Diffusion of a heteropolymer in a multi-interface medium

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Abstract

We consider a heteropolymer, consisting of an i.i.d. concatenation of hydrophilic and hydrophobic monomers, in the presence of water and oil arranged in alternating layers. The heteropolymer is modelled by a directed path $(i, S_i)_{i \in \mathbb{N}_0}$, where the vertical component lives on $\mathbb{Z}$, and the layers are horizontal with equal width. The path measure for the vertical component is given by that of simple random walk multiplied by an exponential weight factor that favors matches and disfavors mismatches between the monomers and the medium. We study the vertical motion of the heteropolymer as a function of its total length n when the width of the layers is d_n and the parameters in the exponential weight factor are such that the heteropolymer tends to stay close to an interface ("localized regime"). In the limit as $n \to \infty$ and under the condition that $\lim_{n\to\infty} d_n / \log\log n = \infty$ and $\lim_{n\to\infty} d_n / \log n = 0$, we show that the vertical motion is a diffusive hopping between neighboring interfaces on a time scale $\exp[\chi d_n(1 + o(1))]$, where χ is computed explicitly in terms of a variational problem. An analysis of this variational problem sheds light on the optimal hopping strategy.

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1 Introduction and main result

1.1 One-interface heteropolymer. We begin by describing the one-interface model that was studied in Bolthausen and den Hollander [3]. This model has two ingredients:

1. $S = (S_i)_{i \in \mathbb{N}_0}$: a simple random walk on $\mathbb{Z}$; P_x, E_x denote its probability law and expectation, given $S_0 = x$.

2. $\omega = (\omega_i)_{i \in \mathbb{N}}$: an i.i.d. sequence of random variables taking the values ± 1 with probability $1/2$ each; $\mathbb{P}, \mathbb{E}$ denote its probability law and expectation.

Fix $\lambda \in [0, \infty)$, $h \in [0, 1)$ and $n \in \mathbb{N}$. Given ω, define a transformed probability law on path space by putting

$$P_x^{(0,n)}(S)(\omega) = \frac{1}{Z_x^{(0,n)}(\omega)} \exp\left\{ \lambda \sum_{i=1}^{n} \Delta(S_i)(\omega_i + h) \right\} P_x(S), \tag{1.1}$$

where $Z_x^{(0,n)}(\omega)$ is the normalizing partition sum and

$$\Delta(S_i) = \begin{cases} \operatorname{sign}(S_i) & \text{if } S_i \neq 0, \\ \Delta(S_{i-1}) & \text{if } S_i = 0. \end{cases} \tag{1.2}$$

We view $P_x^{(0,n)}$ as modelling the following situation. Think of $(i, S_i)_{i \in \mathbb{N}_0}$ as a directed polymer on $\mathbb{Z}^2$, starting at $(0, x)$, consisting of monomers represented by the bonds in the path. The lower half plane is water, the upper half plane is oil. The monomers are of two different types, occurring in a random order indexed by ω. Namely, $\omega_i = -1$ means that monomer i is hydrophilic, $\omega_i = +1$ that it is hydrophobic. Since $\Delta(S_i) = -1$ when monomer i lies in the water and $\Delta(S_i) = +1$ when it lies in the oil, we see that the weight factor in (1.1) encourages matches and discourages mismatches for the first n monomers. For $h = 0$ both types of monomers interact equally strongly with the water and with the oil. For $h \in (0, 1)$, on the other hand, the interaction strength is asymmetric: the hydrophobic monomers interact more strongly with either solvent than the hydrophilic monomers, resulting in the heteropolymer to prefer the oil in the upper half plane over the water in the lower half plane. The parameter λ is the overall interaction strength and plays the role of inverse temperature. [1]

The one-interface model is self-averaging:

Theorem 1.1 ([3], Theorem 1) *For every $\lambda \in [0, \infty)$ and $h \in [0, 1)$ there exists a deterministic number $\phi(\lambda, h)$ such that*

$$\lim_{n \to \infty} \frac{1}{n} \log Z_0^{(0,n)}(\omega) = \phi(\lambda, h) \qquad \mathbb{P} - a.s. \text{ and in } L^1(\mathbb{P}). \tag{1.3}$$

[1] Note that the second line in (1.2) makes the interaction act on bonds rather than on sites. Also note that (1.1) makes perfect sense for $\lambda, h \in \mathbb{R}$ but that only the indicated range of λ, h is relevant.

The function ϕ is the *specific free energy* of the heteropolymer. It is continuous, nondecreasing and convex in both variables, and satisfies $\phi(\lambda,h) \geq \lambda h$. This lower bound comes from the following estimate, which uses the strong law of large numbers for ω:

$$\begin{aligned} Z_0^{(0,n)}(\omega) &\geq E_0\Big[\exp\Big\{\lambda\sum_{i=1}^n \Delta(S_i)(\omega_i+h)\Big\}\ 1\{S_i>0\ \forall 1\leq i\leq n\}\Big] \\ &= \exp\Big\{\lambda hn + \lambda\sum_{i=1}^n \omega_i\Big\} P_0\left[S_i>0\ \forall 1\leq i\leq n\right] \\ &= \exp\{\lambda hn + o(n)\}\, O\left(n^{-1/2}\right), \qquad \mathbb{P}-a.s. \end{aligned} \tag{1.4}$$

Let

$$\begin{aligned} \mathcal{D} &= \{(\lambda,h)\colon\ \phi(\lambda,h)=\lambda h\}, \\ \mathcal{L} &= \{(\lambda,h)\colon\ \phi(\lambda,h)>\lambda h\}. \end{aligned} \tag{1.5}$$

In view of (1.4), intuitively, $\mathcal{D}$ corresponds to the situation where the heteropolymer moves away from the interface in the upward direction ("delocalized regime"), while $\mathcal{L}$ corresponds to the situation where the heteropolymer stays close to the interface and manages to place more than half of its monomers in their preferred medium ("localized regime"). It turns out that both these situations occur:

Theorem 1.2 ([3], Theorem 2, Equation (0.8)(iii) and Corollary 1) *For every $\lambda \in (0,\infty)$ there exists an $h_c(\lambda) \in (0,1)$ such that the heteropolymer is*

$$\begin{array}{ll} \textit{localized} & \textit{if } 0 \leq h < h_c(\lambda), \\ \textit{delocalized} & \textit{if } h \geq h_c(\lambda). \end{array} \tag{1.6}$$

Moreover, $\lambda \mapsto h_c(\lambda)$ is continuous and non-decreasing on $[0,\infty)$, with $h_c(\lambda) \sim C_1\lambda$ as $\lambda \downarrow 0$ and $1-h_c(\lambda) \sim C_2/\lambda$ as $\lambda \to \infty$, for some $C_1, C_2 > 0$.

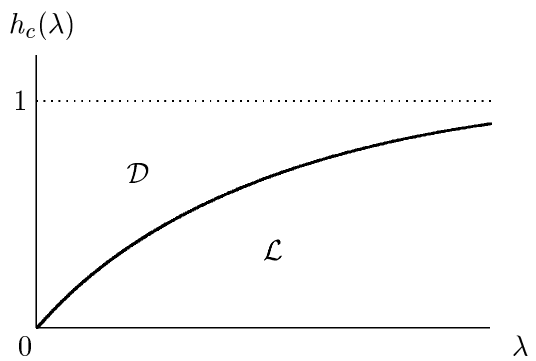

Fig. 1. Qualitative picture of $\lambda \mapsto h_c(\lambda)$.

In Biskup and den Hollander [2] various path properties were derived that confirm the above intuitive description. In the delocalized regime $\mathcal{D}$ the heteropolymer intersects the interface with zero frequency in the limit as $n \to \infty$ ([2], Theorem 4). In the localized regime $\mathcal{L}$, however, this frequency is strictly positive, and the excursions away from the interface are exponentially bounded both in length and in height ([2], Theorem 3). In Albeverio and Zhou [1] it was proved that for $\lambda \in (0, \infty)$ and $h = 0$ both the maximal length and the maximal height of an excursion are of order $\log n$ ([1], Theorem 5.3 and Theorem 6.1). The same holds true throughout the localized regime $\mathcal{L}$ by the estimates in [2].

The one-interface model defined in (1.1–1.2) was introduced in Garel, Huse, Leibler and Orland [5], and early studies include Sinai [13] ($h = 0$) and Grosberg, Izrailev and Nechaev [6] (ω periodic). Recent results on related one-interface models appear in Maritan, Riva and Trovato [10], Martin, Causo and Whittington [11], and Orlandini, Rechnitzer and Whittington [12].

1.2 Multi-interface heteropolymer. In the present paper we study a version of the above model where the water and the oil are arranged in alternating horizontal layers. For $n \in \mathbb{N}$, we choose a layer thickness $d_n \in 2\mathbb{N}$ (an even number for reasons of parity). The interfaces separating the layers are located at the heights

$$\partial D_n = d_n \mathbb{Z}, \tag{1.7}$$

while the (+1)-layers resp. the (−1)-layers span the heights

$$D_n^+ = \bigcup_{k \in \mathbb{N}_0} d_n \left[(2k, 2k+1) \cap \mathbb{Z}\right], \qquad D_n^- = \bigcup_{k \in \mathbb{N}_0} d_n \left[(2k-1, 2k) \cap \mathbb{Z}\right]. \tag{1.8}$$

In analogy with (1.1–1.2), the probability law of the heteropolymer is defined as

$$\hat{P}_{x,d_n}^{(0,n)}(S)(\omega) = \frac{1}{\hat{Z}_{x,d_n}^{(0,n)}(\omega)} \exp\left\{\lambda \sum_{i=1}^{n} \Delta_{d_n}(S_i)(\omega_i + h)\right\} P_x(S), \tag{1.9}$$

where $\hat{Z}_{x,d_n}^{(0,n)}(\omega)$ is the normalizing partition sum and

$$\Delta_{d_n}(S_i) = \begin{cases} +1 & \text{if } S_i \in D_n^+, \\ -1 & \text{if } S_i \in D_n^-, \\ \Delta_{d_n}(S_{i-1}) & \text{if } S_i \in \partial D_n. \end{cases} \tag{1.10}$$

Here, we use the hat-superscript to distinguish the multi-interface model from the one-interface model.

Our first result is a comparison of the multi-interface model with the one-interface model on the level of the specific free energy.

Theorem 1.3 *For every $\lambda \in [0, \infty)$, $h \in [0, 1)$ and for every sequence (d_n) such that $\lim_{n\to\infty} d_n = \infty$,*

$$\lim_{n\to\infty} \frac{1}{n} \log \hat{Z}_{0,d_n}^{(0,n)}(\omega) = \phi(\lambda, h) \qquad \mathbb{P} - a.s. \text{ and in } L^1(\mathbb{P}). \tag{1.11}$$

This result says that for any diverging layer width the two models have the same specific free energy and hence the same phase diagram (see Theorem 1.1 and Fig. 1). Intuitively, this result is plausible: as the interfaces move apart, the heteropolymer "gets to see only one interface at a time". We will see that the limit $d_n \to \infty$ makes the multi-interface model tractable.

1.3 Path behavior in the localized regime. We now come to the main result of this paper. Our goal is to analyze the path behavior for the multi-interface model in the localized regime $\mathcal{L}$, in particular, we want to describe how fast the heteropolymer hops between the interfaces.

For technical reasons we will not analyze the jump process between the interfaces of the layers, but rather between the middle lines of the layers, i.e., $\partial D_n + d_n/2$. The reason is that the oil/water medium is symmetric with respect to these middle lines. Let us therefore introduce the stopping times

$$\hat{\tau}(\sigma) = \inf\{i \in \mathbb{N}_0 :\ S_i = \sigma d_n/2\} \wedge \inf\{i \in \mathbb{N}_0 :\ S_i = -\sigma 3d_n/2\}, \qquad \sigma = \pm 1. \tag{1.12}$$

If $S_0 = 0$, then $\hat{\tau}(\sigma)$ is the first time that S hits a middle line of a σ-layer. Furthermore, let us define $\tau_0 = 0$ and the stopping times

$$\begin{aligned} \tau_1 &= \inf\{i \in \mathbb{N}_0 :\ |S_i - S_0| = d_n\}, \\ \tau_{k+1} &= \tau_1 \circ \theta_{\tau_k} + \tau_k \ \text{ for } k \in \mathbb{N}, \end{aligned} \tag{1.13}$$

where θ_i denotes the time-shift by i. If $S_0 \in \partial D_n + d_n/2$, then τ_k is the k-th jump time between the middle lines of the layers. In terms of these quantities, the number of hits of middle lines up to time t after time $\hat{\tau}(+1)$ is given by

$$N_t = \left[\sup\left\{k \in \mathbb{N}_0 :\ \tau_k \circ \theta_{\hat{\tau}(+1)} + \hat{\tau}(+1) \le t\right\} + 1\right] 1_{\{\hat{\tau}(+1) \le t\}}, \qquad t \in [0,n], \tag{1.14}$$

and the vertical displacement at time t relative to the height at time $\hat{\tau}(+1)$ is given by

$$\tilde{S}_t = \left(S_{\tau_{N_t - 1} \circ \theta_{\hat{\tau}(+1)} + \hat{\tau}(+1)} - S_{\hat{\tau}(+1)}\right) 1_{\{\hat{\tau}(+1) \le t\}}, \qquad t \in [0,n]. \tag{1.15}$$

Similar formulas can be written down with $\hat{\tau}(-1)$ instead of $\hat{\tau}(+1)$, but we choose to follow the jumps starting from the first hit of a middle line of a $(+1)$-layer.

Theorem 1.4 *Let $(\lambda, h) \in \mathcal{L}$. Fix a sequence (d_n) such that*

$$\begin{aligned} &\text{(I)} \quad \lim_{n\to\infty} d_n / \log\log n = \infty, \\ &\text{(II)} \quad \lim_{n\to\infty} d_n / \log n = 0. \end{aligned} \tag{1.16}$$

Then there exists a constant $\chi(\lambda, h) \in (0,\infty)$ such that, under the annealed measure $\mathbb{E} \otimes \hat{P}_{0,d_n}^{(0,n)}$, the process $(\tilde{S}_t)_{t\in[0,n]}$ is a simple random walk on $d_n\mathbb{Z}$ with i.i.d. random waiting times whose variance at time un satisfies

$$\lim_{n\to\infty} \frac{1}{d_n} \log\left[\frac{1}{d_n^2\, un} \mathrm{Var}_{\mathbb{E}\otimes \hat{P}_{0,d_n}^{(0,n)}}(\tilde{S}_{un})\right] = -\chi(\lambda, h), \qquad u \in (0,1). \tag{1.17}$$

Equation (1.17) says that

$$e^{\chi(\lambda,h)d_n(1+o(1))} = \text{average jump time between middle lines of layers.} \tag{1.18}$$

1.4 Discussion of Theorem 1.4 and analysis of $\chi(\lambda, h)$. We begin by explaining the two conditions in (1.16). The results cited in Section 1.1 for the one-interface model, in combination with Theorem 1.3, tell us that in the localized regime $\mathcal{L}$ the heteropolymer is tied down to the interfaces in ∂D_n. The excursions away from ∂D_n have a typical length of order one and a maximal length and height of order $\log n$. Condition (II) therefore guarantees that the heteropolymer jumps between the interfaces many times prior to time n (i.e., the medium is "not too macroscopic"). On the other hand, condition (I) guarantees that the heteropolymer does not jump too frequently, so that between jumps it stabilizes near an interface (i.e., the medium is "not too microscopic"). We do not know whether $\log\log n$ is optimal as a lower bound, but it is important in our proof.

The proof of Theorem 1.4 shows that there exist constants $\chi_\sigma(\lambda, h)$, $\sigma \pm 1$, such that

$$e^{\chi_\sigma(\lambda,h)d_n(1+o(1))} = \text{average crossing time of } \sigma\text{-layers}, \tag{1.19}$$

which implies that

$$\chi(\lambda, h) = \chi_{-1}(\lambda, h) \vee \chi_{+1}(\lambda, h). \tag{1.20}$$

In the course of the proof of Theorem 1.4 we give an explicit description of $\chi_\sigma(\lambda, h)$ in terms of a variational problem involving a one-interface model with one neutral solvent (see (4.1–4.2) and (4.18)). An analysis of this variational problem leads to the following qualitative picture.

