leftrightarrow u_i\} \circ \{w_i \leftrightarrow z_{i+1}\}, \tag{4.3}$$

$$F_2(v_{i-1},t_i,z_i,u_i,w_i,z_{i+1}) = \{v_{i-1} \leftrightarrow w_i\} \circ \{w_i \leftrightarrow t_i\} \circ \{t_i \leftrightarrow z_i\} \\ \circ \{t_i \leftrightarrow u_i\} \circ \{z_i \leftrightarrow u_i\} \circ \{w_i \leftrightarrow z_{i+1}\}, \tag{4.4}$$

$$F(v_{i-1},t_i,z_i,u_i,w_i,z_{i+1}) = F_1(v_{i-1},t_i,z_i,u_i,w_i,z_{i+1}) \cup F_2(v_{i-1},t_i,z_i,u_i,w_i,z_{i+1}). \tag{4.5}$$

The events F_0, F_1 and F_2 are depicted in Figure 3.

By the definition of E' in (3.24),

$$E'(v_{N-1},x;\tilde{C}_{N-1})_N \subset \bigcup_{z_N \in \tilde{C}_{N-1}} \bigcup_{t_N \in \mathbb{V}} F_0(v_{N-1},t_N,z_N,x). \tag{4.6}$$

In fact, viewing the connection from v_{N-1} to x as a string of sausages beginning at v_{N-1} and ending at x, for the event E' to occur there must be a vertex $z_N \in \tilde{C}_{N-1}$ that lies on the last sausage, on a path from v_{N-1} to x. (In fact, both "sides" of the sausage must contain a vertex in $\tilde{C}_{N-1}$, but we do not need or use this.) This leads to (4.6), with t_N representing the other endpoint of the sausage that terminates at x.

Assume, for the moment, that $N \geq 2$. The condition in (4.6) that $z_N \in \tilde{C}_{N-1}$ is a condition on the graph $N-1$ that must be satisfied in conjunction with the event $E'(v_{N-2},u_{N-1};\tilde{C}_{N-2})_{N-1}$. It

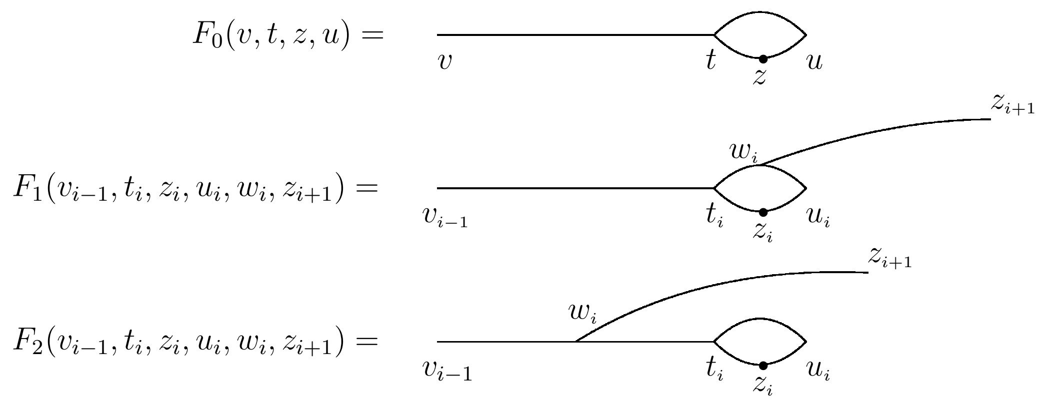

Figure 3: Diagrammatic representations of the events $F_0(v,t,z,u), F_1(v_{i-1},t_i,z_i,u_i,w_i,z_{i+1})$ and $F_2(v_{i-1},t_i,z_i,u_i,w_i,z_{i+1})$. Lines indicate disjoint connections.

is not difficult to see that for $i \in \{1,\dots,N-1\}$,

$$E'(v_{i-1},u_i;\tilde{C}_{i-1})_i \cap \{z_{i+1} \in \tilde{C}_i\} \subset \bigcup_{z_{i+1}\in\tilde{C}_i} \bigcup_{t_i,w_i\in\mathbb{V}} F(v_{i-1},t_i,z_i,u_i,w_i,z_{i+1})_i. \tag{4.7}$$

See Figure 4 for a depiction of the inclusions in (4.6) and (4.7). Further details are given in [13, Lemma 2.5] or [21, Lemma 5.5.8].

With an appropriate treatment for graph 0, (4.6) and (4.7) lead to

$$\begin{aligned} &\{0 \Leftrightarrow u_0\}_0 \cap \Big(\bigcap_{i=1}^{N-1} E'(v_{i-1},u_i;\tilde{C}_{i-1})_i\Big) \cap E'(v_{N-1},x;\tilde{C}_{N-1})_N \\ &\quad\subset \bigcup_{\vec{t},\vec{w},\vec{z}} \Big(F_0(z_0,w_0,u_0,0)_0 \cap \Big(\bigcap_{i=1}^{N-1} F(v_{i-1},t_i,z_i,u_i,w_i,z_{i+1})_i\Big) \cap F_0(v_{N-1},t_N,z_N,x)_N\Big), \end{aligned} \tag{4.8}$$

where $\vec{t}=(t_1,\dots,t_N)$, $\vec{w}=(w_0,\dots,w_{N-1})$ and $\vec{z}=(z_1,\dots,z_N)$. Therefore,

$$\Pi^{(N)}(0,x) \le \sum \mathbb{P}_p(F_0(z_1,w_0,u_0,0)) \prod_{i=1}^{N-1} \mathbb{P}_p(F(v_{i-1},t_i,z_i,u_i,w_i,z_{i+1}))\mathbb{P}_p(F_0(v_{N-1},t_N,z_N,x)), \tag{4.9}$$

where the summation is over $z_1,\dots,z_N,t_1,\dots,t_N,w_0,\dots,w_{N-1},(u_0,v_0),\dots,(u_{N-1},v_{N-1})$. The probability in (4.9) factors because the $N+1$ percolation models are now independent. Each probability in (4.9) can be estimated using the BK inequality. The result is that each of the connections $\{x \leftrightarrow y\}$ present in the events F and F_0 is replaced by a two-point function $\tau_p(x,y)$. This results in a large sum of two-point functions.

To organize a large sum of this form, we let

$$\tilde{\tau}_p(0,x) = p\Omega(D * \tau_p)(0,x), \tag{4.10}$$

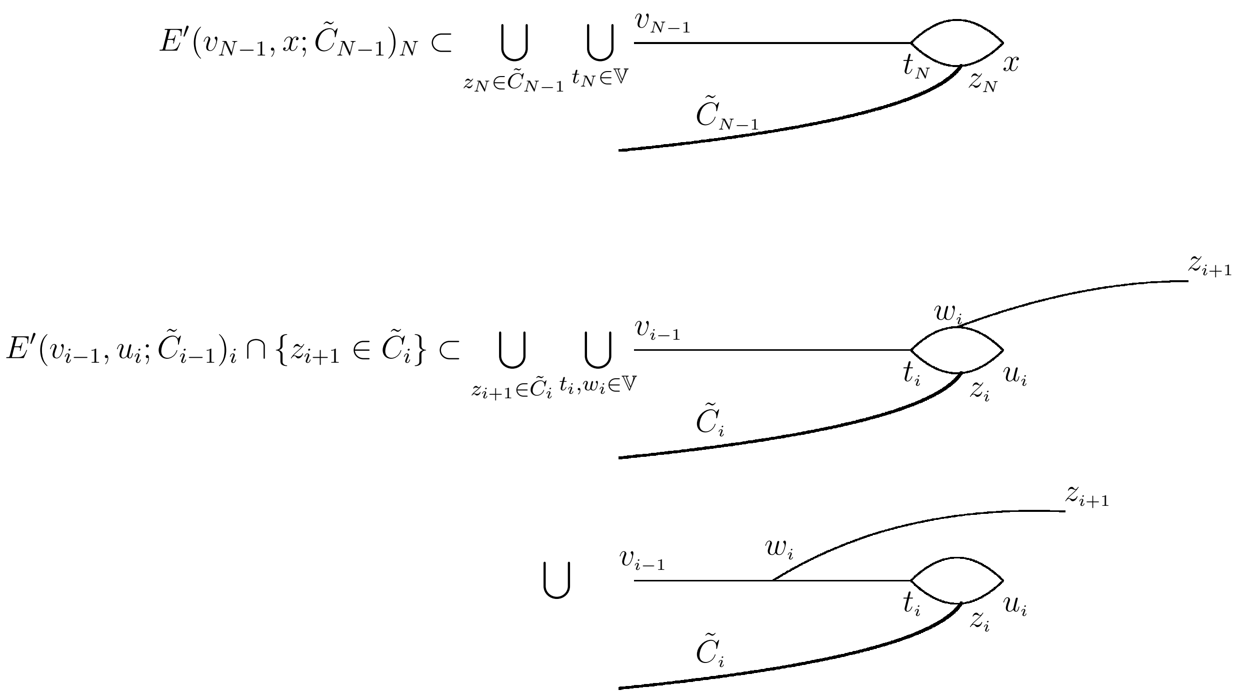

Figure 4: Diagrammatic representations of the inclusions in (4.6) and (4.7).

and define

$$A_3(s, u, v) = \tau_p(s, v)\tau_p(s, u)\tau_p(u, v), \tag{4.11}$$

$$B_1(s, t, u, v) = \tilde{\tau}_p(t, v)\tau_p(s, u), \tag{4.12}$$

$$\begin{aligned} B_2(u, v, s, t) &= \tau_p(u, v)\tau_p(u, t)\tau_p(v, s)\tau_p(s, t) \\ &\quad + \sum_{a \in \mathbb{V}} \tau_p(s, a)\tau_p(a, u)\tau_p(a, t)\delta_{v,s}\tau_p(u, t). \end{aligned} \tag{4.13}$$

The two terms in B_2 arise from the two events F_1 and F_2 in (4.5). We will write them as $B_2^{(1)}$ and $B_2^{(2)}$, respectively. The above quantities are represented diagrammatically in Figure 5. In the diagrams, a line joining x and y represents $\tau_p(x, y)$. In addition, small bars are used to distinguish a line that represents $\tilde{\tau}_p$, as in B_1.

Application of the BK inequality yields

$$\mathbb{P}_p(F_0(z_1, w_0, u_0, 0)) \leq A_3(0, u_0, w_0)\tau_p(w_0, z_1), \tag{4.14}$$

$$\sum_{v_{N-1}} p\Omega D(u_{N-1}, v_{N-1})\mathbb{P}_p(F_0(v_{N-1}, t_N, z_N, x)) \leq \frac{B_1(w_{N-1}, u_{N-1}, z_N, t_N)}{\tau_p(w_{N-1}, z_N)} A_3(x, t_N, z_N). \tag{4.15}$$

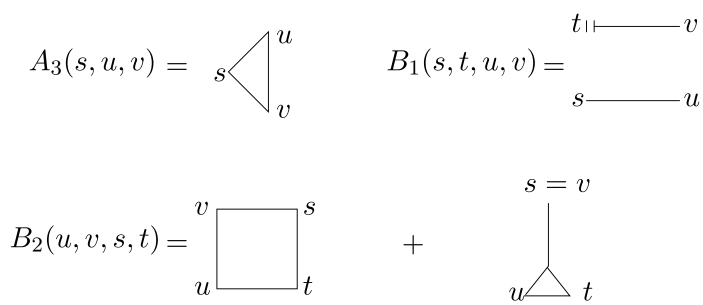

Figure 5: Diagrammatic representations of $A_3(s,u,v)$, $B_1(s,t,u,v)$ and $B_2(u,v,s,t)$.

