Moment series inequalities for the discrete-time bulk service queue

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Abstract

We consider a discrete-time bulk service queueing model. The mean and variance of the stationary queue length can be expressed by means of moment series: series over the zeros in the closed unit disk of the characteristic equation. We represent these moment series in terms of moments of random variables related to the unused service capacity and use these representations to prove simple and sharp bounds on the moment series. We pay considerable attention to the case in which the arrivals follow a Poisson distribution, for which additional properties are proved leading to even sharper bounds. The Poisson case serves as a pilot study for a broader range of distributions.

keywords: bulk service queue, discrete-time, zeros, moment inequalities

1 Introduction and motivation

We consider a discrete-time queueing model with bulk service as defined by the recursion

$$X_{n+1} = \max\{X_n - s, 0\} + A_n. \tag{1}$$

Here, time is assumed to be slotted, X_n denotes the queue length at the beginning of slot n, A_n denotes the number of newly arriving customers during slot n, and $s \geq 2$ denotes the fixed number of customers that can be served during one slot. The number of new customers arriving per slot is assumed to be i.i.d. according to a discrete random variable A with $a_j = P(A = j)$, and probability generating function (pgf)

$$A(z) = \sum_{j=0}^{\infty} a_j z^j, \tag{2}$$

that we assume to be analytic in an open set containing the closed unit disk $|z| \leq 1$. The model described by (1) has a wide range of applications, including ATM switching elements [3], data

transmission over satellites [13], high performance serial busses [9], and cable access-networks [4].

Let X denote the random variable following the stationary distribution of the Markov chain defined by the recursion (1), with

$$x_j = P(X = j) = \lim_{n \to \infty} P(X_n = j), \quad j = 0, 1, 2, \ldots, \tag{3}$$

that exists under the assumption that $E(A) < s$. It follows that the pgf of X is given by (see e.g. [3])

$$X(z) = \frac{A(z) \sum_{j=0}^{s-1} x_j(z^s - z^j)}{z^s - A(z)}, \tag{4}$$

as an analytic function in an open set containing the closed unit disk $|z| \leq 1$. The expression (4) is of indeterminate form, but the s unknowns $x_0, \ldots, x_{s-1}$ can be determined by consideration of the zeros of the denominator in (4) that lie in the closed unit disk (see e.g. [2, 14]). With Rouché's theorem, it can be shown that there are exactly s of these zeros. Thus by analyticity, the numerator of $X(z)$ should vanish at each of the zeros, yielding s equations. One of the zeros equals 1, and leads to a trivial equation. However, the normalization condition $X(1) = 1$ provides an additional equation. Using l'Hôpital's rule, this condition is found to be ($\mu_A = E(A)$)

$$s - \mu_A = \sum_{j=0}^{s-1} x_j(s - j), \tag{5}$$

which equates two expressions for the mean unused service capacity.

The s roots of $A(z) = z^s$ in $|z| \leq 1$ are denoted by $z_0 = 1, z_1, \ldots, z_{s-1}$. By writing the summation in (4) as $C(z-1) \prod_{k=1}^{s-1}(z - z_k)$ with C a constant, and using (5) to derive the value of C, it follows that

$$\prod_{k=1}^{s-1} \frac{z - z_k}{1 - z_k} = \frac{1}{s - \mu_A} \sum_{j=0}^{s-1} x_j \frac{z^s - z^j}{z - 1}, \tag{6}$$

so that (4) can be written as

$$X(z) = \frac{A(z)(s - \mu_A)}{z^s - A(z)}(z - 1) \prod_{k=1}^{s-1} \frac{z - z_k}{1 - z_k}, \quad |z| \leq 1. \tag{7}$$

Expectations and variances are denoted throughout by appending the involved random variable to μ and σ^2, respectively. Accordingly,

$$E(A) = \mu_A = A'(1); \quad \sigma_A^2 = A''(1) + A'(1) - (A'(1))^2, \tag{8}$$

and similarly for X. Explicit expressions for the mean and variance of the steady-state queue length can be obtained by taking derivatives of $X(z)$. There holds (see e.g. [8])

$$\mu_X = \frac{\sigma_A^2}{2(s - \mu_A)} + \frac{1}{2}\mu_A - \frac{1}{2}(s - 1) + \sum_{k=1}^{s-1} \frac{1}{1 - z_k}, \tag{9}$$

$$\sigma_X^2 = \sigma_A^2 + \frac{A'''(1) - s(s-1)(s-2)}{3(s - \mu_A)} + \frac{A''(1) - s(s-1)}{2(s - \mu_A)}$$

$$+ \left(\frac{A''(1) - s(s-1)}{2(s - \mu_A)}\right)^2 - \sum_{k=1}^{s-1} \frac{z_k}{(1 - z_k)^2}. \tag{10}$$

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In this study we are interested in bounding the moment series

$$\sum_{k=1}^{s-1} \frac{1}{1-z_k}, \quad \sum_{k=1}^{s-1} \frac{z_k}{(1-z_k)^2}, \tag{11}$$

which we call μ-series and σ^2-series, respectively. Evidently, both series are real since the zeros z_k are either real or come in conjugate pairs.

In [4] the bounds

$$\frac{1}{2}(s-1) \le \sum_{k=1}^{s-1} \frac{1}{1-z_k} \le \frac{1}{2}(s-1) + \frac{1}{2}\min\{\mu_A, s-1\}, \tag{12}$$

have been shown to hold for the μ-series. The proof of these bounds was based on the representation

$$\sum_{k=1}^{s-1} \frac{1}{1-z_k} = \frac{1}{2}(s-1) + \sum_{j=0}^{s-1} x_j \frac{j(s-j)}{2(s-\mu_A)}, \tag{13}$$

and identity (5). In this paper we extend and complete the approach adopted in [4] and derive relatively simple bounds for the μ-series and the σ^2-series.

These bounds have a number of advantages over the series: they provide insight, depend on the arrival distribution only through the first three moments, and do not require numerical procedures. Moreover, the bounds on the series yield bounds on the mean and variance of the queue length with the same advantages. As such, there is an obvious connection with bounds obtained in the context of the $G/G/1$ queue. More precisely, one can think of X_n as being the sojourn time of the n-th customer in the $G/G/1$ queue, with A_{n-1} its service requirement, and s the deterministic and integer-valued interarrival time between customer n and $n+1$. This model is also referred to as the $D/G/1$ queue (see e.g. Servi [10]). As such, the discrete-time bulk service queueing model fits into the framework of the more general $G/G/1$ queue (see e.g. Wolff [12]). A result for the $G/G/1$ queue, known as Kingman's upper bound (see [6]), would for the $D/G/1$ queue, comparing with (12), yield the first two terms of the upper bound on the μ-series, i.e. $\frac{1}{2}(s-1) + \frac{1}{2}\mu_A$. The $\min\{\mu_A, s-1\}$ term at the right-hand side of (12) is due to the discreteness of A. Moreover, the discreteness of s makes that explicit expressions for the moments of X can be derived, and relations between bounds and moment series can be established.

The main purpose of this paper is to exploit both the discreteness of the process in (1) and the explicit expressions for the moment series to obtain results that are sharper than those obtained for the more general $G/G/1$ queue. In particular, for the Poisson distribution, the general bounds are combined with specific properties of the zeros leading to even sharper bounds. Additionally, the results give insight as to exactly when the bounds are attained. In Section 2, we give a detailed account of the main results, along with an overview of the paper.

We use this opportunity to alert the reader to some other results concerning the model in (1) obtained by us recently. In [5] we present analytic expressions of the Spitzer type (that is, involving the power series coefficients of $A^l(z)$ for $l = 1, 2, \ldots$, see [1], formulas (8)-(9)) for both $\mu_X - \mu_A$ and $\sigma_X^2 - \sigma_A^2$ and for the boundary probabilities x_j, $j = 0, 1, \ldots, s$. This allows us to give analytic formulas for the μ-series, σ^2-series, as well as to present a recursive scheme, based on (4) and the boundary probabilities, to compute all x_j with $j > s$. Furthermore, for a wide class of allowed distributions, among which the Poisson case in Sec. 6, we present in

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[5] an explicit Fourier series representation for the roots z_k, $k = 0, 1, \ldots, s$. These results are useful from various, including the numerical, point of view, but they shed not much light on the actual behaviour of the two series in terms of the first few moments of the distribution of A. The present paper is entirely focussed on establishing results of the latter type.

2 Overview and results

We first define two auxiliary random variables Y and W that take values in $\{0, 1, \ldots, s\}$ as

$$P(Y = j) = \frac{x_j}{\sum_{i=0}^s x_i}, \quad P(W = j) = \frac{(s - j)x_j}{s - \mu_A}, \quad j = 0, 1, \ldots, s, \tag{14}$$

and $P(Y = j) = P(W = j) = 0$, $j = s + 1, s + 2, \ldots$. These random variables are studied in detail in Sec. 3. There holds, in particular,

$$\mu_Y \leq \mu_A; \quad 0 \leq \mu_W \leq s - 1, \tag{15}$$

with equality in the first inequality if and only if A is concentrated on $\{0, 1, \ldots, s\}$. We also prove in Sec. 3 the representations

$$\sum_{k=1}^{s-1} \frac{1}{1 - z_k} = \frac{1}{2}(s - 1) + \frac{1}{2}\mu_A - \frac{\sigma_A^2}{2(s - \mu_A)} + (\mu_X - \mu_A), \tag{16}$$

$$= \frac{1}{2}(s - 1) + \frac{1}{2}\mu_Y - \frac{\sigma_Y^2}{2(s - \mu_Y)} \tag{17}$$

$$= \frac{s(s - 1) - Y''(1)}{2(s - \mu_Y)} = \frac{s^2 - E(Y^2)}{2(s - \mu_Y)} - \frac{1}{2} \tag{18}$$

$$= \frac{1}{2}(s - 1) + \frac{1}{2}\mu_W, \tag{19}$$

for the μ-series. We note here that (13) and (19) are identical.

From (16-19) one can obtain various inequalities for the μ-series, as well as insights into the matter when equality occurs in these. For instance, in (12) the first inequality follows at once from (19) and the fact that $\mu_W \geq 0$. Also, the second inequality in (12) follows from (17) and (19) and the fact that $\mu_Y \leq \mu_A$, $\mu_W \leq s - 1$. Furthermore, the cases of equality in either bound in (12) can easily be settled by using results, given in Sec. 3, on the relation between concentration properties of Y and W on the one hand, and of A on the other.

