Characterizations and Examples of Hidden Regular Variation

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Abstract: Random vectors on the positive orthant whose distributions possess *hidden regular variation* are a subclass of those whose distributions are multivariate regularly varying. The concept is an elaboration of the *coefficient of tail dependence* of Ledford and Tawn(1996, 1997). We provide characterizations and examples of such distribution in terms of mixture models and product models.

Keywords: heavy tails, regular variation, Pareto tails, coefficient of tail dependence, hidden regular variation, asymptotic independence.

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1. Introduction.

A refinement of the class of multivariate regularly varying distributions, which we call *hidden regular variation*, is a semi-parametric subfamily of the full family of distributions possessing multivariate regular variation and asymptotic independence. Various cases of *hidden regular variation* have received considerable attention recently as part of the program to statistically distinguish asymptotic independence from dependence. See Campos et al. (2002), Coles et al. (1999), de Haan and de Ronde (1998), Draisma et al. (2001), Ledford and Tawn (1996, 1997), Peng (1999), Poon et al. (2001), Resnick (2002a), Stărică (1999, 2000). In particular, *hidden regular variation* is based on the analysis of the coefficient of tail dependence of Ledford and Tawn (1996, 1997).

Both multivariate regular variation and hidden regular variation are crucial to defining semi-parametrically specified dependence structures for heavy tailed models where it is important to consider assumptions beyond independence. These concepts enter into analyses because there are often patterns of dependence among large values of the components of a random vector which are not apparent by using summaries of dependence based on the center of the distributions such as cross correlations. See de Haan and de Ronde (1998), Resnick (2002b,c).

A treatment of hidden regular variation, placing it in relation to the concepts of asymptotic independence and second order regular variation, was given in Resnick (2002b) where examples and characterizations were given. Here we continue discussion by providing further characterizations and canonical representations. The characterizations when the hidden angular measure is infinite are not completely satisfactory. These canonical representations are dependent on a notion of multivariate tail equivalence. We also discuss how influential hidden regular variation can be in assessing the behavior of products of components of a random vector possessing hidden regular variation.

1.1. **Outline.**

The rest of this section reviews notation (Subsection 1.2) and the polar coordinate transformation (Subsection 1.3). Section 2 defines multivariate regular variation of a probability distribution on $\mathbb{R}_+^d$, $d \geqslant 1$ and we phrase our definitions in terms of vague convergence of measures. The orientation to measures is most natural in a multivariate context where $d > 1$ and when considering changes of coordinate systems. Section 2 also defines precisely *hidden regular variation*.

Section 3 defines a notion of tail equivalence for multivariate distributions. The one dimensional concept, considered in Resnick (1971), has a natural definition which says that if two distributions F, G are tail equivalent, meaning for some $c > 0$, $1 - F(x) \sim c(1 - G(x))$, as $x \to \infty$, then extremal behavior of i.i.d. samples from F is essentially the same as for i.i.d. samples from G. How to define a multivariate analogue depends on the purpose to which the concept will be put. We propose a definition and based on this definition, we characterize distributions with hidden regular variation with finite hidden angular measure as tail equivalent to certain mixtures.

In Section 4, we give a characterization and representation of distributions with finite hidden angular measure. Section 5, on the other hand, considers the case of infinite hidden angular measure. We give a characterization for this case and several important examples but the characterization has deficiencies. Finally, in Section 6, we study the influence of hidden regular variation on the behavior of the distribution of the product of two asymptotically independent random variables.

1.2. **Notation.**

For simplicity we will generally assume that random vectors have non-negative components. Set

$$\mathbb{E} := [0, \infty]^d \setminus \{\mathbf{0}\}$$

so that the origin is excluded from $\mathbb{E}$. Compact subsets of $\mathbb{E}$ are compact sets of $[0, \infty]^d$ which do not intersect the origin; see the discussion in Resnick (2002b). In some applications, for instance in finance, it is natural to consider the cone $[-\infty, \infty]^d \setminus \{\mathbf{0}\}$. We leave it to the reader to make the modest changes necessary to generalize to this case, by considering the orthants individually.

Vectors are denoted by bold letters, capitals for random vectors and lower case for non-random vectors. For example: $\boldsymbol{x} = (x^{(1)}, \ldots, x^{(d)}) \in \mathbb{R}^d$. Operations between vectors should be interpreted componentwise so that for two vectors $\boldsymbol{x}$ and $\boldsymbol{z}$

$$\boldsymbol{x} < \boldsymbol{z} \text{ means } x^{(i)} < z^{(i)},\ i = 1,\dots,d, \qquad \boldsymbol{x} \leqslant \boldsymbol{z} \text{ means } x^{(i)} \leqslant z^{(i)},\ i = 1,\dots,d,$$

$$\boldsymbol{x} = \boldsymbol{z} \text{ means } x^{(i)} = z^{(i)},\ i = 1,\dots,d, \qquad \boldsymbol{z}\boldsymbol{x} = (z^{(1)}x^{(1)},\dots,z^{(d)}x^{(d)}),$$

$$\boldsymbol{x} \bigvee \boldsymbol{z} = (x^{(1)} \vee z^{(1)},\dots,x^{(d)} \vee z^{(d)}), \qquad \frac{\boldsymbol{x}}{\boldsymbol{z}} = \Big(\frac{x^{(1)}}{z^{(1)}},\dots,\frac{x^{(d)}}{z^{(d)}}\Big),$$

and so on. Also define $\mathbf{0} = (0,\dots,0)$. For a real number c, denote as usual $c\boldsymbol{x} = (cx^{(1)},\dots,cx^{(d)})$. We denote the rectangles (or the higher dimensional intervals) by

$$[\boldsymbol{a},\boldsymbol{b}] = \{\boldsymbol{x} \in \mathbb{E} : \boldsymbol{a} \leqslant \boldsymbol{x} \leqslant \boldsymbol{b}\}.$$