For F_1 and F_2, application of the BK inequality yields

$$\sum_{v_{i-1}} p\Omega D(u_{i-1},v_{i-1})\mathbb{P}_p(F_1(v_{i-1},t_i,z_i,u_i,w_i,z_{i+1})) \leq \frac{B_1(w_{i-1},u_{i-1},z_i,t_i)}{\tau_p(w_{i-1},z_i)} B_2^{(1)}(z_i,t_i,w_i,u_i)\tau_p(w_i,z_{i+1}), \tag{4.16}$$

$$\sum_{v_{i-1},t_i} p\Omega D(u_{i-1},v_{i-1})\mathbb{P}_p(F_2(v_{i-1},t_i,z_i,u_i,w_i,z_{i+1})) \leq \frac{B_1(w_{i-1},u_{i-1},z_i,w_i)}{\tau_p(w_{i-1},z_i)} B_2^{(2)}(z_i,w_i,w_i,u_i)\tau_p(w_i,z_{i+1}). \tag{4.17}$$

Since the first and the third argument of $B_2^{(2)}$ are equal by (4.13), we can combine (4.16)–(4.17) to obtain

$$\sum_{v_{i-1},t_i} p\Omega D(u_{i-1},v_{i-1})\mathbb{P}_p(F(v_{i-1},t_i,z_i,u_i,w_i,z_{i+1})) \leq \sum_{t_i} \frac{B_1(w_{i-1},u_{i-1},z_i,t_i)}{\tau_p(w_{i-1},z_i)} B_2(z_i,t_i,w_i,u_i)\tau_p(w_i,z_{i+1}). \tag{4.18}$$

Upon substitution of the bounds on the probabilities in (4.14), (4.15) and (4.18) into (4.9), the ratios of two-point functions form a telescoping product that disappears. After relabeling the summation indices, (4.9) becomes

$$\Pi^{(N)}(0,x) \leq \sum_{\vec{s},\vec{t},\vec{u},\vec{v}} A_3(0,s_1,t_1) \prod_{i=1}^{N-1} \Big[B_1(s_i,t_i,u_i,v_i)B_2(u_i,v_i,s_{i+1},t_{i+1})\Big] \times B_1(s_N,t_N,u_N,v_N)A_3(u_N,v_N,x). \tag{4.19}$$

The bound (4.19) is valid for $N \geq 1$, and the summation is over all $s_1,\dots,s_N$, $t_1,\dots,t_N$, $(u_1,v_1),\dots,(u_N,v_N)$. For $N=1,2$, the right side is represented diagrammatically in Figure 6. In the diagrams, unlabeled vertices are summed over $\mathbb{V}$.

4.2 The diagrammatic bounds

We now specialize to the case $\mathbb{G} = \mathbb{T}_{r,n}$, making use of the additive structure and the $x \mapsto -x$ symmetry of the torus. We will write $\tau_p(y-x)$ in place of $\tau_p(x,y)$, and similarly for D and $\Pi^{(N)}$.

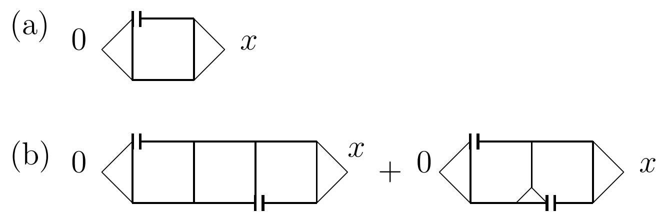

Figure 6: The diagrams bounding (a) $\Pi^{(1)}(0,x)$ and (b) $\Pi^{(2)}(0,x)$.

The upper bounds we prove are in terms of various quantities related to the triangle diagram. Let

$$T_p(x) = \sum_{y,z,u\in\mathbb{V}} \tau_p(y)\tau_p(z-y)p\Omega D(u)\tau_p(x+z-u) = (\tau_p * \tau_p * \tilde{\tau}_p)(x), \tag{4.20}$$

$$T_p = \max_{x\in\mathbb{V}} T_p(x), \tag{4.21}$$

$$T'_p = \max_{x\in\mathbb{V}} \sum_{y,z\in\mathbb{V}} \tau_p(y)\tau_p(z-y)\tau_p(x-z) = \max_{x\in\mathbb{V}}(\tau_p * \tau_p * \tau_p)(x), \tag{4.22}$$

and, for $k \in \mathbb{T}^*_{r,n}$, let

$$W_p(y;k) = \sum_{x\in\mathbb{V}} [1-\cos(k\cdot x)]\tilde{\tau}_p(x)\tau_p(x+y), \tag{4.23}$$

$$W_p(k) = \max_{y\in\mathbb{V}} W_p(y;k). \tag{4.24}$$

Recall that $B_2^{(2)}$ denotes the second term of (4.13). For $k \in \mathbb{T}^*_{r,n}$, we also define

$$\begin{aligned} H_p(a_1,a_2;k) = \sum_{u_{i-1},v_i,v_{i-1},s_i,t_i} & [1-\cos(k\cdot(t_i-u_{i-1}))] B_1(0,a_1,u_{i-1},v_{i-1}) \\ & \times B_2^{(2)}(u_{i-1},v_{i-1},s_i,t_i) B_1(s_i,t_i,v_i,v_i+a_2), \end{aligned} \tag{4.25}$$

and

$$H_p(k) = \max_{a_1,a_2\in\mathbb{V}} H_p(a_1,a_2;k). \tag{4.26}$$

The remainder of this section is devoted to the proof of the following lemma.

Lemma 4.1. *For $N=0$,*

$$\sum_{x\in\mathbb{V}} \Pi^{(0)}(x) \le T_p, \tag{4.27}$$

$$\sum_{x\in\mathbb{V}} [1-\cos(k\cdot x)]\Pi^{(0)}(x) \le W_p(0;k). \tag{4.28}$$

For $N \ge 1$,

$$\sum_{x\in\mathbb{V}} \Pi^{(N)}(x) \le T'_p(2T_pT'_p)^N, \tag{4.29}$$

$$\begin{aligned} \sum_{x\in\mathbb{V}} [1-\cos(k\cdot x)]\Pi^{(N)}(x) \le (4N+3)\Big[& T'_p W_p(k)(2T_p + [1+p\Omega]NT'_p)(2T_pT'_p)^{N-1} \\ & + (N-1)(T_p^2 W_p(k) + H_p(k))(T'_p)^2(2T_pT'_p)^{N-2}\Big], \end{aligned} \tag{4.30}$$

and, for $N=1$, (4.30) can also be replaced by

$$\sum_{x\in\mathbb{V}}[1-\cos(k\cdot x)]\Pi^{(1)}(x)\le W_p(0;k)+31T_pT_p'W_p(k). \tag{4.31}$$

4.2.1 Proof of (4.27)–(4.28)

By (3.15) and the BK inequality,

$$\Pi^{(0)}(x)=\mathbb{P}(0\Leftrightarrow x)-\delta_{0,x}\le\tau_p(x)^2-\delta_{0,x}. \tag{4.32}$$

For $x\neq 0$, the event $\{0\leftrightarrow x\}$ is the union over neighbors y of the origin of $\{\{0,y\}\text{ occupied}\}\circ\{y\leftrightarrow x\}$. Thus, by the BK inequality,

$$\tau_p(x)\le p\Omega(D*\tau_p)(x)=\tilde{\tau}_p(x)\quad(x\neq 0). \tag{4.33}$$

Therefore,

$$\sum_{x\in\mathbb{V}}\Pi^{(0)}(x)\le\sum_{x\in\mathbb{G}}\tau_p(x)\tilde{\tau}_p(x)\le T_p(0). \tag{4.34}$$

Similarly,

$$\sum_{x\in\mathbb{V}}[1-\cos(k\cdot x)]\Pi^{(0)}(x)\le W_p(0;k). \tag{4.35}$$

This proves (4.27)–(4.28).

4.2.2 Proof of (4.29)

For $N\ge 1$, let

$$\Psi^{(N)}(s_{N+1},t_{N+1})=\sum_{\vec{s},\vec{t},\vec{u},\vec{v}}A_3(0,s_1,t_1)\prod_{i=1}^{N}\Big[B_1(s_i,t_i,u_i,v_i)B_2(u_i,v_i,s_{i+1},t_{i+1})\Big]. \tag{4.36}$$

For convenience, we define $\Psi^{(0)}(x,y)=A_3(0,x,y)$, so that

$$\Psi^{(N)}(x,y)=\sum_{u_N,v_N,s_N,t_N}\Psi^{(N-1)}(s_N,t_N)B_1(s_N,t_N,u_N,v_N)B_2(u_N,v_N,x,y),\quad(N\ge 1). \tag{4.37}$$

Since

$$\sum_{x}A_3(u_N,v_N,x)\le\sum_{x,y}B_2(u_N,v_N,x,y), \tag{4.38}$$

it follows that

$$\sum_{x}\Pi^{(N)}(x)\le\sum_{x,y}\Psi^{(N)}(x,y), \tag{4.39}$$

and bounds on $\Pi^{(N)}$ can be obtained from bounds on $\Psi^{(N)}$. We prove bounds on $\Psi^{(N)}$, and hence on $\Pi^{(N)}$, by induction on N.

The induction hypothesis is that

$$\sum_{x,y}\Psi^{(N)}(x,y)\le T_p'(2T_pT_p')^N. \tag{4.40}$$

For $N = 0$, (4.40) is true since

$$\sum_{x,y} A_3(0, x, y) \le T'_p. \tag{4.41}$$

If we assume (4.40) is valid for $N - 1$, then by (4.37),

$$\sum_{x,y} \Psi^{(N)}(x, y) \le \Big(\sum_{s_N,t_N} \Psi^{(N-1)}(s_N, t_N) \Big)\Big(\max_{s_N,t_N} \sum_{u_N,v_N,x,y} B_1(s_N, t_N, u_N, v_N) B_2(u_N, v_N, x, y) \Big), \tag{4.42}$$

and (4.40) then follows once we prove that

$$\max_{s,t} \sum_{u,v,x,y} B_1(s, t, u, v) B_2(u, v, x, y) \le 2T_p T'_p. \tag{4.43}$$

It remains to prove (4.43). There are two terms, due to the two terms in (4.13), and we bound each term separately. The first term is bounded as

$$\begin{aligned}
&\max_{s,t} \sum_{u,v,x,y} \tilde{\tau}_p(v-t)\tau_p(u-s)\tau_p(y-u)\tau_p(x-v)\tau_p(v-u)\tau_p(x-y) \\
&\quad = \max_{s,t} \sum_{u,v} \tilde{\tau}_p(v-t)\tau_p(u-s)\tau_p(v-u)\Big(\sum_{x,y} \tau_p(y-u)\tau_p(x-v)\tau_p(x-y) \Big) \\
&\quad \le T'_p \max_{s,t} \sum_{u,v} \tilde{\tau}_p(v-t)\tau_p(u-s)\tau_p(v-u) \\
&\quad = T_p T'_p.
\end{aligned} \tag{4.44}$$

The second term is bounded similarly by

$$\begin{aligned}
&\max_{s,t} \sum_{u,v,x,y,a} \tilde{\tau}_p(v-t)\tau_p(u-s)\delta_{v,x}\tau_p(y-u)\tau_p(x-a)\tau_p(u-a)\tau_p(y-a) \\
&\quad = \max_{s,t} \sum_{a,y,u} \Big((\tilde{\tau}_p * \tau)(a-t)\tau_p(u-s) \Big)\Big(\tau_p(y-u)\tau_p(u-a)\tau_p(y-a) \Big) \\
&\quad = \max_{s,t} \sum_{y',a'} T_p(a'+s-t)\tau_p(y')\tau_p(a')\tau_p(y'-a') \\
&\quad \le \Big(\max_{a',s,t} T_p(a'+s-t) \Big)\Big(\sum_{y',a'} \tau_p(y')\tau_p(a')\tau_p(y'-a') \Big) = T_p T'_p,
\end{aligned} \tag{4.45}$$

where $a' = a - u$, $y' = y - u$. This completes the proof of (4.43) and hence of (4.29).

4.2.3 Proof of (4.30)

Next, we estimate $\sum_x [1 - \cos(k \cdot x)]\Pi^{(N)}(x)$. In a term in (4.19), there is a sequence of $2N + 1$ two-point functions along the "top" of the diagram, such that the sum of the displacements of these two-point functions is exactly equal to x. For example, in Figure 6(a) there are three displacements along the top of the diagram, and in Figure 6(b) there are five in the first diagram and four in the second. We regard the second diagram as also having five displacements, with the understanding that the third is constrained to vanish. With a similar general convention, each of the 2^{N-1} diagrams bounding $\Pi^{(N)}$ has $2N + 1$ displacements along the top of the diagram. We denote these displacements by $d_1, \ldots, d_{2N+1}$, so that $x = \sum_{j=1}^{2N+1} d_j$. We will argue as follows to distribute the factor $1 - \cos(k \cdot x)$ among the displacements d_j.