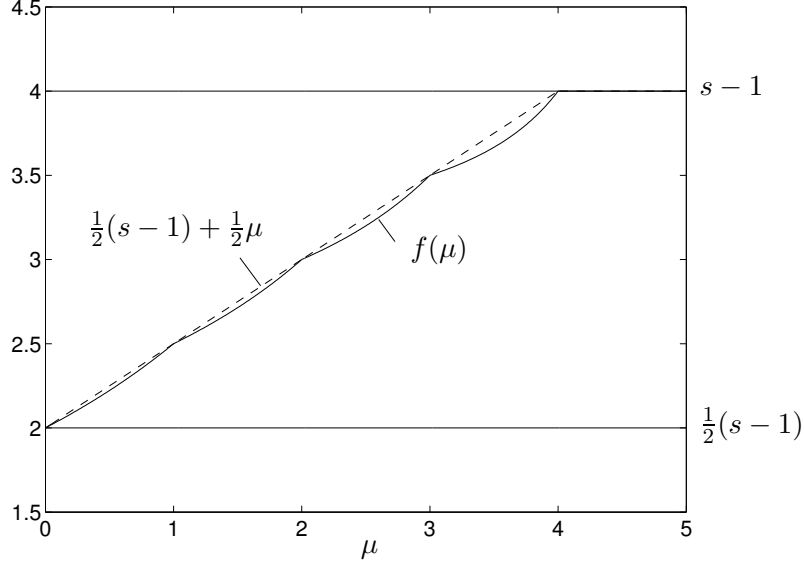We show the following bounds on the μ-series in Sec. 4.

Theorem 2.1. *(i) We have*

$$\sum_{k=1}^{s-1} \frac{1}{1 - z_k} \geq \frac{1}{2}(s - 1) + \frac{1}{2}\mu_A - \frac{\sigma_A^2}{2(s - \mu_A)}, \tag{20}$$

and there is equality if and only if A is concentrated on $\{0, 1, \ldots, s\}$.
(ii) Define $f : [0, s) \to [0, \infty)$ by

$$f(\mu) = \frac{1}{2}(s - 1) + \frac{1}{2}\mu - \frac{\langle \mu \rangle - \langle \mu \rangle^2}{2(s - \mu)}, \tag{21}$$

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Figure 1: Universal bounds for the μ-series, $s = 5$.

where we have defined $\langle \mu \rangle = \mu - \lfloor \mu \rfloor$ and $\lfloor \mu \rfloor = $ largest integer $\leq \mu$. Then we have

$$\sum_{k=1}^{s-1} \frac{1}{1 - z_k} \leq f(\mu_A), \tag{22}$$

and there is equality if and only if A is concentrated on $\{j, j+1\}$ with $j = 0, 1, \ldots, s-2$ or A is concentrated on $\{s-1, s, s+1, \ldots\}$.

In Sec. 4 we present somewhat sharper forms of Thm. 2.1 that explicitly involve μ_Y and σ_Y^2. The result in Thm. 2.1(i) presents a sharpening of the first inequality in (12) in case that $\sigma_A^2 \leq \mu_A(s - \mu_A)$. The inequality in Thm. 2.1(ii) is a refinement of the second inequality in (12) in which the discrete nature of the involved random variables is taken into account. In Fig. 1, we have plotted the graphs of both $f(\mu)$ and $\mu \to \frac{1}{2}(s-1) + \frac{1}{2}\min\{\mu, s-1\}$ for $s = 5$. As one sees, the graph of f hangs down from the second graph as a sort of guirlande with nodes at all integers $\mu = 0, 1, \ldots, s-1$.

We show in Sec. 3 the representations

$$\sum_{k=1}^{s-1} \frac{z_k}{(1 - z_k)^2} = \frac{A'''(1) - s(s-1)(s-2)}{3(s - \mu_A)} + \frac{A''(1) - s(s-1)}{2(s - \mu_A)}$$

$$+ \left(\frac{A''(1) - s(s-1)}{2(s - \mu_A)} \right)^2 - (\sigma_X^2 - \sigma_A^2) \tag{23}$$

$$= \frac{Y'''(1) - s(s-1)(s-2)}{3(s - \mu_Y)} + \frac{Y''(1) - s(s-1)}{2(s - \mu_Y)} + \left(\frac{Y''(1) - s(s-1)}{2(s - \mu_Y)} \right)^2 \tag{24}$$

$$= \frac{1}{4} \left(\frac{s^2 - E(Y^2)}{s - \mu_Y} \right)^2 - \frac{1}{3} \frac{s^3 - E(Y^3)}{s - \mu_Y} + \frac{1}{12} \tag{25}$$

$$= -\frac{1}{12}(s - \mu_W)^2 - \frac{1}{3}\sigma_W^2 + \frac{1}{12}, \tag{26}$$

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for the σ^2-series. In Sec. 5 we show the following result.

Theorem 2.2. *We have*

$$\frac{-s^2}{3(4 - \mu_A/s)} + \frac{1}{12} \leq \sum_{k=1}^{s-1} \frac{z_k}{(1 - z_k)^2} \leq -\frac{1}{12}(s - \mu_A)^2 + \frac{1}{12}. \tag{27}$$

Theorem 2.2 should be considered as a counterpart of the bounds in (12) for the μ-series. In Sec. 5 we present a more precise and sharper result in which the σ^2-series is bounded in terms of μ_Y and σ_Y^2, and from which one can infer the cases of equality in (27). This requires a result, communicated to us by E. Verbitskiy, on the extreme values of the third central moment of a random variable taking all real values between 0 and s, whose mean and variance are prescribed. The bounds in Thm. 2.2 disregard the discrete nature of the involved random variable, and, indeed, there is again a guirlande phenomenon that is detailed in Sec. 5. The bounds in (27) can be sharpened somewhat by using (26). Indeed, we have

$$-\frac{1}{9}(s - \frac{1}{2})^2 \leq \sum_{k=1}^{s-1} \frac{z_k}{(1 - z_k)^2} \leq 0, \tag{28}$$

and this improves the bounds in (27) when $\mu_A \uparrow s$.

Theorem 2.3. *(i) We have*

$$\sum_{k=1}^{s-1} \frac{z_k}{(1 - z_k)^2} \leq \frac{A'''(1) - s(s-1)(s-2)}{3(s - \mu_A)} + \frac{A''(1) - s(s-1)}{2(s - \mu_A)} + \left(\frac{A''(1) - s(s-1)}{2(s - \mu_A)}\right)^2, \tag{29}$$

and there is equality if and only if A is concentrated on $\{0, 1, \ldots, s\}$.
(ii) Defining $h : [0, s) \to [0, \infty)$ by

$$h(\mu) = \begin{cases} 0, & 0 \leq \mu \leq 2, \\ \mu(\mu - 1)(\mu - 2), & \mu > 2, \end{cases} \tag{30}$$

there holds

$$\sum_{k=1}^{s-1} \frac{z_k}{(1 - z_k)^2} \geq \frac{h(\mu_A) - h(s)}{3(s - \mu_A)} + \frac{A''(1) - s(s-1)}{2(s - \mu_A)} + \left(\frac{A''(1) - s(s-1)}{2(s - \mu_A)}\right)^2. \tag{31}$$

Here σ_A^2 and μ_A must be constrained according to

$$\sigma_A^2 \leq (s - \mu_A)(\mu_A + 2s - 4). \tag{32}$$

There is equality in (31) if and only if A is concentrated on $\{0, 1, 2\}$ or on $\{j\}$ with $j = 2, \ldots, s - 1$.

The proof of this result uses the representation (23) together with $\sigma_X^2 \geq \sigma_A^2$ for Thm. 2.3(i), and representation (24) in conjunction with Jensen's inequality and $\mu_Y \leq \mu_A$ for Thm. 2.3(ii).

In Sec. 6 we study in considerable detail the Poisson distribution, for which

$$a_j = e^{-\lambda}\frac{\lambda^j}{j!}, \quad j = 0, 1, \ldots; \qquad A(z) = e^{\lambda(z-1)}, \tag{33}$$

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where $0 \leq \lambda < s$. The roots $z_0, z_1, \ldots, z_{s-1}$ now occur on, what we call, the generalized Szegö curve

$$\mathcal{S}_\theta = \{z \in \mathbb{C} \mid |z| \leq 1, \ |z| = |e^{\theta(z-1)}|\}, \quad \theta := \lambda/s, \tag{34}$$

see [5, 11]. It is shown in Sec. 6 that $\mathrm{Re}[z(1-z)^{-2}] \leq 0$ for $z \in \mathcal{S}_\theta$. Moreover, $\mathcal{S}_\theta$ allows a parametrization $z_\theta(\alpha)$, $\alpha \in [0, 2\pi]$, with $z_\theta(\alpha)$ the unique solution z in $|z| \leq 1$ of the equation

$$z = e^{i\alpha} e^{\theta(z-1)}. \tag{35}$$

Consequently, we have $z_k = z_\theta(2\pi k/s)$, $k = 0, 1, \ldots, s-1$, and in Sec. 6 we give an explicit Fourier series representation of $z_\theta(\alpha)$, $\alpha \in [0, 2\pi]$, which allows convenient computation of all z_k's. It is shown, furthermore, in Sec. 6 that both the μ-series and σ^2-series increase in $\theta \in [0, 1)$. The Thms. 2.1 and 2.3 lead in this case to the inequalities

$$\frac{1}{2}(s-1) + \frac{1}{2}\lambda - \frac{\lambda}{2(s-\lambda)} \leq \sum_{k=1}^{s-1} \frac{1}{1-z_k} \leq \frac{1}{2}(s-1) + \frac{1}{2}\lambda, \quad 0 \leq \lambda < s, \tag{36}$$

and

$$\sum_{k=1}^{s-1} \frac{z_k}{(1-z_k)^2} \geq -\frac{1}{12}(s-\lambda)^2 - \frac{1}{2}\lambda + \frac{(s+2\lambda)s}{12(s-\lambda)^2} - \frac{\lambda(\lambda - 2/3)}{s - \lambda}, \tag{37}$$

$$\sum_{k=1}^{s-1} \frac{z_k}{(1-z_k)^2} \leq -\frac{1}{12}(s-\lambda)^2 - \frac{1}{2}\lambda + \frac{(s+2\lambda)s}{12(s-\lambda)^2}, \tag{38}$$

where the inequality in (37) holds for a range of λ slightly smaller than $[0, s)$. In particular, it can be shown from these bounds that the μ-series and σ^2-series exhibit to leading order a $\frac{1}{2}(s-1) + \frac{1}{2}\lambda$ and a $-\frac{1}{12}(s-\lambda)^2 - \frac{1}{2}\lambda$ behaviour, respectively, on λ-ranges $[0, \lambda_1(s)]$ and $[0, \lambda_2(s)]$ where $s - \lambda_1(s) = \mathcal{O}(s^{1/2})$, $s - \lambda_2(s) = \mathcal{O}(s^{2/3})$ as $s \to \infty$.