Higher dimensional rectangles with one or both endpoints open are defined analogously, for example,

$$(\boldsymbol{a},\boldsymbol{b}] = \{\boldsymbol{x} \in \mathbb{E} : \boldsymbol{a} < \boldsymbol{x} \leqslant \boldsymbol{b}\}.$$

Complements are taken with respect to $\mathbb{E}$, so that for $\boldsymbol{x} > \mathbf{0}$,

$$[\mathbf{0},\boldsymbol{x}]^c = \mathbb{E} \setminus [\mathbf{0},\boldsymbol{x}] = \{\boldsymbol{y} \in \mathbb{E} : \bigvee_{i=1}^{d} \frac{y^{(i)}}{x^{(i)}} > 1\}.$$

For $i = 1,\dots,d$, define the basis vectors

$$\boldsymbol{e}_i = (0,\dots,0,1,0,\dots,0)$$

so that the axes originating at $\mathbf{0}$ are $\mathbb{L}_i := \{t\boldsymbol{e}_i, t > 0\}$, $i = 1,\dots,d$. Then define the cone

$$\mathbb{E}_0 = \mathbb{E} \setminus \bigcup_{i=1}^{d} \mathbb{L}_i = \{\boldsymbol{s} \in \mathbb{E} : \text{ For some } 1 \leqslant i < j \leqslant d, s^{(i)} \wedge s^{(j)} > 0\}.$$

If $d = 2$, we have $\mathbb{E}_0 = (0,\infty]^2$. The cone $\mathbb{E}_0$ consists of points of $\mathbb{E}$ such that at most $d-2$ coordinates are 0.

1.3. **The polar coordinate transformation.** It is sometimes illuminating to consider multivariate regular variation for the distribution of a random vector after a polar coordinate transformation. Suppose $\|\cdot\| : \mathbb{R}^d \mapsto [0,\infty)$ is a *norm* on $\mathbb{R}^d$. The most useful norms for us are the usual Euclidean L_2 norm, the L_p norm for $p > 0$ and the L_∞ norm: $\|\boldsymbol{x}\| = \bigvee_{i=1}^{n} |x^{(i)}|$. Assume the norm has been scaled so that $\|\boldsymbol{e}_i\| = 1$ for $i = 1,\dots,d$. Given a chosen norm $\|\cdot\|$, the points at unit distance from the origin $\mathbf{0}$ are

$$\aleph := \{\boldsymbol{x} \in \mathbb{E} : \|\boldsymbol{x}\| = 1\}.$$

For the purpose of hidden regular variation, we need to look at a smaller subcone $\mathfrak{C}$ of $\mathbb{E}$ and the restriction of $\aleph$ to $\mathfrak{C}$ is denoted by $\aleph_{\mathfrak{C}} = \aleph \cap \mathfrak{C}$. Recall norms on $\mathbb{R}^d$ are all topologically equivalent in that convergence in one norm implies convergence in another.

Now fix a norm. Define the polar coordinate transformation $T : [0,\infty)^d \setminus \{\mathbf{0}\} \mapsto (0,\infty) \times \aleph$ by

$$T(\boldsymbol{x}) = \left(\|\boldsymbol{x}\|, \frac{\boldsymbol{x}}{\|\boldsymbol{x}\|}\right) =: (r,\boldsymbol{a}),$$

and the inverse transformation $T^{\leftarrow} : (0,\infty) \times \aleph \mapsto [0,\infty]_+^d \setminus \{\mathbf{0}\}$ by

$$T^{\leftarrow}(r,\boldsymbol{a}) = r\boldsymbol{a}.$$

Think of $\boldsymbol{a} \in \aleph$ as defining a direction and r telling how far in direction $\boldsymbol{a}$ to proceed. Since we excluded $\mathbf{0}$ from the domain of T, both T and $T^{\leftarrow}$ are continuous bijections.

When $d = 2$, it is customary, but not obligatory, to write $T(\boldsymbol{x}) = (r,\theta)$, where $\boldsymbol{x} = (r\cos\theta, r\sin\theta)$, with $0 \leqslant \theta \leqslant \pi/2$, rather than the more consistent notation $T(\boldsymbol{X}) = \big(r,(\cos\theta,\sin\theta)\big)$. For a random vector $\boldsymbol{X}$ in $\mathbb{R}^d$ we sometimes write $T(\boldsymbol{X}) = (R_X, \boldsymbol{\Theta}_X)$. When there is no risk of confusion, we suppress the subscript.

2. Multivariate regular variation and hidden regular variation.

2.1. **Multivariate regular variation for distributions of random vectors.** Multivariate regular variation is usually either defined by convergence of a family of functions defined on the cone $\mathfrak{C} \subseteq \mathbb{E}$ or as vague convergence of measures defined on the Borel subsets of the cone. Behavior on the boundary of the cone can be crucial. See Balkema (1973), Basrak et al. (2002), Davis and Hsing (1995), Davis and Mikosch (1998), de Haan et al. (1984), Meerschaert and Scheffler (2001), Resnick (1986), Resnick (1987, Chapter 5). The most common cones used are $\mathbb{E}$ itself and $\mathbb{E}_0$.

Consider a d-dimensional non-negative random vector $\boldsymbol{Y} = (Y^{(1)}, \dots, Y^{(d)})$ whose range includes the cone $\mathfrak{C}$. We suppose that all one-dimensional marginal distributions are the same. This can be achieved by marginal transformations of the components. (While this method of transformation is a simple theoretical procedure, associated statistical difficulties should not be neglected.) The distribution of $\boldsymbol{Y}$ is *multivariate regularly varying on the cone* $\mathfrak{C}$ with limit measure ν, a non-zero Radon measure defined on Borel subsets of $\mathfrak{C}$, if there exists a function $b(t) \uparrow \infty$ as $t \to \infty$, such that

$$t\,\mathrm{P}\left[\frac{\boldsymbol{Y}}{b(t)} \in \cdot\right] \xrightarrow{\mathrm{v}} \nu \tag{2.1}$$