Let $t = \sum_{j=1}^{J} t_j$. Taking the real part of the telescoping sum

$$1 - e^{it} = \sum_{j=1}^{J} [1 - e^{it_j}] e^{i \sum_{m<j} t_m} \tag{4.46}$$

leads to the bound

$$1 - \cos t \le \sum_{j=1}^{J} [1 - \cos t_j] + \sum_{j=1}^{J} \sin t_j \sin \sum_{m<j} t_m. \tag{4.47}$$

It follows from the identity $\sin(x+y) = \sin x \cos y + \cos x \sin y$ that $|\sin(x+y)| \le |\sin x| + |\sin y|$. Applying this recursively gives

$$1 - \cos t \le \sum_{j=1}^{J} [1 - \cos t_j] + \sum_{j=1}^{J} \sum_{m=1}^{j-1} |\sin t_j| |\sin t_m|. \tag{4.48}$$

In the last term we use $|ab| \le (a^2 + b^2)/2$, and then $1 - \cos^2 a \le 2[1 - \cos a]$, to obtain

$$\begin{aligned} 1 - \cos t &\le \sum_{j=1}^{J} [1 - \cos t_j] + \frac{1}{2} \sum_{j=1}^{J} \sum_{m=1}^{j-1} [\sin^2 t_j + \sin^2 t_m] \\ &\le \sum_{j=1}^{J} [1 - \cos t_j] + J \sum_{j=1}^{J} \sin^2 t_j \\ &= \sum_{j=1}^{J} [1 - \cos t_j] + J \sum_{j=1}^{J} [1 - \cos^2 t_j] \\ &\le (2J+1) \sum_{j=1}^{J} [1 - \cos t_j]. \end{aligned} \tag{4.49}$$

We apply (4.49) with $t = k \cdot x = \sum_{j=1}^{2N+1} k \cdot d_j$ to obtain a sum of $2N+1$ diagrams like the ones for $\Pi^{(N)}(x)$, except now in the j^{th} term the j^{th} line in the top of the diagram represents $[1 - \cos(k \cdot d_j)] \tau_p(d_j)$ rather than $\tau_p(d_j)$.

We distinguish three cases: (a) the displacement d_j is in a line of A_3, (b) the displacement d_j is in a line of B_1, (c) the displacement d_j is in a line of B_2.

Case (a): the displacement is in a line of A_3. We consider the case where the weight $[1 - \cos(k \cdot d_j)]$ falls on the last of the factors A_3 in (4.19). This contribution is equal to

$$\sum_{u,v} \Psi^{(N-1)}(u,v) \sum_{w,x,y} B_1(u,v,w,y) \tau_p(y-w) [1 - \cos(k \cdot (x-y))] \tau_p(x-y) \tau_p(x-w). \tag{4.50}$$

Applying (4.33) to $\tau_p(x-y)$, we have

$$\max_{u,v} \sum_{w,x,y} B_1(u,v,w,y) \tau_p(y-w) [1 - \cos(k \cdot (x-y))] \tau_p(x-y) \tau_p(x-w) \le T_p W_p(k). \tag{4.51}$$

It then follows from (4.40) that (4.50) is bounded above by $T'_p (2 T_p T'_p)^{N-1} T_p W_p(k)$. By symmetry, the same bound applies when the weight falls into the first factor of A_3, i.e, when we have a factor $[1 - \cos(k \cdot d_1)]$. Thus case (a) leads to an upper bound

$$2 T'_p (2 T_p T'_p)^{N-1} T_p W_p(k). \tag{4.52}$$

Case (b): the displacement is in a line of B_1. Suppose that the factor $[1-\cos(k\cdot d_j)]$ falls on the i^{th} factor B_1 in (4.19). Depending on i, it falls either on $\tilde{\tau}_p$ or on τ_p in (4.12). We write the right side of (4.19) with the extra factor as

$$\sum_x \sum_{s_i,t_i,u_i,v_i} \Psi^{(i)}(s_i,t_i)\tilde{B}_1(s_i,t_i,u_i,v_i)\Psi^{(N-i-1)}(u_i-x,v_i-x), \tag{4.53}$$

where either

$$\tilde{B}_1(s,t,u,v) = [1-\cos(k\cdot(u-s))]\tilde{\tau}_p(u-s)\tau_p(v-t), \tag{4.54}$$

or

$$\tilde{B}_1(s,t,u,v) = \tilde{\tau}_p(u-s)[1-\cos(k\cdot(v-t))]\tau_p(v-t). \tag{4.55}$$

For (4.54), we let $a_1 = t_i - s_i$, $a_2 = v_i - u_i$, and $x' = u_i - x$. With this notation, the contribution to (4.53) due to (4.54) is bounded above by

$$\begin{aligned}
&\Big(\sum_{s_i,a_1}\Psi^{(i)}(s_i,s_i+a_1)\Big)\Big(\sum_{x',a_2}\Psi^{(N-i-1)}(x',x'+a_2)\Big)\Big(\max_{s_i,a_1,a_2}\sum_{u_i}\tilde{B}_1(s_i,s_i+a_1,u_i,u_i+a_2)\Big)\\
&\quad = \Big(\sum_{s_i,t_i}\Psi^{(i)}(s_i,t_i)\Big)\Big(\sum_{x,y}\Psi^{(N-i)}(x,y)\Big)W_p(k)\\
&\quad \le T_p'(2T_pT_p')^iT_p'(2T_pT_p')^{N-i-1}W_p(k) = T_p'(2T_pT_p')^{N-1}T_p'W_p(k),
\end{aligned} \tag{4.56}$$

where we used (4.40). For (4.55), we use (4.33) for $\tau_p(v-t)$ and also obtain an additional factor $p\Omega$ from $\tilde{\tau}_p(u-s)$. Since there are N choices of factors B_1, case (b) leads to an upper bound

$$[1+p\Omega]NT_p'(2T_pT_p')^{N-1}T_p'W_p(k). \tag{4.57}$$

Case (c): the displacement is in a line of B_2. It is sufficient to estimate

$$\begin{aligned}
\max \sum_{\substack{s_{i-1},t_{i-1},u_{i-1},v_{i-1}\\ s_i,t_i,u_i,v_i,x}} &\Psi^{(i-1)}(s_{i-1},t_{i-1})\Psi^{(N-i-1)}(u_i-x,v_i-x)[1-\cos(k\cdot d_i)]\\
&\times B_1(s_{i-1},t_{i-1},u_{i-1},v_{i-1})B_2(u_{i-1},v_{i-1},s_i,t_i)B_1(s_i,t_i,u_i,v_i),
\end{aligned} \tag{4.58}$$

where the maximum is over the choices $d_i = s_i - v_{i-1}$ or $d_i = t_i - u_{i-1}$. We consider separately the contributions due to $B_2^{(1)}$ and $B_2^{(2)}$ of (4.13), beginning with $B_2^{(2)}$.

Recall the definition of $H(a_1,a_2;k)$ in (4.25). The contribution to (4.58) due to $B_2^{(2)}$ can be rewritten as

$$\begin{aligned}
&\sum_{s_{i-1},a_1,a_2,x'}\Psi^{(i-1)}(s_{i-1},s_{i-1}+a_1)\Psi^{(N-i-1)}(x',x'+a_2)H(a_1,a_2;k)\\
&\qquad \le H_p(k)\Big(\sum_{x,y}\Psi^{(i-1)}(x,y)\Big)\Big(\sum_{x,y}\Psi^{(N-i-1)}(x,y)\Big)\\
&\qquad \le H_p(k)(T_p')^2(2T_pT_p')^{N-2}.
\end{aligned} \tag{4.59}$$

Since there are $N-1$ factors B_2 to choose, this contribution to case (c) contributes at most

$$(N-1)H_p(k)(T_p')^2(2T_pT_p')^{N-2}. \tag{4.60}$$

It is not difficult to check that the contribution to case (c) due to $B_2^{(1)}$ is at most

$$(N-1)(T_p^2W_p(k))(T_p')^2(2T_pT_p')^{N-2}. \tag{4.61}$$

The desired estimate (4.30) then follows from (4.49), (4.52), (4.57) and (4.60)–(4.61).

4.2.4 Proof of (4.31)

Recall from (4.19) that

$$\Pi_p^{(1)}(x) \leq \sum_{s,t,u,v} A_3(0,s,t)B_1(s,t,u,v)A_3(u,v,x). \tag{4.62}$$

We define $A_3'(u,v,x)$ by

$$A_3(u,v,x) = \delta_{u,x}\delta_{v,x} + A_3'(u,v,x). \tag{4.63}$$

Then we have

$$\begin{aligned}\sum_{x\in\mathbb{V}}[1-\cos(k\cdot x)]\Pi_p^{(1)}(x) \leq \sum_{x\in\mathbb{V}}&[1-\cos(k\cdot x)]B_1(0,0,x,x)\\ &+\sum_{x\in\mathbb{V}}\sum_{s,t,u,v}[1-\cos(k\cdot x)]A_3'(0,s,t)B_1(s,t,u,v)A_3(u,v,x)\\ &+\sum_{x\in\mathbb{V}}\sum_{s,t,u,v}[1-\cos(k\cdot x)]B_1(0,0,u,v)A_3'(u,v,x).\end{aligned} \tag{4.64}$$

The first term equals $W_p(0;k)$. The second and third terms are bounded above by $7\cdot 3T_pT_p'W_p(k)$ and $5\cdot 2T_pW_p(k) \leq 10T_pT_p'W_p(k)$, respectively, using (4.49) (with $J=3$ and $J=2$) and the methods of Section 4.2.3.

5 Analysis of the lace expansion

In this section, we use the lace expansion to prove the triangle condition of Theorem 1.3. The analysis is similar in spirit to the analysis of [13], but it has been simplified and reorganized, and it differs significantly in detail from the presentation of [13]. Specific improvements include: (i) We have reduced the number of functions in the bootstrap argument from five to three (cf. [13, Proposition 4.3]), and in the bootstrap we work directly with the Fourier transform of the two-point fuction rather than with the triangle and related diagrams. (ii) We work with $1-\cos(k\cdot x)$ directly, rather than expanding the cosine to second order. (iii) Our treatment of $H_p(k)$ in Lemma 5.5 below is simpler than the corresponding treatment of [13, Section 4.4.3(e)].

We work in this section on an arbitrary torus $\mathbb{T}_{r,n}$ with $r \geq 2$, assuming that Assumption 1.1 is satisfied. As usual, we write the degree of the torus as Ω, and we abbreviate $p_c(\mathbb{T}_{r,n})$ to p_c.

Our analysis actually uses a slightly weaker assumption than the one stated in Assumption 1.1. Instead of (1.26), we will assume in the proof that

$$\frac{1}{V}\sum_{k\in\mathbb{T}_{r,n}^*:\,k\neq 0}\frac{|\hat{D}(k)|^2}{[1-\mu\hat{D}(k)]^3} \leq \beta \tag{5.1}$$

holds uniformly in $\mu \in [0, 1-\frac{1}{2}\lambda^{-1}V^{-1/3}]$. Equation (5.1) is strictly weaker than (1.26), but not in a significant way. The analogue of (2.28) with μ inserted in the denominator follows from (5.1) in the same way that (2.28) follows from (1.26).