Theorem 1.5 *For every $\lambda \in (0, \infty)$:*
(i) $\chi_{-1}(\lambda, 0) = \chi_{+1}(\lambda, 0)$.
(ii) On $[0, h_c(\lambda))$, $h \mapsto \chi_{-1}(\lambda, h)$ is continuous and non-decreasing, while $h \mapsto \chi_{+1}(\lambda, h)$ is continuous and non-increasing.
(iii) $\lim_{h\uparrow h_c(\lambda)} \chi_{+1}(\lambda, h) = 0$.

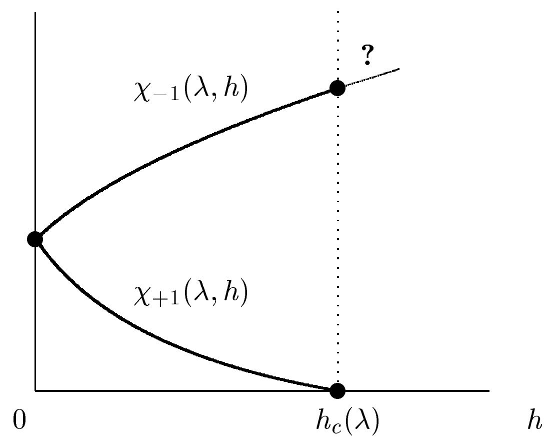

Fig. 2. Qualitative picture of $h \mapsto \chi_\sigma(\lambda, h)$ for $\sigma \pm 1$ and $\lambda \in (0, \infty)$.

In view of (1.20) and Theorem 1.5(i–ii), we have

$$\chi(\lambda, h) = \chi_{-1}(\lambda, h). \tag{1.21}$$

i.e., the variance in (1.17) is dominated by the average crossing time of the (-1)-layers, which is at least as long as the average crossing time of the $(+1)$-layers. This comes from the fact that the heteropolymer prefers to wander off into the $(+1)$-layers as soon as $h > 0$. In view of Theorem 1.5(iii), on the phase transition line separating $\mathcal{L}$ from $\mathcal{D}$ (see Fig. 1) the average crossing time of the $(+1)$-layers vanishes on time scale e^{d_n}. We have no control over how the average crossing time of the (-1)-layers behaves in the delocalized regime $\mathcal{D}$, but we expect it to be smooth across the phase transition line (see the dotted line in Fig. 2).

We are unable to prove strict monotonicity of $h \mapsto \chi_\sigma(\lambda, h)$, as suggested in Fig. 2.

1.5 Some future challenges. Here is a list of some open problems that merit closer investigation:

(1) Is there a version of (1.17) for the quenched rather than the annealed model, i.e., for $\mathbb{P}$-a.s. all ω with respect to $\hat{P}^{(0,n)}_{0,d_n}(\omega)$? We expect that the answer is yes, with the same χ, because of the ergodic theorem for ω. A proof can probably be worked out with the help of the "decoupling of excursions" argument in Section 3.

(2) What can we say about the hopping in the delocalized regime $\mathcal{D}$? Since the crossing of the (-1)-layers is harder than in the localized regime $\mathcal{L}$, we expect the jump process to further slow down (see the dotted line in Fig. 2).

(3) What happens when the layer widths are random, say, layer k has width $Y_k d_n$ with $(Y_k)_{k\in\mathbb{Z}}$ i.i.d. random variables that are bounded away from 0 and ∞? The underlying jump process between layers will be a random walk in random environment.

(4) The present paper is a first attempt to move away from the simple geometry of a single flat interface. We are ultimately interested in situations where the two media mix "as droplets of oil floating around in water". Can anything be said for such more complicated models? A toy model in this direction is studied in den Hollander and Whittington [7].

1.6 Outline. In Section 2 we prove Theorem 1.3 and derive a number of preparatory lemmas. In Section 3 we provide a decoupling argument through which the probability law of the lengths of the successive excursions can be estimated in terms of that of a single excursion. In Section 4 we give asymptotic estimates for the latter. In Section 5 these estimates are used to prove Theorem 1.4. Theorem 1.5 is proved in Section 6.

2 Proof of Theorem 1.3 and preparations

This section contains the proof of Theorem 1.3 as well as three technical lemmas (Lemmas 2.1–2.3) that will be needed along the way. In Section 2.1 we look at partition sums, in Section 2.2 at excursion lengths.

2.1 Asymptotic behavior of partition sums. We start with the proof of Theorem 1.3, which together with Theorem 1.2 shows that Fig. 1 is the phase diagram also for the multi-interface model. Throughout the sequel we write

$$H_{n,d_n}(\omega, S) = \lambda \sum_{i=1}^{n} \Delta_{d_n}(S_i)(\omega_i + h) \tag{2.1}$$

to denote the Hamiltonian of the multi-interface heteropolymer defined in (1.9) and

$$H_n(\omega, S) = \lambda \sum_{i=1}^{n} \Delta(S_i)(\omega_i + h) \tag{2.2}$$

to denote the Hamiltonian of the one-interface heteropolymer defined in (1.1).

Proof of Theorem 1.3. The proof is based on a folding argument applied to the random walk S. We assume that $S_0 = 0$.

Define $\eta_0 = 0$ and

$$\begin{aligned} \eta_1 &= \inf\{i \in \mathbb{N}\colon\ S_i \in \partial D_n \setminus \{S_0\}\}, \\ \eta_{k+1} &= \eta_1 \circ \theta_{\eta_k} + \eta_k, \qquad k \in \mathbb{N}, \end{aligned} \tag{2.3}$$

i.e., η_k is the k-th crossing time of a layer, and

$$\begin{aligned} \ell_1 &= \sup\{0 < i < \eta_1\colon\ |S_i - S_{\eta_1}| = |S_{\eta_0} - S_{\eta_1}|/2\}, \\ \ell_{k+1} &= \ell_1 \circ \theta_{\eta_k} + \eta_k, \qquad k \in \mathbb{N}, \end{aligned} \tag{2.4}$$

i.e., ℓ_k is the last hitting time of a middle line of a layer prior to time η_k. Also define

$$\mathcal{N}_n = \sup\{k \in \mathbb{N}_0\colon\ \eta_k \le n\}, \tag{2.5}$$

i.e., the number of layer crossings up to time n. Obviously, $\mathcal{N}_n \le n/d_n$.

Next, define a folding map $S \mapsto S^*$ for $S = (S_i)_{i=0}^n$ and $S^* = (S_i^*)_{i=0}^n$ as follows. Put $S_i^{(0)} = S_i$ for $0 \le i \le n$. For $1 \le k \le \mathcal{N}_n$, define recursively

$$S_i^{(k)} = \begin{cases} S_i^{(k-1)} & \text{if } 1 \le i \le \ell_k, \\ 2S_{\ell_k}^{(k-1)} - S_i^{(k-1)} & \text{if } i > \ell_k, \end{cases} \tag{2.6}$$

and set $S^* = S^{(\mathcal{N}_n)}$. Thus, we successively reflect the tail of the path at the heights $\pm d_n/2$. The important observation is that $\sup_{0 \le i \le n} |S_i^*| < d_n$, $S_{\eta_k}^* = 0$ for $0 \le k \le \mathcal{N}_n$, and

$$\Delta_{d_n}(S_i) = \Delta(S_i^*) \quad \text{for } 0 \le i \le n, \tag{2.7}$$

the latter implying, via (2.1–2.2), that

$$H_{n,d_n}(\omega,S) = H_n(\omega,S^*). \tag{2.8}$$

Therefore we need only worry about how many paths S are mapped onto a single path S^*.

To that end, define $R_0 = 0$ and

$$\begin{array}{lcl} R_1 & = & \inf\{i \in \mathbb{N}\colon\ S_i \in \partial D_n\}, \\ R_{k+1} & = & R_1 \circ \theta_{R_k} + R_k, \qquad k \in \mathbb{N}, \end{array} \tag{2.9}$$

i.e., R_k is the k-th hitting time of an interface. Pick any path S^* with $\sup_{0\le i\le n}|S_i^*| < d_n$. We can defold S^* whenever

$$\sup_{R_{k-1}(S^*)\le i<R_k(S^*)} |S_i^*| = d_n/2, \qquad 1 \le k \le \mathcal{N}_n(S^*). \tag{2.10}$$

But this event can occur at most n/d_n times and therefore S^* is the image of at most $2^{n/d_n}$ paths S. Hence, using (2.8) we get

$$\begin{array}{rcl} \hat{Z}_{0,d_n}^{(0,n)}(\omega) & = & \sum\limits_S \exp\{H_{n,d_n}(\omega,S)\}\, 2^{-n} \\ & \le & 2^{n/d_n} \sum\limits_{S^*\colon\ \sup_{0\le i\le n}|S_i^*|<d_n} \exp\{H_n(\omega,S^*)\}\, 2^{-n} \\ & \le & 2^{n/d_n} Z_0^{(0,n)}(\omega) \end{array} \tag{2.11}$$

and similarly

$$\begin{array}{rcl} \hat{Z}_{0,d_n}^{(0,n)}(\omega) & \ge & \sum\limits_{S\colon\ \sup_{0\le i\le n}|S_i|<d_n} \exp\{H_{n,d_n}(\omega,S)\}\, 2^{-n} \\ & = & \sum\limits_{S^*\colon\ \sup_{0\le i\le n}|S_i^*|<d_n} \exp\{H_n(\omega,S^*)\}\, 2^{-n} \\ & \ge & 2^{-n/d_n} \sum\limits_S \exp\{H_n(\omega,S)\}\, 2^{-n} \\ & = & 2^{-n/d_n} Z_0^{(0,n)}(\omega). \end{array} \tag{2.12}$$

Since $\lim_{n\to\infty} d_n = \infty$, Theorem 1.3 is now a consequence of Theorem 1.1 and (2.11–2.12).

■

We next consider the partition sum for the multi-interface model up to time $2n$ restricted to the endpoint S_{2n} lying in an interface:

$$\hat{Z}_{0,\partial D_n,d_n}^{(0,2n)}(\omega) = E_0\Big[\exp\{H_{2n,d_n}(\omega,S)\}, S_{2n} \in \partial D_n\Big]. \tag{2.13}$$

The following lemma says that this restriction has no effect on the specific free energy.

Lemma 2.1 *For every $\lambda \in [0,\infty)$, $h \in [0,1)$ and for every sequence (d_n) such that $\lim_{n\to\infty} d_n = \infty$,*

$$\lim_{n\to\infty} \frac{1}{2n} \log \hat{Z}^{(0,2n)}_{0,\partial D_n,d_n}(\omega) = \phi(\lambda,h) \qquad \mathbb{P}-a.s. \text{ and in } L^1(\mathbb{P}). \tag{2.14}$$

Proof of Lemma 2.1. Using the stopping time R_1 defined in (2.9), we have for $m \in \mathbb{N}$,

$$\begin{aligned} \hat{Z}^{(0,2m)}_{0,d_n}(\omega) &= \hat{Z}^{(0,2m)}_{0,\partial D_n,d_n}(\omega) + \sum_{k=0}^{m-1} \hat{Z}^{(0,2k)}_{0,\partial D_n,d_n}(\omega) E_0\left[\exp\left\{H_{2m-2k,d_n}(\theta_{2k}\omega,S)\right\}, R_1 > 2m-2k\right] \\ &= \hat{Z}^{(0,2m)}_{0,\partial D_n,d_n}(\omega) + \sum_{k=0}^{m-1} \hat{Z}^{(0,2k)}_{0,\partial D_n,d_n}(\omega) E_0\left[\exp\left\{H_{2m-2k,d_n}(\theta_{2k}\omega,S)\right\}, R_1 = 2m-2k\right] \\ &\qquad \times \frac{P_0\left[R_1 > 2m-2k\right]}{P_0\left[R_1 = 2m-2k\right]} \\ &\leq \hat{Z}^{(0,2m)}_{0,\partial D_n,d_n}(\omega) + \sum_{k=0}^{m-1} \frac{\hat{Z}^{(0,2k)}_{0,\partial D_n,d_n}(\omega) E_0\left[\exp\left\{H_{2m-2k,d_n}(\theta_{2k}\omega,S)\right\}, R_1 = 2m-2k\right]}{P_0\left[R_1 = 2m-2k\right]}. \end{aligned} \tag{2.15}$$

Here, the first line is a renewal relation, while the second line uses that the excursion starting from an interface at time $2k$ stays within a single layer until time $2m$. Next we apply the folding argument from the proof of Theorem 1.3: for $l \in \mathbb{N}$ we fold the part of the path that lies between $\inf\{i > 0\colon\ S_i = d_n/2\}$ and $\sup\{0 < i < 2l\colon\ S_i = d_n/2\}$ into the layer $(0,d_n)\cap\mathbb{N}$ with symmetry axis $d_n/2$. Then we get

$$\begin{aligned} P_0\left[R_1 = 2l\right] &\geq P_0\left[R_1 = 2l, S_{2l} = 0\right] \\ &= 2P_0\left[S_i \in (0,d_n) \text{ for } 0 < i < 2l,\ S_{2l} = 0\right] \\ &\geq 2 \times 2^{-2l/d_n} P_0\left[S_i > 0 \text{ for } 0 < i < 2l,\ S_{2l} = 0\right] \\ &\geq 2^{-2l/d_n} c_1^{-1} l^{-3/2}. \end{aligned} \tag{2.16}$$

Substituting (2.16) into the last line of (2.15) to estimate the denominator of the summand, we find the second inequality in

$$\hat{Z}^{(0,2m)}_{0,\partial D_n,d_n} \leq \hat{Z}^{(0,2m)}_{0,d_n} \leq (1 + c_1 2^{2m/d_n} m^{3/2}) \hat{Z}^{(0,2m)}_{0,\partial D_n,d_n}. \tag{2.17}$$

The first inequality is trivial. Together with Theorem 1.3 this yields the claim.

■

2.2 Estimates on large excursions. Our next lemma is a large deviation result for the restricted partition sum defined in (2.13) and for the successive excursion lengths, both in the localized regime $\mathcal{L}$. Recall from (1.5) that $\phi(\lambda,h) > \lambda h$ for $(\lambda,h) \in \mathcal{L}$.

Lemma 2.2 *Assume that* $(\lambda,h)\in\mathcal{L}$ *and* $\lim_{n\to\infty} d_n=\infty$.
(i) For all $\varepsilon\in(0,\phi(\lambda,h)-\lambda h)$ *there exists a* $\delta_\varepsilon>0$ *such that*

$$\mathbb{P}\left[\frac{1}{2n}\log\hat{Z}^{(0,2n)}_{0,\partial D_n,d_n}(\omega)<\phi(\lambda,h)-\varepsilon\right]\le\mathcal{O}(1)\exp\left\{-\delta_\varepsilon n\right\},\qquad n\to\infty. \tag{2.18}$$

(ii) There exists a constant $\kappa>0$ *such that, for all* $K\in\mathbb{N}$ *and all* $m_k\in 2\mathbb{N}$ $(1\le k\le K)$ *with* $\sum_{k=1}^K m_k\le n$,

$$\mathbb{E}\otimes\hat{P}^{(0,n)}_{0,d_n}\left[R_k-R_{k-1}=m_k \text{ for } 1\le k\le K\right]\le\mathcal{O}(1)\prod_{k=1}^K\exp\{-\kappa m_k\}. \tag{2.19}$$

Proof of Lemma 2.2. The proof is similar to that of Lemmas 3 and 4 in Biskup and den Hollander [2]. Lemma 2.1 is needed for the proof of (i), while (i) is needed for the proof of (ii). The reader is referred to [2] for details.

■

We close this section with an estimate on the maximal excursion length away from an interface. For $l,n\in\mathbb{N}$, define

$$\mathcal{R}^{d_n}_l=\sup_{k\in\mathbb{N}}\left\{(R_k\wedge l)-(R_{k-1}\wedge l)\right\}, \tag{2.20}$$

i.e., the maximal excursion length up to time l.