in $M_+(\mathfrak{C})$, the space of Radon measures on $\mathfrak{C}$. Here convergence is vague convergence of measures and $M_+(\mathfrak{C})$ has the vague topology. See, for example, Kallenberg (1983), Karr (1986), Neveu (1977), Resnick (1987). Condition (2.1) implies that there exists a non-negative constant $\alpha \geqslant 0$ such that for relatively compact sets $B \subseteq \mathfrak{C}$

$$\nu(cB) = c^{-\alpha}\nu(B), \quad c > 0, \tag{2.2}$$

which is called the homogeneity property of ν. Henceforth, assume that $\alpha > 0$. The function $b(\cdot)$ is necessarily regularly varying of index $1/\alpha$ which we write as $b \in RV_{1/\alpha}$ and we can choose b to be strictly increasing, and continuous. We shall always assume that b is strictly continuous and increasing. There are equivalent formulations in terms of polar coordinates (cf. Basrak, 2000, Basrak et al., 2002, Resnick, 2002b,c). The regular variation condition (2.1) is equivalent to

$$t\,\mathrm{P}\left[\left(\frac{R}{b(t)}, \boldsymbol{\Theta}\right) \in \cdot\right] \xrightarrow{\mathrm{v}} c\,\nu_\alpha \times S, \tag{2.3}$$

where $c > 0$, ν_α is a Radon measure on $(0, \infty]$ given by

$$\nu_\alpha((x, \infty]) = x^{-\alpha}, x > 0,$$

S is a Radon measure on $\aleph_{\mathfrak{C}}$ (and can be chosen to be a probability measure if $\aleph_{\mathfrak{C}}$ is compact), and convergence is in $M_+\big((0, \infty] \times \aleph_{\mathfrak{C}}\big)$.

Asymptotic independence occurs when $\mathfrak{C} = \mathbb{E}$ and $\nu(\mathbb{E}_0) = 0$ or equivalently when S concentrates uniformly on the set $\{\boldsymbol{e}_i, 1 \leqslant i \leqslant d\}$.

2.2. **Hidden regular variation for distributions of random vectors.** The random vector $\boldsymbol{Y}$ has a distribution possessing *hidden regular variation* if there is a subcone $\mathfrak{C}_0 \subset \mathfrak{C}$ and the distribution has regular variation on $\mathfrak{C}$ but also regular variation of lower order on $\mathfrak{C}_0$. So in addition to (2.1), we assume that there exists a non-decreasing function $b_0(t) \uparrow \infty$ such that $b(t)/b_0(t) \to \infty$ and there exists a measure $\nu_0 \neq 0$ which is Radon on $\mathfrak{C}_0$ and such that

$$t\,\mathrm{P}\left[\frac{\boldsymbol{Y}}{b_0(t)} \in \cdot\right] \xrightarrow{\mathrm{v}} \nu_0 \tag{2.4}$$

on $M_+(\mathfrak{C}_0)$. Then there exists $\alpha_0 \geqslant \alpha$ such that $b_0 \in RV_{1/\alpha_0}$ and ν_0 and α_0 satisfy the analogue of (2.2). We can again assume that b_0 is strictly increasing, continuous and further assume that $b(1) = 1$ and hence $b^{\leftarrow}(1) = 1$.

When $\mathfrak{C} = \mathbb{E}$ and $\mathfrak{C}_0 = \mathbb{E}_0$, then (cf. Resnick, 2002b) hidden regular variation implies asymptotic independence. In this case, the name *hidden* regular variation is justified by the fact that the relatively crude normalization necessary for convergence on the axes is too large and obscures structure in the interior of the cone. A normalization of smaller order, $b_0(\cdot)$, is necessary on the smaller cone. Also, as has previously been noted by several authors including de Haan and de Ronde (1998, pg. 41)), Resnick (1987, pg. 297), Sibuya (1960), asymptotic independence can be quite different from independence. The measure ν_0 of hidden regular variation is a measure of the dependence in asymptotic independence.

There are possible variants of hidden regular variation. First, the normalization in (2.4) of each component of the random vector is by the same function. One could allow different normalizations $b_i(t),\ i = 1, \dots, d$, provided $b(t)/b_i(t) \to \infty$ for $i = 1, \dots, d$. The case of different normalizing functions can be reduced to (2.4) by monotone transformations. Second, the definition of hidden regular variation uses two cones $\mathfrak{C} \supset \mathfrak{C}_0$. In principle, one could have more cones $\mathfrak{C} \supset \mathfrak{C}_0 \supset \mathfrak{C}_1 \supset \cdots \supset \mathfrak{C}_k$ with regular variation of progressively smaller order present in each. An simple example where $d = 3 = k$ is given in Example 5.1 below. A characterization of hidden regular variation when $(\mathfrak{C}, \mathfrak{C}_0) = (\mathbb{E}, \mathbb{E}_0)$ is given in Resnick (2002b). It uses max- and min-linear combinations of the form $\bigvee_{i=1}^d s_i Y^{(i)}$ and $\bigwedge_{i=1}^d a_i Y^{(i)}$. See also the work of Coles et al. (1999).

The hidden regular variation condition (2.4) has an equivalent form in polar coordinates:

$$t\,\mathrm{P}\left[\left(\frac{R}{b_0(t)}, \boldsymbol{\Theta}\right) \in \cdot\right] \xrightarrow{\mathrm{v}} c_0\, \nu_{\alpha_0} \times S_0, \tag{2.5}$$

where $c_0 > 0$, S_0 is a Radon measure on $\aleph_0 := \aleph_{\mathfrak{C}_0}$ and convergence is in $M_+\big((0,\infty] \times \aleph_0\big)$. The measure ν_0 in (2.4) can be either finite or infinite on $\{\boldsymbol{x} \in \mathfrak{C}_0 : \|\boldsymbol{x}\| > 1\}$ and equivalently, the measure S_0 in (2.5) can be either finite or infinite on $\aleph_0$. See Examples 1 and 2 in Resnick (2002b).