5.1 The bootstrap argument

Taking the Fourier transform of (3.8) and solving for $\hat{\tau}_p(k)$ gives

$$\hat{\tau}_p(k) = \frac{1 + \hat{\Pi}_M(k) + \hat{R}_M(k)}{1 - p\Omega\hat{D}(k)[1 + \hat{\Pi}_M(k)]}, \tag{5.2}$$

for all $k \in \mathbb{T}^*_{r,n}$ and all $M = 0, 1, 2, \ldots$. Recall from (2.4) that $\hat{C}_\mu(k) = [1-\mu\Omega\hat{D}(k)]^{-1}$. As explained in Section 3.1, we would like to compare $\hat{\tau}_p(k)$ with $\hat{C}_\mu(k)$, with $\mu\Omega$ equal to $p\Omega[1 + \hat{\Pi}_M(0)]$. We know that $\hat{\tau}_p(0) = \chi(p) > 0$, but we do not yet know that $1 + \hat{\Pi}_M(0) + \hat{R}_M(0)$ is positive and thus we cannot yet be sure that the denominator of (5.2) is positive when $k = 0$. We therefore do not yet know that our choice of μ is less than Ω^{-1}. To safeguard against the possibility that $p\Omega[1 + \hat{\Pi}_M(0)] \geq 1$ (later we will see that this possibility is not realized), we define $\mu_p^{(M)}$ by

$$\mu_p^{(M)}\Omega = \min\{1 - \frac{1}{2}\lambda^{-1}V^{-1/3}, p\Omega[1 + \hat{\Pi}_M(0)]^+\}, \tag{5.3}$$

where $x^+ = \max\{x, 0\}$. We will prove that for all M sufficiently large (depending on p), and for all $p \leq p_c$,

$$\max_{k\in\mathbb{T}^*_{r,n}} \frac{\hat{\tau}_p(k)}{\hat{C}_{\mu_p^{(M)}}(k)} \leq 3. \tag{5.4}$$

The inequality (5.4) is the key ingredient in the proof of Theorem 1.3.

The proof of (5.4) is based on the following elementary lemma. The lemma states that under an appropriate continuity assumption, if an inequality implies a stronger inequality, then in fact the stronger inequality must hold. This kind of bootstrap argument has been applied repeatedly in lace expansion analyses, and goes back to [24] in this context.

Lemma 5.1 (The bootstrap). *Let f be a continuous function on the interval $[p_1, p_2]$, and assume that $f(p_1) \leq 3$. Suppose for each $p \in (p_1, p_2)$ that if $f(p) \leq 4$ then in fact $f(p) \leq 3$. Then $f(p) \leq 3$ for all $p \in [p_1, p_2]$.*

Proof. By hypothesis, $f(p)$ cannot be strictly between 3 and 4 for any $p \in [p_1, p_2)$. Since $f(p_1) \leq 3$, it follows by continuity that $f(p) \leq 3$ for all $p \in [p_1, p_2]$. □

We will apply Lemma 5.1 with $p_1 = 0$, $p_2 = p_c$, and

$$f(p) = \max\{f_1(p), f_2(p), f_3(p)\}, \tag{5.5}$$

where

$$f_1(p) = p\Omega, \quad f_2(p) = \max_{k\in\mathbb{T}^*_{r,n}} \frac{\hat{\tau}_p(k)}{\hat{C}_{\mu_p^{(M)}}(k)}, \tag{5.6}$$

$$f_3(p) = \max_{\substack{k,l\in\mathbb{T}^*_{r,n}\\ k\neq 0}} \frac{\hat{C}_1(k)}{X} \frac{\hat{\tau}_p(l) - \frac{1}{2}(\hat{\tau}_p(l-k) + \hat{\tau}_p(l+k))}{\hat{C}_{\mu_p^{(M)}}(l-k)\hat{C}_{\mu_p^{(M)}}(l) + \hat{C}_{\mu_p^{(M)}}(l)\hat{C}_{\mu_p^{(M)}}(l+k) + \hat{C}_{\mu_p^{(M)}}(l-k)\hat{C}_{\mu_p^{(M)}}(l+k)}, \tag{5.7}$$

with the constant X equal to $X = 675$ (the exact value of X is not important).

We benefit from working on a finite graph in using f_2 and f_3 as defined above. As we will argue below, it is easy to establish continuity in p of the quantities to the right of the maxima in the definitions of f_2 and f_3, and hence the continuity of f_2 and f_3 follows immediately. However, on an infinite graph k and l lie in an infinite set, the maxima must be replaced by suprema, and the continuity for each fixed k, l does not necessarily imply the continuity of the suprema. Different functions were used in [13] for this reason.

The expression $\hat{\tau}_p(l) - \frac{1}{2}(\hat{\tau}_p(l-k) + \hat{\tau}_p(l+k))$ in (5.7) is closely related to a discrete second derivative of $\hat{\tau}_p(l)$, and we discuss this now in more detail as a preliminary for what follows. Given a function $\hat{f}$ on $\mathbb{T}^*_{r,n}$ and $k, l \in \mathbb{T}^*_{r,n}$, let

$$\partial_k^+ \hat{f}(l) = \hat{f}(l+k) - \hat{f}(l), \tag{5.8}$$

$$\partial_k^- \hat{f}(l) = \hat{f}(l) - \hat{f}(l-k), \tag{5.9}$$

and $\Delta_k \hat{f}(l) = \partial_k^- \partial_k^+ \hat{f}(l)$. Then

$$-\frac{1}{2}\Delta_k \hat{f}(l) = \hat{f}(l) - \frac{1}{2}(\hat{f}(l+k) + \hat{f}(l-k)). \tag{5.10}$$

In particular, $-\frac{1}{2}\Delta_k \hat{\tau}_p(l)$ appears in the numerator of $f_3(p)$.

The following will be useful in computations involving Δ_k. Let g be a symmetric function on the torus, meaning $g(x) = g(-x)$. Then the Fourier transform of g is actually the cosine series $\hat{g}(l) = \sum_x g(x)\cos(l \cdot x)$. We define

$$\hat{g}^{\cos}(l,k) = \sum_x g(x)\cos(l \cdot x)\cos(k \cdot x) = \frac{1}{2}[\hat{g}(l-k) + \hat{g}(l+k)], \tag{5.11}$$

$$\hat{g}^{\sin}(l,k) = \sum_x g(x)\sin(l \cdot x)\sin(k \cdot x) = \frac{1}{2}[\hat{g}(l-k) - \hat{g}(l+k)]. \tag{5.12}$$

Then

$$-\frac{1}{2}\Delta_k \hat{g}(l) = \hat{g}(l) - \hat{g}^{\cos}(l,k), \tag{5.13}$$

and, assuming only for (5.14) that $g(x) \geq 0$,

$$\frac{1}{2}|\Delta_k \hat{g}(l)| = |\hat{g}(l) - \hat{g}^{\cos}(l,k)| \leq \hat{g}(0) - \hat{g}(k). \tag{5.14}$$

By definition,

$$\hat{g}(l+k)\hat{g}(l-k) = \sum_{x,y} g(x)g(y)\cos((l+k)\cdot x)\cos((l-k)\cdot y). \tag{5.15}$$

Using symmetry and the identity $\cos(a+b) = \cos a \cos b - \sin a \sin b$, this becomes

$$\hat{g}(l+k)\hat{g}(l-k) = \hat{g}^{\cos}(l,k)^2 - \hat{g}^{\sin}(l,k)^2. \tag{5.16}$$

5.2 Bounds on $\Pi^{(N)}$

The verification of the main hypothesis of Lemma 5.1 for f of (5.5) relies crucially on the bounds on $\Pi^{(N)}$ given in the following lemma.

Lemma 5.2 (Bounds on the lace expansion). *Let $N = 0, 1, 2, \ldots$, and assume that Assumption 1.1 holds. For each $K > 0$, there is a constant $\bar{c}_K$ such that if $f(p)$ of (5.5) obeys $f(p) \leq K$, then*

$$\sum_{x \in \mathbb{T}_{r,n}} \Pi^{(N)}(x) \leq [\bar{c}_K(\lambda^3 \vee \beta)]^{(N-1)\vee 1} \tag{5.17}$$

and

$$\sum_{x \in \mathbb{T}_{r,n}} [1 - \cos(k \cdot x)]\Pi^{(N)}(x) \leq [1 - \hat{D}(k)][\bar{c}_K(\lambda^3 \vee \beta)]^{(N-1)\vee 1}. \tag{5.18}$$

Lemma 5.2 will follow from Lemma 4.1 combined with the following three lemmas. For these three lemmas, we recall the quantities defined in (4.20)–(4.26) and also define

$$T_p^{(\alpha)} = \frac{1}{V} \sum_{k \in \mathbb{T}^*_{r,n}} |\hat{D}(k)|^\alpha \hat{\tau}_p(k)^3. \tag{5.19}$$

Lemma 5.3. *Fix $p \in (0, p_c)$, assume that $f(p)$ of (5.5) obeys $f(p) \leq K$, and assume that Assumption 1.1 holds. There is a constant c_K, independent of p, such that*

$$T_p^{(2)} \leq c_K(\lambda^3 \vee \beta), \qquad T_p \leq c_K(\lambda^3 \vee \beta), \qquad T'_p \leq 1 + c_K(\lambda^3 \vee \beta). \tag{5.20}$$

The bound on $T_p^{(2)}$ also applies if $\hat{\tau}_p(k)^3$ is replaced by $\hat{\tau}_p(k)$ or $\hat{\tau}_p(k)^2$ in (5.19). In addition, λ^3 can be replaced by $V^{-1}\chi(p)^3$ in each of the above bounds.

Proof. We begin with $T_p^{(2)}$. We extract the term due to $k = 0$ in (5.19) and use $f_2(p) \leq K$ to obtain

$$T_p^{(2)} \leq V^{-1}\chi(p)^3 + V^{-1} \sum_{k \neq 0} |\hat{D}(k)|^2 K^3 \hat{C}_{\mu_p^{(M)}}(k)^3. \tag{5.21}$$

The first term obeys $V^{-1}\chi(p)^3 \leq V^{-1}\chi(p_c)^3 = \lambda^3$, and the desired result follows from (5.1). The final statement of the lemma follows from the fact that the quantity with $\hat{\tau}_p(k)$ or $\hat{\tau}_p(k)^2$ is easily bounded in x-space by the quantity with $\hat{\tau}_p(k)^3$.

For T_p, we extract the term in (4.20) due to $y = z = 0$ and $u = x$, which is $p\Omega D(x) \leq K\beta$, using $f_1(p) \leq K$ and (1.25). This gives

$$T_p(x) \leq K\beta + \sum_{u,y,z:(y,z-y,x+z-u)\neq(0,0,0)} \tau_p(y)\tau_p(z-y)KD(u)\tau_p(x+z-u). \tag{5.22}$$

Therefore, by (4.33),

$$T_p \leq K\beta + 3K^2 \max_x \sum_{y,z \in \mathbb{T}_{r,n}} \tau_p(y)(D * \tau_p)(z-y)(D * \tau_p)(x+z), \tag{5.23}$$

where the factor 3 comes from the 3 factors τ_p whose argument can differ from 0. In terms of the Fourier transform, this gives

$$T_p \leq K\beta + 3K^2 \max_x V^{-1} \sum_{k \in \mathbb{T}^*_{r,n}} \hat{D}(k)^2 \hat{\tau}_p(k)^3 e^{-ik\cdot x} \leq K\beta + 3K^2 T_p^{(2)}. \tag{5.24}$$

Our bound on $T_p^{(2)}$ then gives the desired estimate for T_p.