Lemma 2.3 *Assume that* $(\lambda,h)\in\mathcal{L}$ *and* $\lim_{n\to\infty} d_n=\infty$. *There exist* $\kappa>0$ *and* $c_2>0$ *such that for all* $\zeta>0$,

$$\mathbb{E}\otimes\hat{P}^{(0,n)}_{0,d_n}\left[\mathcal{R}^{d_n}_l>\zeta\log n\right]\le c_2 n^{1-\kappa\zeta},\qquad n\to\infty. \tag{2.21}$$

Proof of Lemma 2.3. For $\zeta>0$ and $n\in\mathbb{N}$ we have

$$\begin{aligned}
&\mathbb{E}\otimes\hat{P}^{(0,n)}_{0,d_n}\left[\mathcal{R}^{d_n}_n>\zeta\log n\right]\\
&\quad\le\sum_{i=0}^{n-\zeta\log n-1}\mathbb{E}\otimes\hat{P}^{(0,n)}_{0,d_n}\left[S_i\in\partial D_n,(R_1\circ\theta_i)\wedge(n-i)>\zeta\log n\right]\\
&\quad=\sum_{i=0}^{n-\zeta\log n-1}\sum_{k=\zeta\log n+1}^{n-i}\mathbb{E}\otimes\hat{P}^{(0,n)}_{0,d_n}\left[S_i\in\partial D_n,R_1\circ\theta_i=k\right]\\
&\quad\le\sum_{i=0}^{n-\zeta\log n-1}\sum_{k=\zeta\log n+1}^{n-i}\mathbb{E}\otimes\hat{P}^{(0,n)}_{0,d_n}\left[R_1\circ\theta_i=k\mid S_i\in\partial D_n\right]\\
&\quad=\sum_{i=0}^{n-\zeta\log n-1}\sum_{k=\zeta\log n+1}^{n-i}\mathbb{E}\otimes\hat{P}^{(0,n-i)}_{0,d_n}\left[R_1=k\right].
\end{aligned} \tag{2.22}$$

Using Lemma 2.2(ii), we see that there exist constants $\kappa > 0$ and $c_3 > 0$ such that for all $\zeta > 0$,

$$\begin{aligned}\mathbb{E} \otimes \hat{P}_0^{(0,n)} \left[\mathcal{R}_n^{d_n} > \zeta \log n\right] &\leq \sum_{i=0}^{n-\zeta \log n - 1} \sum_{k=\zeta \log n+1}^{n-i} c_3 \exp\{-\kappa k\} \\ &\leq nc_3\kappa^{-1}n^{-\kappa\zeta}, \qquad n \to \infty. \end{aligned} \tag{2.23}$$

This proves the claim with $c_2 = c_3\kappa^{-1}$.

■

3 Decoupling of excursions

This section contains three further lemmas (Lemmas 3.1–3.3) in which we estimate the probability law of the lengths of the successive excursions in terms of that of a single excursion. The latter will be estimated in Section 4. In Section 3.1 we look at the effect of adding a bridge point, in Section 3.2 we derive the decoupling estimates.

3.1 Adding a bridge point. We begin by estimating how much it costs to do an additional hitting of an interface. For $\zeta > 0$ and $l, n \in \mathbb{N}$, define the event (recall (2.20))

$$\mathcal{A}_l^n = \left\{\mathcal{R}_l^{d_n} \leq \zeta \log n\right\}. \tag{3.1}$$

Lemma 3.1 *Assume that* $\lim_{n\to\infty} d_n = \infty$. *For all* $\zeta > 0$ *there exists* $c_4 > 0$ *such that for all* $b \in 2\mathbb{N}$,

$$\hat{P}_{0,d_n}^{(0,n)} \left[\mathcal{A}_n^n\right] \leq c_4 d_n^3 \hat{P}_{0,d_n}^{(0,n)} \left[\mathcal{A}_n^n, S_b \in \partial D_n\right], \qquad n \to \infty. \tag{3.2}$$

Proof of Lemma 3.1. For $b \in 2\mathbb{N}$, define

$$L^b = \sup\{0 \leq k \leq b\colon\ S_k \in \partial D_n\}, \tag{3.3}$$

i.e., the last hitting time of an interface prior to time b. For simplicity we assume that $b \leq n - \zeta \log n$. Then we have

$$\begin{aligned} E_0 \left[\exp\{H_{n,d_n}(\omega, S)\}, \mathcal{A}_n^n\right] &= E_0 \left[\exp\{H_{n,d_n}(\omega, S)\}, \mathcal{A}_n^n, L^b = b\right] \\ &+ \sum_{2\leq l+r\leq \zeta \log n} E_0 \left[\exp\{H_{n,d_n}(\omega, S)\}, \mathcal{A}_n^n, b - L^b = l, R_1 \circ \theta_b = r\right]. \end{aligned} \tag{3.4}$$

The case $b > n - \zeta \log n$ is analogous, but we have to restrict the sum in (3.4) to $r \leq n - b$. Let us estimate the last term in the above inequality for fixed l, r. The important observation is that, on the event $\{b - L^b = l, R_1 \circ \theta_b = r\}$, $\Delta_{d_n}(S_i)$ has the same sign for all $b - l < i \leq b + r$. If we want to do an additional hitting of an interface in this interval, then all we have to do is to make sure that the hitting of the interface is in fact a reflection at the interface, since this does not change the sign of $\Delta_{d_n}(S_i)$ and hence leaves the Hamiltonian $H_{n,d_n}(\omega, S)$ in (2.1) invariant. Consequently,

$$
\begin{aligned}
&E_0\Big[\exp\{H_{n,d_n}(\omega,S)\}, \mathcal{A}_n^n, b-L^b=l, R_1\circ\theta_b=r\Big]\\
&\quad\le 2E_0\Big[\exp\{H_{n,d_n}(\omega,S)\}, \mathcal{A}_n^n, b-L^b=l, R_1\circ\theta_b=r, S_{L^b}=S_{R_1\circ\theta_b}\Big]\\
&\quad= 2\sum_{z\in\partial D_n} E_0\left[\exp\{H_{b-l,d_n}(\omega,S)\}, \mathcal{A}_{b-l}^n, S_{b-l}=z\right]\\
&\qquad\qquad\times E_z\left[\exp\{H_{l+r,d_n}(\theta_{b-l}\omega,S)\}, \mathcal{A}_{l+r}^n, R_1=l+r, S_{l+r}=z\right]\\
&\qquad\qquad\times E_z\Big[\exp\{H_{n-(b+r),d_n}(\theta_{b+r}\omega,S)\}, \mathcal{A}_{n-(b+r)}^n\Big]\\
&\quad= E_0\Big[\exp\{H_{n,d_n}(\omega,S)\}, \mathcal{A}_n^n, R_1\circ\theta_{b-l}=l, R_2\circ\theta_{b-l}=l+r,\\
&\qquad\qquad\qquad\qquad S_{b-l}=S_b=S_{b+r}\in\partial D_n\Big]\\
&\qquad\qquad\times\frac{2P_0\left[R_1=l+r, S_{l+r}=0\right]}{P_0\left[R_1=l, R_2=l+r, S_l=S_{l+r}=0\right]}.
\end{aligned}
\tag{3.5}
$$

The inequality uses the fact that paths may be reflected in middle lines because the medium is symmetric with respect to middle lines. A standard calculation for simple random walk gives that

$$
P_0\left[R_1=m, S_m=0\right]=\frac{1}{d_n}\sum_{|k|<d_n}\cos^m\left(\frac{\pi k}{d_n}\right)\sin^2\left(\frac{\pi k}{d_n}\right)
\tag{3.6}
$$

(see e.g. Hughes [9], equation (3.291)). It is therefore easily seen that the ratio in the last term in (3.5) is bounded above by $c_4 d_n^3$ uniformly in l, r. Inserting this bound into (3.5) and the resulting estimate into (3.4), we obtain

$$
\begin{aligned}
&E_0\left[\exp\{H_{n,d_n}(\omega,S)\}, \mathcal{A}_n^n\right]\\
&\quad\le c_4 d_n^3\bigg(E_0\Big[\exp\{H_{n,d_n}(\omega,S)\}, \mathcal{A}_n^n, L^b=b\Big]\\
&\qquad+\sum_{2\le l+r\le\zeta\log n} E_0\Big[\exp\{H_{n,d_n}(\omega,S)\}, \mathcal{A}_n^n, R_1\circ\theta_{b-l}=l, R_2\circ\theta_{b-l}=l+r,\\
&\qquad\qquad\qquad\qquad S_{b-l}=S_b=S_{b+r}\in\partial D_n\Big]\bigg)\\
&\quad= c_4 d_n^3 E_0\Big[\exp\{H_{n,d_n}(\omega,S)\}\mathcal{A}_n^n, S_b\in\partial D_n\Big].
\end{aligned}
\tag{3.7}
$$

Divide by $E_0[\exp\{H_{n,d_n}(\omega,S)\}]$ to get the claim (recall (1.9)).

■

3.2 Decoupling estimates for excursion times. We now come to our two main decoupling estimates. For fixed ω, the successive excursion lengths are dependent. Lemmas 3.2–3.3 below show that, under the annealed measure $\mathbb{E} \otimes \hat{P}^{(0,n)}_{0,d_n}$, they can be decoupled at the price of an error term. Recall (1.12–1.13).

Lemma 3.2 *Assume that* $\lim_{n\to\infty} d_n = \infty$. *For all* $\zeta > 2/\kappa$ *there exists* $c_5 > 0$ *such that for all* $N \in \mathbb{N}$ *and* $l_i \geq d_n/2$ $(0 \leq i \leq N)$ *with*

$$\sum_{i=0}^{N} (l_i + \zeta \log n) \leq n \tag{3.8}$$

the following is true as $n \to \infty$*:*

$$\begin{aligned} &\mathbb{E} \otimes \hat{P}^{(0,n)}_{0,d_n} \left[\{\hat{\tau}(+1) \leq l_0\} \cap \bigcap_{i=1}^{N} \{\tau_i \circ \theta_{\hat{\tau}(+1)} - \tau_{i-1} \circ \theta_{\hat{\tau}(+1)} \leq l_i\} \right] \\ &\quad \leq c_2 n^{1-\kappa\zeta} + \inf_{I_N \subset \{0,\ldots,N\}} \prod_{i \in I_N} \left(c_5 d_n^6 \, \mathbb{E} \otimes \hat{P}^{(0,n)}_{0,d_n} \left[\hat{\tau}\left((-1)^i\right) \leq l_i, \mathcal{A}^n_n \right] + c_2 n^{1-\kappa\zeta} \right). \end{aligned} \tag{3.9}$$

Proof of Lemma 3.2. After applying Lemma 2.3, we can restrict ourselves to events contained in $\mathcal{A}_n$. Fix any $I_N \subset \{0,\ldots,N\}$. Throughout the proof we assume that n is large enough.

First we consider the case $0 \in I_N$. Using the inequality

$$\hat{Z}^{(0,n)}_{0,d_n}(\omega) \geq \hat{Z}^{(0,2m)}_{0,\partial D_n,d_n}(\omega) \hat{Z}^{(0,n-2m)}_{0,d_n}(\theta_{2m}\omega), \qquad 0 \leq 2m \leq n, \tag{3.10}$$

the independence on disjoint time intervals and Lemma 2.3, we have (variables with the wrong parity automatically cancel)

$$\begin{aligned} &\mathbb{E} \otimes \hat{P}^{(0,n)}_{0,d_n} \left[\{\hat{\tau}(+1) \leq l_0\} \cap \bigcap_{i=1}^{N} \{\tau_i \circ \theta_{\hat{\tau}(+1)} - \tau_{i-1} \circ \theta_{\hat{\tau}(+1)} \leq l_i\}, \mathcal{A}^n_n \right] \\ &\leq \sum_{t_0=d_n/2}^{l_0} \sum_{r_0=d_n/2}^{\zeta \log n} \mathbb{E} \otimes \hat{P}^{(0,t_0+r_0)}_{0,\partial D_n,d_n} \left[\hat{\tau}(+1) = t_0, R_1 \circ \theta_{t_0} = r_0, \mathcal{A}^n_{t_0+r_0} \right] \\ &\qquad \times \mathbb{E} \otimes \hat{P}^{(0,n-(t_0+r_0))}_{0,d_n} \Bigg[\\ &\qquad\qquad \{\hat{\tau}(-1) \leq l_1 - r_0\} \cap \bigcap_{i=2}^{N} \{\tau_{i-1} \circ \theta_{\hat{\tau}(-1)} - \tau_{i-2} \circ \theta_{\hat{\tau}(-1)} \leq l_i\}, \mathcal{A}^n_{n-(t_0+r_0)} \Bigg] \\ &\leq \Bigg(\sum_{t_0=d_n/2}^{l_0} \sum_{r_0=d_n/2}^{\zeta \log n} \mathbb{E} \left[\hat{P}^{(0,t_0+r_0)}_{0,\partial D_n,d_n} \left[\hat{\tau}(+1) = t_0, R_1 \circ \theta_{t_0} = r_0, \mathcal{A}^n_{t_0+r_0} \right] \hat{P}^{(0,n)}_{0,d_n} \left[\mathcal{A}^n_n \right] \right] \\ &\qquad + c_2 n^{1-\kappa\zeta} \Bigg) \times \sup_{d_n \leq t_0 \leq l_0 + \zeta \log n} \mathbb{E} \otimes \hat{P}^{(0,n-t_0)}_{0,d_n} \Bigg[\\ &\qquad\qquad \{\hat{\tau}(-1) \leq l_1\} \cap \bigcap_{i=2}^{N} \{\tau_{i-1} \circ \theta_{\hat{\tau}(-1)} - \tau_{i-2} \circ \theta_{\hat{\tau}(-1)} \leq l_i\}, \mathcal{A}^n_{n-t_0} \Bigg]. \end{aligned} \tag{3.11}$$

Note that the term under the supremum is of the same type as the one in the left-hand side of (3.11) but with $\hat{\tau}(+1)$ replaced by $\hat{\tau}(-1)$ due to a change of layer (recall (1.12)). To the first term on the right-hand side we can apply Lemma 3.1. Indeed, choose $b = t_0 + r_0$, to estimate

$$\begin{aligned} \hat{P}^{(0,n)}_{0,d_n}[\mathcal{A}^n_n] &\leq c_4 d_n^3\, \hat{P}^{(0,n)}_{0,d_n}[\mathcal{A}^n_n, S_{t_0+r_0} \in \partial D_n] \\ &\leq c_4 d_n^3\, \frac{\hat{Z}^{(0,t_0+r_0)}_{0,\partial D_n,d_n}(\omega) E_0\left[\exp\{H_{n-(t_0+r_0),d_n}(\theta_{t_0+r_0}\omega, S)\}, \mathcal{A}^n_{n-(t_0+r_0)}\right]}{\hat{Z}^{(0,n)}_{0,d_n}(\omega)}, \end{aligned} \tag{3.12}$$

which gives

$$\begin{aligned} &\sum_{t_0=d_n/2}^{l_0} \sum_{r_0=d_n/2}^{\zeta \log n} \mathbb{E}\left[\hat{P}^{(0,t_0+r_0)}_{0,\partial D_n,d_n}\left[\hat{\tau}(+1) = t_0, R_1 \circ \theta_{t_0} = r_0, \mathcal{A}^n_{t_0+r_0}\right] \hat{P}^{(0,n)}_{0,d_n}[\mathcal{A}^n_n]\right] \\ &\quad \leq c_4 d_n^3\, \mathbb{E} \otimes \hat{P}^{(0,n)}_{0,d_n}[\hat{\tau}(+1) \leq l_0, \mathcal{A}^n_n]. \end{aligned} \tag{3.13}$$

Next we consider the case $0 \notin I_N$. Define $k_0 = \inf\{k \in \mathbb{N}_0; k \in I_N\}$. For $k \in \mathbb{N}_0$, define $\bar{l}_k = \sum_{i=0}^k l_i$. Then, using (3.12), we obtain as in (3.11),

$$\begin{aligned} &\mathbb{E} \otimes \hat{P}^{(0,n)}_{0,d_n}\left[\{\hat{\tau}(+1) \leq l_0\} \cap \bigcap_{i=1}^N \{\tau_i \circ \theta_{\hat{\tau}(+1)} - \tau_{i-1} \circ \theta_{\hat{\tau}(+1)} \leq l_i\}, \mathcal{A}^n_n\right] \\ &\leq \left(\sum_{t=0}^{\bar{l}_{k_0-1}} \sum_{r=d_n/2}^{\zeta \log n} \mathbb{E}\left[\hat{P}^{(0,t+r)}_{0,\partial D_n,d_n}\left[\tau_{k_0-1} \circ \theta_{\hat{\tau}(+1)} = t, R_1 \circ \theta_{t_0} = r, \mathcal{A}^n_{t+r}\right] \hat{P}^{(0,n)}_{0,d_n}[\mathcal{A}^n_n]\right] \right. \\ &\qquad \left. + c_2 n^{1-\kappa\zeta}\right) \times \sup_{0 \leq t \leq \bar{l}_{k_0-1} + \zeta \log n} \mathbb{E} \otimes \hat{P}^{(0,n-t)}_{0,d_n}\Bigg[\\ &\{\hat{\tau}((-1)^{k_0}) \leq l_{k_0}\} \cap \bigcap_{i=k_0+1}^N \{\tau_{i-k_0} \circ \theta_{\hat{\tau}((-1)^{k_0})} - \tau_{i-k_0-1} \circ \theta_{\hat{\tau}((-1)^{k_0})} \leq l_i\}, \mathcal{A}^n_{n-t}\Bigg] \\ &\leq (c_4 d_n^3 + c_2 n^{2-\kappa\zeta}) \times \sup_{n-\bar{l}_{k_0-1}+\zeta \log n \leq n_0 \leq n} \mathbb{E} \otimes \hat{P}^{(0,n_0)}_{0,d_n}\Bigg[\\ &\qquad \{\hat{\tau}((-1)^{k_0}) \leq l_{k_0}\} \cap \bigcap_{i=k_0+1}^N \{\tau_{i-k_0} \circ \theta_{\hat{\tau}((-1)^{k_0})} - \tau_{i-k_0-1} \circ \theta_{\hat{\tau}((-1)^{k_0})} \leq l_i\}, \mathcal{A}^n_{n_0}\Bigg]. \end{aligned} \tag{3.14}$$

Iterating the above decoupling argument, we obtain the claim for $c_5 = 2c_4^2$. Note that $c_2 n^{2-\kappa\zeta} \leq c_4 d_n^3$ for large n because $\zeta > 2/\kappa$.