We will call the measure S the *angular measure* or *spectral measure*. S_0 is called the *hidden angular measure* or *hidden spectral measure*. Keep in mind these are quantities defined by an asymptotic procedure.

The *coefficient of tail dependence* is defined by Ledford and Tawn (1996, 1997, 1998) and is a somewhat weaker concept than our definition of hidden regular variation. Schlather (2001) points out some curiosities stemming from the formulation. When hidden regular variation is present, the coefficient of tail dependence is α_0^{-1}. Related work is in Coles et al. (1999), reviewed and discussed more fully in Heffernan (2000), where two limits $(\chi, \bar{\chi})$ are assumed to exist in the context $d = 2$. When hidden regular variation is assumed as in (2.1) and (2.4), we have

$$\begin{aligned}\chi &:= \lim_{t\to\infty} P[Y^{(2)} > t | Y^{(1)} > t] = 0\\ \bar{\chi} &:= \lim_{t\to\infty} \frac{2\log P[Y^{(1)} > t]}{\log P[Y^{(1)} > t, Y^{(2)} > t]} - 1 = \frac{2\alpha - \alpha_0}{\alpha_0}.\end{aligned}$$

3. Multivariate tail equivalence.

Two probability distributions F, G on $\mathbb{R}_+$ are tail equivalent (see Resnick, 1971) if they have the same right endpoint x_0 and for some $c \in (0, \infty)$,

$$\lim_{s\uparrow x_0} \frac{1 - G(s)}{1 - F(s)} = c.$$

Then, upper extremes of i.i.d. samples from F behave asymptotically in the same way as upper extremes of i.i.d. samples from G. In the heavy tailed case, this definition is equivalent to supposing that there exists a function $b(t) \uparrow \infty$ such that for some $\alpha > 0$, as $t \to \infty$

$$tF(b(t)\cdot) \xrightarrow{\mathrm{v}} \nu_\alpha, \quad tG(b(t)\cdot) \xrightarrow{\mathrm{v}} c\nu_\alpha.$$

Now suppose $\boldsymbol{Y}$ and $\boldsymbol{Z}$ are $[0,\infty)^d$-valued random vectors with distributions F, G respectively. In the multivariate regular variation context, we say that F and G (or by abuse of language $\boldsymbol{Y}$ and $\boldsymbol{Z}$) *are tail equivalent on the cone* $\mathfrak{C} \subset \mathbb{E}$ if there exists a scaling function $b(t) \uparrow \infty$ such that

$$t\,\mathrm{P}\left[\frac{\boldsymbol{Y}}{b(t)} \in \cdot\right] = tF(b(t)\cdot) \xrightarrow{\mathrm{v}} \nu \quad \text{and} \quad t\,\mathrm{P}\left[\frac{\boldsymbol{Z}}{b(t)} \in \cdot\right] = tG(b(t)\cdot) \xrightarrow{\mathrm{v}} c\nu \tag{3.1}$$

in $M_+(\mathfrak{C})$, for some constant $c > 0$ and non-zero Radon measure ν on $\mathfrak{C}$. We shall write

$$\boldsymbol{Y} \overset{te(\mathfrak{C})}{\sim} \boldsymbol{Z}.$$

If $\{\boldsymbol{Y}_n, n \geqslant 1\}$ is an i.i.d. sample from F and $\{\boldsymbol{Z}_n, n \geqslant 1\}$ is an i.i.d. sample from G, then

$$\sum_{i=1}^{n} \epsilon_{\boldsymbol{Y}_i/b(n)} \Rightarrow \text{ PRM } (\nu), \quad \text{and} \quad \sum_{i=1}^{n} \epsilon_{\boldsymbol{Z}_i/b(n)} \Rightarrow \text{ PRM } (c\nu), \tag{3.2}$$

in $M_p(\mathfrak{C})$, the space of Radon point measures on $\mathfrak{C}$. Here PRM(ν) means a Poisson random measure with points in $\mathfrak{C}$ whose mean measure is ν. The two convergence statements in (3.2) mean that extremes of each sample when taken in $\mathfrak{C}$ will asymptotically have the same properties. See Resnick (1987).

4. Mixture characterization when the hidden angular measure is finite.

We now show that a regularly varying distribution on $\mathbb{E}$ possessing hidden regular variation on $\mathbb{E}_0$ is tail equivalent on $\mathbb{E}$ and $\mathbb{E}_0$ to a mixture model, provided the hidden angular measure is finite. The following definitions facilitate our presentation.

Suppose $\boldsymbol{Y}$ is a random vector whose distribution is multivariate regularly varying on $\mathfrak{C}$ and satisfies (2.1). Call $\boldsymbol{Y}$ *extremally dependent* on $\mathfrak{C}_0$, if the limit measure ν is non-zero on $\mathfrak{C}_0$. (If $(\mathfrak{C}, \mathfrak{C}_0 = \mathbb{E}, \mathbb{E}_0)$, then this means that $\boldsymbol{Y}$ is not asymptotically independent.)

A multivariate regularly varying random vector $\boldsymbol{Y}$ is called *completely asymptotically independent*, if its support is bounded away from $\boldsymbol{\infty}$. Then for some $t > 0$, the support is contained in $[t\boldsymbol{1}, \boldsymbol{\infty}]^c$ and hence cannot have any hidden regular variation, though it will surely be asymptotically independent. Therefore, an exactly independent random vector, that is, a vector with independent components, cannot be completely asymptotic independent (cf. Resnick, 2002b, Example 2). A completely asymptotically independent vector is tail equivalent on the cone $\mathbb{E}$ to a random variable $\boldsymbol{Y}$ which is a mixture of regularly varying distributions concentrating on the axes. This means $\boldsymbol{Y}$ can be represented as follows: Let $I, \{X_i, i = 1, \dots, d\}$ be independent and $\{X_i\}$ i.i.d. with one-dimensional distribution $F(x)$ with $\bar{F}(x) \in RV_{-\alpha}$. Then $\boldsymbol{Y}$ is tail equivalent on $\mathbb{E}$ to $\sum_{j=1}^{d} 1_{[I=j]} X_j \boldsymbol{e}_j$. Also note that the parameters χ, and $\bar{\chi}$ (cf. Coles et al., 1999, Heffernan, 2000) will have values 0 and ∞ respectively. The two concepts of extremal dependence and complete asymptotic independence represent two extreme situations of hidden regular variation.