The bound on T_p' is a consequence of $T_p' \leq 1 + 3T_p$. Here the term 1 is due to the contribution to (4.22) with $y = z - y = x - z = 0$, so that $x = y = z = 0$. If at least one of $y, z - y, x - z$ is nonzero, then we can use (4.33) for the corresponding two-point function. $\square$

Lemma 5.4. *Fix $p \in (0, p_c)$, assume that $f(p)$ of (5.5) obeys $f(p) \leq K$, and assume that Assumption 1.1 holds. There is a constant c_K, independent of p, such that*

$$W_p(0;k) \leq c_K[1 - \hat{D}(k)](\lambda^3 \vee \beta), \qquad W_p(k) \leq c_K[1 - \hat{D}(k)]. \tag{5.25}$$

Proof. For the bound on $W_p(0;k)$, we use (4.33) to obtain

$$\begin{aligned} \tilde{\tau}_p(x) &\leq p\Omega D(x) + \sum_{v : v \neq x} p\Omega D(v)\tau(x - v). \\ &\leq p\Omega D(x) + [p\Omega]^2 (D * D * \tau_p)(x). \end{aligned} \tag{5.26}$$

We insert (5.26) into the definition (4.23) of $W_p(0;k)$ to get

$$W_p(0;k) \leq p\Omega \sum_x [1 - \cos(k \cdot x)] D(x) \tau_p(x) + [p\Omega]^2 \sum_x [1 - \cos(k \cdot x)] \tau_p(x) (D * D * \tau_p)(x). \tag{5.27}$$

We begin with the first term in (5.27), which receives no contribution from $x = 0$. Using (4.33) and (5.26), we obtain

$$\begin{aligned} &p\Omega \sum_{x \neq 0} [1 - \cos(k \cdot x)] D(x) \tau_p(x) \\ &\quad \leq [p\Omega]^2 \sum_x [1 - \cos(k \cdot x)] D(x)^2 + [p\Omega]^2 \sum_x [1 - \cos(k \cdot x)] D(x) \sum_{v \neq x} D(v) \tau_p(x - v) \\ &\quad \leq [p\Omega]^2 \sum_x [1 - \cos(k \cdot x)] D(x)^2 + [p\Omega]^3 \sum_x [1 - \cos(k \cdot x)] D(x) (D * D)(x) \\ &\qquad + [p\Omega]^3 \sum_x [1 - \cos(k \cdot x)] D(x) (D * D * \tau_p)(x). \end{aligned} \tag{5.28}$$

The first term on the right side is bounded by $K^2 \beta [1 - \hat{D}(k)]$, by (1.25). The second term can be bounded similarly, using $\max_x (D * D)(x) \leq \beta$. For the last term in (5.28), we use Parseval's identity, together with the fact that the Fourier transform of $[1 - \cos(k \cdot x)] D(x)$ is $\hat{D}(l) - \hat{D}^{\cos}(k, l)$, to obtain

$$\sum_x [1 - \cos(k \cdot x)] D(x) (D * D * \tau_p)(x) = \frac{1}{V} \sum_{l \in \mathbb{T}^*_{r,n}} [\hat{D}(l) - \hat{D}^{\cos}(k, l)] \hat{D}(l)^2 \hat{\tau}_p(l). \tag{5.29}$$

Applying (5.14) and the bound on $T^{(2)}$ (with $\hat{\tau}_p(k)^3$ replaced by $\hat{\tau}_p(k)$) yields

$$\sum_x [1 - \cos(k \cdot x)] D(x) (D * D * \tau_p)(x) \leq [1 - \hat{D}(k)] \frac{1}{V} \sum_{l \in \mathbb{T}^*_{r,n}} \hat{D}(l)^2 \hat{\tau}_p(l) \leq c_K (\lambda^3 \vee \beta) [1 - \hat{D}(k)]. \tag{5.30}$$

This completes the bound on the first term of (5.27).

For the second term in (5.27), we again use Parseval to obtain

$$\sum_x [1-\cos(k\cdot x)]\tau_p(x)(D*D*\tau_p)(x) = \frac{1}{V}\sum_{l\in\mathbb{T}^*_{r,n}} \Big[\hat{\tau}_p(l) - \frac{1}{2}(\hat{\tau}_p(l+k)+\hat{\tau}_p(l-k))\Big]\hat{D}(l)^2\hat{\tau}_p(l). \quad (5.31)$$

We apply the assumed bounds on $f_2(p)$ and $f_3(p)$ to obtain

$$\begin{aligned}
&\sum_x [1-\cos(k\cdot x)]\tau_p(x)(D*D*\tau_p)(x) \\
&\qquad \le c\lambda^3 + K^2X[1-\hat{D}(k)]\frac{1}{V}\sum_{l\neq 0}\hat{D}(l)^2\hat{C}_{\mu_p^{(M)}}(l) \\
&\qquad\quad \times \Big[\hat{C}_{\mu_p^{(M)}}(l-k)\hat{C}_{\mu_p^{(M)}}(l) + \hat{C}_{\mu_p^{(M)}}(l)\hat{C}_{\mu_p^{(M)}}(l+k) + \hat{C}_{\mu_p^{(M)}}(l-k)\hat{C}_{\mu_p^{(M)}}(l+k)\Big]. \quad (5.32)
\end{aligned}$$

For the last term in (5.32), we set

$$C_{\mu,k}(x) = \cos(k\cdot x)C_\mu(x). \quad (5.33)$$

Then

$$|C_{\mu,k}(x)| \le C_\mu(x), \quad (5.34)$$

and, recalling (5.11),

$$\hat{C}_{\mu,k}(l) = \sum_{x\in\mathbb{V}}\cos(k\cdot x)\cos(l\cdot x)C_\mu(x) = \hat{C}^{\cos}(k,l). \quad (5.35)$$

Also, by (5.16),

$$\hat{C}_{\mu_p^{(M)}}(l-k)\hat{C}_{\mu_p^{(M)}}(l+k) = \hat{C}^{\cos}_{\mu_p^{(M)}}(k,l)^2 - \hat{C}^{\sin}_{\mu_p^{(M)}}(k,l)^2 \le \hat{C}^{\cos}_{\mu_p^{(M)}}(k,l)^2. \quad (5.36)$$

Therefore, using (5.34) and Parseval's identity,

$$\begin{aligned}
\frac{1}{V}\sum_{l\in\mathbb{T}^*_{r,n}}\hat{C}_{\mu_p^{(M)}}(l)\hat{D}(l)^2\hat{C}_{\mu_p^{(M)}}(l-k)\hat{C}_{\mu_p^{(M)}}(l+k) &\le \frac{1}{V}\sum_{l\in\mathbb{T}^*_{r,n}}\hat{C}_{\mu_p^{(M)}}(l)\hat{D}(l)^2\hat{C}^{\cos}_{\mu_p^{(M)}}(k,l)^2 \\
&= (D*D*C_{\mu_p^{(M)}}*C_{\mu_p^{(M)},k}*C_{\mu_p^{(M)},k})(0) \\
&\le (D*D*C_{\mu_p^{(M)}}*C_{\mu_p^{(M)}}*C_{\mu_p^{(M)}})(0). \quad (5.37)
\end{aligned}$$

Moreover, by (5.1),

$$(D*D*C_{\mu_p^{(M)}}*C_{\mu_p^{(M)}}*C_{\mu_p^{(M)}})(0) = \frac{1}{V}\sum_{l\in\mathbb{T}^*_{r,n}}\hat{D}(l)^2\hat{C}_{\mu_p^{(M)}}(l)^3 \le 8(\lambda^3\vee\beta), \quad (5.38)$$

where the λ^3 arises from the $l=0$ term together with the fact that $1-\mu \ge \frac{1}{2}\lambda^{-1}V^{-1/3}$. This proves the desired bound on the last term in (5.32).

To bound the sum of the remaining terms in (5.32), we consider

$$\frac{1}{V}\sum_{l\in\mathbb{T}^*_{r,n}}\hat{D}^2(l)\hat{C}_{\mu_p^{(M)}}(l)^2\Big[\hat{C}_{\mu_p^{(M)}}(l-k)+\hat{C}_{\mu_p^{(M)}}(l+k)\Big]. \quad (5.39)$$

Applying (5.11), (5.35) and (5.1), (5.39) equals

$$\frac{2}{V}\sum_{l\in\mathbb{T}^*_{r,n}}\hat{D}^2(l)\hat{C}_{\mu_p^{(M)}}(l)^2\hat{C}^{\cos}_{\mu_p^{(M)}}(k,l) = 2(D*D*C_{\mu_p^{(M)}}*C_{\mu_p^{(M)}}*C_{\mu_p^{(M)},k})(0)$$
$$\leq 2(D*D*C_{\mu_p^{(M)}}*C_{\mu_p^{(M)}}*C_{\mu_p^{(M)}})(0) \leq 16(\lambda^3\vee\beta). \qquad (5.40)$$

This completes the bound on the second term of (5.27), and thus the proof that $W_p(0;k) \leq c_K(\lambda^3\vee\beta)[1-\hat{D}(k)]$.

Finally, we estimate $W_p(k)$. Note that no factor $\lambda^3\vee\beta$ appears in the desired bound. By definition,

$$W_p(k) = p\Omega\max_{y\in\mathbb{V}}\sum_{x,v\in\mathbb{V}}[1-\cos(k\cdot x)]D(v)\tau_p(x-v)\tau_p(x+y). \qquad (5.41)$$

Let

$$D_k(x) = [1-\cos(k\cdot x)]D(x), \qquad \tau_{p,k}(x) = [1-\cos(k\cdot x)]\tau_p(x). \qquad (5.42)$$

Applying (4.49) with $t = k\cdot v + k\cdot(x-v)$, we obtain

$$W_p(k) \leq 5p\Omega\max_{y\in\mathbb{V}}\sum_{x,v\in\mathbb{V}}[1-\cos(k\cdot v)]D(v)\tau_p(x-v)\tau_p(y-x)$$
$$+ 5p\Omega\max_{y\in\mathbb{V}}\sum_{x,v\in\mathbb{V}}D(v)[1-\cos(k\cdot(x-v))]\tau_p(x-v)\tau_p(y-x)$$
$$\leq 5K\max_{y\in\mathbb{V}}(D_k*\tau_p*\tau_p)(y) + 5K\max_{y\in\mathbb{V}}(D*\tau_{p,k}*\tau_p)(y). \qquad (5.43)$$

For the first term, we have

$$(D_k*\tau_p*\tau_p)(y) = \frac{1}{V}\sum_{l\in\mathbb{T}^*_{r,n}}e^{-il\cdot y}\hat{D}_k(l)\hat{\tau}_p(l)^2 \leq \frac{K^2}{V}\sum_{l\in\mathbb{T}^*_{r,n}}\hat{D}_k(l)\hat{C}_{\mu_p^{(M)}}(l)^2. \qquad (5.44)$$

It follows from (5.14) that for all $k,l\in\mathbb{T}^*_{r,n}$

$$|\hat{D}_k(l)| = |\hat{D}(l)-\hat{D}^{\cos}(k,l)| \leq [1-\hat{D}(k)], \qquad (5.45)$$

and hence, by (2.28),

$$\max_{y\in\mathbb{V}}(D_k*\tau_p*\tau_p)(y) \leq [1-\hat{D}(k)]\frac{K^2}{V}\sum_{l\in\mathbb{T}^*_{r,n}}\hat{C}_{\mu_p^{(M)}}(l)^2 \leq c_K(\lambda^3\vee 1)[1-\hat{D}(k)], \qquad (5.46)$$

where the λ^3 arises from the $l=0$ term.