In Sec. 7 we display the sets

$$\frac{1}{2} \leq \mathrm{Re}\left[\frac{1}{1-z}\right] \leq 1; \quad \mathrm{Re}\left[\frac{z}{(1-z)^2}\right] \leq 0, \tag{39}$$

where we restrict to z with $|z| \leq 1$. The bounds (12) on the μ-series and (28) on the σ^2-series give rise to the somewhat imprecise statement that the zeros $z_1, \ldots, z_{s-1}$ of $z^s - A(z)$ exhibit on the average a preference for the two regions in (39). However, this statement cannot be made more pertinent since we will show that there are no universal zero-free regions.

In Sec. 8 we present further examples of distributions A to illustrate the bounds on the μ-series and σ^2-series.

3 Representations of the μ-series and σ^2-series

In this section we take a closer look at the random variables Y and W as defined by (14), and we show the representations (16-19) and (23-26) of the μ-series and σ^2-series they give rise to.

We note that $Y(z)$ has degree s and that the roots of $Y(z) = z^s$ are precisely $z_0 = 1, z_1, \ldots, z_{s-1}$. The latter statement follows from the fact that the numerator $A(z) \sum_{j=0}^{s} x_j (z^s -$

z^j) at the right-hand side of (4) has to cancel the s zeros of the denominator $z^s - A(z)$ within the closed unit disk $|z| \leq 1$ (when $A(0) = 0$ some trivial modifications are required). As a consequence, the random variables Y and A give rise to the same μ-series and σ^2-series while $P(Y > s) = 0$. It follows from (5) that

$$s - \mu_A = (s - \mu_Y)P(X \leq s), \tag{40}$$

and thus $\mu_Y \leq \mu_A$ with equality if and only if $P(X > s) = 0$. From the process definition we see furthermore that

$$A = X = Y \quad \Leftrightarrow \quad P(A > s) = 0. \tag{41}$$

We now show the representations (16-19) and (23-26) in Sec. 2. The representations (16), (23) follow at once upon rewriting (9), (10). The representations (17), (24) follow from the observation that A and Y yield the same μ-series and σ^2-series, and the fact that $P(Y > s) = 0$, so that (17), (24) result from consideration of the process definition and application of (16), (23) with Y instead of A. The proof of (18) is a straightforward consequence of the fact that

$$Y''(1) = E[Y(Y-1)] = \sigma_Y^2 + \mu_Y^2 - \mu_Y. \tag{42}$$

Representation (25) follows from (24) and the fact that

$$\frac{Y'''(1) - s(s-1)(s-2)}{3(s - \mu_Y)} = \frac{E[Y(Y-1)(Y-2) - s(s-1)(s-2)]}{3(s - \mu_Y)}$$
$$= \frac{E[(Y^3 - s^3) - 3(Y^2 - s^2) + 2(Y - s)]}{3(s - \mu_Y)}. \tag{43}$$

Finally, we show the representations (19), (26). The former follows from

$$\frac{s^2 - E(Y^2)}{s - \mu_Y} = \frac{1}{s - \mu_Y}\sum_{j=0}^{s}(s^2 - j^2)P(Y = j) = \frac{1}{(s - \mu_Y)P(X \leq s)}\sum_{j=0}^{s}(s+j)(s-j)x_j$$
$$= \frac{s - \mu_A}{(s - \mu_Y)P(X \leq s)}E(s + W) = s + \mu_W, \tag{44}$$

where we have used the definitions of Y and W together with (40). Similarly, we have

$$\frac{s^3 - E(Y^3)}{s - \mu_Y} = E(s^2 + sW + W^2) = s^2 + s\mu_W + E(W^2), \tag{45}$$

and (26) follows upon some administration.

We shall now be concerned with the question how certain concentration properties of Y (and W) are reflected by corresponding properties of A. The result given below is vital in Secs. 4, 5 for settling cases of equality in our theorems.

Definition 3.1. *Let B be a random variable with values in $\{0, 1, \ldots\}$ and let $\mathcal{S}$ be a subset of $\{0, 1, \ldots\}$. We say that B is concentrated on $\mathcal{S}$ when $P(B \notin \mathcal{S}) = 0$.*

According to this definition we have that Y is concentrated on $\{0, 1, \ldots, s\}$ while W is concentrated on $\{0, 1, \ldots, s - 1\}$. Moreover, we have the following result.

Lemma 3.2. *(i) Let $j = 0, 1, \ldots, s - 1$. Then Y concentrated on $\{j\} \Leftrightarrow A$ concentrated on $\{j\}$.*

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(ii) Let $j = 0, 1, \ldots, s-2$. Then Y concentrated on $\{j, j+1\}$ $\Leftrightarrow$ A concentrated on $\{j, j+1\}$.

(iii) Y concentrated on $\{s-1, s\}$ $\Leftrightarrow$ A concentrated on $\{s-1, s, s+1, \ldots\}$.

(iv) Y concentrated on $\{0, s\}$ $\Leftrightarrow$ W concentrated on $\{0\}$ $\Leftrightarrow$ A concentrated on $\{0, s, 2s, \ldots\}$.

For reasons of brevity we omit the proof of Lemma 3.2. It follows by a careful analysis from the process definition.

4 Bounds for the μ-series

In this section we prove (the claims associated with) Thm. 2.1. From the process definition in (1) we see that $\mu_X \geq \mu_A$. So

$$\sum_{k=1}^{s-1} \frac{1}{1-z_k} \geq \frac{1}{2}(s-1) + \frac{1}{2}\mu_A - \frac{\sigma_A^2}{2(s-\mu_A)}, \tag{46}$$

with equality if and only if A is concentrated on $\{0, \ldots, s\}$. We further see from representation (19) that

$$\sum_{k=1}^{s-1} \frac{1}{1-z_k} \geq \frac{1}{2}(s-1), \tag{47}$$

and there is equality if and only if A is concentrated on $\{0, s, 2s, \ldots\}$. Next we consider the representation (17) in which the μ-series is expressed in terms of the mean and variance of Y. Observe that for any random variable B concentrated on $\{0, \ldots, s\}$ with mean μ the smallest value of σ_B^2 is given by $\langle\mu\rangle - \langle\mu\rangle^2$ (as defined in Thm. 2.1), and is assumed when

$$P(B = \lfloor\mu\rfloor) = 1 - \langle\mu\rangle, \quad P(B = \lfloor\mu\rfloor + 1) = \langle\mu\rangle. \tag{48}$$

The function f as defined by (21) is strictly increasing in $\mu \in [0, s-1]$, and constant, $s-1$, for $\mu \in [s-1, s)$. We thus have

$$\sum_{k=1}^{s-1} \frac{1}{1-z_k} \leq f(\mu_Y) \leq f(\mu_A) \leq \frac{1}{2}(s-1) + \frac{1}{2}\min\{\mu_A, s-1\}. \tag{49}$$

In the first inequality there is equality if and only if $\mu_Y = 0, 1, \ldots, s-1$ and Y is concentrated on $\{\mu_Y\}$, or μ_Y is non-integer and Y is concentrated on $\{\lfloor\mu_Y\rfloor, \lfloor\mu_Y\rfloor + 1\}$. In the second inequality there is equality if and only if $\mu_Y < s-1$ and $\mu_A = \mu_Y$, or $s-1 \leq \mu_Y < s$. In the third inequality there is equality if and only if $\mu_A = 0, 1, \ldots, s-2$ or $\mu_A \geq s-1$. The inequalities (46-47) together with the second inequality in (49) prove Theorem 2.1. And also the case of equality in the second inequality in (12) is settled now: it holds if and only if A is concentrated on $\{j\}$ with $j = 0, 1, \ldots, s-2$ or A is concentrated on $\{s-1, s, s+1, \ldots\}$.

5 Bounds for the σ^2-series

In this section we prove Thms. 2.2-2.3. We first derive bounds for the σ^2-series that depend on the mean and the variance of Y, from which we derive bounds that depend on μ_A. We consider the representation (25) in which the σ^2-series is expressed in terms of μ_Y, σ_Y^2 and

$E(Y^3)$. We are interested in the smallest and largest value of (25) under the condition that μ_Y and σ_Y^2 take prescribed values. For convenience we assume Y takes, not necessarily integer, values between 0 and s, and that $0 < \mu_Y < s$. Under these assumptions, we have

$$0 < \theta := \frac{\mu_Y}{s} < 1, \quad 0 \leq \omega := \frac{\sigma_Y^2}{\mu_Y(s - \mu_Y)} \leq 1, \tag{50}$$

and equality in the last inequality occurs if and only if Y is concentrated on $\{0, s\}$. We start by presenting a lemma.

Lemma 5.1. *Let D be a random variable with values in $[-c, d]$, where $c \geq 0$, $d \geq 0$, and assume that $\mu_D = 0$, $\sigma_D^2 = \sigma^2$ is fixed. Then the minimum and maximum value of $E(D^3)$ are given by*

$$\frac{\sigma^4}{c} - c\sigma^2, \quad d\sigma^2 - \frac{\sigma^4}{d}, \tag{51}$$

respectively. The minimum and maximum value occur when D is concentrated on $\{-c, \sigma^2/c\}$ and $\{-\sigma^2/d, d\}$, respectively.

The proof of this result follows from Thm. 2.4 in [7], as was kindly communicated to us by E. Verbitskiy.

We next present three results from which Thm. 2.2 follows. In Thms. 5.2-5.4 the random variable Y is allowed to take non-integer values in $[0, s]$ and θ, ω are as in (50).

Theorem 5.2. *We have*

$$\sum_{k=1}^{s-1} \frac{z_k}{(1 - z_k)^2} \geq -\frac{1}{12}s^2(1 - \theta + \theta\omega)^2 + \frac{1}{12} - \frac{1}{3}s^2(1 - \omega)\theta\omega, \tag{52}$$

$$\sum_{k=1}^{s-1} \frac{z_k}{(1 - z_k)^2} \leq -\frac{1}{12}s^2(1 - \theta + \theta\omega)^2 + \frac{1}{12}. \tag{53}$$

The lower bound is assumed if and only if Y is concentrated on

$$\{0, \mu_Y + \frac{\sigma_Y^2}{\mu_Y}\} = \{0, s\omega + s(1 - \omega)\theta\}, \tag{54}$$

and the upper bound is assumed if and only if Y is concentrated on

$$\{\mu_Y - \frac{\sigma_Y^2}{s - \mu_Y}, s\} = \{s(1 - \omega)\theta, s\}. \tag{55}$$

Theorem 5.3. *We have*

$$-\frac{s^2}{3(4 - \theta)} + \frac{1}{12} \leq \sum_{k=1}^{s-1} \frac{z_k}{(1 - z_k)^2} \leq -\frac{1}{12}s^2(1 - \theta)^2 + \frac{1}{12}. \tag{56}$$

The lower bound is assumed if and only if Y is concentrated on the set in (54) with $\omega = (3 - \theta)/(4 - \theta)$, and the upper bound is assumed if and only if Y is concentrated on the set in (55) with $\omega = 0$.