4.1. Characterization for finite hidden angular measure.

The next Theorem summarizes how tail behavior of the distributions on $\mathbb{E}$ possessing hidden regular variation on $\mathbb{E}_0$ with finite hidden angular measure can be characterized in terms of tail equivalence.

Theorem 4.1. *Suppose F is a multivariate regularly varying distribution on $\mathbb{E}$ with hidden regular variation on $\mathbb{E}_0$, finite hidden angular measure, and scaling functions $b(t), b_0(t)$ with $b(t)/b_0(t) \to \infty$. Then F is tail equivalent on both the cones $\mathbb{E}$ and $\mathbb{E}_0$ to a distribution which is mixture of*

(i) *a completely asymptotically independent distribution on $\mathbb{E}$ (with no hidden regular variation) whose marginal distributions have scaling function $b(t)$,*
   and
(ii) *an extremally dependent distribution on $\mathbb{E}$ with scaling function $b_0(t)$. (The tails of this extremally dependent distribution are thus lighter than those of the completely asymptotically independent distribution.)*

Conversely, if F is tail equivalent to a mixture as above with $b(t)/b_0(t) \to \infty$, then F is multivariate regularly varying distribution on $\mathbb{E}$ and has hidden regular variation on $\mathbb{E}_0$ with finite hidden angular measure and with scaling functions b and b_0.

Proof. Let $\boldsymbol{Y}$ be a d-dimensional random vector distributed as F, which is multivariate regularly varying on $\mathbb{E}$ with hidden regular variation on $\mathbb{E}_0$. Suppose $\boldsymbol{Y}$ has finite hidden angular measure which, without loss of generality, we take to be a probability measure. Then $\boldsymbol{Y}$ satisfies the conditions (2.3) and (2.5) with $c_0 = 1$. Extend the hidden angular measure S_0, defined in (2.5), to $\aleph$ by defining it to be 0 on $\aleph \setminus \aleph_0$. Recall that the one-dimensional marginal distributions of $\boldsymbol{Y}$ are identical. Take i.i.d. random variables $X_1, \dots, X_d$ having the same distribution as Y_1 and independent of the independent pair R_0 and $\mathbf{\Theta}_0$ (all defined on the same probability space), where $\mathbf{\Theta}_0$ is distributed as S_0 and R_0 is distributed as F_0 given by

$$\overline{F_0}(x) = \begin{cases} 1, & x \leqslant 1 \\ \frac{1}{b_0^{\leftarrow}(x)}, & x \geqslant 1 \end{cases}, \tag{4.1}$$

since $b_0^{\leftarrow}(1) = 1$. We further take an integer valued random variable I independent of all the rest (and defined on the same probability space as before), which takes value 0 with probability $\frac{1}{2}$ and values $1, 2, \ldots, d$ with probability $\frac{1}{2d}$ each. Define $\boldsymbol{V}$ to be the random vector by transforming the pair $(R_0, \mathbf{\Theta}_0)$ to Cartesian coordinates, that is, $\boldsymbol{V} = R_0 \mathbf{\Theta}_0$. Finally define

$$\boldsymbol{Z} = \mathbb{1}_{\{I=0\}} \boldsymbol{V} + \sum_{i=1}^{d} \mathbb{1}_{\{I=i\}} X_i \boldsymbol{e}_i, \tag{4.2}$$

and we claim that the distribution of $\boldsymbol{Z}$ is tail equivalent to F on both $\mathbb{E}$ and $\mathbb{E}_0$.

First we check that $\boldsymbol{V}$ is multivariate regularly varying on $\mathbb{E}$ with scaling function $b_0(t)$ and hence also on $\mathbb{E}_0$. We check this using polar coordinates through the convergence in (2.3). For any Borel set $\Lambda \subset \aleph$ and $x > 0$,

$$t\,\mathrm{P}\left[\frac{R_V}{b_0(t)} > x, \mathbf{\Theta}_V \in \Lambda\right] = t\,\mathrm{P}\left[\frac{R_0}{b_0(t)} > x\right] S_0(\Lambda) = t\frac{1}{b_0^{\leftarrow}(b_0(t)x)} S_0(\Lambda) \sim x^{-\alpha_0} S_0(\Lambda). \tag{4.3}$$

Also since $S_0 \not\equiv 0$, $S_0(\aleph \setminus \aleph_0) = 0$, $\boldsymbol{V}$ is extremally dependent. So $\boldsymbol{Z}$ is a mixture of a multivariate regularly varying random vector $\boldsymbol{V}$ which is extremally dependent and a random vector whose distribution is multivariate regularly varying and completely asymptotically independent.

Next observe that

$$\mathbf{\Theta}_Z = \mathbb{1}_{\{I=0\}} \mathbf{\Theta}_0 + \sum_{i=1}^{d} \mathbb{1}_{\{I=i\}} \boldsymbol{e}_i \tag{4.4}$$

and

$$R_Z = \mathbb{1}_{\{I=0\}} R_0 + \sum_{i=1}^{d} \mathbb{1}_{\{I=i\}} X_i. \tag{4.5}$$