The remaining term to estimate in (5.43) is

$$\max_{y\in\mathbb{V}}(D*\tau_{p,k}*\tau_p)(y) = \max_{y\in\mathbb{V}}\frac{1}{V}\sum_{l\in\mathbb{T}^*_{r,n}}e^{-il\cdot y}\hat{D}(l)\hat{\tau}_p(l)\hat{\tau}_{p,k}(l). \qquad (5.47)$$

Since

$$\hat{\tau}_{p,k}(l) = \hat{\tau}_p(l) - \frac{1}{2}(\hat{\tau}_p(l+k)+\hat{\tau}_p(l-k)), \qquad (5.48)$$

we can use the bound on $f_3(p)$ to obtain

$$\max_{y\in\mathbb{V}}(D * \tau_{p,k} * \tau_p)(y)$$
$$\leq K^2 X[1-\hat{D}(k)]\frac{1}{V}\sum_{l\in\mathbb{T}^*_{r,n}} |\hat{D}(l)|\hat{C}_{\mu_p^{(M)}}(l)$$
$$\times\left[\hat{C}_{\mu_p^{(M)}}(l-k)\hat{C}_{\mu_p^{(M)}}(l)+\hat{C}_{\mu_p^{(M)}}(l)\hat{C}_{\mu_p^{(M)}}(l+k)+\hat{C}_{\mu_p^{(M)}}(l-k)\hat{C}_{\mu_p^{(M)}}(l+k)\right].$$

The above sums can all be bounded using the methods employed for the previous term. For example, the last term can be estimated using $|\hat{D}(l)| \leq 1$, (5.36), (5.34) and (2.28), by

$$\begin{aligned}\frac{1}{V}\sum_{l\in\mathbb{T}^*_{r,n}} \hat{C}_{\mu_p^{(M)}}(l)\hat{C}_{\mu_p^{(M)}}(l-k)\hat{C}_{\mu_p^{(M)}}(l+k) &\leq \max_{k\in\mathbb{T}^*_{r,n}}\frac{1}{V}\sum_{l\in\mathbb{T}^*_{r,n}} |\hat{D}(l)|\hat{C}_{\mu_p^{(M)}}(l)\hat{C}^{\cos}_{\mu_p^{(M)}}(l,k)^2 \\ &= \max_{k\in\mathbb{T}^*_{r,n}}(C_{\mu_p^{(M)}} * C_{\mu_p^{(M)},k} * C_{\mu_p^{(M)},k})(0) \\ &\leq (C_{\mu_p^{(M)}} * C_{\mu_p^{(M)}} * C_{\mu_p^{(M)}})(0) = S_p^{(0)}, \end{aligned} \tag{5.49}$$

where we define

$$S_p^{(\alpha)} = V^{-1}\sum_{l\in\mathbb{T}^*_{r,n}} |\hat{D}(l)|^\alpha \hat{C}_{\mu_p^{(M)}}(l)^3. \tag{5.50}$$

□

Lemma 5.5. *Fix $p \in (0, p_c)$, assume that $f(p)$ of (5.5) obeys $f(p) \leq K$, and assume that Assumption 1.1 holds. There is a constant c_K, independent of p, such that*

$$H_p(k) \leq c_K(\lambda^3 \vee \beta)[1-\hat{D}(k)]. \tag{5.51}$$

Proof. Recall the definition of $H_p(a_1, a_2; k)$ in (4.25), and let $d_i = t_i - u_{i-1}$. In terms of the Fourier transform, and recalling (5.42),

$$\begin{aligned}H(a_1,a_2;k) = \frac{1}{V^3}\sum_{l_1,l_2,l_3\in\mathbb{T}^*_{r,n}} & e^{-il_1\cdot a_1}e^{-il_2\cdot a_2}\hat{D}(l_1)\hat{\tau}_p(l_1)^2\hat{D}(l_2)\hat{\tau}_p(l_2)^2\hat{\tau}_{p,k}(l_3) \\ & \times \hat{\tau}_p(l_1-l_2)\hat{\tau}_p(l_2-l_3)\hat{\tau}_p(l_1-l_3). \end{aligned} \tag{5.52}$$

We use $f(p) \leq K$ to replace $\hat{\tau}_p(k)$ by $K\hat{C}_{\mu_p^{(M)}}(k)$ and (recalling (5.48)) $\hat{\tau}_{p,k}(l_3)$ by

$$KX[1-\hat{D}(k)]\left[\hat{C}_{\mu_p^{(M)}}(l_3-k)\hat{C}_{\mu_p^{(M)}}(l_3)+\hat{C}_{\mu_p^{(M)}}(l_3)\hat{C}_{\mu_p^{(M)}}(l_3+k)+\hat{C}_{\mu_p^{(M)}}(l_3-k)\hat{C}_{\mu_p^{(M)}}(l_3+k)\right]. \tag{5.53}$$

This gives an upper bound for (5.52) consisting of a sum of 3 terms.

The last of these terms can be bounded by

$$\begin{aligned}K^8X[1-\hat{D}(k)]\frac{1}{V^3}\sum_{l_1,l_2,l_3\in\mathbb{T}^*_{r,n}} & |\hat{D}(l_1)|\hat{C}_{\mu_p^{(M)}}(l_1)^2|\hat{D}(l_2)|\hat{C}_{\mu_p^{(M)}}(l_2)^2 \\ & \times \hat{C}_{\mu_p^{(M)}}(l_3-k)\hat{C}_{\mu_p^{(M)}}(l_3+k)\hat{C}_{\mu_p^{(M)}}(l_1-l_2)\hat{C}_{\mu_p^{(M)}}(l_2-l_3)\hat{C}_{\mu_p^{(M)}}(l_1-l_3). \end{aligned} \tag{5.54}$$

Using Hölder's inequality with $p = 3$ and $q = 3/2$, (5.54) is bounded above by a K-dependent constant times

$$[1-\hat{D}(k)]\Big(\frac{1}{V^3}\sum_{l_1,l_2,l_3}|\hat{D}(l_1)|^{3/2}\hat{C}_{\mu_p^{(M)}}(l_1)^3|\hat{D}(l_2)|^{3/2}\hat{C}_{\mu_p^{(M)}}(l_2)^3\hat{C}_{\mu_p^{(M)}}(l_3+k)^{3/2}\hat{C}_{\mu_p^{(M)}}(l_1-l_3)^{3/2}\Big)^{2/3}$$
$$\times\Big(\frac{1}{V^3}\sum_{l_1,l_2,l_3}\hat{C}_{\mu_p^{(M)}}(l_1-l_2)^3\hat{C}_{\mu_p^{(M)}}(l_2-l_3)^3\hat{C}_{\mu_p^{(M)}}(l_3-k)^3\Big)^{1/3}. \tag{5.55}$$

The Cauchy–Schwarz inequality implies that for all k and l_1,

$$\sum_{l_3}\hat{C}_{\mu_p^{(M)}}(l_3+k)^{3/2}\hat{C}_{\mu_p^{(M)}}(l_1-l_3)^{3/2} \le S_p^{(0)}, \tag{5.56}$$

with $S_p^{(0)}$ given by (5.50). Therefore (5.55) is bounded above by

$$[1-\hat{D}(k)]\Big(S_p^{(0)}\Big)^{5/3}\Big(S_p^{(3/2)}\Big)^{4/3}. \tag{5.57}$$

To complete the proof, we note that by Hölder's inequality,

$$S_p^{(3/2)} \le \Big(S_p^{(2)}\Big)^{3/4}\Big(S_p^{(0)}\Big)^{1/4}. \tag{5.58}$$

Thus (5.57) is bounded above by $[1-\hat{D}(k)]S_p^{(2)}(S_p^{(0)})^2$. The latter factor can be bounded using (2.28), and the former with (5.1). This gives a bound of the desired form, with the λ^3 arising as usual from the $l = 0$ term of $S_p^{(2)}$.

Routine bounds can be used to deal with the other two terms in a similar fashion. □

Proof of Lemma 5.2. This is an immediate consequence of Lemmas 4.1 and 5.3–5.5. The bound (4.31) is used for (5.18) when $N = 1$ (as (4.30) is not sufficient). □

The following consequence of Lemma 5.2 plays an essential role in completing the bootstrap argument.

Lemma 5.6. *Let $M = 0, 1, 2, \ldots$, and assume that Assumption 1.1 holds. If $f(p)$ of (5.5) obeys $f(p) \le K$, then, for λ and β sufficiently small,*

$$\sum_{x\in\mathbb{T}_{r,n}}|\hat{\Pi}_M(x)| \le 3\bar{c}_K(\lambda^3\vee\beta), \tag{5.59}$$

$$\sum_{x\in\mathbb{T}_{r,n}}[1-\cos(k\cdot x)]|\hat{\Pi}_M(x)| \le 3\bar{c}_K(\lambda^3\vee\beta)[1-\hat{D}(k)], \tag{5.60}$$

and for M sufficiently large (depending on p, K, r, n),

$$\sum_{x\in\mathbb{T}_{r,n}}|R_M(x)| \le (\lambda^3\vee\beta), \tag{5.61}$$

$$\sum_{x\in\mathbb{T}_{r,n}}[1-\cos(k\cdot x)]|R_M(x)| \le (\lambda^3\vee\beta)[1-\hat{D}(k)]. \tag{5.62}$$

Proof. The bounds (5.59)–(5.60) are immediate consequences of Lemma 5.2 (the factor 3 comes from summing the geometric series).

For the remainder term $R_M(x)$, we conclude from (3.37) that

$$|R_M(x)| \leq p\Omega \sum_{u,v} \Pi^{(M)}(u) D(v-u) \tau_p(x-v), \tag{5.63}$$

and hence (5.61) is bounded above by $p\Omega\hat{\Pi}^{(M)}(0)\chi(p)$. This can be made less than $\lambda^3 \vee \beta$ by taking M sufficiently large, by Lemma 5.2. For (5.62), we apply (4.49) with $J = 3$ to obtain

$$\begin{aligned} &\sum_{x\in\mathbb{T}_{r,n}} [1-\cos(k\cdot x)]|R_M(x)| \qquad (5.64)\\ &\leq 7p\Omega[1-\hat{D}(k)]\hat{\Pi}_p^{(M)}(0)\chi(p) + 7p\Omega\Big[\hat{\Pi}_p^{(M)}(0) - \hat{\Pi}_p^{(M)}(k)\Big]\chi(p) + 7p\Omega\hat{\Pi}_p^{(M)}(0)\Big[\hat{\tau}_p(0) - \hat{\tau}_p(k)\Big]. \end{aligned}$$

By Lemma 5.2, we can choose M large enough that $14K\hat{\Pi}_p^{(M)}(0)\chi(p) \leq \frac{1}{3}(\lambda^3 \vee \beta)$. The second term can be treated similarly. For the last term, we apply the bound $f_3(p) \leq 3$ for $l = 0$ and again choose M large, with a final appeal to Lemma 5.2. ∎

5.3 The bootstrap argument completed

We now show that f of (5.5) obeys the assumptions of Lemma 5.1, with $p_1 = 0$ and $p_2 = p_c$.

To see that $f(0) \leq 3$, we note that $\hat{\tau}_0(k) = 1$, $\mu_0^{(M)} = 0$ and hence $\hat{C}_{\mu_0^{(M)}}(k) = 1$, so that $f_2(0) = 1$. Since $f_1(0) = f_3(0) = 0$, we have $f(0) = 1 < 3$.

Next, we verify the continuity of f. Continuity of f_1 is clear. For f_2, since $\mathbb{T}_{r,n}$ is finite it follows that $\hat{\tau}_p(k)$ is a polynomial in p and hence is continuous. Similarly, $\hat{\Pi}_M(0)$ is a polynomial in p. Therefore $\mu_p^{(M)}$ is continuous in p, and hence $\hat{C}_{\mu_p^{(M)}}(k)$ also is, since $\hat{C}_\mu(k)$ is continuous in μ. The numerator and denominator in the definition of f_2 are therefore both continuous. There is no division by zero, since the denominator is positive when $\mu_p^{(M)} < 1$, by (2.4). The maximum over k is a maximum over a finite set, so f_2 is continuous. Similarly, f_3 is continuous, and thus f is continuous.

The remaining hypothesis of Lemma 5.1 is the substantial one, and requires the detailed information about Π_M and R_M provided by Lemma 5.6. We fix $p < p_c$ and prove that $f(p) \leq 4$ implies $f(p) \leq 3$. By the assumption that $f(p) \leq 4$, the hypotheses of Lemma 5.2 are satisfied with $K = 4$. Therefore, assuming M is sufficiently large, the bounds (5.59)–(5.62) hold, with $\bar{c}_K$ replaced by $\bar{c}_4$.

Let

$$\lambda_p^{(M)}\Omega = p\Omega[1 + \hat{\Pi}_M(0)]. \tag{5.65}$$

We now show that $\lambda_p^{(M)}\Omega \in [0, 1 - \frac{1}{2}\lambda^{-1}V^{-1/3}]$, and hence $\mu_p^{(M)} = \lambda_p^{(M)}$. By (5.2) with $k = 0$,

$$\chi(p)[1 - \lambda_p^{(M)}\Omega] = 1 + \hat{\Pi}_M(0) + \hat{R}_M(0). \tag{5.66}$$

Therefore,

$$1 - \lambda_p^{(M)}\Omega \geq \chi^{-1}(p)\left[1 - |\Pi_M(0)| - |\hat{R}_M(0)|\right] \geq \chi^{-1}(p)\Big[1 - (3\bar{c}_4 + 1)(\lambda^3 \vee \beta)\Big]. \tag{5.67}$$

Since $\chi(p) \le \chi(p_c) = \lambda V^{1/3}$, for $\lambda^3 \vee \beta$ sufficiently small it follows that

$$\lambda_p^{(M)}\Omega \le 1 - \frac{1}{2}\lambda^{-1}V^{-1/3}. \tag{5.68}$$

In addition, when λ and β are sufficiently small,

$$\lambda_p^{(M)}\Omega = p\Omega[1 + \hat{\Pi}_M(0)] \ge p\Omega\Big[1 - 3\bar{c}_4(\lambda^3 \vee \beta)\Big] \ge 0. \tag{5.69}$$

This proves that $\mu_p^{(M)}\Omega = \lambda_p^{(M)}\Omega = p\Omega[1 + \hat{\Pi}_M(0)]$.