■

Lemma 3.3 *Assume that* $\lim_{n\to\infty} d_n = \infty$. *For all* $\zeta > 0$ *there exists* $c_4 > 0$ *such that for all*

$N \in \mathbb{N}$ and $l_i \in \mathbb{N}$ $(0 \le i \le N)$ the following is true as $n \to \infty$:

$$\mathbb{E} \otimes \hat{P}^{(0,n)}_{0,d_n} \left[\{\hat{\tau}(+1) \wedge n > l_0\} \cap \bigcap_{i=1}^{N} \{(\tau_i \circ \theta_{\hat{\tau}(+1)}) \wedge n - (\tau_{i-1} \circ \theta_{\hat{\tau}(+1)}) \wedge n > l_i\} \right] \tag{3.15}$$
$$\le c_2 n^{1-\kappa\zeta} + \prod_{i=0}^{N} \left(c_4 d_n^3 \, \mathbb{E} \otimes \hat{P}^{(0,n)}_{0,d_n} \left[\hat{\tau}\left((-1)^i\right) \wedge n > l_i - \zeta \log n, \mathcal{A}_n^n \right] + c_2 n^{1-\kappa\zeta} \right).$$

Proof of Lemma 3.3. The proof is similar to that of Lemma 3.2. Therefore we only indicate where the two proofs differ. Abbreviate $n_{t_0,r_0} = n - (t_0 + r_0)$. Then, as in (3.11), we have

$$\mathbb{E} \otimes \hat{P}^{(0,n)}_{0,d_n} \left[\{\hat{\tau}(+1) \wedge n > l_0\} \cap \bigcap_{i=1}^{N} \{(\tau_i \circ \theta_{\hat{\tau}(+1)}) \wedge n - (\tau_{i-1} \circ \theta_{\hat{\tau}(+1)}) \wedge n > l_i\}, \mathcal{A}_n^n \right]$$
$$\le \sum_{t_0=l_0+1}^{n} \sum_{r_0=d_n/2}^{\zeta \log n \wedge (n-t_0)} \mathbb{E} \otimes \hat{P}^{(0,t_0+r_0)}_{0,\partial D_n, d_n} \left[\hat{\tau}(+1) \wedge n = t_0, R_1 \circ \theta_{t_0} \wedge (n - t_0) = r_0, \mathcal{A}^n_{t_0+r_0} \right]$$
$$\times \sup_{\substack{l_0 \le t_0 \le n \\ d_n/2 \le r_0 \le \zeta \log n \wedge (n-t_0)}} \mathbb{E} \otimes \hat{P}^{(0,n_{t_0,r_0})}_{0,d_n} \Bigg[\{\hat{\tau}(-1) \wedge n_{t_0,r_0} > l_1 - r_0\} \cap \tag{3.16}$$
$$\bigcap_{i=2}^{N} \{(\tau_{i-1} \circ \theta_{\hat{\tau}(-1)}) \wedge n_{t_0,r_0} - (\tau_{i-2} \circ \theta_{\hat{\tau}(-1)}) \wedge n_{t_0,r_0} > l_i\}, \mathcal{A}^n_{n_{t_0,r_0}} \Bigg].$$

Now we can deduce the claim in the same way as for Lemma 3.2, using in addition that $\{\hat{\tau}(-1) \wedge k > l_1 - r_0\} \subset \{\hat{\tau}(-1) \wedge n > l_1 - \zeta \log n\}$ for $d_n/2 \le r_0 \le \zeta \log n$ and $0 \le k \le n$.

■

Lemmas 3.2–3.3 provide an upper bound for the probability that the lengths of the first N excursions from the middle line of a $(+1)$-layer do not exceed, respectively, exceed $l_1, \dots, l_N$, for N arbitrary. These bounds will be used in Section 5 to prove Theorem 1.4.

4 The first-passage time

In the previous section we have decoupled the excursions. In the present section we derive the key estimates that involve a single excursion. In Section 4.1 we look at a one-interface model with one neutral solvent, which plays a key role in the variational problem for $\chi(\lambda, h)$ in Theorem 1.4 that will be introduced in Section 4.2. In Section 4.3 we use this variational problem to derive upper and lower bounds for the first-passage time.

4.1 A one-interface model with one neutral solvent. For $m \in 2\mathbb{N}$ and $\sigma = \pm 1$, define (recall (2.13))

$$Y^{(0,m)}_{0,\partial D_n, d_n}(\omega, \sigma) = \frac{\exp\{\sigma\lambda \sum_{i=1}^{m} (\omega_i + h)\}}{\hat{Z}^{(0,m)}_{0,\partial D_n, d_n}(\omega)}. \tag{4.1}$$

Lemma 4.1 *For every* $(\lambda, h) \in \mathcal{L}$ *and* $\sigma = \pm 1$ *there exists a deterministic number* $\mu_\sigma(\lambda, h) \in (0, \infty)$ *such that, for every sequence* (d_n) *with* $\lim_{n\to\infty} d_n = \infty$,

$$\lim_{n\to\infty} \frac{1}{2n} \log \mathbb{E}\left[Y_{0,\partial D_n, d_n}^{(0,2n)}(\omega, \sigma)\right] = -\mu_\sigma(\lambda, h). \tag{4.2}$$

Proof of Lemma 4.1. For $m \in 2\mathbb{N}$ and $\sigma = \pm 1$, define

$$Y^{(0,m)}(\omega, \sigma) = \frac{1}{E_0\left[\exp\left\{\lambda \sum_{i=1}^{m} (\Delta(S_i) - \sigma)(\omega_i + h)\right\}, S_m = 0\right]}. \tag{4.3}$$

Note that the interaction is neutral for the σ-layers ($\sigma = \pm 1$). Using the folding argument from the proof of Theorem 1.3, we see that

$$2^{-2n/d_n} Y^{(0,2n)}(\omega, \sigma) \leq Y_{0,\partial D_n, d_n}^{(0,2n)}(\omega, \sigma) \leq 2^{2n/d_n} Y^{(0,2n)}(\omega, \sigma), \tag{4.4}$$

so it suffices to prove that

$$\lim_{n\to\infty} \frac{1}{2n} \log \mathbb{E}\left[Y^{(0,2n)}(\omega, \sigma)\right] = -\mu_\sigma(\lambda, h). \tag{4.5}$$

For $m, l \in 2\mathbb{N}$ we have, using the independence of ω on disjoint time intervals,

$$\begin{aligned}
&\log \mathbb{E}\left[Y^{(0,m+l)}(\omega, \sigma)\right] \\
&= \log \mathbb{E}\left[\frac{1}{E_0\left[\exp\left\{\lambda \sum_{i=1}^{m+l} (\Delta(S_i) - \sigma)(\omega_i + h)\right\}, S_{m+l} = 0\right]}\right] \\
&\leq \log \mathbb{E}\left[\frac{1}{E_0\left[\exp\left\{\lambda \sum_{i=1}^{m+l} (\Delta(S_i) - \sigma)(\omega_i + h)\right\}, S_m = S_{m+l} = 0\right]}\right] \\
&= \log \mathbb{E}\left[Y^{(0,m)}(\omega, \sigma)\right] + \log \mathbb{E}\left[Y^{(0,l)}(\omega, \sigma)\right].
\end{aligned} \tag{4.6}$$

Hence $m \mapsto \log \mathbb{E}\left[Y^{(0,m)}(\omega, \sigma)\right]$ is a subadditive sequence, which implies (4.5) with

$$-\mu_\sigma(\lambda, h) = \inf_{n \geq 1} \frac{1}{2n} \log \mathbb{E}\left[Y^{(0,2n)}(\omega, \sigma)\right]. \tag{4.7}$$

It remains to prove that $\mu_\sigma(\lambda, h) \in (0, \infty)$. Using Chebychev's inequality, we see that

$$\begin{aligned}
&\log \mathbb{E}\left[Y^{(0,2n)}(\omega, \sigma)\right] \\
&\geq -\log \mathbb{E}\left[E_0\left[\exp\left\{\lambda \sum_{i=1}^{2n} (\Delta(S_i) - \sigma)(\omega_i + h)\right\}, S_{2n} = 0\right]\right] \\
&\geq -4\lambda(1 + h)n,
\end{aligned} \tag{4.8}$$

so $\mu_\sigma(\lambda,h)\le 2\lambda(1+h)$. On the other hand,

$$\begin{aligned}
&\mathbb{E}\left[Y^{(0,2n)}(\omega,-1)+Y^{(0,2n)}(\omega,+1)\right]\\
&\quad = 2\mathbb{E}\left[\frac{(1/2)\exp\left\{-\lambda\sum_{i=1}^{2n}(\omega_i+h)\right\}+(1/2)\exp\left\{\lambda\sum_{i=1}^{2n}(\omega_i+h)\right\}}{E_0\left[\exp\left\{\lambda\sum_{i=1}^{2n}\Delta(S_i)(\omega_i+h)\right\},S_{2n}=0\right]}\right] \qquad (4.9)\\
&\quad = \frac{2}{P_0[T_1=2n]}\mathbb{E}\left[P_{0,0}^{(0,2n)}\left[T_1=2n\right]\right],
\end{aligned}$$

where $T_1=\inf\{k\in\mathbb{N}\colon\ S_k=0\}$ and $P_{0,0}^{(0,2n)}$ is the path measure defined as

$$P_{0,0}^{(0,2n)}(S)=\frac{1}{Z_{0,0}^{(0,2n)}}\exp\left\{\lambda\sum_{i=1}^{2n}\Delta(S_i)(\omega_i+h)\right\}1_{\{S_{2n}=0\}}P_0(S), \qquad (4.10)$$

with $Z_{0,0}^{(0,2n)}$ the normalizing partition sum. From Biskup and den Hollander [2], Lemma 4, we know that for every $(\lambda,h)\in\mathcal{L}$ there exists a $\kappa>0$ such that

$$\mathbb{E}\left[P_{0,0}^{(0,2n)}\left[T_1=2n\right]\right]\le\exp\left\{-2\kappa n\right\},\qquad n\to\infty. \qquad (4.11)$$

Moreover, we know that $P_0[T_1=2n]\ge(c_6 n)^{-3/2}$. Hence (4.9) yields

$$\begin{aligned}
\frac{1}{2n}\log\mathbb{E}\left[Y^{(0,2n)}(\omega,\sigma)\right] &\le \frac{1}{2n}\log\mathbb{E}\left[Y^{(0,2n)}(\omega,-1)+Y^{(0,2n)}(\omega,+1)\right]\\
&\le \frac{1}{2n}\left[\log 2+(3/2)\log(c_6 n)-2\kappa n\right],\qquad n\to\infty. \qquad (4.12)
\end{aligned}$$

So $\mu_\sigma(\lambda,h)\ge\kappa>0$.

■

4.2 Variational formula for $\chi(\lambda,h)$.

Lemma 4.2 *Assume that* $\lim_{n\to\infty}d_n=\infty$. *For* $y\ge 1$ *and every nonnegative sequence* (ε_n) *such that* $\lim_{n\to\infty}\varepsilon_n=0$,

$$\lim_{n\to\infty}\frac{1}{d_n}\log P_0\left[R_1\ge yd_n,\max_{1\le i\le yd_n}S_i\ge d_n/2,S_{yd_n}\le\varepsilon_n d_n\right]=-I(y), \qquad (4.13)$$

where

$$I(y)=\frac{y+1}{2}\log\frac{y+1}{y}+\frac{y-1}{2}\log\frac{y-1}{y}. \qquad (4.14)$$

Proof of Lemma 4.2. This is an elementary large deviation estimate for simple random walk, based on a combinatorial expression similar to (3.6). Indeed, $I(y)$ is y times the relative entropy of $\frac{y+1}{2y}\delta_{+1}+\frac{y-1}{2y}\delta_{-1}$ with respect to $\frac{1}{2}\delta_{+1}+\frac{1}{2}\delta_{-1}$.

■

We note that the rate function $y \mapsto I(y)$ is strictly decreasing with $\lim_{y \downarrow 1} I(y) = \log 2$ and $\lim_{y\to\infty} I(y) = 0$.

For $\varepsilon_1 \in (0, \mu_\sigma(\lambda, h))$, we next define

$$\chi_\sigma(\lambda, h, \varepsilon_1) = \min_{y \geq 1} \{y [\mu_\sigma(\lambda, h) - \varepsilon_1] + I(y)\}. \tag{4.15}$$

Let $y_\sigma(\varepsilon_1)$ denote the maximizer, i.e.,

$$\chi_\sigma(\lambda, h, \varepsilon_1) = y_\sigma(\varepsilon_1) [\mu_\sigma(\lambda, h) - \varepsilon_1] + I(y_\sigma(\varepsilon_1)). \tag{4.16}$$

We have

$$\begin{aligned} &\lim_{\varepsilon_1 \downarrow 0} \chi_\sigma(\lambda, h, \varepsilon_1) = \chi_\sigma(\lambda, h, 0) = \chi_\sigma(\lambda, h), \\ &\lim_{\varepsilon_1 \downarrow 0} y_\sigma(\varepsilon_1) = y_\sigma(0) = y_\sigma, \end{aligned} \tag{4.17}$$

and

$$\chi_\sigma(\lambda, h) = \min_{y \geq 1} \{y\mu_\sigma(\lambda, h) + I(y)\} \in [\mu_\sigma(\lambda, h), \mu_\sigma(\lambda, h) + \log 2]. \tag{4.18}$$

Define

$$\chi(\lambda, h) = \max \{\chi_{-1}(\lambda, h), \chi_{+1}(\lambda, h)\}. \tag{4.19}$$

The quantity $\chi(\lambda, h)$ will be analyzed in Section 6.

4.3 First-passage time. In this section we derive upper and lower bounds for the first-passage time involving $\chi(\lambda, h)$ (Lemmas 4.3–4.4). It is now that conditions (1.16)(I) and (1.16)(II) come into play.