Then for Λ_0, a compact subset of $\aleph_0$ (and hence $\boldsymbol{e}_i \notin \Lambda_0$, $i = 1, \ldots, d$), we have that

$$\begin{aligned} t\,\mathrm{P}\left[\frac{R_Z}{b_0(t)} > x, \mathbf{\Theta}_Z \in \Lambda_0\right] &= t\,\mathrm{P}\left[I = 0, \frac{R_0}{b_0(t)} > x, \mathbf{\Theta}_0 \in \Lambda_0\right] + \sum_{i=1}^{d} t\,\mathrm{P}\left[I = i, \frac{X_i}{b_0(t)} > x, \boldsymbol{e}_i \in \Lambda_0\right] \\ &= \frac{1}{2} T\,\mathrm{P}[R_0 > b_0(t)x] S_0(\Lambda_0) + 0 \\ &\sim \frac{1}{2} x^{-\alpha_0} S_0(\Lambda_0), \end{aligned} \tag{4.6}$$

by (4.3), since $\boldsymbol{e}_i \notin \Lambda_0$, for $i = 1, 2, \ldots, d$. Therefore,

$$t\,\mathrm{P}\left[\left(\frac{R_Z}{b_0(t)}, \mathbf{\Theta}_Z\right) \in \cdot\right] \xrightarrow{\mathrm{v}} \frac{1}{2} \nu_{\alpha_0} \times S_0 \text{ on } (0, \infty] \times \aleph_0.$$

Again, for Λ, a subset of $\aleph$, we have that

$$\begin{aligned} t\,\mathrm{P}\left[\frac{R_Z}{b(t)} > x, \mathbf{\Theta}_Z \in \Lambda\right] &= t\,\mathrm{P}\left[I = 0, \frac{R_0}{b(t)} > x, \mathbf{\Theta}_0 \in \Lambda\right] + \sum_{i=1}^{d} t\,\mathrm{P}\left[I = i, \frac{X_i}{b(t)} > x, \boldsymbol{e}_i \in \Lambda\right] \\ &= \frac{1}{2} t\,\mathrm{P}[R_0 > b(t)x] S_0(\Lambda) + \frac{1}{2} t\,\mathrm{P}[X_1 > b(T)x] \frac{1}{d} \sum_{i=1}^{d} \mathbb{1}_{\{\boldsymbol{e}_i\}}(\Lambda) \end{aligned}$$

$$\to 0+\frac{c^*}{2}x^{-\alpha}\frac{1}{d}\sum_{i=1}^{d}\mathbb{1}_{\{\boldsymbol{e}_i\}}(\Lambda), \tag{4.7}$$

where $c^*=\nu((1,\infty]\times[0,\infty]^{d-1})$. The 0 in the last line of the previous display results from $b_0^{\leftarrow}(b(t))\to\infty$ since $b(t)/b_0(t)\to\infty$ implies $b^{\leftarrow}(t)/b_0^{\leftarrow}(t)\to 0$ which in turn implies $1/b_0^{\leftarrow}(b(s))\to 0$ as $s\to\infty$. Thus

$$t\,\mathrm{P}\left[\left(\frac{R_Z}{b(t)},\boldsymbol{\Theta}_Z\right)\in\cdot\right]\xrightarrow{\mathrm{v}}\frac{c^*}{2}\nu_\alpha\times S \text{ on } (0,\infty]\times\aleph,$$

where S is the uniform distribution on $\{\boldsymbol{e}_1,\dots,\boldsymbol{e}_d\}$. Therefore, $\boldsymbol{Y}$ and $\boldsymbol{Z}$ are asymptotically equivalent on both the cones $\mathbb{E}$ and $\mathbb{E}_0$.

Conversely, suppose $\boldsymbol{Y}$ has hidden regular variation and (i) and (ii) hold with $b(t)/b_0(t)\to\infty$. Since a completely asymptotically independent distribution cannot have a non-zero hidden angular measure, and the scaling function $b_0(t)$ of the extremally dependent part is of smaller order than that of the completely asymptotically independent part, the hidden angular measure of F is a multiple of the hidden angular measure of the extremally dependent part, which is finite. □

4.2. **Significance of a finite angular measure on $\mathbb{E}_0$.** Before considering distributions with infinite hidden angular measure, we highlight the significance of having a finite angular measure. Having a finite hidden angular measure on $\mathbb{E}_0$ means that regular variation on $\mathbb{E}_0$ can be extended to regular variation of the same order on the full cone $\mathbb{E}$ and that marginals are regularly varying.

Theorem 4.2. *Suppose $\boldsymbol{V}$ is regularly varying on $\mathbb{E}_0$, with index α_0, scaling function $b_0(t)$, limit measure ν_0, angular measure S_0 on $\aleph_0$. The following are equivalent:*

(i) *S_0 is finite on $\aleph_0$.*

(ii) *There exists a random vector $\boldsymbol{V}_*$ defined on $\mathbb{E}_0$ such that*

$$\boldsymbol{V}_*\overset{te(\mathbb{E}_0)}{\sim}\boldsymbol{V}$$

on $\mathbb{E}_0$ and for $i=1,\dots,d$

$$t\,\mathrm{P}\left[V_*^{(i)}>b_0(t)x\right]\to cx^{-\alpha_0},\quad t\to\infty$$

for some $c>0$, so that each component $V_*^{(i)}$ has regularly varying tail probabilities with index α_0.

(iii) *There exists a random vector $\boldsymbol{V}_*$ defined on $\mathbb{E}_0$ such that*

$$\boldsymbol{V}_*\overset{te(\mathbb{E}_0)}{\sim}\boldsymbol{V}$$

on $\mathbb{E}_0$ and $\boldsymbol{V}_*$ is regularly varying on the full cone $\mathbb{E}$ with scaling function $b_0(t)$ and limit measure ν and

$$\nu|_{\mathbb{E}_0}=\nu_0;$$

that is, the restriction of ν to $\mathbb{E}_0$ is ν_0.