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Theorem 5.4. *We have*

$$-\frac{1}{9}s^2 + \frac{1}{12} \leq \sum_{k=1}^{s-1} \frac{z_k}{(1-z_k)^2} \leq \frac{1}{12}. \tag{57}$$

The lower bound is assumed if and only if Y is concentrated on the set in (54) with $\omega = (3-\theta)/(4-\theta) \to \frac{2}{3}$ and $\theta \uparrow 1$, and the upper bound is assumed if and only if Y is concentrated on the set in (55) with $\omega = 0$ and $\theta \uparrow 1$.

Proofs. It is convenient to combine the proofs of the above results. We rewrite representation (25) using

$$E(Y^2) = \sigma_Y^2 + \mu_Y^2, \quad E(Y^3) = m_Y^3 + 3\mu_Y \sigma_Y^2 + \mu_Y^3, \tag{58}$$

where $m_Y^3 = E((Y - \mu_Y)^3)$. This yields

$$\sum_{k=1}^{s-1} \frac{z_k}{(1-z_k)^2} = -\frac{1}{12}(s - \mu_Y)^2 - \frac{1}{2}\sigma_Y^2 + \left(\frac{\sigma_Y^2}{2(s - \mu_Y)}\right)^2 + \frac{m_Y^3}{3(s - \mu_Y)} + \frac{1}{12}. \tag{59}$$

We then use Lemma 5.1 with $D = Y - \mu_Y$, $c = \mu_Y$, $d = s - \mu_Y$ and some administration, to see that

$$\sum_{k=1}^{s-1} \frac{z_k}{(1-z_k)^2} \geq -\frac{1}{12}\left(s - \mu_Y + \frac{\sigma_Y^2}{s - \mu_Y}\right)^2 + \frac{1}{12} - \frac{s\sigma_Y^2}{3(s - \mu_Y)}\left(1 - \frac{\sigma_Y^2}{\mu_Y(s - \mu_Y)}\right), \tag{60}$$

$$\sum_{k=1}^{s-1} \frac{z_k}{(1-z_k)^2} \leq -\frac{1}{12}\left(s - \mu_Y + \frac{\sigma_Y^2}{s - \mu_Y}\right)^2 + \frac{1}{12}, \tag{61}$$

with equality if and only if Y is concentrated on $\{0, \mu_Y + \sigma_Y^2/\mu_Y\}$ and on $\{\mu_Y - \sigma_Y^2/(s - \mu_Y), s\}$, respectively. The inequalities in (60) and (61) can be written succinctly, in terms of θ, ω as (52) and (53), respectively, and this shows Thm. 5.2.

For fixed $\theta \in (0, 1)$, the minimum of (52) equals $-s^2/(4(3 - \theta)) + 1/12$ and occurs uniquely at $\omega = (3 - \theta)/(4 - \theta)$. The maximum of (53) equals $-s^2(1 - \theta)^2/12 + 1/12$ and occurs uniquely at $\omega = 0$. This shows Thm. 5.3.

Finally, the minimum of the first member of (56) equals $-\frac{1}{9}s^2 + 1/12$ and occurs uniquely when $\omega = (3 - \theta)/(4 - \theta) \to 2/3$ and $\theta \uparrow 1$, while the maximum of the third member of (56) equals $1/12$ and occurs uniquely when $\omega = 0$ and $\theta \uparrow 1$. This then also shows Thm. 5.4. $\square$

The bounds in Thm. 2.2 are in terms of μ_A. They can be obtained straightforwardly from Thm. 5.3 by noting that $\mu_Y \leq \mu_A$ and the fact that the first member in (56) is decreasing in θ while the third member in (56) is increasing in θ. A corresponding result for the inequalities in (52) and (53) is unlikely to hold since the relation between σ_Y^2 and σ_A^2 seems much more awkward. Note once more that $Y = A$ when A is concentrated on $\{0, 1, \ldots, s\}$, and then Thms. 5.2-5.4 hold with Y replaced by A.

In Thms. 5.2-5.4 the discrete nature of the random variables has been disregarded. Accordingly, the two bounds in (52) and (53) are achieved by some integer-valued Y if and only if

$$\mu_Y + \frac{\sigma_Y^2}{\mu_Y} = s\omega + s(1 - \omega)\theta \in \mathbb{Z}, \tag{62}$$

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$$\mu_Y - \frac{\sigma_Y^2}{s - \mu_Y} = s(1 - \omega)\theta \in \mathbb{Z}, \tag{63}$$

respectively. In general, when these integrality conditions are not met, slight improvement of the bounds in Thm. 5.2 can be achieved by invoking an appropriate discrete version of Lemma 5.1 in Formula (59). This then gives rise to two guirlanded (μ, σ)- or (θ, ω)-surfaces, with contact curves described by (62) and (63), just as we had a guirlanded graph in Thm. 2.1 for the upper bound for the μ-series (since the lower bound is constant and achievable by Y concentrated on $\{0, s\}$ no guirlande phenomenon occurs for the lower bound of the μ-series).

A slight improvement of the upper bound in (56) can be obtained by observing that $\sigma_Y^2 \geq \langle\mu_Y\rangle - \langle\mu_Y\rangle^2$ when Y is integer-valued. Thus we find, see (61), in a similar fashion as in Sec. 4 for the μ-series

$$
\begin{aligned}
\sum_{k=1}^{s-1} \frac{z_k}{(1 - z_k)^2} &\leq -\frac{1}{12}\left(s - \mu_Y + \frac{\langle\mu_Y\rangle - \langle\mu_Y\rangle^2}{s - \mu_Y}\right)^2 + \frac{1}{12} \\
&= -\frac{1}{12}(2s - 1 - 2f(\mu_Y))^2 + \frac{1}{12} \\
&\leq -\frac{1}{12}(2s - 1 - 2f(\mu_A))^2 + \frac{1}{12} =: g(\mu_A) \leq 0,
\end{aligned}
\tag{64}
$$

with f as in Thm. 2.1.

We may also observe the bounds

$$-\frac{1}{9}\left(s - \frac{1}{2}\right)^2 \leq \sum_{k=1}^{s-1} \frac{z_k}{(1 - z_k)^2} \leq 0, \tag{65}$$

and their simple proofs from the representation (26) in terms of W. Indeed, consider an arbitrary random variable C concentrated on $\{0, 1, \ldots, s - 1\}$ with mean μ and variance σ^2. When μ is fixed, the minimum value of

$$-\frac{1}{12}(s - \mu)^2 - \frac{1}{3}\sigma^2 + \frac{1}{12} \tag{66}$$

occurs when C is concentrated on $\{0, s - 1\}$ and equals

$$-\frac{1}{9}\left(s - \frac{1}{2}\right)^2 + \frac{1}{4}\left(\mu - \frac{1}{3}(s - 2)\right)^2 \geq -\frac{1}{9}\left(s - \frac{1}{2}\right)^2. \tag{67}$$

Similarly, the maximum value of (66) occurs when C is concentrated on $\{\mu\}$ or on $\{\lfloor\mu\rfloor, \lfloor\mu\rfloor + 1\}$ (μ non-integer) and equals

$$-\frac{1}{12}(s - \mu)^2 - \frac{1}{3}(\langle\mu\rangle - \langle\mu\rangle^2) + \frac{1}{12} \leq 0, \tag{68}$$

with equality if and only if $\mu = s - 1$.

In Fig. 2 we have plotted the bounds in (56), (64) and (65) for $s = 5$ and $0 \leq \mu_A < s$. Observe that the graph of g hangs down from $-\frac{1}{12}(s - \mu)^2 + \frac{1}{12}$ as a guirlande with nodes at all integers $\mu = 0, 1, \ldots, s - 1$.

We conclude this section by proving Thm. 2.3. Theorem 2.3(i) follows at once from (23) and the fact that $\sigma_X^2 \geq \sigma_A^2$, with equality if and only if A is concentrated on $\{0, 1, \ldots, s\}$. As to Thm. 2.3(ii) we start from the representation (24) in which we write

$$Y'''(1) = E(Y(Y - 1)(Y - 2)) = E(h(Y)), \tag{69}$$

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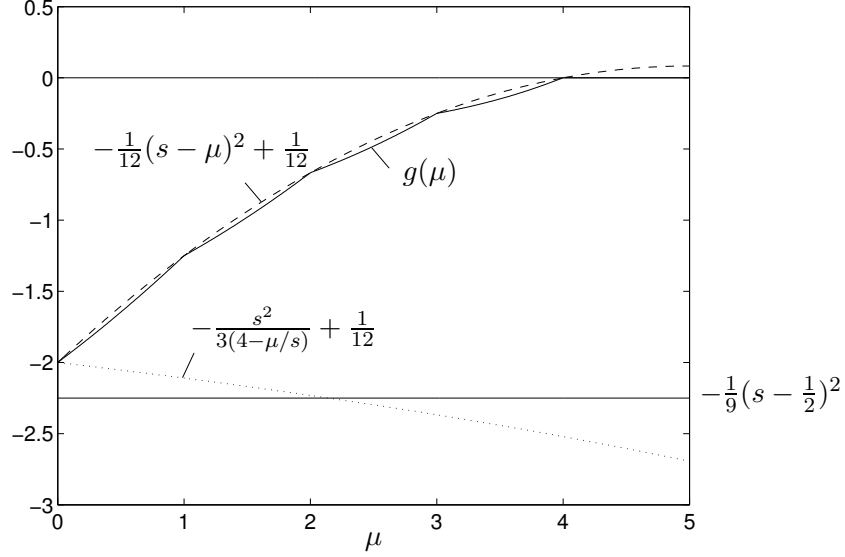
Figure 2: Universal bounds for the σ^2-series, $s = 5$.