Lemma 4.3 *Assume* (1.16)(I). *For all* $\varepsilon_2 > 0$ *and* $l \in \mathbb{N}$,

$$\mathbb{E} \otimes \hat{P}_{0,d_n}^{(0,n)} [\hat{\tau}(\sigma) \leq l, \mathcal{A}_n^n] \leq l \exp\{-\chi_\sigma(\lambda, h) d_n + \varepsilon_2 d_n\}, \qquad n \to \infty. \tag{4.20}$$

Proof of Lemma 4.3. We borrow an argument from the proof of Lemma 6.2 in Albeverio and Zhou [1]. For $\sigma = \pm 1$ and $\varepsilon \in (0, 1/2)$, we estimate (recall (1.12))

$$\begin{aligned} &\mathbb{E} \otimes \hat{P}_{0,d_n}^{(0,n)} [\hat{\tau}(\sigma) \leq l, \mathcal{A}_n^n] \\ &\leq \sum_{k=0}^{l-1} \sum_{j=\lfloor \frac{1}{\varepsilon}+1 \rfloor}^{\lceil \frac{\zeta \log n}{\varepsilon d_n} - 1 \rceil} \sum_{p=0,1} \mathbb{E} \otimes \hat{P}_{0,d_n}^{(0,n)} \Big[S_k = p(-\sigma d_n), R_1 \circ \theta_k \in [(j-1)\varepsilon d_n, (j+1)\varepsilon d_n], \\ &\qquad\qquad \max_{1 \leq i \leq R_1 \circ \theta_k} (-1)^p \sigma S_{k+i} \geq d_n/2 + p d_n, \mathcal{A}_n^n \Big]. \end{aligned} \tag{4.21}$$

Let us first consider the case $p = 0$ and $\sigma = 1$. We estimate

$$\begin{aligned}
&\mathbb{E} \otimes \hat{P}^{(0,n)}_{0,d_n}\Big[S_k = 0, R_1 \circ \theta_k \in [(j-1)\varepsilon d_n, (j+1)\varepsilon d_n], \max_{1 \le i \le R_1 \circ \theta_k} S_{k+i} \ge d_n/2, \mathcal{A}_n^n\Big] \\
&= 2\ \mathbb{E} \otimes \hat{P}^{(0,n)}_{0,d_n}\Big[S_k = 0, R_1 \circ \theta_k \in [(j-1)\varepsilon d_n, (j+1)\varepsilon d_n], \\
&\qquad\qquad \max_{1 \le i \le R_1 \circ \theta_k} S_{k+i} \ge d_n/2, S_{k+R_1\circ\theta_k} = 0, \mathcal{A}_n^n\Big] \\
&\le 2\ \mathbb{E} \otimes \hat{P}^{(0,n)}_{0,d_n}\Big[S_k = 0, R_1 \circ \theta_k \in [(j-1)\varepsilon d_n, (j+1)\varepsilon d_n], \\
&\qquad\qquad \max_{1 \le i \le (j-1)\varepsilon d_n} S_{k+i} \ge d_n/2, S_{k+(j-1)\varepsilon d_n} \le 2\varepsilon d_n, \mathcal{A}_n^n\Big] \\
&\le 2 \sum_{m=\lfloor (j-1)\varepsilon d_n \rfloor}^{\lfloor (j+1)\varepsilon d_n \rfloor} \mathbb{E}\left[Y^{(0,m)}_{0,\partial D_n, d_n}(\omega, +1)\right] \\
&\qquad\qquad \times P_0\left[R_1 \ge (j-1)\varepsilon d_n, \max_{1 \le i \le (j-1)\varepsilon d_n} S_i \ge d_n/2, S_{(j-1)\varepsilon d_n} \le 2\varepsilon d_n\right].
\end{aligned} \tag{4.22}$$

In the last line we recall (4.1) and use that after time k the path returns to the interface for the first time at time $k+m$ (the inequality is uniform in k). The cases $p = 1$ and/or $\sigma = -1$ are analogous. Inserting the estimates into (4.21), we obtain

$$\begin{aligned}
&\mathbb{E} \otimes \hat{P}^{(0,n)}_{0,d_n}\left[\hat{\tau}\,(\sigma) \le l, \mathcal{A}_n^n\right] \\
&\quad \le 4 \sum_{k=0}^{l-1} \sum_{j=\lfloor \frac{1}{\varepsilon}+1 \rfloor}^{\lceil \frac{\zeta \log n}{\varepsilon d_n} - 1 \rceil} \sum_{m=\lfloor (j-1)\varepsilon d_n \rfloor}^{\lfloor (j+1)\varepsilon d_n \rfloor} \mathbb{E}\left[Y^{(0,m)}_{0,\partial D_n, d_n}(\omega, \sigma)\right] \\
&\quad\quad \times P_0\left[R_1 \ge (j-1)\varepsilon d_n, \max_{1 \le i \le (j-1)\varepsilon d_n} S_i \ge d_n/2, S_{(j-1)\varepsilon d_n} \le 2\varepsilon d_n\right].
\end{aligned} \tag{4.23}$$

Next we use Lemmas 4.1–4.2. Pick $\varepsilon_2 > 0$, and pick $\varepsilon_1 \in (0, \mu_\sigma(\lambda, h))$ so small that $|\chi_\sigma(\lambda, h, \varepsilon_1) - \chi_\sigma(\lambda, h, 0)| \le \varepsilon_2/3$. Furthermore, pick $\varepsilon = \varepsilon_n$ such that $\varepsilon_n \to 0$ and $\varepsilon_n d_n \to \infty$ as $n \to \infty$. Then

$$\begin{aligned}
&\mathbb{E} \otimes \hat{P}^{(0,n)}_{0,d_n}\left[\hat{\tau}\,(\sigma) \le l, \mathcal{A}_n^n\right] \\
&\le 4 \sum_{k=0}^{l-1} \sum_{j=\lfloor \frac{1}{\varepsilon_n}+1 \rfloor}^{\lceil \frac{\zeta \log n}{\varepsilon_n d_n} - 1 \rceil} (2\varepsilon_n d_n + 1) \exp\Big\{-(j-1)\varepsilon_n d_n \left[\mu_\sigma(\lambda, h) - \varepsilon_1\right] - d_n I\left((j-1)\varepsilon_n\right) + d_n \varepsilon_2/3\Big\} \\
&\le 4 \sum_{k=0}^{l-1} \sum_{j=\lfloor \frac{1}{\varepsilon_n}+1 \rfloor}^{\lceil \frac{\zeta \log n}{\varepsilon d_n} - 1 \rceil} 3\varepsilon_n d_n \exp\left\{-d_n \chi_\sigma(\lambda, h, \varepsilon_1) + d_n \varepsilon_2/3\right\} \\
&\le 12 l\, \zeta \log n\, \exp\left\{-d_n \chi_\sigma(\lambda, h, 0) + d_n 2\varepsilon_2/3\right\}, \qquad n \to \infty.
\end{aligned} \tag{4.24}$$

■

For $\delta > 0$, define

$$t_n = \exp\{\delta d_n\}. \tag{4.25}$$

To prove the next lemma, we chop our time horizon n into intervals of length t_n.

Lemma 4.4 *Assume* (1.16)(I) *and* (1.16)(II). *For all* $\varepsilon_3 > 0$ *and* $l \in \mathbb{N}$,

$$\mathbb{E} \otimes \hat{P}^{(0,n)}_{0,d_n} [\hat{\tau}(\sigma) \wedge n > lt_n, \mathcal{A}^n_n] \leq \big(1 - \exp\{-d_n \chi_\sigma(\lambda, h) - d_n \varepsilon_3\}\big)^l, \qquad n \to \infty. \tag{4.26}$$

Proof of Lemma 4.4. Throughout the proof we assume that n is large enough. For $x, y \in \mathbb{Z}$, define (recall (1.9))

$$\hat{P}^{(0,n)}_{x,y,d_n}(S)(\omega) = \hat{P}^{(0,n)}_{x,d_n}(S \mid S_{t_n} = y)(\omega). \tag{4.27}$$

For $l \in \mathbb{N}$ we have (putting $x_0 = 0$)

$$\begin{aligned}
&\mathbb{E} \otimes \hat{P}^{(0,n)}_{0,d_n} [\hat{\tau}(\sigma) \wedge n > lt_n, \mathcal{A}^n_n] \\
&\quad = \mathbb{E} \otimes \hat{P}^{(0,n)}_{0,d_n} \left[\bigcap_{k=0}^{l-1} \left\{ \max_{1 \leq i \leq t_n} |\sigma S_{kt_n+i} + d_n/2| < d_n \right\}, \mathcal{A}^n_n \right] \\
&\quad \leq \mathbb{E} \Bigg[\frac{1}{\hat{Z}^{(0,n)}_{0,d_n}(\omega)} \sum_{x_1,\dots,x_l:\ |\sigma x_i + d_n/2| < d_n\ \forall 1 \leq i \leq l} \ \prod_{k=0}^{l-1} \\
&\qquad \left(\hat{P}^{(0,t_n)}_{x_k,x_{k+1},d_n} \left[\max_{1 \leq i \leq t_n} |\sigma S_i + d_n/2| < d_n, \mathcal{A}^n_n \right] \hat{Z}^{(0,t_n)}_{x_k,x_{k+1},d_n} \right) (\theta_{kt_n}\omega) \hat{Z}^{(0,n-lt_n)}_{x_l,d_n}(\theta_{lt_n}\omega) \Bigg] \\
&\quad \leq \left(\mathbb{E} \left[\max_{x,y:\ |\sigma x + d_n/2|, |\sigma y + d_n/2| < d_n} \hat{P}^{(0,t_n)}_{x,y,d_n} \left[\max_{1 \leq i \leq t_n} |\sigma S_i + d_n/2| < d_n, \mathcal{A}^n_n \right] (\omega) \right] \right)^l .
\end{aligned} \tag{4.28}$$

To estimate the right-hand side, we fix x, y such that $|\sigma x + d_n/2| < d_n$ and $|\sigma y + d_n/2| < d_n$. Define $\bar{P}^{(0,t_n)}_{x,y,d_n}$ to be the following measure for the random walk S:

$$\bar{P}^{(0,t_n)}_{x,y,d_n}(S)(\omega) = \hat{P}^{(0,t_n)}_{x,d_n} \left(S \;\middle|\; \mathcal{A}^n_{t_n}, \max_{1 \leq i \leq t_n} |\sigma S_i + d_n/2| < 3d_n/2, S_{t_n} = y \right)(\omega), \tag{4.29}$$

i.e., the path is conditioned to start at x at time 0, to end at y at time t_n, to stay inside the height interval $(-3d_n/2, d_n/2)$ and to not make excursions longer than $\zeta \log t_n$. Then

$$\begin{aligned}
\hat{P}^{(0,t_n)}_{x,y,d_n} \left[\max_{1 \leq i \leq t_n} |\sigma S_i + d_n/2| < d_n, \mathcal{A}^n_n \right](\omega) &\leq \bar{P}^{(0,t_n)}_{x,y,d_n} \left[\max_{1 \leq i \leq t_n} |\sigma S_i + d_n/2| < d_n \right](\omega) \\
&= 1 - \bar{P}^{(0,t_n)}_{x,y,d_n} \left[\max_{1 \leq i \leq t_n} |\sigma S_i + d_n/2| \geq d_n \right](\omega).
\end{aligned} \tag{4.30}$$

Our goal is to estimate the last term on the right-hand side.

Let

$$\begin{aligned}
I_n &= [t_n/2, t_n/2 + 3\zeta \log n], \\
\mathcal{C}_{I_n} &= \{S\colon \text{ there are no } i_1, i_2, i_3 \in I_n \text{ with } i_1 < i_2 < i_3 \text{ such that} \\
&\qquad\quad S_{i_1} \in \partial D_n,\ |S_{i_2} - S_{i_1}| = d_n/2,\ |S_{i_3} - S_{i_2}| = d_n/2\},
\end{aligned} \tag{4.31}$$

i.e., $\mathcal{C}_{I_n}$ is the event that there are no two half-crossings of a layer in the time interval I_n. Define $s_n = 2\lfloor y_\sigma d_n/2 \rfloor$ (recall (4.17)) and note that $s_n < \zeta \log n$ for large n by (1.16)(II). Define

$$\alpha_1(n) = t_n/2 + \zeta \log n, \qquad \alpha_2(n) = t_n/2 + \zeta \log n + s_n, \tag{4.32}$$

and note that $[\alpha_1(n)-\zeta\log n, \alpha_2(n)+\zeta\log n]\subset I_n\subset[0,t_n]$ for large n. We have

$$\begin{aligned}
&\bar P^{(0,t_n)}_{x,y,d_n}\left[\max_{1\le i\le t_n}|\sigma S_i+d_n/2|\ge d_n\right](\omega)\\
&\ge \bar P^{(0,t_n)}_{x,y,d_n}\left[\max_{1\le i\le t_n}|\sigma S_i+d_n/2|\ge d_n\,\middle|\, S_{\alpha_1(n)}=S_{\alpha_2(n)}\in\{0,-\sigma d_n\}\right](\omega) \qquad (4.33)\\
&\qquad\times \bar P^{(0,t_n)}_{x,y,d_n}\left[S_{\alpha_1(n)}=S_{\alpha_2(n)}\in\{0,-\sigma d_n\}\,\middle|\,\mathcal{C}_{I_n}\right](\omega)\ \bar P^{(0,t_n)}_{x,y,d_n}\left[\mathcal{C}_{I_n}\right](\omega).
\end{aligned}$$

Let us first look at the second term on the right-hand side of (4.33), which we write as

$$\begin{aligned}
&\bar P^{(0,t_n)}_{x,y,d_n}\left[S_{\alpha_1(n)}=S_{\alpha_2(n)}\in\{0,-\sigma d_n\}\,\middle|\,\mathcal{C}_{I_n}\right](\omega)\\
&=\bar P^{(0,t_n)}_{x,y,d_n}\left[S_{\alpha_1(n)}=S_{\alpha_2(n)}\,\middle|\,S_{\alpha_1(n)}\in\{0,-\sigma d_n\},\mathcal{C}_{I_n}\right](\omega)\,\bar P^{(0,t_n)}_{x,y,d_n}\left[S_{\alpha_1(n)}\in\{0,-\sigma d_n\}\,\middle|\,\mathcal{C}_{I_n}\right](\omega).
\end{aligned} \tag{4.34}$$

To the second term on the right-hand side of (4.34) we can apply the same argument as in the proof of Lemma 3.1, to obtain

$$\begin{aligned}
&E^{(0,t_n)}_{x,y,d_n}\left[\exp\{H_{t_n}(\omega,S)\},\max_{1\le i\le t_n}|\sigma S_i+d_n/2|<3d_n/2,\mathcal{A}_{t_n},\mathcal{C}_{I_n}\right]\\
&\le c_4d_n^3\,E^{(0,t_n)}_{x,y,d_n}\Big[\exp\{H_{t_n}(\omega,S)\},S_{\alpha_1(n)}\in\{0,-\sigma d_n\},\\
&\qquad\qquad \max_{1\le i\le t_n}|\sigma S_i+d_n/2|<3d_n/2,\mathcal{A}_{t_n},\mathcal{C}_{I_n}\Big].
\end{aligned} \tag{4.35}$$

Here we use the event $\mathcal{C}_{I_n}$ to avoid having to do the first step of (3.5), since this step does not apply when the endpoint of the path is fixed. Dividing the two sides of (4.35), we obtain

$$\bar P^{(0,t_n)}_{x,y,d_n}\left[S_{\alpha_1(n)}\in\{0,-\sigma d_n\}\,\middle|\,\mathcal{C}_{I_n}\right](\omega)\ge\left(c_4d_n^3\right)^{-1}. \tag{4.36}$$

The first term on the right-hand side of (4.34) we treat in a similar way. Combining the two estimates, we obtain

$$\bar P^{(0,t_n)}_{x,y,d_n}\left[S_{\alpha_1(n)}=S_{\alpha_2(n)}\in\{0,-\sigma d_n\}\,\middle|\,\mathcal{C}_{I_n}\right](\omega)\ge\left(c_4d_n^3\right)^{-2}. \tag{4.37}$$

Let us next look at the first term on the right-hand side of (4.33). This term we can estimate by

$$\begin{aligned}
&\bar P^{(0,t_n)}_{x,y,d_n}\left[\max_{1\le i\le t_n}|\sigma S_i+d_n/2|\ge d_n\,\middle|\,S_{\alpha_1(n)}=S_{\alpha_2(n)}\in\{0,-\sigma d_n\}\right](\omega)\\
&\ge\frac12 Y^{(0,s_n)}_{0,\partial D_n,d_n}(\theta_{\alpha_1(n)}\omega,\sigma)\,P_0\left[\max_{1\le i\le s_n}S_i\ge d_n/2,R_1=s_n,S_{R_1}=0\right].
\end{aligned} \tag{4.38}$$

Inserting (4.30), (4.33), (4.37–4.38) into the right-hand side of (4.29) and using Lemma 4.5 below, we see that for $\varepsilon_3>0$

$$\begin{aligned}
&\mathbb{E}\left[\min_{x,y:\ |\sigma x+d_n/2|,|\sigma y+d_n/2|<d_n}\bar P^{(0,t_n)}_{x,y,d_n}\left[\max_{1\le i\le t_n}|\sigma S_i+d_n/2|\ge d_n\right](\omega)\right]\\
&\ge e^{-d_n\varepsilon_3/4}\left(c_4d_n^3\right)^{-2}\mathbb{E}\left[Y^{(0,s_n)}_{0,\partial D_n,d_n}(\omega,\sigma)\right]P_0\left[\max_{1\le i\le s_n}S_i\ge d_n/2,R_1=s_n,S_{R_1}=0\right].
\end{aligned} \tag{4.39}$$