(iv) *There exists a random vector $\boldsymbol{V}_*$ defined on $\mathbb{E}_0$ such that*

$$\boldsymbol{V}_*\overset{te(\mathbb{E}_0)}{\sim}\boldsymbol{V}$$

on $\mathbb{E}_0$ such that for any $\boldsymbol{s}\in[\boldsymbol{0},\boldsymbol{\infty})\setminus\{\boldsymbol{0}\}$ and any $\boldsymbol{a}\in(\boldsymbol{0},\boldsymbol{\infty}]\setminus\bigcup_{i=1}^{d}\{\boldsymbol{e}_i^{-1}\}$

$$\bigvee_{i=1}^{d}s_iV^{(i)}\quad\textit{and}\quad\bigwedge_{i=1}^{d}a_iV^{(i)},$$

are tail equivalent on $[0,\infty)$ and have regularly varying tail probabilities of index α_0.

Remark 4.1. We cannot replace $\boldsymbol{V}_*$ with $\boldsymbol{V}$ and still hope the assertions to hold. In (4.2), we have $\boldsymbol{Z}$ regularly varying on $\mathbb{E}_0$ but the marginal distributions are not regularly varying with index α_0.

Proof of Theorem 4.2. (i)⇒(iii): S_0 is defined on $\mathbb{E}_0$. It can be extended to all of $\mathbb{E}$ by setting $S_0(\mathbf{e}_i) = 0$. Now repeat the construction of $R_0, \boldsymbol{\Theta}_0$ given prior to and including (4.1) and this time define $\boldsymbol{V}_* = R_0\boldsymbol{\Theta}_0$ on $\mathbb{E}$ which will be regularly varying on $\mathbb{E}$ with scaling function $b_0(t)$, limit measure ν with $\nu|_{\mathbb{E}_0} = \nu_0$.

(iii)⇒(i): If $\boldsymbol{V}_*$ is regularly varying on $\mathbb{E}$, then

$$t\,\mathrm{P}[\|\boldsymbol{V}_*\| > b_0(t)] \to \nu\{\boldsymbol{x} \in \mathbb{E} : \|\boldsymbol{x}\| > 1\} < \infty.$$

Therefore

$$\nu\{\boldsymbol{x} \in \mathbb{E}_0 : \|\boldsymbol{x}\| > 1\} = \nu_0\{\boldsymbol{x} \in \mathbb{E}_0 : \|\boldsymbol{x}\| > 1\} = S_0(\aleph_0) < \infty.$$

(iii)⇒(ii): Since

$$t\,\mathrm{P}\left[\frac{\boldsymbol{V}_*}{b_0(t)} \in \cdot\right] \to \nu$$

on $\mathbb{E}$, we have that

$$t\,\mathrm{P}\left[\frac{\boldsymbol{V}_*^{(1)}}{b_0(t)} > x\right] = t\,\mathrm{P}\left[\frac{\boldsymbol{V}_*}{b_0(t)} \in (x,\infty] \times [0,\infty]^{d-1}\right] \to \nu\left((x,\infty] \times [0,\infty]^{d-1}\right).$$

This implies marginal regular variation.

(ii)⇒(iii): Suppose $\boldsymbol{V}_*$ is regularly varying on $\mathbb{E}_0$ and the one dimensional marginal tails are also regularly varying. For $\boldsymbol{x} \in \mathbb{E}$ such that at least 2 components are strictly positive, we have by inclusion-exclusion

$$\begin{aligned} t\,\mathrm{P}\left(\left[\frac{\boldsymbol{V}_*}{b_0(t)} \leqslant \boldsymbol{x}\right]^c\right) =& t\,\mathrm{P}\left(\bigcup_{i=1}^{d}\left[\frac{\boldsymbol{V}_*^{(i)}}{b_0(t)} > x^{(i)}\right]\right) \\ =& t\sum_{i=1}^{d}\mathrm{P}\left[\frac{\boldsymbol{V}_*^{(i)}}{b_0(t)} > x^{(i)}\right] - t\sum_{1\leqslant i<j\leqslant d}\mathrm{P}\left[\frac{\boldsymbol{V}_*^{(i)}}{b_0(t)} > x^{(i)}, \frac{\boldsymbol{V}_*^{(j)}}{b_0(t)} > x^{(j)}\right] + \\ & \cdots + (-1)^{d+1}t\,\mathrm{P}\left[\frac{\boldsymbol{V}_*}{b_0(t)} > \boldsymbol{x}\right]. \end{aligned}$$

Convergence of the first sum results from one-dimensional marginal convergence. Convergence of the other terms results from vague convergence on $\mathbb{E}_0$.

To understand (iv), see Theorem 1 of Resnick (2002b). □

5. Hidden regular variation when the hidden angular measure is infinite.

Techniques used in the previous section when the hidden angular measure was assumed finite need modification when the hidden angular measure is infinite. We discuss the differences between the two cases, provide tools for discussing regular variation on a subcone of $\mathbb{E}$, give a broad class of examples for this infinite case and provide a characterization.

5.1. Regular variation on $\mathbb{E}_0$.

When the hidden angular measure was finite, regular variation on $\mathbb{E}_0$ could be extended to all of $\mathbb{E}$ with the same index; see Theorem 4.2. The situation when the hidden angular measure is infinite, is rather different. If regular variation can be extended to all of $\mathbb{E}$, then the order of the extension is different.

Theorem 5.1. *Suppose* $\boldsymbol{V}$ *is defined on* $\mathbb{E}$*, has equal marginal distributions, and is regularly varying on* $\mathbb{E}_0$*, with index* α_0*, scaling function* $b_0(t) \in RV_{1/\alpha_0}$*, limit measure* ν_0*, infinite angular measure* S_0 *on* $\aleph_0$*. Then* $\boldsymbol{V}$ *is also regularly varying on* $\mathbb{E}$ *iff there exists* $\alpha \leqslant \alpha_0$ *such that*

$$\mathrm{P}\left[V^{(i)} > x\right] \in RV_{-\alpha}, \quad i = 1,\dots,d,$$

and with the quantile function

$$b(t) = \left(\frac{1}{\mathrm{P}[V^{(i)} > \cdot]}\right)^{\leftarrow}(t) \tag{5.1}$$