with h given in (30). In (69) the last identity follows from the fact that Y is integer-valued. The function h is convex on $[0, \infty)$ and strictly convex on $[2, \infty)$, whence by Jensen's inequality there holds

$$E(h(Y)) \geq h(E(Y)) = h(\mu_Y), \tag{70}$$

with equality if and only if Y is concentrated on $\{0, 1, 2\}$ or Y is concentrated on $\{j\}$ with $j = 2, 3, \ldots, s-1$. Next we observe from convexity of h that the function $(h(\mu) - h(s))/(s - \mu)$ is strictly decreasing in $\mu \in [0, s)$. Hence, as $\mu_Y \leq \mu_A$, we have

$$\frac{Y'''(1) - s(s-1)(s-2)}{3(s - \mu_Y)} \geq \frac{h(\mu_Y) - h(s)}{3(s - \mu_Y)} \geq \frac{h(\mu_A) - h(s)}{3(s - \mu_A)}, \tag{71}$$

with equality if and only $\mu_A = \mu_Y$. We next turn to the quantity

$$\left(\frac{s(s-1) - Y''(1)}{2(s - \mu_Y)}\right)^2 - \frac{s(s-1) - Y''(1)}{2(s - \mu_Y)}, \tag{72}$$

that occurs at the right-hand side of (24). We note from (18) that

$$\frac{s(s-1) - Y''(1)}{2(s - \mu_Y)} \geq \frac{1}{2}(s - 1). \tag{73}$$

Furthermore, we have from (18) and Thm. 2.1(i) that

$$\frac{s(s-1) - Y''(1)}{2(s - \mu_Y)} \geq \frac{1}{2}(s - 1) + \frac{1}{2}\mu_A - \frac{\sigma_A^2}{2(s - \mu_A)} = \frac{s(s-1) - A''(1)}{2(s - \mu_A)}. \tag{74}$$

Denoting the far left-hand side of (74) by x_Y and the far right-hand side of (74) by x_A we have $x_Y \geq \frac{1}{2}(s - 1)$ and $x_A \geq \frac{1}{2}(s - 1)$, whence

$$(x_Y^2 - x_Y) - (x_A^2 - x_A) = (x_Y - x_A)(x_Y + x_A - 1) \geq 0, \tag{75}$$

whenever $x_A \geq -\frac{1}{2}(s - 1) + 1$. This latter condition can be worked out to yield constraint (32). Hence, under this constraint, (29) follows. The cases with equality easily follow from what has been said in connection with occurrence of equality in (70) and (71).

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6   Special results for the Poisson distribution

In this section we consider the case that A is distributed according to a Poisson distribution, see (33), for which we prove monotonicity of the μ-series and σ^2-series. This facilitates a sharpening of the lower bounds for both series. We have

$$\mu_A = \sigma_A^2 = \lambda; \qquad A^{(k)}(1) = \lambda^k, \tag{76}$$

with $A^{(k)}(1)$ the k-th derivative of $A(z)$ evaluated at $z = 1$. The roots $z_0, z_1, \ldots, z_{s-1}$ lie on the so-called generalized Szegö curve, as defined by (34). In Fig. 3 some examples of $\mathcal{S}_\theta$ are plotted.

We now introduce two useful parameterizations of $\mathcal{S}_\theta$. First, we represent a point z on $\mathcal{S}_\theta$ as

$$z = r_\theta(\varphi)e^{i\varphi}, \quad 0 \leq \varphi \leq 2\pi, \tag{77}$$

where $0 \leq r_\theta(\varphi) \leq 1$. In (34) and (77) we allow $\theta = 1$, i.e. $\lambda = s$. There holds

$$r_\theta(\varphi) = \exp\{\theta(r_\theta(\varphi)\cos\varphi - 1)\}, \quad 0 \leq \varphi \leq 2\pi. \tag{78}$$

It follows that

$$\frac{d}{d\theta}(\theta r_\theta(\varphi)) = \frac{(1-\theta)r_\theta(\varphi)}{1 - \theta r_\theta(\varphi)\cos\varphi} \geq 0, \tag{79}$$

$$\frac{d}{d\theta}(r_\theta(\varphi)) = \frac{-\theta r_\theta^2(\varphi)\sin\varphi}{1 - \theta r_\theta(\varphi)\cos\varphi} \leq 0, \tag{80}$$

which yields the result that for $0 \leq \theta \leq 1$

$$\theta r_\theta(\varphi) \leq r_1(\varphi) \leq r_\theta(\varphi), \quad 0 \leq \varphi \leq 2\pi. \tag{81}$$

It thus holds that the interior of $\mathcal{S}_1$ is a root-free region for any $\theta \leq 1$. Moreover, there holds

$$\max\left\{0, \frac{\cos\varphi}{1 + |\sin\varphi|}\right\} \leq r_1(\varphi) \leq \frac{1}{1 + |\sin\varphi|}, \quad 0 \leq \varphi \leq 2\pi. \tag{82}$$

In Sec. 7 we shall see that this implies that $\mathrm{Re}[z(1-z)^{-2}] \leq 0$ for all $z \in \mathcal{S}_\theta$ and all $\theta \leq 1$.

A second parameterization of $\mathcal{S}_\theta$ is obtained by solving for $\alpha \in [0, 2\pi]$ the equation
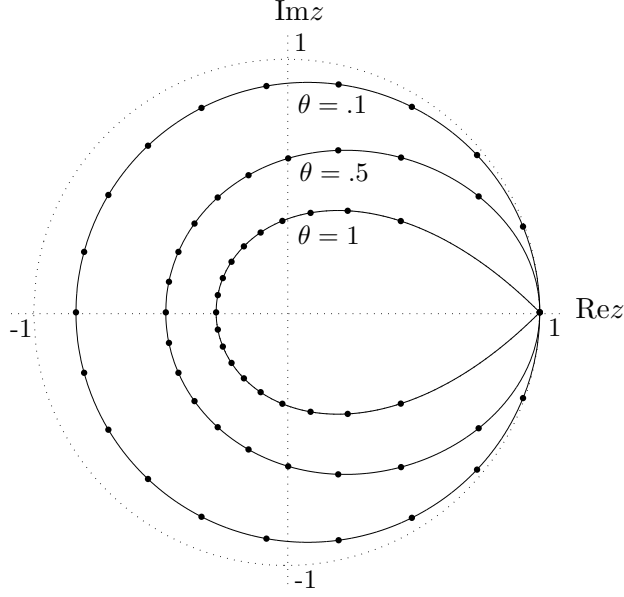
$$ze^{\theta(1-z)} = e^{i\alpha}. \tag{83}$$

Denoting the solution of (83) by $z_\theta(\alpha)$, we have the following Fourier series representation, see [5] where this is done for more general A as well,

$$z_\theta(\alpha) = \sum_{l=1}^{\infty} e^{-l\theta}\frac{(\theta l)^{l-1}}{l!}e^{il\alpha}, \quad \alpha \in [0, 2\pi]. \tag{84}$$

This allows convenient computation of all z_k's, since

$$z_k = z_{k,\theta} = z_\theta(2\pi k/s), \quad k = 0, 1, \ldots, s - 1. \tag{85}$$

Using the parametrizations of $\mathcal{S}_\theta$, we derive the following results.

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Figure 3: $\mathcal{S}_\theta$ for $\theta = .1, .5., 1$. The roots $z_0, \ldots, z_{19}$ ($s = 20$) are indicated as dots.

Lemma 6.1. *For any z on the generalized Szegö curve $\mathcal{S}_\theta$, it holds that*

$$Re\left[\frac{z}{(1-z)(1-\theta z)}\right] \leq 0, \tag{86}$$

with equality if and only if $z \to 1$.

Proof. With $z = re^{i\varphi}$, we get

$$\begin{aligned}
Re\left[\frac{z}{(1-z)(1-\theta z)}\right] &= \frac{r}{|1-z|^2|1-\theta z|^2}Re[e^{i\varphi}(1-re^{-i\varphi})(1-\theta re^{-i\varphi})] \\
&= \frac{r}{|1-z|^2|1-\theta z|^2}(\cos\varphi - (1+\theta)r + \theta r^2\cos\varphi),
\end{aligned}$$

and it suffices to show that, omitting the subindex θ in r_θ for notational convenience,

$$g(\varphi) := (1 + \theta r^2(\varphi))\cos\varphi - (1+\theta)r(\varphi) \leq 0, \tag{87}$$

with equality if and only if $\varphi = 0$. Here it is evidently sufficient to consider the case that $\cos\varphi > 0$, $\varphi \geq 0$, i.e. $\varphi \in [0, \frac{1}{2}\pi)$. There is indeed equality in (87) when $\varphi = 0$ since $r(0) = 1$. It follows from (78) that

$$r'(\varphi) = \frac{-\theta r^2(\varphi)\sin\varphi}{1 - \theta r(\varphi)\cos\varphi}, \tag{88}$$

and hence

$$\begin{aligned}
g'(\varphi) &= -(1 + \theta r^2(\varphi))\sin\varphi + (2\theta r(\varphi)\cos\varphi - 1 - \theta)r'(\varphi)) \\
&= -(1 + \theta r^2(\varphi))\sin\varphi - \frac{(2\theta r(\varphi)\cos\varphi - 1 - \theta)\theta r^2(\varphi)\sin\varphi}{1 - \theta r(\varphi)\cos\varphi} \\
&= \frac{-\sin\varphi}{1 - \theta r(\varphi)\cos\varphi}[(1 + \theta r^2(\varphi))(1 - \theta r(\varphi)\cos\varphi) + (2\theta r(\varphi)\cos\varphi - 1 - \theta)\theta r^2(\varphi))] \\
&= \frac{-\sin\varphi}{1 - \theta r(\varphi)\cos\varphi}[1 - \theta r(\varphi)\cos\varphi - \theta^2 r^2(\varphi)(1 - r(\varphi)\cos\varphi)]. \tag{89}
\end{aligned}$$

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Now, as $\cos\varphi > 0$ and $\theta \le 1$,

$$
\begin{aligned}
1 - \theta r(\varphi)\cos\varphi - \theta^2 r^2(\varphi)(1 - r(\varphi)\cos\varphi) &\ge 1 - r(\varphi)\cos\varphi - \theta^2 r^2(\varphi)(1 - r(\varphi)\cos\varphi) \\
&= (1 - r(\varphi)\cos\varphi)(1 - \theta^2 r^2(\varphi)) \ge 0, \qquad (90)
\end{aligned}
$$

with equality in the last inequality if and only if $\varphi = 0$. Thus $g'(\varphi) < 0$ for $\varphi > 0$, and it follows that (87) is ≤ 0 with equality if and only if $\varphi = 0$. This completes the proof. $\qquad\square$

Lemma 6.2. *The μ-series in case of $A(z) = e^{\theta s(z-1)}$ is increasing in $\theta \in [0,1)$.*

Proof. From

$$
z_\theta(\alpha) = e^{\theta(z_\theta(\alpha)-1)}, \quad \frac{dz_\theta(\alpha)}{d\theta} = \frac{z_\theta(\alpha)(z_\theta(\alpha)-1)}{1 - \theta z_\theta(\alpha)}, \qquad (91)
$$

we obtain

$$
\frac{d}{d\theta}(1 - z_\theta(\alpha))^{-1} = \frac{1}{(1 - z_\theta(\alpha))^2}\frac{dz_\theta(\alpha)}{d\theta} = \frac{-z_\theta(\alpha)}{(1 - z_\theta(\alpha))(1 - \theta z_\theta(\alpha))}. \qquad (92)
$$