5.3.1 The improved bounds on $f_1(p)$ and $f_2(p)$

First, we improve the bound on $f_1(p)$. We have already shown in (5.68) that $\mu_p^{(M)}\Omega \le 1$. Therefore, by (5.17),

$$f_1(p) = p\Omega = \frac{\mu_p^{(M)}\Omega}{1 + \hat{\Pi}_M(0)} \le \frac{1}{1 - 3\bar{c}_4(\lambda^3 \vee \beta)} \le 3 \tag{5.70}$$

when λ and β are small enough.

To improve the bound on $f_2(p)$, we write (5.2) as $\hat{\tau} = \hat{N}/\hat{F}$, with

$$\hat{N}(k) = 1 + \hat{\Pi}_M(k) + \hat{R}_M(k), \qquad \hat{F}(k) = 1 - p\Omega\hat{D}(k)[1 + \hat{\Pi}_M(k)]. \tag{5.71}$$

This yields

$$\begin{aligned}\frac{\hat{\tau}_p(k)}{\hat{C}_{\mu_p^{(M)}}(k)} &= \hat{N}(k) + \hat{\tau}_p(k)[1 - \mu_p^{(M)}\Omega\hat{D}(k) - \hat{F}(k)] \\ &= [1 + \hat{\Pi}_M(k) + \hat{R}_M(k)] + \hat{\tau}_p(k)p\Omega\hat{D}(k)[\hat{\Pi}_M(k) - \hat{\Pi}_M(0)]. \end{aligned} \tag{5.72}$$

By Lemma 5.6, and by our assumptions that $\hat{\tau}_p(k) \le 4\hat{C}_{\mu_p^{(M)}}(k)$ and $p\Omega \le 4$, it follows from (5.72) that

$$\frac{\hat{\tau}_p(k)}{\hat{C}_{\mu_p^{(M)}}(k)} \le 1 + \Big(3\bar{c}_4 + 1 + 4^2 3\bar{c}_4\hat{C}_{\mu_p^{(M)}}(k)[1 - \hat{D}(k)]\Big)(\lambda^3 \vee \beta). \tag{5.73}$$

Since

$$0 \le \hat{C}_{\mu_p^{(M)}}(k)[1 - \hat{D}(k)] = 1 + \frac{\mu_p^{(M)}\Omega - 1}{1 - \mu_p^{(M)}\Omega\hat{D}(k)}\hat{D}(k) \le 2, \tag{5.74}$$

it follows from (5.73) that

$$\frac{\hat{\tau}_p(k)}{\hat{C}_{\mu_p^{(M)}}(k)} \le 1 + (3\bar{c}_4 + 1 + 96\bar{c}_4)(\lambda^3 \vee \beta). \tag{5.75}$$

This is less than 3, if $\lambda^3 \vee \beta$ is small enough.

5.3.2 The improved bound on $f_3(p)$

This is the most substantial part of the argument. Let g be a symmetric function on the torus, meaning $g(x) = g(-x)$, and let

$$\hat{G}(k) = \frac{1}{1 - \hat{g}(k)}. \tag{5.76}$$

Then $\hat{G}$ obeys the identity in the following lemma.

Lemma 5.7. *For all $k, l \in \mathbb{T}^*$,*

$$-\frac{1}{2}\Delta_k G(l) = \hat{G}(l-k)\hat{G}(l)\hat{G}(l+k)\Big[[\hat{g}(l) - \hat{g}^{\cos}(l,k)][1 - \hat{g}^{\cos}(l,k)] - \hat{g}^{\sin}(l,k)^2\Big]. \tag{5.77}$$

Proof. Let $\hat{g}_\pm = \hat{g}(l \pm k)$ and write $\hat{g} = \hat{g}(l)$. Direct computation using (5.10) gives

$$\begin{aligned}
-\frac{1}{2}\Delta_k G(l) &= \frac{1}{2}\hat{G}(l)\hat{G}(l+k)\hat{G}(l-k)\Big[[2\hat{g} - \hat{g}_+ - \hat{g}_-] + [2\hat{g}_+\hat{g}_- - \hat{g}\hat{g}_- - \hat{g}\hat{g}_+]\Big] \\
&= \hat{G}(l)\hat{G}(l+k)\hat{G}(l-k)\Big[[\hat{g}(l) - \hat{g}^{\cos}(l,k)] + [\hat{g}_+\hat{g}_- - \hat{g}(l)\hat{g}^{\cos}(l,k)]\Big],
\end{aligned} \tag{5.78}$$

using (5.11) in the last step. By (5.16),

$$\hat{g}_-\hat{g}_+ = \hat{g}^{\cos}(l,k)^2 - \hat{g}^{\sin}(l,k)^2. \tag{5.79}$$

Substitution in (5.78) gives (5.77). □

We will use (5.77) to improve the bound on $f_3(p)$. For this, we recall the definitions of $\hat{N}$ and $\hat{F}$ in (5.71) and write $\hat{\tau}_p(l)$ as

$$\hat{\tau}_p(l) = \frac{\hat{N}(l)}{\hat{F}(l)} = \frac{1}{1 - \hat{g}(l)} \tag{5.80}$$

with

$$\hat{g}(l) = 1 - \frac{\hat{F}(l)}{\hat{N}(l)} = 1 - \frac{1}{\hat{N}(l)}\Big\{1 - \mu_p^{(M)}\Omega\hat{D}(l) + p\Omega\hat{D}(l)[\hat{\Pi}_M(0) - \hat{\Pi}_M(l)]\Big\}. \tag{5.81}$$

By Lemma 5.6,

$$|\hat{N}(l) - 1| \le 4\bar{c}_4(\lambda^3 \vee \beta). \tag{5.82}$$

In particular, $\hat{N}(l) > 0$. Since $\hat{\tau}_p(l) \ge 0$ (as proved in [3]), it follows that $\hat{F}(l) > 0$. Lemma 5.6 and (5.74) then imply that

$$\begin{aligned}
0 \le \hat{F}(l) &\le [1 - \mu_p^{(M)}\Omega\hat{D}(l)] + 3^2\bar{c}_4(\lambda^3 \vee \beta)[1 - \hat{D}(l)] \\
&\le [1 + 18\bar{c}_4(\lambda^3 \vee \beta)][1 - \mu_p^{(M)}\Omega\hat{D}(l)].
\end{aligned} \tag{5.83}$$

Since $f_2(p) \le 3$, (5.77) implies that

$$\begin{aligned}
\hat{\tau}_p(l) - \frac{1}{2}(\hat{\tau}_p(l+k) + \hat{\tau}_p(l-k)) &= -\frac{1}{2}\Delta_k\hat{\tau}_p(l) \\
&\le 3^3\hat{C}_{\mu_p^{(M)}}(l-k)\hat{C}_{\mu_p^{(M)}}(l)\hat{C}_{\mu_p^{(M)}}(l+k)\Big[|\hat{g}(l) - \hat{g}^{\cos}(l,k)||1 - \hat{g}^{\cos}(l,k)| + \hat{g}^{\sin}(l,k)^2\Big].
\end{aligned} \tag{5.84}$$

We will prove the following three inequalities:

$$|1-\hat{g}^{\cos}(l,k)| \leq \frac{1}{\hat{C}_{\mu_p^{(M)}}(l-k)} + \frac{1}{\hat{C}_{\mu_p^{(M)}}(l+k)}, \tag{5.85}$$

$$|\hat{g}^{\sin}(l,k)|^2 \leq 75[1-\hat{D}(k)]\frac{1}{\hat{C}_{\mu_p^{(M)}}(l)}, \tag{5.86}$$

$$|\hat{g}(l)-\hat{g}^{\cos}(l,k)| \leq 4[1-\hat{D}(k)]. \tag{5.87}$$

These inequalities are sufficient to improve the bound on $f_3(p)$, since they imply that the right side of (5.84) is bounded above by

$$3[1-\hat{D}(k)]\hat{C}_{\mu_p^{(M)}}(l-k)\hat{C}_{\mu_p^{(M)}}(l)\hat{C}_{\mu_p^{(M)}}(l+k)\Big[\frac{36}{\hat{C}_{\mu_p^{(M)}}(l-k)} + \frac{36}{\hat{C}_{\mu_p^{(M)}}(l+k)} + \frac{675}{\hat{C}_{\mu_p^{(M)}}(l)}\Big]. \tag{5.88}$$

Recalling that $\hat{C}_1(k) = [1-\hat{D}(k)]^{-1}$ and $X = 675$, this gives $f_3(p) \leq 3$.

To prove (5.85), we use (5.11) to write

$$1-\hat{g}^{\cos}(l,k) = \frac{1}{2}\frac{\hat{F}(l-k)}{\hat{N}(l-k)} + \frac{1}{2}\frac{\hat{F}(l+k)}{\hat{N}(l+k)}. \tag{5.89}$$

The desired estimate (5.85) then follows from (5.82)–(5.83).

To prove (5.86), we use (5.81) and (5.12) to see that

$$\begin{aligned} |\hat{g}^{\sin}(l,k)|^2 &= \left| -\frac{\hat{F}^{\sin}(l,k)}{\hat{N}(l-k)} + \frac{\hat{F}(l+k)\hat{N}^{\sin}(l,k)}{\hat{N}(l-k)\hat{N}(l+k)} \right|^2 \\ &\leq 2\frac{|\hat{F}^{\sin}(l,k)|^2}{\hat{N}(l-k)^2} + 2\frac{\hat{F}(l+k)^2|\hat{N}^{\sin}(l,k)|^2}{\hat{N}(l-k)^2\hat{N}(l+k)^2}. \end{aligned} \tag{5.90}$$

To deal with the first term on the right side of (5.90), we use (5.71) and (5.12) to obtain

$$\hat{F}^{\sin}(l,k) = -p\Omega\left[\hat{D}^{\sin}(l,k)[1+\hat{\Pi}_M(l-k)] + \hat{D}(l+k)\hat{\Pi}_M^{\sin}(l,k)\right]. \tag{5.91}$$

By the Cauchy-Schwarz inequality and the elementary estimate $1-\cos^2 t \leq 2[1-\cos t]$,

$$\begin{aligned} \hat{D}^{\sin}(k,l)^2 &\leq \sum_x \sin(k\cdot x)^2 D(x) \sum_y \sin(l\cdot y)^2 D(y) \\ &= \sum_x [1-\cos(k\cdot x)^2] D(x) \sum_y [1-\cos(l\cdot y)^2] D(y) \\ &\leq 4[1-\hat{D}(k)][1-\hat{D}(l)]. \end{aligned} \tag{5.92}$$

The estimate (5.92) can also be applied to $\hat{\Pi}_M^{\sin}(l,k)$ or $\hat{R}_M^{\sin}(l,k)$ using Lemma 5.6, but now the upper bound contains a small factor proportional to $\lambda^3 \vee \beta$. The factor $\hat{F}(l+k)$ is at most 5, by (5.71). This shows that the contributions to (5.90) other than that from (5.92) are of the same form but with an arbitrarily small prefactor. In addition, the factor $1 - \hat{D}(l)$ can be bounded above by $2\hat{C}_{\mu_p^{(M)}}(l)^{-1}$, by (5.74). Also, we have $p\Omega \leq 3$. Therefore, as required,

$$\hat{g}^{\sin}(k,l)^2 \leq 3^2 \cdot 4 \cdot 2(1+o(1))[1-\hat{D}(k)]\hat{C}_{\mu_p^{(M)}}(l)^{-1}. \tag{5.93}$$