Finally, use Lemmas 4.1–4.2. Then we have for large n (recall that $s_n = 2\lfloor y_\sigma d_n/2 \rfloor$),

$$\begin{aligned}
&\mathbb{E}\left[\min_{x,y:\ |\sigma x + d_n/2|, |\sigma y + d_n/2| < d_n} \bar{P}^{(0,t_n)}_{x,y,d_n}\left[\max_{1\le i\le t_n} |\sigma S_i + d_n/2| \geq d_n\right](\omega)\right] \\
&\quad \geq \left(c_4 d_n^3\right)^{-2} \exp\Big\{ - d_n \left[y_0(0)\mu_\sigma(\lambda,h) + I(y_0(0))\right] - d_n(3\varepsilon_3/4)\Big\} \\
&\quad \geq \exp\left\{-d_n\chi_\sigma(\lambda,h) - d_n\varepsilon_3\right\}.
\end{aligned} \tag{4.40}$$

■

4.4 Two inequalities. In the proof of Lemma 4.4 we have used the following:

Lemma 4.5 *Let* $(\lambda,h)\in\mathcal{L}$, *and assume* (1.16)(I) *and* (1.16)(II). *Let* $\bar{P}^{(0,t_n)}_{x,y,d_n}(\omega)$ *be the path measure defined in* (4.29). *Then for* $\sigma = \pm 1$*:*
(i)

$$\begin{aligned}
&\mathbb{E}\left[Y^{(0,n)}_{0,\partial D_n,d_n}(\omega,\sigma) \min_{x,y:\ |\sigma x + d_n/2|, |\sigma y + d_n/2| < d_n} \bar{P}^{(0,t_n)}_{x,y,d_n}\left[\mathcal{C}_{I_n}\right](\omega)\right] \\
&\quad \geq \mathbb{E}\left[Y^{(0,n)}_{0,\partial D_n,d_n}(\sigma)\right] \mathbb{E}\left[\min_{x,y:\ |\sigma x + d_n/2|, |\sigma y + d_n/2| < d_n} \bar{P}^{(0,t_n)}_{x,y,d_n}\left[\mathcal{C}_{I_n}\right](\omega)\right].
\end{aligned} \tag{4.41}$$

(ii)

$$\mathbb{E}\left[\min_{x,y:\ |\sigma x + d_n/2|, |\sigma y + d_n/2| < d_n} \bar{P}^{(0,t_n)}_{x,y,d_n}\left[\mathcal{C}_{I_n}\right](\omega)\right] \geq \exp\{-\varepsilon d_n\} \qquad \forall\, \varepsilon > 0,\ n \geq n_0(\varepsilon). \tag{4.42}$$

Proof of Lemma 4.5. (i) We will prove that

$$\begin{aligned}
&\text{(a)}\quad \omega \mapsto Y^{(0,n)}_{0,\partial D_n,d_n}(\omega,\sigma), \\
&\text{(b)}\quad \omega \mapsto \min_{x,y:\ |\sigma x + d_n/2|, |\sigma y + d_n/2| < d_n} \bar{P}^{(0,t_n)}_{x,y,d_n}\left[\mathcal{C}_{I_n}\right](\omega),
\end{aligned} \tag{4.43}$$

are both non-decreasing when $\sigma = +1$ and both non-increasing when $\sigma = -1$. The claim will then follow from the FKG-inequality applied to $\mathbb{P}$ (see Fortuin, Kasteleyn and Ginibre [4]).

We give the proof for $\sigma = +1$. The proof for $\sigma = -1$ is analogous.

(a) Fix $1 \le j \le n$. Let ω, ω' be such that $\omega_i = \omega'_i$ for $1 \le i \le n$ with $i \neq j$ and $\omega_j = -1, \omega'_j = +1$. We have from (4.1) that

$$Y^{(0,2n)}_{0,\partial D_n,d_n}(\omega',+1) \geq Y^{(0,2n)}_{0,\partial D_n,d_n}(\omega,+1) \tag{4.44}$$

if and only

$$\hat{Z}^{(0,2n)}_{0,\partial D_n,d_n}(\omega') \leq e^{2\lambda}\hat{Z}^{(0,2n)}_{0,\partial D_n,d_n}(\omega). \tag{4.45}$$

With the help of the relation (recall (2.1))

$$H_{2n,d_n}(\omega',S) = H_{2n,d_n}(\omega,S) + 2\lambda\Delta_{d_n}(S_j), \tag{4.46}$$

the inequality in (4.45) amounts to (recall (2.13))

$$\begin{aligned} &E_0\Big[\exp\{H_{2n,d_n}(\omega',S)\}, S_{2n}\in\partial D_n\Big] \\ &\qquad \le E_0\Big[\exp\{H_{2n,d_n}(\omega',S)\}\, e^{2\lambda[1-\Delta_{d_n}(S_j)]}, S_{2n}\in\partial D_n\Big], \end{aligned} \tag{4.47}$$

which is trivially true because $\Delta_{d_n}(S_j)\in\{-1,+1\}$.

(b) Let

$$\mathcal{B}_n^{x,y} = \big\{S\colon\ S_0=x,\ S_{t_n}=y,\ S_i\in(-3d_n/2,d_n/2)\ \forall\, 1\le i\le t_n\big\}\cap\mathcal{A}_{t_n}^n. \tag{4.48}$$

Then

$$\bar P_{x,y,d_n}^{(0,t_n)}[\mathcal{C}_{I_n}](\omega) = \frac{\sum_{S\in\mathcal{C}_{I_n}\cap\mathcal{B}_n^{x,y}}\exp\{H_{t_n,d_n}(\omega,S)\}}{\sum_{S\in\mathcal{B}_n^{x,y}}\exp\{H_{t_n,d_n}(\omega,S)\}}. \tag{4.49}$$

Fix $x,y\in(-3d_n/2,d_n/2)$. Pick ω,ω' as in the proof of (a). Then

$$\bar P_{x,y,d_n}^{(0,t_n)}[\mathcal{C}_{I_n}](\omega') \ge \bar P_{x,y,d_n}^{(0,t_n)}[\mathcal{C}_{I_n}](\omega) \tag{4.50}$$

if and only if

$$\sum_{S_1\in\mathcal{B}_n^{x,y}}\ \sum_{S_2\in\mathcal{B}_n^{x,y}} \rho(S_1)(\omega)\rho(S_2)(\omega)\ f(S_1)[g(S_1)-g(S_2)]\ge 0, \tag{4.51}$$

where we abbreviate

$$\rho(S)(\omega) = \frac{\exp\{H_{t_n,d_n}(\omega,S)\}}{\sum_{S\in\mathcal{B}_n^{x,y}}\exp\{H_{t_n,d_n}(\omega,S)\}}, \qquad S\in\mathcal{B}_n^{x,y}, \tag{4.52}$$

and

$$f(S)=1\{S\in\mathcal{C}_{I_n}\}, \qquad g(S)=e^{2\lambda\Delta_{d_n}(S_j)}, \qquad S\in\mathcal{B}_n^{x,y}. \tag{4.53}$$

Here we have again used (4.46). What (4.51) says is that under the probability measure $\rho(\omega)$ the functions f and g are positively correlated:

$$\rho(\omega)[fg]\ge\rho(\omega)[f]\rho(\omega)[g]. \tag{4.54}$$

We will prove (4.54) with the help of the FKG-inequality. In order to do so, we need a partial ordering on paths. To achieve this, we first reflect paths in the middle line at height $-d_n/2$. To that end we rewrite (4.54) as

$$\tilde\rho(\omega)[fg]\ge\tilde\rho(\omega)[f]\tilde\rho(\omega)[g] \tag{4.55}$$

with

$$\tilde\rho(S)(\omega) = \frac{\exp\{H_{t_n,d_n}(\omega,S)\}2^{N(S)}}{\sum_{S\in\tilde{\mathcal{B}}_n^{x,y}}\exp\{H_{t_n,d_n}(\omega,S)\}2^{N(S)}}, \qquad S\in\tilde{\mathcal{B}}_n^{x,y}, \tag{4.56}$$

where

$$\begin{aligned} \tilde{\mathcal{B}}_n^{x,y} &= \big\{S\colon\ S_0=x,\ S_{t_n}=y,\ S_i\in[-d_n/2,d_n/2)\ \forall\,1\le i\le t_n\big\}\cap\mathcal{A}_{t_n}^n, \\ N(S) &= \sum_{i=1}^{t_n} 1\{S_i=-d_n/2\}. \end{aligned} \tag{4.57}$$

Here we use that f, g are invariant under the reflection (recall (4.31) and the symmetry of the medium with respect to middle lines), and now also $x, y \in [-d_n/2, d_n/2)$.

On the set $\tilde{\mathcal{B}}_n^{x,y}$ there is a natural partial ordering:

$$S_1 \le S_2 \qquad \text{if and only if} \qquad [S_1]_i \le [S_2]_i \; \forall\, 1 \le i \le t_n. \tag{4.58}$$

Let $S_1 \vee S_2$ and $S_1 \wedge S_2$ denote the pointwise maximum, respectively, pointwise minimum of S_1 and S_2. Then

$$\begin{aligned} H_{t_n,d_n}(\omega, S_1 \vee S_2) + H_{t_n,d_n}(\omega, S_1 \wedge S_2) &= H_{t_n,d_n}(\omega, S_1) + H_{t_n,d_n}(\omega, S_2), \\ N(S_1 \vee S_2) + N(S_1 \wedge S_2) &= N(S_1) + N(S_2), \end{aligned} \tag{4.59}$$

because, for each i, either $[S_1 \vee S_2]_i = [S_1]_i$ and $[S_1 \wedge S_2]_i = [S_2]_i$ or vice versa. Consequently,

$$\tilde{\rho}(\omega)(S_1 \vee S_2)\tilde{\rho}(\omega)(S_1 \wedge S_2) = \tilde{\rho}(\omega)(S_1)\tilde{\rho}(\omega)(S_2), \qquad \forall\, S_1, S_2, \tag{4.60}$$

i.e., $\tilde{\rho}(\omega)$ satisfies the convexity condition needed for the FKG-inequality.

Now, both $S \mapsto f(S)$ and $S \mapsto g(S)$ are non-decreasing in the partial ordering defined by (4.58). Hence we conclude that (4.55) indeed holds, and therefore also (4.50). Since x, y were fixed arbitrarily, the same is true when in (4.50) we take the minimum over x, y. Since ω was fixed arbitrarily, this completes the proof of (b) in (4.43).

(ii) We give the proof for $\sigma = +1$. The proof for $\sigma = -1$ is analogous. We will prove that

$$\begin{aligned} &\text{(a)} \quad \min_{x,y \in (-3d_n/2, d_n/2)} \bar{P}^{(0,t_n)}_{x,y,d_n}\left[\mathcal{C}_{I_n}\right](\omega) \ge \min_{x,y \in [-d_n,0]} \bar{P}^{(0,t_n),I_n}_{x,y,d_n}\left[\mathcal{C}_{I_n}\right](\omega) \qquad \forall\, \omega, \\ &\text{(b)} \quad \mathbb{E}\left[\min_{x,y \in [-d_n,0]} \bar{P}^{(0,t_n),I_n}_{x,y,d_n}\left[\mathcal{C}_{I_n}\right](\omega)\right] \ge \exp\{-\varepsilon d_n\} \qquad \forall\, \varepsilon > 0,\ n \ge n_0(\varepsilon), \end{aligned} \tag{4.61}$$

where

$$\bar{P}^{(0,t_n),I_n}_{x,y,d_n}(S)(\omega) = \hat{P}^{(0,t_n),I_n}_{x,d_n}\left(S \mid \mathcal{A}^n_{t_n},\, S_i \in [-d_n, 0]\; \forall\, 1 \le i \le t_n,\, S_{t_n} = y\right)(\omega) \tag{4.62}$$

with $\hat{P}^{(0,t_n),I_n}_{x,d_n}$ the same probability measure as in (1.9) but with the interaction "switched off" outside I_n, i.e., with (2.1) replaced by

$$H^{I_n}_{t_n,d_n}(\omega, S) = \lambda \sum_{i \in I_n} \Delta_{d_n}(S_i)(\omega_i + h). \tag{4.63}$$

(a) By (4.50), the left-hand side of (4.61)(a) is non-decreasing in ω. Therefore we get a lower bound by putting $\omega_i = -1$ for all $1 \le i \le n$ except $i \in I_n$. Hence

$$\bar{P}^{(0,t_n)}_{x,y,d_n}\left[\mathcal{C}_{I_n}\right](\omega) \ge \frac{\sum_{S \in \mathcal{C}_{I_n} \cap \mathcal{B}^{x,y}_n} \exp\{H^{I_n}_{t_n,d_n}(\omega, S) - 2\lambda(1-h)N^{I_n}_{+}(S)\}}{\sum_{S \in \mathcal{B}^{x,y}_n} \exp\{H^{I_n}_{t_n,d_n}(\omega, S) - 2\lambda(1-h)N^{I_n}_{+}(S)\}}, \tag{4.64}$$

where we recall (4.48) and define

$$N_+^{I_n}(S) = \sum_{1\le i\le t_n,\, i\notin I_n} 1\{\Delta_{d_n}(S_i) = +1\} \tag{4.65}$$

to be the number of bonds in the path over the time interval $(0,t_n)\setminus I_n$ that fall in a $(+1)$-layer. Next, we do the reflection in the middle line at height $-d_n/2$, which gives

$$\text{r.h.s. } (4.64) = \tilde\rho(\mathcal{C}_{I_n})(\omega) \tag{4.66}$$

with

$$\tilde\rho(S) = \frac{\exp\{H_{t_n,d_n}^{I_n}(\omega,S) - 2\lambda(1-h)N_+^{I_n}(S)\}2^{N(S)}}{\sum_{S\in\tilde{\mathcal{B}}_n^{x,y}}\exp\{H_{t_n,d_n}^{I_n}(\omega,S) - 2\lambda(1-h)N_+^{I_n}(S)\}2^{N(S)}}, \qquad S\in\tilde{\mathcal{B}}_n^{x,y}, \tag{4.67}$$

where we recall (4.57).

Our next step is to remove the $N_+^{I_n}(S)$ with the help of the Holley-inequality (see Holley [8]). To that end, let

$$\mathcal{K}_n = \{S\colon\ N_+^{I_n}(S) = 0\} \tag{4.68}$$

and define

$$\hat\rho(S) = \frac{1\{S\in\mathcal{K}_n\}\ \exp\{H_{t_n,d_n}^{I_n}(\omega,S)\}\,2^{N(S)}}{\sum_{S\in\tilde{\mathcal{B}}_n^{x,y}} 1\{S\in\mathcal{K}_n\}\ \exp\{H_{t_n,d_n}^{I_n}(\omega,S)\}\,2^{N(S)}}, \qquad S\in\tilde{\mathcal{B}}_n^{x,y}. \tag{4.69}$$

We observe that $\tilde\rho$ is stochastically larger than $\hat\rho$ in the partial ordering defined by (4.58), i.e.,

$$\tilde\rho(S_1\vee S_2)\hat\rho(S_1\wedge S_2) \ge \tilde\rho(S_1)\hat\rho(S_2), \qquad \forall\, S_1,S_2. \tag{4.70}$$

Indeed, if $S_2\in\mathcal{K}_n$, then $S_1\wedge S_2\in\mathcal{K}_n$ and $N_+^{I_n}(S_1\vee S_2) = N_+^{I_n}(S_1)$. Together with (4.59), this proves (4.70). Since $S\mapsto 1\{S\in\mathcal{C}_{I_n}\}$ is non-decreasing in the partial ordering, as was noted below (4.60), it follows from the Holley-inequality that

$$\tilde\rho(\mathcal{C}_{I_n}) \ge \hat\rho(\mathcal{C}_{I_n}). \tag{4.71}$$

Finally, we undo the reflection by removing the weight factor $2^{N(S)}$, to obtain

$$\hat\rho(\mathcal{C}_{I_n}) = \frac{\sum_{S\in\mathcal{C}_{I_n}\cap\mathcal{B}_n^{x,y}\cap\mathcal{K}_n}\exp\{H_{t_n,d_n}^{I_n}(\omega,S)\}}{\sum_{S\in\mathcal{B}_n^{x,y}\cap\mathcal{K}_n}\exp\{H_{t_n,d_n}^{I_n}(\omega,S)\}}, \tag{4.72}$$

which is equal to the right-hand side of (4.61)(a).