Applying Lemma 6.1 then shows that the real part of (92) is ≥ 0 for each point on $\mathcal{S}_\theta$, and thus for all roots $z_1, \ldots, z_{s-1}$. $\qquad\square$

Lemma 6.3. *The σ^2-series in case of $A(z) = e^{\theta s(z-1)}$ is increasing in $\theta \in [0,1)$.*

Proof. It is readily seen that

$$
\frac{d}{d\theta}\left(\frac{z_\theta(\alpha)}{(1 - z_\theta(\alpha))^2}\right) = \frac{-z_\theta(\alpha)}{(1 - z_\theta(\alpha))(1 - \theta z_\theta(\alpha))} \cdot \frac{1 + z_\theta(\alpha)}{1 - z_\theta(\alpha)}, \qquad (93)
$$

and thus

$$
\begin{aligned}
\mathrm{Re}\left[\frac{d}{d\theta}\left(\frac{z_\theta(\alpha)}{(1 - z_\theta(\alpha))^2}\right)\right] &= \mathrm{Re}\left[\frac{-z_\theta(\alpha)}{(1 - z_\theta(\alpha))(1 - \theta z_\theta(\alpha))}\right]\mathrm{Re}\left[\frac{1 + z_\theta(\alpha)}{1 - z_\theta(\alpha)}\right] \\
&\quad - \mathrm{Im}\left[\frac{-z_\theta(\alpha)}{(1 - z_\theta(\alpha))(1 - \theta z_\theta(\alpha))}\right]\mathrm{Im}\left[\frac{1 + z_\theta(\alpha)}{1 - z_\theta(\alpha)}\right]. \qquad (94)
\end{aligned}
$$

First note that with $z = re^{i\varphi}$

$$
\begin{aligned}
\mathrm{Im}\left[\frac{z}{(1-z)(1-\theta z)}\right] &= \frac{r}{|1-z|^2|1-\theta z|^2}\cdot\mathrm{Im}[e^{i\varphi}(1 - re^{-i\varphi})(1 - \theta re^{-i\varphi})] \\
&= \frac{r(1-\theta r^2)}{|1-z|^2|1-\theta z|^2}\cdot\sin\varphi. \qquad (95)
\end{aligned}
$$

Furthermore, we have

$$
\frac{1+z}{1-z} = \frac{1}{|1-z|^2}(1 - r^2 + 2ir\sin\varphi), \qquad (96)
$$

whence

$$
\mathrm{Re}\left[\frac{1+z}{1-z}\right] = \frac{1 - r^2}{|1-z|^2}, \quad \mathrm{Im}\left[\frac{1+z}{1-z}\right] = \frac{2r}{|1-z|^2}\cdot\sin\varphi. \qquad (97)
$$

Altogether, this shows that both members at the right-hand side of (94) are ≥ 0, and thus the real part of (93) is ≥ 0 for each point on $\mathcal{S}_\theta$, including all roots $z_1, \ldots, z_{s-1}$. $\qquad\square$

Combining the monotonicity of the μ-series and σ^2-series, as proven in Lemma 6.2 and Lemma 6.3, and the bounds in Thms. 2.1 and 2.3 yields the following results.
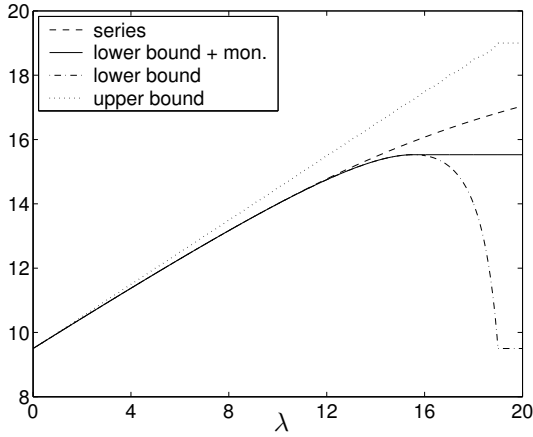
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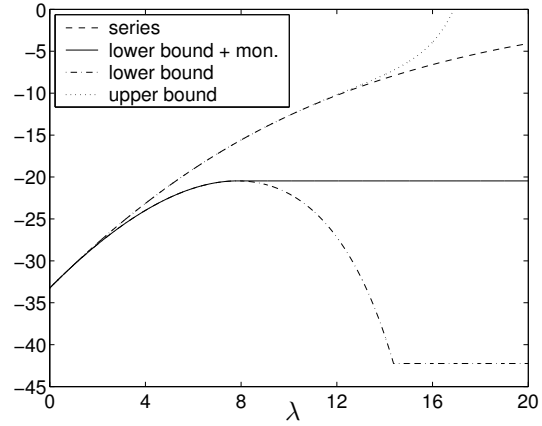
Figure 4: The μ-series, Poisson, $s = 20$.

Figure 5: The σ^2-series, Poisson, $s = 20$.
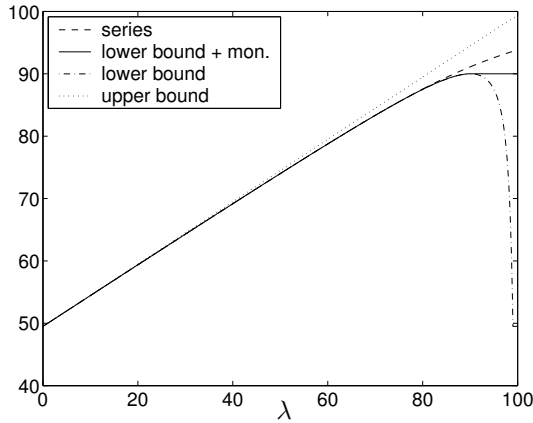
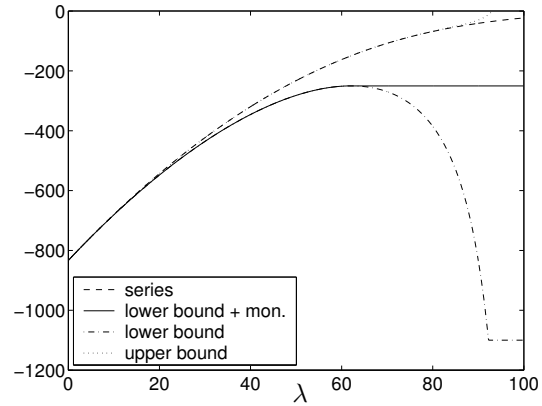

Figure 6: The μ-series, Poisson, $s = 100$.

Figure 7: The σ^2-series, Poisson, $s = 100$.

Theorem 6.4. *For A distributed according to the Poisson distribution, i.e. $A(z) = e^{\lambda(z-1)}$, that satisfies $\lambda < s$, the corresponding μ-series can be bounded as*

$$\sum_{k=1}^{s-1} \frac{1}{1 - z_k} \;\geq\; \frac{1}{2}(s-1) + m_1(\lambda), \tag{98}$$

$$\sum_{k=1}^{s-1} \frac{1}{1 - z_k} \;\leq\; \frac{1}{2}(s-1) + \frac{1}{2}\lambda - \frac{\langle\lambda\rangle - \langle\lambda\rangle^2}{2(s-\lambda)}, \tag{99}$$

where $m_1(\lambda) = \max\{\frac{\tau}{2} + \frac{\tau}{2(s-\tau)} \mid 0 \leq \tau \leq \lambda\}$.

Theorem 6.5. *For A distributed according to the Poisson distribution, i.e. $A(z) = e^{\lambda(z-1)}$, that satisfies $\lambda < s$, and when Cond. (32) holds, the corresponding σ^2-series can be bounded*

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as

$$\sum_{k=0}^{s-1} \frac{z_k}{(1-z_k)^2} \geq m_2(\lambda), \tag{100}$$

$$\sum_{k=0}^{s-1} \frac{z_k}{(1-z_k)^2} \leq -\frac{1}{12}(s-\lambda)^2 - \frac{1}{2}\lambda + \frac{s(s+2\lambda)}{12(s-\lambda)^2}, \tag{101}$$

where $m_2(\lambda) = \max\{-\frac{1}{12}(s-\tau)^2 - \frac{1}{2}\tau + \frac{s(s+2\tau)}{12(s-\tau)^2} - \frac{\tau}{s-\tau}(\tau - \frac{2}{3}) \mid 0 \leq \tau \leq \lambda\}.$

The functions $m_1(\lambda)$ and $m_2(\lambda)$ are strictly increasing for $\lambda \in [0, s - \sqrt{s}]$ and $\lambda \in [0, \lambda_2(s)]$, respectively, where $\lambda_2(s)$ is a point close to $s - (6(s^2 - \frac{1}{2}s))^{1/3}$.

Fig. 4 and Fig. 6 display the μ-series and the bounds in Thm. 6.4 for $s = 20$ and $s = 100$, respectively, with $\frac{1}{2}(s-1)$ as an overall lower bound. The more general lower bound arising from Thm. 2.1 is also plotted.

Fig. 5 and Fig. 7 display the σ^2-series and the bounds in Thm. 6.5 for $s = 20$ and $s = 100$, respectively, where $-\frac{1}{9}(s - \frac{1}{2})^2$ holds as an overall lower bound and as the lower bound when condition (32), i.e. $\lambda \leq 19.64$ for $s = 20$ and $\lambda \leq 99.66$ for $s = 100$, is not met. The more general lower bound arising from Thm. 2.3 is also plotted.

In Figs. 4-7 it is nicely demonstrated that the lower bound is sharpened substantially when monotonicity can be proven. We conjecture that monotonicity of the μ-series and σ^2-series can be shown for distributions of A other than Poisson, e.g. the binomial and geometric distribution. Moreover, in the Poisson case, it should be possible to establish concavity of the μ-series and σ^2-series as a function of θ with the techniques developed in [5].