Finally, we estimate $\hat{g}(l) - \hat{g}^{\cos}(l,k) = -\frac{1}{2}\Delta_k \hat{g}(l)$ and prove (5.87). For this, we use $\Delta_k = \partial_k^- \partial_k^+$ in conjunction with the quotient and product rules

$$\partial_k^+ \frac{b(l)}{d(l)} = \frac{\partial_k^+ b(l)}{d(l)} - \frac{b(l)\partial_k^+ d(l)}{d(l)d(l+k)}, \tag{5.94}$$

$$\partial_k^- \frac{b(l)}{d(l)} = \frac{\partial_k^- b(l)}{d(l)} - \frac{b(l-k)\partial_k^- d(l)}{d(l)d(l-k)}, \tag{5.95}$$

$$\partial_k^+ [\hat{f}(l)\hat{h}(l)] = \partial_k^+ \hat{f}(l)\hat{h}(l+k) + \hat{f}(l)\partial_k^+ \hat{h}(l), \tag{5.96}$$

$$\partial_k^- [\hat{f}(l)\hat{h}(l)] = \partial_k^- \hat{f}(l)\hat{h}(l) + \hat{f}(l-k)\partial_k^- \hat{h}(l). \tag{5.97}$$

This gives

$$\begin{aligned} -\frac{1}{2}\Delta_k \hat{g}(l) &= -\frac{1}{2}\Delta_k \frac{\hat{F}(l)}{\hat{N}(l)} \\ &= \frac{-\frac{1}{2}\Delta_k \hat{F}(l)}{\hat{N}(l)} + \frac{1}{2}\frac{\partial_k^+ \hat{F}(l-k)\partial_k^- \hat{N}(l)}{\hat{N}(l)\hat{N}(l-k)} + \frac{1}{2}\frac{\partial_k^- \hat{F}(l)\partial_k^+ \hat{N}(l)}{\hat{N}(l)\hat{N}(l+k)} \\ &\quad + \frac{1}{2}\frac{\hat{F}(l-k)\Delta_k \hat{N}(l)}{\hat{N}(l)\hat{N}(l+k)} - \frac{1}{2}\frac{\hat{F}(l-k)\partial_k^+ \hat{N}(l-k)\partial_k^- [\hat{N}(l)\hat{N}(l+k)]}{\hat{N}(l-k)\hat{N}(l)^2\hat{N}(l+k)}. \end{aligned} \tag{5.98}$$

The denominators are all as close to 1 as desired, by (5.82), and we need to estimate the numerators.

The first term on the right side of (5.98) is the main term. Its numerator is equal to

$$\begin{aligned} -\frac{1}{2}\Delta_k \hat{F}(l) &= p\Omega \frac{1}{2}\Delta_k \hat{D}(l)[1+\hat{\Pi}_M(l+k)] + \frac{1}{2}p\Omega \partial_k^+ \hat{D}(l-k)\partial_k^- \hat{\Pi}_M(l+k) \\ &\quad + \frac{1}{2}p\Omega \partial_k^- \hat{D}(l)\partial_k^+ \hat{\Pi}_M(l) + p\Omega \hat{D}(l-k)[\frac{1}{2}\Delta_k \hat{\Pi}_M(l)]. \end{aligned} \tag{5.99}$$

We bound the factors $p\Omega$ by 3. The factor $|\frac{1}{2}\Delta_k \hat{D}(l)|$ is bounded above by $1 - \hat{D}(k)$, by (5.14). The last term on the right side of (5.99) is bounded by a small multiple of $1 - \hat{D}(k)$, by (5.14) and Lemma 5.6. For the cross terms, we use

$$\begin{aligned} |\partial_k^\pm D(l)| &\leq \sum_x D(x)|\mathrm{Re}\{e^{il\cdot x}[e^{\pm ik\cdot x} - 1]\}| \\ &\leq \sum_x D(x)\Big[[1-\cos(k\cdot x)] + |\sin(k\cdot x)||\sin(l\cdot x)|\Big] \\ &\leq [1-\hat{D}(k)] + 2[1-\hat{D}(k)]^{1/2}[1-\hat{D}(l)]^{1/2} \\ &\leq [1-\hat{D}(k)] + 2^{3/2}[1-\hat{D}(k)]^{1/2}, \end{aligned} \tag{5.100}$$

using (5.92) for the third inequality. Applying Lemma 5.6, similar estimates apply to $\partial_k^\pm \hat{\Pi}_M$ and $\partial_k^\pm \hat{R}_M$, but with a small factor. The two cross terms in (5.99) are therefore bounded by a small multiple of $1-\hat{D}(k)$. Writing $o(1)$ to denote a multiple of $\lambda^3 \vee \beta$, we have shown that the first term on the right side of (5.98) is bounded above by $3[1+o(1)][1-\hat{D}(k)]$.

It is sufficient to show that the remaining terms in (5.98) are at most $o(1)[1-\hat{D}(k)]$. The fourth term on the right side of (5.98) obeys this bound, using (5.83) to bound $\hat{F}(l-k)$ by a constant, and (5.14) and Lemma 5.6 to bound $\Delta_k \hat{N}(l) = \Delta_k \hat{\Pi}_M(l) + \Delta_k \hat{R}_M(l)$ by $o(1)[1-\hat{D}(k)]$.

The remaining three terms in (5.98) each contain a product of a derivative of $\hat{F}$ with a derivative of $\hat{N}$, or a product of two derivatives of $\hat{N}$ (using (5.97) for the last term). Other factors of $\hat{F}$ or $\hat{N}$ are bounded by harmless constants. The above arguments imply that $\partial_k^\pm \hat{N}(l)$ is bounded by $o(1)\{[1-\hat{D}(k)] + 2^{3/2}[1-\hat{D}(k)]^{1/2}\}$, as in (5.100) but with a small factor. By the definition of $\hat{F}$ in (5.71) and by the product rule (5.96), we have

$$\partial_k^+ \hat{F}(l) = -p\Omega \partial_k^\pm \hat{D}(l)[1+\hat{\Pi}_M(l+k)] - p\Omega \hat{D}(l) \partial_k^+ \hat{\Pi}_M(l), \tag{5.101}$$

which is bounded by a multiple of the right side of (5.100) (with no small factor). The same bound is obeyed by $\partial_k^- \hat{F}(l)$. Although the derivative of $\hat{F}$ does not produce a small factor, it is accompanied by a derivative of $\hat{N}$ which does provide the desired factor $o(1)$. Thus each of the remaining three terms in (5.98) is at most $o(1)[1-\hat{D}(k)]$.

This completes the proof that (5.98) is bounded above by $4[1-\hat{D}(k)]$. Therefore, as required, we have shown that $f_3(p) \le 3$.

5.3.3 Conclusion

This completes the verification of the hypotheses of Lemma 5.1. Thus the conclusion of the lemma applies, and $f(p) \le 3$ for all $p \le p_c$. In particular, (5.4) and the bounds of Lemmas 5.2–5.6 all hold. This implies that we may take the limit $M \to \infty$ in (5.2) and (5.72) so that, in particular,

$$\hat{\tau}_p(k) = \frac{1+\hat{\Pi}_p(k)}{1 - p\Omega \hat{D}(k)[1+\hat{\Pi}_p(k)]}, \tag{5.102}$$

where Π_p denotes $\Pi_{M=\infty}$.

5.4 The triangle condition

Proof of Theorem 1.3. By definition, $1 \le \nabla_p(x,x) \le T_p'$. Since $f(p) \le 3$, it follows from Lemma 5.3 that $\nabla_p(x,x) \le 1 + c_3(V^{-1}\chi(p)^3 \vee \beta)$, for $p \le p_c$. This gives (1.27) for $x=y$. For $x \ne y$ it follows from (4.33) that

$$\nabla_p(x,y) = (\tau_p * \tau_p * \tau_p)(x,y) \le 3T_p(y-x) \qquad (x \ne y), \tag{5.103}$$

where the factor 3 arises since there are three factors τ_p that could have a nonzero argument and hence permit application of (4.33). The desired bound then follows from Lemma 5.3. This also proves (1.10) with $a_0 = 3c_3(\lambda^3 \vee \beta)$. □

6 Asymptotics for $\hat{\tau}_p(k)$

In this section, we show that it is possible to extend (1.30) to an asymptotic formula for $\hat{\tau}_p(k)$ when $p \leq p_c$, for all $k \in \mathbb{T}^*_{r,n}$. We will not use this result elsewhere in the paper.

Let $\epsilon = \Omega(p_c - p) \geq 0$ and $\epsilon_0 = \lambda^{-1}V^{-1/3}$. The result of [7, Theorem 1.2 i)] gives the following extension of Theorem 1.5 i) to all $p \leq p_c$:

$$\frac{1}{\epsilon_0 + \epsilon} \leq \chi(p) \leq \frac{1}{\epsilon_0 + [1 - a_0]\epsilon}. \tag{6.1}$$

As we have seen in Section 5.4, $a_0 \leq O(\lambda^3 \vee \beta)$. Let

$$m_p\Omega = 1 - \epsilon - \epsilon_0 = 1 - \Omega(p_c - p) - \lambda^{-1}V^{-1/3}. \tag{6.2}$$

Theorem 6.1 (Asymptotics for the two-point function). *For $p \leq p_c$,*

$$\hat{\tau}_p(k) = (1 + O(\lambda^3 \vee \beta))\hat{C}_{m_p}(k) = \frac{1 + O(\lambda^3 \vee \beta)}{1 - m_p\Omega\hat{D}(k)}, \tag{6.3}$$

with the error term uniform in $k \in \mathbb{T}^*_{r,n}$ and $p \leq p_c$.

Proof. For $k = 0$, (6.3) is a consequence of (6.1). We therefore assume $k \neq 0$ henceforth.

We may now take the limit $M \to \infty$ in (5.72). Writing $\hat{\Pi}_p(0)$ for $\hat{\Pi}_\infty(0)$ and also

$$\mu_p = \mu_p^{(\infty)} = p[1 + \hat{\Pi}_p(0)], \tag{6.4}$$

this gives

$$\frac{\hat{\tau}_p(k)}{\hat{C}_{\mu_p}(k)} - 1 = \hat{\Pi}_p(k) + \hat{\tau}_p(k)p\Omega\hat{D}(k)\Big[\hat{\Pi}_p(k) - \hat{\Pi}_p(0)\Big]. \tag{6.5}$$

By (5.4), (5.60)–(5.18) and (5.74), this leads to

$$\Big|\frac{\hat{\tau}_p(k)}{\hat{C}_{\mu_p}(k)} - 1\Big| = O(\lambda^3 \vee \beta). \tag{6.6}$$

Since

$$\frac{\hat{\tau}_p(k)}{\hat{C}_{m_p}(k)} - 1 = \left(\frac{\hat{\tau}_p(k)}{\hat{C}_{\mu_p}(k)} - 1\right)\frac{\hat{C}_{\mu_p}(k)}{\hat{C}_{m_p}(k)} + \left(\frac{\hat{C}_{\mu_p}(k)}{\hat{C}_{m_p}(k)} - 1\right) \tag{6.7}$$

and since

$$\Big|\frac{\hat{C}_{\mu_p}(k)}{\hat{C}_{m_p}(k)} - 1\Big| = \frac{|(\mu_p - m_p)\Omega\hat{D}(k)|}{1 - \mu_p\Omega\hat{D}(k)} \leq \frac{|\mu_p - m_p|\Omega}{1 - \mu_p\Omega}, \tag{6.8}$$

it suffices to show that

$$\frac{|\mu_p - m_p|\Omega}{1 - \mu_p\Omega} = O(\lambda^3 \vee \beta). \tag{6.9}$$

But by definition and (5.102),

$$\mu_p\Omega = 1 - [1 + \hat{\Pi}_p(0)]\chi(p)^{-1}. \tag{6.10}$$

Also, by (6.1),

$$m_p\Omega = 1 - [1 + O(\lambda^3 \vee \beta)]\chi(p)^{-1}. \tag{6.11}$$

Therefore,

$$\frac{|\mu_p - m_p|\Omega}{1-\mu_p\Omega} = \frac{O(\lambda^3 \vee \beta)\chi(p)^{-1}}{[1+\hat{\Pi}_p(0)]\chi(p)^{-1}} = O(\lambda^3 \vee \beta), \tag{6.12}$$

as required. □

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