(b) The effect of "switching off" the interaction outside I_n is that the path measure in $(0,t_n)\setminus I_n$ is that of simple random walk. As we will see shortly, this fact will allow us to control the conditioning that appears in (4.62).

Recall (4.31). Define

$$\begin{array}{lcl}\mathcal{D}^1_{t_n} &=& \{S\colon \exists i\in[t_n/2-(\zeta/2)\log n, t_n/2]\colon\ S_i\in\{0,-d_n\}\},\\ \mathcal{D}^2_{t_n} &=& \{S\colon \exists i\in[t_n/2+(3\zeta)\log n, t_n/2+(7\zeta/2)\log n]\colon\ S_i\in\{0,-d_n\}\},\\ \mathcal{D}_{I_n} &=& \{S\colon \exists i\in[t_n/2, t_n/2+(\zeta/2)\log n]\ \exists j\in[t_n/2+(5\zeta/2)\log n, t_n/2+(3\zeta)\log n]\colon\\ && \quad S_i\in\{0,-d_n\},\, S_j\in\{0,-d_n\}\}.\end{array} \tag{4.73}$$

Then

$$(\mathcal{A}^1_{t_n}\cap\mathcal{A}^2_{t_n}\cap\mathcal{A}_{I_n}\cap\mathcal{D}^1_{t_n}\cap\mathcal{D}^2_{t_n}\cap\mathcal{D}_{I_n})\subset\mathcal{A}_{t_n}\subset(\mathcal{A}^1_{t_n}\cap\mathcal{A}^2_{t_n}\cap\mathcal{A}_{I_n}) \tag{4.74}$$

with $\mathcal{A}^i_{t_n}$ the event that no excursion in the left-half ($i=1$) resp. the right-half ($i=2$) of $(0,t_n)\setminus I_n$ exceeds $\zeta\log n$, and $\mathcal{A}_{I_n}$ the same in I_n. With this observation we can estimate

$$\begin{aligned}&\min_{x,y\in[-d_n,0]}\bar{P}^{(0,t_n),I_n}_{x,y,d_n}[\mathcal{C}_{I_n}](\omega)\\ &\geq \min_{x,y\in[-d_n,0]}\frac{\begin{array}{l}\sum_{x',y'\in[-d_n,0]}P_x(\mathcal{A}^1_{t_n}\cap\mathcal{B}^1_{t_n}\cap\mathcal{D}^1_{t_n}\cap\{S_{t_n/2}=x'\})\\ \quad\times\hat{P}^{I_n}_{x',d_n}(\mathcal{C}_{I_n}\cap\mathcal{A}_{I_n}\cap\mathcal{B}_{I_n}\cap\mathcal{D}_{I_n}\cap\{S_{t_n/2+3\zeta\log n}=y'\})(\omega)\\ \quad\times P_{y'}(\mathcal{A}^2_{t_n}\cap\mathcal{B}^2_{t_n}\cap\mathcal{D}^2_{t_n}\cap\{S_{t_n}=y\})\end{array}}{\begin{array}{l}\sum_{x',y'\in[-d_n,0]}P_x(\mathcal{A}^1_{t_n}\cap\mathcal{B}^1_{t_n}\cap\{S_{t_n/2}=x'\})\\ \quad\times\hat{P}^{I_n}_{x',d_n}(\mathcal{A}_{I_n}\cap\mathcal{B}_{I_n}\cap\{S_{t_n/2+3\zeta\log n}=y'\})(\omega)\\ \quad\times P_{y'}(\mathcal{A}^2_{t_n}\cap\mathcal{B}^2_{t_n}\cap\{S_{t_n}=y\})\end{array}}\end{aligned} \tag{4.75}$$

with $\mathcal{B}^i_{t_n}$ the event that the path stays confined to $[-d_n,0]$ in the left-half ($i=1$) resp. the right-half ($i=2$) of $(0,t_n)\setminus I_n$, and $\mathcal{B}_{I_n}$ the same in I_n. Here, $P_x(S)$ is the path measure for simple random walk and $\hat{P}^{I_n}_{x,d_n}(S)(\omega)$ is the path measure for the heteropolymer in I_n (as in (1.9)), both for the path starting from x.

Next, we estimate

$$\text{r.h.s. (4.75)}\geq I\times II(\omega)\times III \tag{4.76}$$

with

$$\begin{array}{lcl}I &=& \min_{x,x'\in[-d_n,0]}P_x(\mathcal{D}^1_{t_n}\mid\mathcal{A}^1_{t_n}\cap\mathcal{B}^1_{t_n}\cap\{S_{t_n/2}=x'\}),\\ II(\omega) &=& \min_{x',y'\in[-d_n,0]}\hat{P}^{I_n}_{x',d_n}(\mathcal{C}_{I_n}\cap\mathcal{D}_{I_n}\mid\mathcal{A}_{I_n}\cap\mathcal{B}_{I_n}\cap\{S_{t_n/2+3\zeta\log n}=y'\})(\omega),\\ III &=& \min_{y,y'\in[-d_n,0]}P_{y'}(\mathcal{D}^2_{t_n}\mid\mathcal{A}^2_{t_n}\cap\mathcal{B}^2_{t_n}\cap\{S_{t_n}=y\}).\end{array} \tag{4.77}$$

Since $t_n\gg d_n^2$ by (4.25), the minimum over x in I and y in III is not felt in the limit of large n. Therefore we get

$$I, III\geq\exp\{-c(d_n/2)^2/2(\zeta/2)\log n\}\qquad\text{for some } c>0, \tag{4.78}$$

the right-hand side being the probability that simple random walk travels a distance $d_n/2$ within time $(\zeta/2)\log n$ in order to hit the interface as required in (4.73). Since $d_n\ll\log n$ by (1.16)(II), the latter is much larger than the bound in the right-hand side of (4.61)(b).

Thus, it remains to bound $\mathbb{E}(II(\omega))$. This is a quantity for the heteropolymer in I_n where all the interaction with $(0,t_n)\setminus I_n$ has vanished. First, we estimate

$$II(\omega) \geq \exp\{-c(d_n/2)^2/2(\zeta/2)\log n\}\, \hat{P}^{I_n}_{0,d_n}\left(\mathcal{C}_{I_n} \mid \mathcal{A}_{I_n}\cap\mathcal{B}_{I_n}\cap\{S_{t_n/2+3\log n}=0\}\right)(\omega) \qquad (4.79)$$

by an argument similar to that in the proof of Lemma 2.1. (The event $\mathcal{D}_{I_n}$ is realized when the path hits 0 at both ends of I_n.) Second, we use that $d_n \gg \log|I_n| \sim \log\log n$ by (1.16)(I), to obtain that

$$\lim_{n\to\infty}\mathbb{E}\left[\hat{P}^{I_n}_{0,d_n}\left(\mathcal{C}_{I_n}\mid\mathcal{A}_{I_n}\cap\mathcal{B}_{I_n}\cap\{S_{t_n/2+3\log n}=0\}\right)(\omega)\right]=1. \qquad (4.80)$$

Indeed, this follows from the result in Albeverio and Zhou [1] cited at the end of Section 1.2, namely, in $\mathbb{P}$-probability the maximal length and the maximal height of an excursion in the interval I_n are of order $\log|I_n|$. (In [1], Theorem 5.3 and Theorem 6.1, this result was proved only for $h=0$, but it carries over to $0<h<h_c(\lambda)$ by similar arguments; see in particular Biskup and den Hollander [2], Theorem 3(e) and Lemma 4.) This, together with (1.16(II)), finishes the proof.

■

5 Proof of Theorem 1.4

In this section we prove Theorem 1.4, which is our main result for the path behavior in the localized regime $\mathcal{L}$. The proof is based on an upper bound (Lemma 5.1) and a lower bound (Lemma 5.2) for the quantity defined in (1.14). The proof relies on Lemmas 3.2–3.3 and 4.3–4.4.

Recall (1.12–1.15). It is clear that, under the annealed measure $\mathbb{E}\otimes\hat{P}^{(0,n)}_{0,d_n}$, $(\tilde{S}_t)_{t\in[0,n]}$ is a simple random walk on $d_n\mathbb{Z}$ with i.i.d. random waiting times, since the jump process

$$\left(S_{\tau_k\circ\theta_{\hat{\tau}(+1)}+\hat{\tau}(+1)} - S_{\tau_{k-1}\circ\theta_{\hat{\tau}(+1)}+\hat{\tau}(+1)}\right)_{k=1}^{N_n-1} \qquad (5.1)$$

is an i.i.d. sequence of random variables taking the values $\pm d_n$ with probability $1/2$ each and the medium D_n is symmetric with respect to the middle lines $\partial D_n + d_n/2$. So it remains to prove (1.17). Since

$$\mathrm{Var}_{\mathbb{E}\otimes\hat{P}^{(0,n)}_{0,d_n}}(\tilde{S}_{un}) = \mathbb{E}\otimes\hat{E}^{(0,n)}_{0,d_n}[\tilde{S}^2_{un}] = d_n^2\ \mathbb{E}\otimes\hat{E}^{(0,n)}_{0,d_n}\left[N_{un}-1\right], \qquad (5.2)$$

the proof of (1.17) amounts to analyzing the asymptotic behavior of the expected number of jumps N_{un}. This will be done in Lemmas 5.1–5.2 below and involves the quantity $\chi(\lambda,h)$ defined in (4.19).

Lemma 5.1 *Let $(\lambda,h)\in\mathcal{L}$. Assume (1.16)(I) and (1.16)(II). For all $\varepsilon_4>0$ and $u\in(0,1)$,*

$$\mathbb{E}\otimes\hat{E}^{(0,n)}_{0,d_n}\left[N_{un}\right] \leq un\exp\left\{-\chi(\lambda,h)d_n+\varepsilon_4 d_n\right\}, \qquad n\to\infty. \qquad (5.3)$$

Proof of Lemma 5.1. Throughout the proof we assume that n is large enough. Let us first resume what we know from Lemmas 3.2 and 4.3. Choose $\kappa > 0$ and $c_2 > 0$ according to Lemma 2.3, $\zeta_1 > 0$, $\zeta = (2+\zeta_1)/\kappa > 2/\kappa$ and $\varepsilon_5 \in (0, \chi_{-1}(\lambda,h) \wedge \chi_{+1}(\lambda,h))$. Then for all $N \in \mathbb{N}$ and $l_i \in \mathbb{N}$ $(0 \le i \le N)$ with $\sum_{i=0}^N (l_i + \zeta \log n) \le n$ we have

$$\mathbb{E} \otimes \hat{P}_{0,d_n}^{(0,n)} \left[\{\hat{\tau}(+1) \le l_0\} \cap \bigcap_{i=1}^N \{\tau_i \circ \theta_{\hat{\tau}(+1)} - \tau_{i-1} \circ \theta_{\hat{\tau}(+1)} \le l_i\} \right] \tag{5.4}$$

$$\le \begin{cases} 0 & \text{if } \inf_i l_i < d_n/2, \\ \left(\delta_n + \inf_{I_N \subset \{0,\dots N\}} \prod_{i \in I_N} \left(l_i \exp\left\{ -\chi_{(-1)^i}(\lambda,h) d_n + \varepsilon_5 d_n \right\} \right) \right) \wedge 1 & \text{otherwise,} \end{cases}$$

where $\delta_n = c_2 n^{1-\kappa\zeta} = c_2 \exp\{-(1+\zeta_1)\log n\}$. Let us next define

$$\begin{aligned} e_{\sigma,n} &= \tfrac{1}{2} \exp\left\{ \chi_\sigma(\lambda,h) d_n - \varepsilon_5 d_n \right\} + d_n/2, \\ p_{\sigma,n} &= \tfrac{1}{2} \left(1 + \exp\left\{ -\chi_\sigma(\lambda,h) d_n + \varepsilon_5 d_n + \log d_n \right\} \right). \end{aligned} \tag{5.5}$$

If we put $I_N^{(1)} = \left\{ 0 \le i \le N \colon\ l_i < e_{(-1)^i,n} \right\}$, then for all $i \in I_N^{(1)}$ we have

$$l_i \exp\left\{ -\chi_{(-1)^i}(\lambda,h) d_n + \varepsilon_5 d_n \right\} \le p_{(-1)^i,n} \le (1-\delta_n)^{-1}, \qquad n \to \infty. \tag{5.6}$$

Therefore (5.4) yields

$$\begin{aligned} &\mathbb{E} \otimes \hat{P}_{0,d_n}^{(0,n)} \left[\{\hat{\tau}(+1) \le l_0\} \cap \bigcap_{i=1}^N \{\tau_i \circ \theta_{\hat{\tau}(+1)} - \tau_{i-1} \circ \theta_{\hat{\tau}(+1)} \le l_i\} \right] \\ &\le \left(\delta_n 1_{\{I_N^{(1)} \neq \emptyset\}} + \prod_{i=0}^N \left(l_i \exp\left\{ -\chi_{(-1)^i}(\lambda,h) d_n + \varepsilon_5 d_n \right\} 1_{i \in I_N^{(1)}} + 1_{i \notin I_N^{(1)}} \right) \right) 1_{\{\inf_i l_i \ge d_n/2\}} \\ &\le \delta_n 1_{\{I_N^{(1)} \neq \emptyset,\ \inf_i l_i \ge d_n/2\}} + \prod_{i=0}^N \left(p_{(-1)^i,n} 1_{\{d_n/2 \le l_i < e_{(-1)^i,n}\}} + 1_{\{l_i \ge e_{(-1)^i,n}\}} \right). \end{aligned} \tag{5.7}$$

For $N \in \mathbb{N}_0$, let $(X_0,\dots,X_N)$ be the random vector in $\mathbb{N}^{N+1}$ with distribution P given by

$$P\left[\bigcap_{i=0}^N \{X_i \le l_i\} \right] = \text{ r.h.s. (5.7)}. \tag{5.8}$$

Define $T_{-1}^{(1)} = 0$ and $T_k^{(1)} = \sum_{i=0}^k X_i$, $k \ge 0$. For $t \ge 0$, define

$$N_t^{(1)} = \sup\left\{ k \in \mathbb{N}_0 \colon\ T_{k-1}^{(1)} \le t \right\}. \tag{5.9}$$

For $k \in \{-1,\dots,K(n,t)\}$ with $K(n,t) = \lfloor \frac{n-t}{\zeta \log n} \rfloor$, we have (recall (1.14))

$$\begin{aligned} \mathbb{E} \otimes \hat{P}_{0,d_n}^{(0,n)} [N_t \ge k+1] &= \mathbb{E} \otimes \hat{P}_{0,d_n}^{(0,n)} \left[\tau_k \circ \theta_{\hat{\tau}(1)} + \hat{\tau}(1) \le t \right] \\ &\le P\left[T_k^{(1)} \le t \right] \\ &= P\left[N_t^{(1)} \ge k+1 \right]. \end{aligned} \tag{5.10}$$

Therefore we obtain from (5.10) that, for $u \in (0,1)$,

$$\begin{aligned}
\mathbb{E} \otimes \hat{E}^{(0,n)}_{0,d_n}[N_{un}] &= \sum_{k=0}^{\lfloor un/(d_n/2) \rfloor} \mathbb{E} \otimes \hat{P}^{(0,n)}_{0,d_n}[N_{un} > k] \\
&= \sum_{k=0}^{K(n,un) \wedge \lfloor un/(d_n/2) \rfloor} \mathbb{E} \otimes \hat{P}^{(0,n)}_{0,d_n}[N_{un} > k] \\
&\quad + \sum_{k=K(n,un) \wedge \lfloor un/(d_n/2) \rfloor + 1}^{\lfloor un/(d_n/2) \rfloor} \mathbb{E} \otimes \hat{P}^{(0,n)}_{0,d_n}[N_{un} > k] \qquad (5.11) \\
&\leq \sum_{k=0}^{\infty} P\left[N^{(1)}_{un} > k\right] + \left(\frac{un}{(d_n/2)} - K(n,un)\right)_+ P\left[N^{(1)}_{un} > K(n,un)\right] \\
&\leq \left(1 + \frac{un}{(d_n/2)K(n,un)} - 1\right) E\left[N^{(1)}_{un}\right] \\
&\leq (\log n) E\left[N^{(1)}_{un}\right]. \qquad (5.12)
\end{aligned}$$

Thus we are left with proving an upper bound for the expectation on the right-hand side of (5.11), which only contains the random variables X_i.