7 Geometric properties of $\mathrm{Re}\left[(1-z)^{-1}\right]$, $\mathrm{Re}\left[z(1-z)^{-2}\right]$

The inequalities presented in Sec. 2 give a considerable amount of information on the location of the roots $z_1, \ldots, z_{s-1}$. Among other things, it raises the question whether there exists a universal root-free region in $|z| < 1$, of which $\mathcal{S}_1$ and its interior in the Poisson case is an example. Thus, does there exist an open set $\mathcal{S}$ contained in the unit disc such that for an arbitrarily distributed A any root z of $A(z) = z^s$ lies outside $\mathcal{S}$? The answer is no. For an allowed A can have its zeros of $A(z)$ anywhere, except on the positive real axis $0 < z < 1$, and by taking s sufficiently large these zeros approximate roots of $A(z) = z^s$ with any desired precision. Evidently, the inequalities in Sec. 2 provide only on-average information that leads to the observations described next. We consider the functions
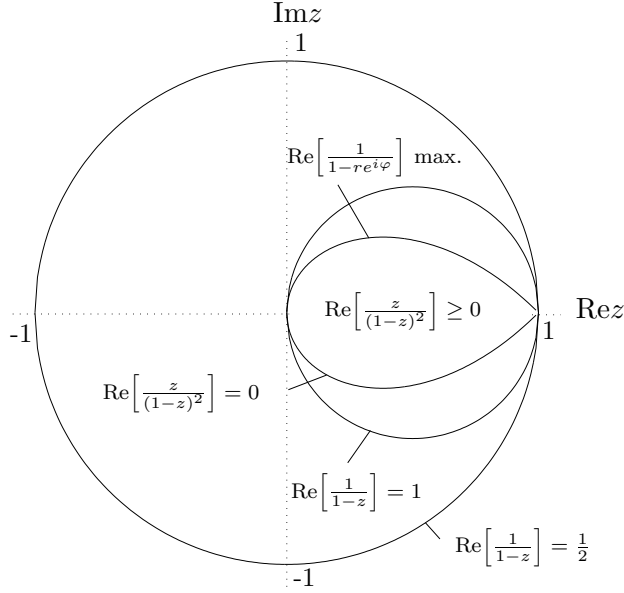
$$\frac{1}{1-z}, \quad \frac{z}{(1-z)^2}. \tag{102}$$

There holds for $z \neq 1$:

$$|z| \leq 1 \quad \Leftrightarrow \quad \mathrm{Re}\left[(1-z)^{-1}\right] \geq \frac{1}{2}. \tag{103}$$

More generally, for $\zeta > 0$ and $z \neq 1$, $z = x + iy$ with real x and y, we have

$$\mathrm{Re}\left[(1-z)^{-1}\right] = \frac{1}{2\zeta} \quad \Leftrightarrow \quad (x - (1-\zeta))^2 + y^2 = \zeta^2. \tag{104}$$

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Figure 8: Geometric properties of the functions (102).

Roughly spoken, Equation (12) leads one to expect that the roots satisfy $\frac{1}{2} \leq \operatorname{Re}\left[(1-z)^{-1}\right] \leq 1$, $|z| \leq 1$, and thus are concentrated mainly in the region,

$$\{z \in \mathbb{C} \mid |z| \leq 1, \ |z - 1/2| \geq 1/2\}, \tag{105}$$

see Fig. 8. For $0 < \varphi \leq \frac{1}{2}\pi$, $z = re^{i\varphi}$, the maximum value of

$$\operatorname{Re}\left[(1 - re^{i\varphi})^{-1}\right] = \frac{1 - r\cos\varphi}{1 - 2r\cos\varphi + r^2}, \quad 0 \leq r \leq 1, \tag{106}$$

equals $\frac{1}{2}(1 + \frac{1}{\sin\varphi})$ and occurs at

$$r = \frac{\cos\varphi}{1 + \sin\varphi} = \frac{1 - \sin\varphi}{\cos\varphi}. \tag{107}$$

For $\frac{1}{2}\pi \leq \varphi \leq \pi$ the maximum value of (106) equals 1 and occurs at $r = 0$. Note that (106) is even and 2π-periodic in φ, and see Fig. 8.

The curve described by (107) can also be generated in connection with $\operatorname{Re}[z(1-z)^{-2}]$. We have for $z \neq 1$, $z = re^{i\varphi}$,

$$\operatorname{Re}\left[z(1-z)^{-2}\right] = \frac{r((1 + r^2)\cos\varphi - 2r)}{(1 - 2r\cos\varphi + r^2)^2}, \tag{108}$$

which is ≥ 0 if and only if, see Fig. 8,

$$0 \leq r \leq \max\left\{0, \frac{\cos\varphi}{1 + |\sin\varphi|}\right\}. \tag{109}$$

It is seen from (81) and (82) that the region described by (109) is zero-free for all allowed values of λ.

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8 More examples

In this section we present more examples of distributions of A to illustrate the behaviour of the μ-series and σ^2-series and the importance of the bounds in Thms. 2.1 and 2.3. The μ-series and σ^2-series can be computed numerically by finding the roots $z_1, \ldots, z_{s-1}$, which is feasible in the cases below. When A is concentrated on $\{0, 1 \ldots, s\}$ we can check these numerical results since the lower bound in Thm. 2.1 coincides with the μ-series and the upper bound in Thm. 2.3 coincides with the σ^2-series in that case. We display the μ-series and σ^2-series, with corresponding lower and upper bounds, for a number of parametrically given A in which μ_A covers the whole range of permitted values below $s = 5$. For these cases we also exhibit explicitly the quantities μ_A, σ_A^2 and $A''(1)$, $A'''(1)$, as required in the various bounds.

For the μ-series we employ the bounds in Thm. 2.1 together with $\frac{1}{2}(s - 1)$ as an overall lower bound. For the σ^2-series we employ the bounds in Thm. 2.3, where the lower bound (31) is only used when condition (32) is satisfied. If not, we use the overall lower bound $-\frac{1}{9}(s - \frac{1}{2})^2$, and the overall upper bound 0. The cases where one can read off equality from the figures are covered by our theorems.

Example 8.1. Let A be uniformly distributed on $\{0, 1, \ldots, n - 1\}$ so that

$$A(z) = \frac{1}{n}(1 + z + \ldots + z^{n-1}) = \frac{1}{n} \frac{z^n - 1}{z - 1}. \tag{110}$$

We have

$$\mu_A = \frac{1}{2}(n - 1), \quad \sigma_A^2 = \frac{1}{12}(n^2 - 1), \tag{111}$$

and for $k = 2, 3, \ldots$

$$A^{(k)}(1) = \frac{1}{k + 1}(n - 1)(n - 2) \cdot \ldots \cdot (n - k). \tag{112}$$

Fig. 9 and Fig. 10 display the μ-series and σ^2-series for $s = 5$, $\mu_A \in [0, s - \frac{1}{2})$, i.e. $1 \le n \le 2s$. As a curiosity we mention that the values of the μ-series and σ^2-series at $n = s, s + 1$ are identical, viz. $\frac{2}{3}(s - 1)$ and $-\frac{1}{18}(s - 1)(s + 2)$, respectively. Condition (32) is satisfied for $\mu_A \le 4.27$.

Example 8.2. Take $a_n = 1 - a$, $a_{n+1} = a$ where $a \in [0, 1]$ and $n = 0, 1, \ldots$, so that

$$A(z) = (1 - a)z^n + az^{n+1}. \tag{113}$$

We have

$$\mu_A = n + a, \quad \sigma_A^2 = a - a^2, \tag{114}$$

and for $k = 2, 3, \ldots$

$$A^{(k)}(1) = n(n - 1) \cdot \ldots \cdot (n - k + 2)(n + 1 - (1 - a)k). \tag{115}$$

Fig. 11 and Fig. 12 display the μ-series and σ^2-series for $s = 5$, $\mu_A \in [0, s)$, i.e. $0 \le n \le s - 1$, $a \in [0, 1)$. Note that the μ-series and its lower and upper bound equal the guirlande upper bound. The graph of the σ^2-series is given by the right-hand side of (64), and coincides with the upper bound. In this case, we have

$$P(W = n) = \frac{(s - n)(1 - a)}{s - n - a}, \quad P(W = n + 1) = \frac{(s - n - 1)a}{s - n - a}, \tag{116}$$
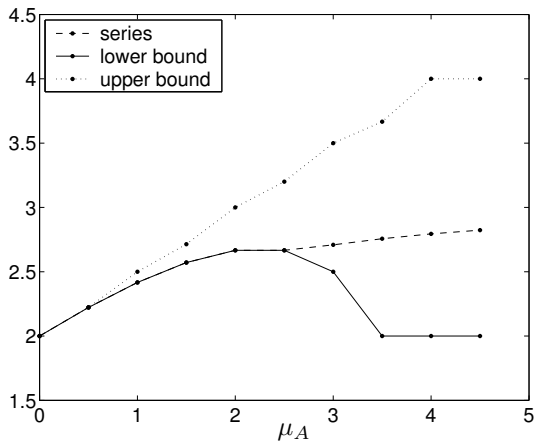
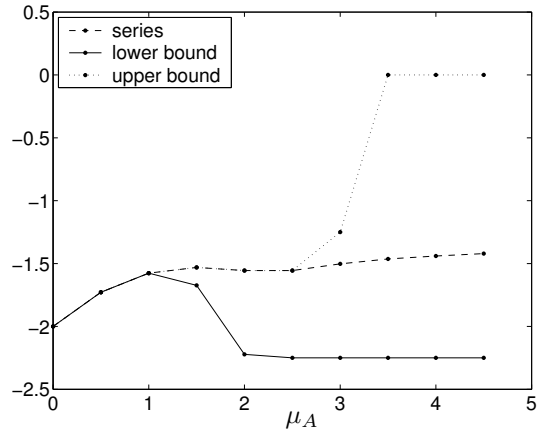**Figure 9**: The μ-series, Ex. 8.1, $s = 5$.

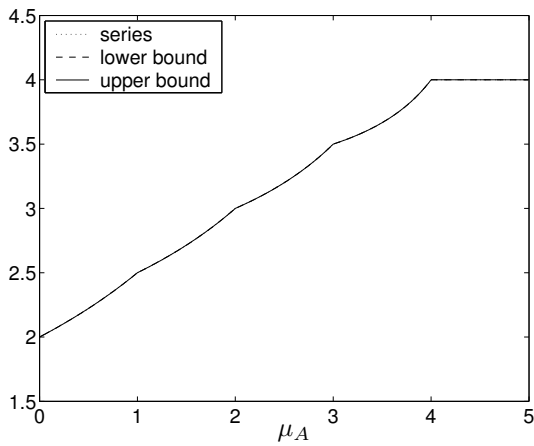**Figure 10**: The σ^2-series, Ex. 8.1, $s = 5$.
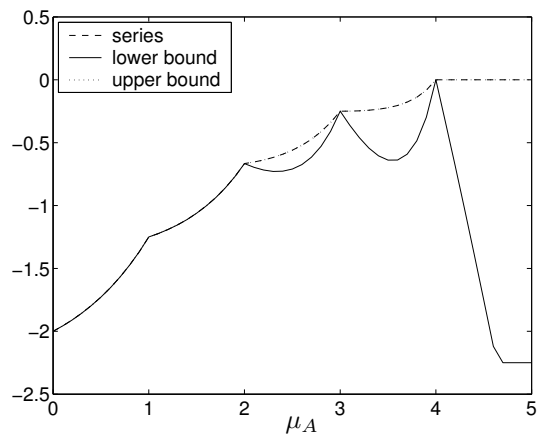

Figure 11: The μ-series, Ex. 8.2, $s = 5$.

Figure 12: The σ^2-series, Ex. 8.2, $s = 5$.
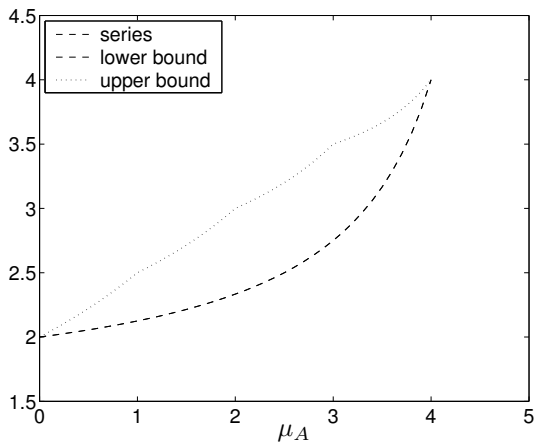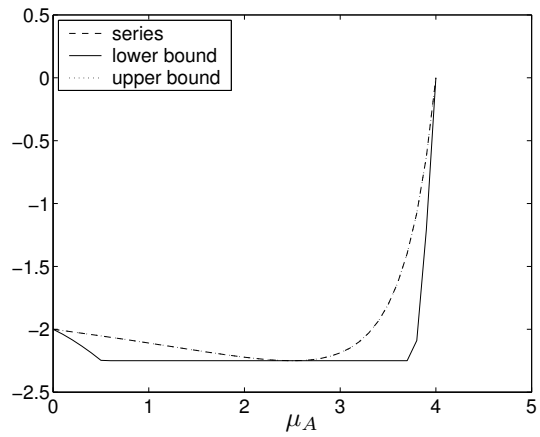

Figure 13: The μ-series, Ex. 8.4, $s = 5$.

Figure 14: The σ^2-series, Ex. 8.4, $s = 5$.

and so there is no need for numerical determination of the roots. Instead, since $X = A = Y$, we could use representation (17) and (26).

Example 8.3. Take $a_0 = 1 - \mu/s$, $a_s = \mu/s$ with $\mu \in [0, s)$, so that

$$A(z) = (1 - \frac{\mu}{s}) + \frac{\mu}{s}z^s. \tag{117}$$

We have

$$\mu_A = \mu, \quad \sigma_A^2 = \mu(s - \mu), \tag{118}$$

and for $k = 2, 3, \ldots$

$$A^{(k)}(1) = \mu(s - 1)(s - 2) \cdot \ldots \cdot (s - k + 1). \tag{119}$$

Note that $z_k = \exp(2\pi i k/s)$, and thus

$$\sum_{k=1}^{s-1} \frac{1}{1 - z_k} = \frac{1}{2}(s - 1), \quad \sum_{k=1}^{s-1} \frac{z_k}{(1 - z_k)^2} = -\frac{1}{12}(s^2 - 1), \tag{120}$$

which can also be found using (17) and (26), and the fact that $P(W = 0) = 1$.

Example 8.4. Take $a_0 = 1 - \mu/(s - 1)$, $a_{s-1} = \mu/(s - 1)$ with $\mu \in [0, s - 1]$, so that

$$A(z) = (1 - \frac{\mu}{s - 1}) + \frac{\mu}{s - 1}z^{s-1}. \tag{121}$$

We have

$$\mu_A = \mu, \quad \sigma_A^2 = \mu(s - 1 - \mu), \tag{122}$$

and for $k = 2, 3, \ldots$

$$A^{(k)}(1) = \mu(s - 2)(s - 3) \cdot \ldots \cdot (s - k). \tag{123}$$

We also compute

$$P(W = 0) = \frac{s(s - 1) - \mu s}{(s - 1)(s - \mu)}, \quad P(W = s - 1) = \frac{\mu}{(s - 1)(s - \mu)}, \tag{124}$$

so that

$$\mu_W = \frac{\mu}{s - \mu}, \quad \sigma_W^2 = \frac{\mu s}{s - \mu}(1 - \frac{1}{s - \mu}) = \mu_W(s - 1 - \mu_W). \tag{125}$$

Therefore,

$$\sum_{k=1}^{s-1} \frac{1}{1 - z_k} = \frac{1}{2}(s - 1) + \frac{1}{2}\mu_W, \quad \sum_{k=1}^{s-1} \frac{z_k}{(1 - z_k)^2} = -\frac{1}{12}s^2 - \frac{1}{2}\mu_W(\frac{1}{3}s - \frac{2}{3}) + \frac{1}{4}\mu_W^2 + \frac{1}{12}, \tag{126}$$

and these quantities are displayed in Fig. 13 and Fig. 14 for $s = 5$, $\mu_A \in [0, s - 1]$. The least value, $-\frac{1}{9}(s - \frac{1}{2})^2$, of the σ^2-series occurs for $\mu_W = \frac{1}{3}(s - 2)$, i.e. for $\mu = s(s - 2)/(s + 1) = 2\frac{1}{2}$. The μ-series and σ^2-series coincide with their lower and upper bounds, respectively.
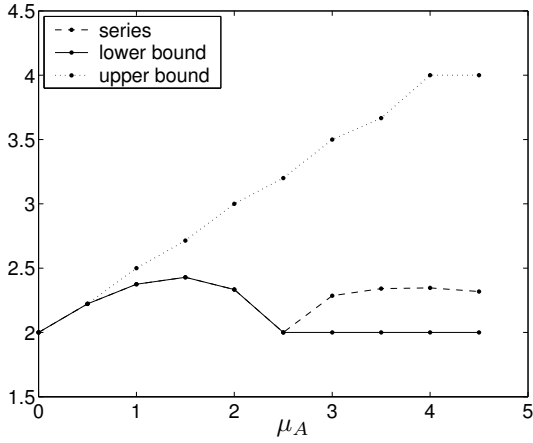
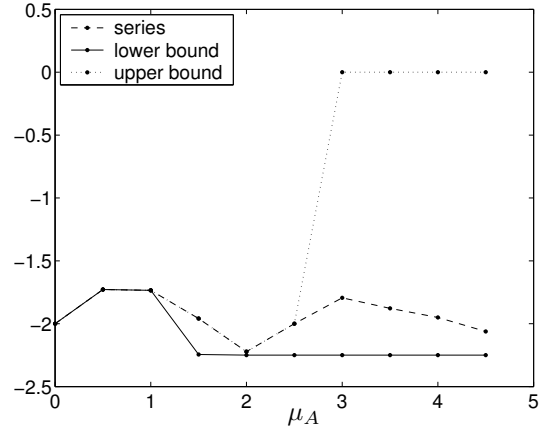**Figure 15**: The μ-series, Ex. 8.5, $s = 5$.

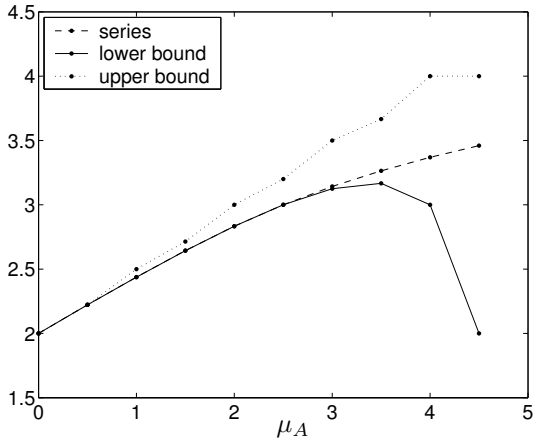**Figure 16**: The σ^2-series, Ex. 8.5, $s = 5$.
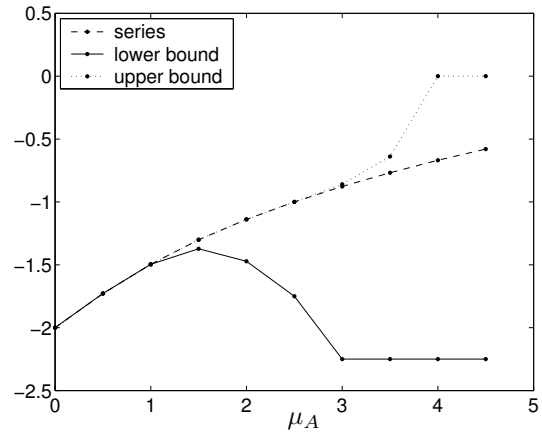

Figure 17: The μ-series, Ex. 8.6, $s = 5$.

Figure 18: The σ^2-series, Ex. 8.6, $s = 5$.

Example 8.5. Take $a_0 = 1/2$, $a_{n-1} = 1/2$ where $n \in [1, 2s]$, so that

$$A(z) = \frac{1}{2} + \frac{1}{2}z^{n-1}. \tag{127}$$

We have

$$\mu_A = \frac{1}{2}(n-1), \quad \sigma_A^2 = \frac{1}{4}(n-1)^2, \tag{128}$$

and for $k = 2, 3, \ldots$

$$A^{(k)}(1) = \frac{1}{2}(n-1)(n-2) \cdot \ldots \cdot (n-k). \tag{129}$$

Fig. 15 and Fig. 16 display the μ-series and σ^2-series for $s = 5$, $\mu_A \in [0, s - \frac{1}{2}]$, i.e. for $1 \le n \le 2s$. Note that the μ-series starts decreasing as a function of $n - 1$ around $n - 1 = s(2 - \sqrt{2})$, which is well before $n - 1 = s$. Condition (32) is satisfied for $\mu_A \le 3.63$.

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Example 8.6. Take a symmetric binomial distributed A,

$$a_j = \frac{1}{2^{n-1}} \binom{n-1}{j}, \quad j = 0, 1, \ldots, n-1; \quad a_j = 0, \quad j = n, n+1, \ldots, \tag{130}$$

so that

$$A(z) = \left(\frac{1+z}{2} \right)^{n-1}. \tag{131}$$

We now have

$$\mu_A = \frac{1}{2}(n-1), \quad \sigma_A^2 = \frac{1}{4}(n-1), \tag{132}$$

and for $k = 2, 3, \ldots$

$$A^{(k)}(1) = \left(\frac{1}{2} \right)^k (n-1)(n-2) \cdot \ldots \cdot (n-k). \tag{133}$$

Fig. 17 and Fig. 18 display the μ-series and σ^2-series for $s = 5$, $\mu_A \in [0, s - \frac{1}{2}]$, i.e. for $1 \leq n \leq 2s$, and we observe a qualitatively similar behaviour for the two series as in the Poisson case, see Sec. 6. Condition (32) is satisfied for $\mu_A \leq 4.77$.

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