To handle $E[N^{(1)}_{un}]$, note that the X_i's do not have the same distribution: even i corresponds to $\sigma = +1$, odd i to $\sigma = -1$. Therefore we need to further simplify the problem. Let $Y_0 = 0$ and $Y_i = X_{2i-2} + X_{2i-1}$, $i \in \mathbb{N}$. Then we have

$$\begin{aligned}
N^{(1)}_{un} &= \sup\left\{k \in \mathbb{N}_0 : \; T^{(1)}_{k-1} \leq un\right\} \\
&\leq 2\sup\left\{k \in \mathbb{N}_0 : \; T^{(1)}_{2k-1} \leq un\right\} \\
&= 2\sup\left\{k \in \mathbb{N}_0 : \; \sum_{i=1}^{k} Y_i \leq un\right\} \qquad (5.13)
\end{aligned}$$

and

$$\begin{aligned}
P\left[\bigcap_{i=1}^{N} \{Y_i \leq l_i\}\right] &= P\left[\bigcap_{i=1}^{N} \{X_{2i-2} + X_{2i+1} \leq l_i\}\right] \\
&\leq \delta_n 1_{\{I^{(2)}_N \neq \emptyset, \; \inf_i l_i \geq d_n\}} + \prod_{i=1}^{N} \left(p 1_{\{d_n \leq l_i < e_n\}} + 1_{\{l_i \geq e_n\}}\right), \qquad (5.14)
\end{aligned}$$

where we introduce

$$1 > p > \lim_{n \to \infty} [p_{+1,n} + p_{-1,n} - p_{+1,n} p_{-1,n}] = 3/4, \qquad (5.15)$$

$$e_n = e_{+1,n} + e_{-1,n} = d_n + \sum_{i=0,1} \tfrac{1}{2} \exp\left\{\chi_{(-1)^i}(\lambda,h) d_n - \varepsilon_5 d_n\right\}, \qquad (5.16)$$

$$I^{(2)}_N = \{i \geq 1 : \; l_i < e_n\}. \qquad (5.17)$$

For $N \in \mathbb{N}_0$, let $(Z_1, \ldots, Z_N)$ be the random vector in $\mathbb{N}^N$ with distribution

$$P\left[\bigcap_{i=1}^{N} \{Z_i \le l_i\}\right] = \text{ r.h.s. (5.14)}. \tag{5.18}$$

Define $T_0^{(2)} = 0$ and $T_k^{(2)} = \sum_{i=1}^k Z_k$, $k \in \mathbb{N}$. For $u \in (0,1)$, define

$$N_{un}^{(2)} = \sup\left\{k \in \mathbb{N}_0 : \; T_k^{(2)} \le un\right\}. \tag{5.19}$$

Using (5.10–5.14), we see that for $u \in (0,1)$,

$$\mathbb{E} \otimes \hat{E}_{0,d_n}^{(0,n)}\left[N_{un}\right] \le 2(\log n)\, E\left[N_{un}^{(2)}\right]. \tag{5.20}$$

Therefore it remains to calculate

$$\begin{aligned}
E\left[N_{un}^{(2)}\right] &= \sum_{k=0}^{\infty} P\left[N_{un}^{(2)} > k\right] \\
&= \sum_{k=0}^{\infty} P\left[T_{k+1}^{(2)} \le un\right] \\
&= \sum_{k=0}^{\infty} P\left[T_{k+1}^{(2)} - (k+1)d_n \le un - (k+1)d_n\right] \\
&= \sum_{k=0}^{\lfloor un/d_n - 1\rfloor} \sum_{l=0}^{\lfloor \frac{un-(k+1)d_n}{e_n - d_n}\rfloor} P\left[T_{k+1}^{(2)} - (k+1)d_n = l(e_n - d_n)\right] \qquad (5.21)\\
&\le \sum_{k=0}^{\lfloor un/d_n - 1\rfloor} \left(\delta_n + \sum_{l=0}^{\lfloor \frac{un-(k+1)d_n}{e_n - d_n}\rfloor} \binom{k+1}{l} p^{k+1-l}(1-p)^l\right) \\
&\le c_2 u n^{-\zeta_1}/d_n + \sum_{l=0}^{\lfloor un/e_n\rfloor + 1} \sum_{k:\; k+1 \ge l} \binom{k+1}{l} p^{k+1-l}(1-p)^l \\
&\le c_2 u n^{-\zeta_1}/d_n + \left(\frac{un}{e_n} + 2\right)(1-p)^{-1}.
\end{aligned}$$

Inserting this into (5.20) and recalling (5.16), we obtain that

$$\mathbb{E} \otimes \hat{E}_{0,d_n}^{(0,n)}\left[N_{un}\right] \le un \exp\left\{-\chi(\lambda,h)d_n + 2\varepsilon_5 d_n\right\}, \tag{5.22}$$

which completes the proof since ε_5 is arbitrary.

■

Lemma 5.2 *Let* $(\lambda, h) \in \mathcal{L}$. *Assume* (1.16)(I) *and* (1.16)(II). *For all* $\varepsilon_6 > 0$ *and* $u \in (0,1)$,

$$\mathbb{E} \otimes \hat{E}_{0,d_n}^{(0,n)}\left[N_{un}\right] \ge un \exp\left\{-\chi(\lambda,h)d_n - \varepsilon_6 d_n\right\}, \qquad n \to \infty. \tag{5.23}$$

Proof of Lemma 5.2. Let us first resume what we know from Lemmas 3.3 and 4.4. Choose $\kappa > 0$ and $c_2 > 0$ according to Lemma 2.3, $\zeta_1 > 0$, $\zeta = (2+\zeta_1)/\kappa > 2/\kappa$, $\varepsilon_7 > 0$ and $\delta > 0$. Then for all $N \in \mathbb{N}$ and $l_i \in \mathbb{N}$ $(0 \le i \le N)$ we have

$$\mathbb{E} \otimes \hat{P}^{(0,n)}_{0,d_n} \left[\{\hat{\tau}(+1) \wedge n > L_0\} \cap \bigcap_{i=1}^{N} \{(\tau_i \circ \theta_{\hat{\tau}(+1)}) \wedge n - (\tau_{i-1} \circ \theta_{\hat{\tau}(+1)}) \wedge n > L_i\} \right] \tag{5.24}$$

$$\le \begin{cases} 0 & \text{if } \sum_i L_i \ge n, \\ \left(\delta_n + \prod_{i=0}^{N} \left(\left(1 - \exp\left\{-\chi_{(-1)^i}(\lambda,h) d_n - \varepsilon_7 d_n\right\}\right)^{l_i} + \delta_n \right) \right) \wedge 1 & \text{if } (*), \\ 1 & \text{otherwise,} \end{cases}$$

where $L_i = l_i \exp\{\delta d_n\} - \zeta \log n$, $\delta_n = c_2 n^{1-\kappa\zeta} = c_2 \exp\{-(1+\zeta_1)\log n\}$, and $(*)$ is the condition

$$\sum_i L_i < n \qquad \text{and} \qquad \inf_i L_i \exp\{-\chi_{(-1)^i}(\lambda,h) d_n - 2\varepsilon_7 d_n - \delta d_n\} \ge 1. \tag{5.25}$$

Again, our goal is to simplify the expression on the right-hand side of (5.25). Under $(*)$ we have

$$\begin{aligned} \left(1 - \exp\left\{-\chi_{(-1)^i}(\lambda,h) d_n - \varepsilon_7 d_n\right\}\right)^{l_i} &\le \exp\left\{-l_i \tfrac{1}{2} \exp\left\{-\chi_{(-1)^i}(\lambda,h) d_n - \varepsilon_7 d_n\right\}\right\} \\ &\le \exp\left\{-\tfrac{1}{2} \exp\left\{\varepsilon_7 d_n\right\}\right\} = o(\delta_n). \end{aligned} \tag{5.26}$$

Note that $\lim_{n\to\infty} \delta_n = 0$.

For $N \in \mathbb{N}$, let $(\hat{X}_1, \ldots, \hat{X}_N)$ be the random vector in $\mathbb{N}^N$ with distribution P given by

$$P\left[\bigcap_{i=1}^{N} \left\{\hat{X}_i > L_i\right\}\right] = \begin{cases} 0 & \text{if } \sum_i L_i \ge n, \\ \delta_n + (2\delta_n)^N & \text{if } \sum_i L_i < n \text{ and } \inf_i \frac{L_i}{e_n} \ge 1, \\ 1 & \text{otherwise,} \end{cases} \tag{5.27}$$

where

$$e_n = \exp\left\{\chi(\lambda,h) d_n + 2\varepsilon_7 d_n + \delta d_n\right\}. \tag{5.28}$$

Define $T^{(3)}_0 = 0$ and $T^{(3)}_k = \sum_{i=1}^k \hat{X}_i$, $k \in \mathbb{N}$. For $t \ge 0$, define

$$N^{(3)}_t = \sup\left\{k \in \mathbb{N}_0 : \; T^{(3)}_k \le t\right\}. \tag{5.29}$$

Using a similar argument as in (5.10), we see that for all $k \in \mathbb{N}_0 \cup \{-1\}$,

$$\mathbb{E} \otimes \hat{P}^{(0,n)}_{0,d_n}\left[N_t \ge k+1\right] \ge P\left[N^{(3)}_t \ge k+1\right], \tag{5.30}$$

and so we obtain, for $u \in (0,1)$,

$$\mathbb{E} \otimes \hat{E}^{(0,n)}_{0,d_n}\left[N_{un}\right] \ge E\left[N^{(3)}_{un}\right]. \tag{5.31}$$

Thus we are left with proving a lower bound for the expectation on the right-hand side of (5.31).

We have

$$\begin{aligned}
E\left[N_{un}^{(3)}\right] &= \sum_{k=0}^{\infty} P\left[N_{un}^{(3)} > k\right] \\
&= \sum_{k=0}^{\infty} P\left[T_{k+1}^{(3)} \leq un\right] \\
&= \sum_{k=0}^{\infty} P\left[T_{k+1}^{(3)} - (k+1)e_n \leq un - (k+1)e_n\right] \\
&= \sum_{k=0}^{\lfloor un/e_n - 1\rfloor} \sum_{l=0}^{\lfloor \frac{un-(k+1)e_n}{n-e_n}\rfloor} P\left[T_{k+1}^{(2)} - (k+1)d_n = l(e_n - d_n)\right].
\end{aligned} \tag{5.32}$$

Since $\lim_{n\to\infty} \frac{un-(k+1)e_n}{n-e_n} = u < 1$, only the term with $l = 0$ contributes asymptotically. But this term can be explicitly written down, so

$$\begin{aligned}
E\left[N_{un}^{(3)}\right] &= \sum_{k=0}^{\lfloor un/e_n - 1\rfloor} \left(1 - \left(\delta_n + \sum_{l=1}^{k+1} \binom{k+1}{l} (-1)^{l+1} (2\delta_n)^l\right)\right) \\
&\geq -\frac{un}{e_n}\delta_n + \sum_{k=0}^{\lfloor un/e_n - 1\rfloor} (1-2\delta_n)^{k-1} \\
&= \frac{1-2\delta_n}{\delta_n}\left[1 - (1-2\delta_n)^{un/e_n}\right](1+o(1)) \\
&= \frac{1-2\delta_n}{\delta_n}\left[1 - \exp\left\{-\frac{un}{e_n} 2\delta_n (1+o(1))\right\}\right] \\
&= (1-2\delta_n)\frac{un}{e_n}(1+o(1)).
\end{aligned} \tag{5.33}$$

Inserting this into (5.31), we obtain that

$$\mathbb{E} \otimes \hat{E}_{0,d_n}^{(0,n)}\left[N_{un}\right] \geq un \exp\left\{-\chi(\lambda,h)d_n - 3\varepsilon_7 d_n - \delta d_n\right\}, \tag{5.34}$$

which finishes the proof since ε_7 and δ are arbitrary.

Combining (5.2) and Lemmas 5.1–5.2, we obtain (1.17) in Theorem 1.4. The bounds $\chi(\lambda, h) \in (0, \infty)$ were already mentioned in (4.18–4.19).

■

6 Proof of Theorem 1.5

In this section we prove Theorem 1.5. Recall the variational problem in (4.18),

$$\chi_\sigma(\lambda, h) = \inf_{y\geq 1}[y\mu_\sigma(\lambda, h) + I(y)], \tag{6.1}$$

where I is the rate function in (4.14) and $\mu_\sigma(\lambda,h)$ is the quantity defined in (4.2). Throughout the proof, $\lambda \in (0,\infty)$ is fixed.

(i) It is immediate from (4.1–4.2) and the symmetry of $\mathbb{P}$ under the reflection $\omega \to -\omega$ that $\mu_{+1}(\lambda,0) = \mu_{-1}(\lambda,0)$. Consequently, $\chi_{+1}(\lambda,0) = \chi_{-1}(\lambda,0)$ via (6.1).

(ii) Return to (4.3). Fix $0 \le h_2 < h_1 < h_c(\lambda)$ and write

$$\exp\left\{\lambda\sum_{i=1}^{m}(\Delta(S_i)-\sigma)(\omega_i+h_1)\right\} = \Theta_\sigma^{(0,m)}(S)\ \exp\left\{\lambda\sum_{i=1}^{m}(\Delta(S_i)-\sigma)(\omega_i+h_2)\right\} \tag{6.2}$$

with

$$\Theta_\sigma^{(0,m)}(S) = \exp\left\{\lambda(h_1-h_2)\sum_{i=1}^{m}(\Delta(S_i)-\sigma)\right\}. \tag{6.3}$$

It follows from (6.3) that, for any S,

$$\begin{aligned} &\exp\{-2\lambda(h_1-h_2)m\} \le \Theta_{+1}^{(0,m)}(S) \le 1,\\ &1 \le \Theta_{-1}^{(0,m)}(S) \le \exp\{2\lambda(h_1-h_2)m\}. \end{aligned} \tag{6.4}$$

Consequently, for any ω,

$$\begin{aligned} &Y^{(0,m)}(\omega,+1)(\lambda,h_2) \le Y^{(0,m)}(\omega,+1)(\lambda,h_1) \le \exp\{2\lambda(h_1-h_2)m\}\,Y^{(0,m)}(\omega,+1)(\lambda,h_2),\\ &Y^{(0,m)}(\omega,-1)(\lambda,h_2)\exp\{-2\lambda(h_1-h_2)m\} \le Y^{(0,m)}(\omega,-1)(\lambda,h_1) \le Y^{(0,m)}(\omega,-1)(\lambda,h_2). \end{aligned} \tag{6.5}$$

Via (4.5), this shows that $h \mapsto \mu_\sigma(\lambda,h)$ is continuous for $\sigma = \pm1$, non-increasing for $\sigma = +1$ and non-decreasing for $\sigma = -1$. Via (6.1), this proves that $h \mapsto \chi_\sigma(\lambda,h)$ is continuous for $\sigma = \pm1$, non-increasing for $\sigma = +1$ and non-decreasing for $\sigma = -1$.

(iii) By Jensen, Theorem 1.1 (see also Bolthausen and den Hollander [3], Lemmas 1 and 2), (4.3) and the strong law of large numbers for ω, we have

$$\begin{aligned} \tfrac{1}{2n}\log\mathbb{E}[Y^{(0,2n)}(\omega,\sigma)] &\ge \tfrac{1}{2n}\mathbb{E}[\log Y^{(0,2n)}(\omega,\sigma)],\\ \lim_{n\to\infty}\tfrac{1}{2n}\log[1/Y^{(0,2n)}](\omega,\sigma) &= \phi(\lambda,h)-\sigma\lambda h \qquad \mathbb{P}-a.s. \text{ and in } L^1(\mathbb{P}). \end{aligned} \tag{6.6}$$

Therefore $\mu_\sigma(\lambda,h) \le \phi(\lambda,h)-\sigma\lambda h$. Hence $\lim_{h\uparrow h_c(\lambda)}\mu_{+1}(\lambda,h) = 0$. Thus $\lim_{h\uparrow h_c(\lambda)}\chi_{+1}(\lambda,h) = 0$, because $\inf_{y\ge1} I(y) = I(\infty) = 0$.

■

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