satisfying $b(t)/b_0(t) \to \infty$ *as* $t \to \infty$*.*

Proof. Only if part: Suppose $\boldsymbol{V}$ is regularly varying on $\mathbb{E}$ with limit measure ν. We may assume the quantile function given in (5.1) is the scaling function of the regular variation. If $\nu(\mathbb{E}_0) > 0$, then for some $\boldsymbol{x} > \boldsymbol{0}$,

$$t\,\mathrm{P}\left[\frac{\boldsymbol{V}}{b_0(t)} > \boldsymbol{x}\right] \to \nu_0((\boldsymbol{x}, \boldsymbol{\infty}]) > 0,$$
$$t\,\mathrm{P}\left[\frac{\boldsymbol{V}}{b(t)} > \boldsymbol{x}\right] \to \nu((\boldsymbol{x}, \boldsymbol{\infty}]) > 0$$

which implies $b(t) \sim c b_0(t)$, for some $c > 0$. This means that regular variation on $\mathbb{E}_0$ is extended to all of $\mathbb{E}$ and, by Theorem 4.2, S_0, the hidden angular measure, is finite. This is a contradiction and hence we conclude we must have $\nu(\mathbb{E}_0) = 0$. But then we have

$$\frac{\mathrm{P}[\boldsymbol{V}/b(t) > \boldsymbol{x}]}{\mathrm{P}[\boldsymbol{V}/b_0(t) > \boldsymbol{x}]} \to 0,$$

which implies $b(t)/b_0(t) \to \infty$ as $t \to \infty$. Thus we conclude $\alpha \leqslant \alpha_0$. Also

$$t\,\mathrm{P}\left[\frac{V^{(1)}}{b(t)} \geqslant x\right] = t\,\mathrm{P}\left[\frac{\boldsymbol{V}}{b(t)} \in [x, \infty] \times [0, \infty]^{d-1}\right] \to \nu\big([x, \infty] \times [0, \infty]^{d-1}\big)$$

which implies $\mathrm{P}[V^{(1)} > x] \in RV_{-\alpha}$.

If part: The converse follows from Corollary 1 in Resnick (2002b) without the assumption of infinite hidden angular measure. □

5.2. **A coordinate system for regular variation on a subcone; the inverse norm transformation.** Consider the map $\boldsymbol{x} \mapsto \boldsymbol{x}^{-1}$ from $[\boldsymbol{0}, \boldsymbol{\infty}] \mapsto [\boldsymbol{0}, \boldsymbol{\infty}]$; that is

$$\left(x^{(1)}, \dots, x^{(d)}\right) \mapsto \left(\frac{1}{x^{(1)}}, \dots, \frac{1}{x^{(d)}}\right).$$

(Here we define $1/\infty = 0$ and $1/0 = \infty$.) Let $\|\cdot\|$ be a norm on $\mathbb{R}^d$. Extend the definition from $[0, \infty)^d$ to $[0, \infty]^d$ by defining $\|\boldsymbol{x}\| = \infty$ if any component $x^{(i)}$ of $\boldsymbol{x}$ is infinite. Now define

$$\|\cdot\|_{\text{INV}} : [0, \infty]^d \mapsto [0, \infty], \quad \text{by} \quad \|\boldsymbol{x}\|_{\text{INV}} = \frac{1}{\|\boldsymbol{x}^{-1}\|},$$

and

$$\aleph_{\text{INV}} = \{\boldsymbol{x} \in \mathbb{E} : \|\boldsymbol{x}\|_{\text{INV}} = 1\}.$$

Note $\|\boldsymbol{x}\|_{\text{INV}} = \infty$ iff $\boldsymbol{x} = \boldsymbol{\infty}$ and $\|\boldsymbol{x}\|_{\text{INV}} = 0$ iff $\wedge_{i=1}^d x^{(i)} = 0$. Furthermore, observe that $\|\cdot\|_{\text{INV}}$ is homogeneous:

$$\|t\boldsymbol{x}\|_{\text{INV}} = t\|\boldsymbol{x}\|_{\text{INV}}$$

and that $\{\boldsymbol{x} \in \mathbb{E} : \|\boldsymbol{x}\|_{\text{INV}} \geqslant 1\}$ is a compact neighborhood of $\boldsymbol{\infty}$. If $\|\boldsymbol{x}\| = \sum_{i=1}^d x^{(i)}$, then

$$\|\boldsymbol{x}\|_{\text{INV}} = \frac{1}{\sum_{i=1}^d (x^{(i)})^{-1}} = \frac{\prod_{i=1}^d x^{(i)}}{\sum_{i=1}^d \prod_{j \neq i} x^{(j)}}.$$

If $\|\boldsymbol{x}\| = \vee_{i=1}^d x^{(i)}$ then $\|\boldsymbol{x}\|_{\text{INV}} = \wedge_{i=1}^d x^{(i)}$. In this case, the set $\{\boldsymbol{x} \in \mathbb{E} : \|\boldsymbol{x}\|_{\text{INV}} \geqslant 1\}$ is the compact hypercube, whose diagonally opposite vertices are $\boldsymbol{1}$ and $\boldsymbol{\infty}$. This has the advantage of being immune to whether the hidden angular measure is infinite or not. Thus, if ν_0 is the limit measure in the definition of regular variation on $\mathbb{E}_0$,

$$S_{\text{INV}}(\Lambda) := \nu_0\left(\left\{\boldsymbol{x} \in \mathbb{E} : \|\boldsymbol{x}\|_{\text{INV}} > 1; \frac{\boldsymbol{x}}{\|\boldsymbol{x}\|_{\text{INV}}} \in \Lambda\right\}\right)$$

is a finite measure on $\aleph_{\text{INV}}$. Because of homogeneity of both $\|\cdot\|_{\text{INV}}$ and ν_0 we also have the product form:

$$\nu_0\left(\left\{\boldsymbol{x} \in \mathbb{E} : \|\boldsymbol{x}\|_{\text{INV}} > r; \frac{\boldsymbol{x}}{\|\boldsymbol{x}\|_{\text{INV}}} \in \Lambda\right\}\right) = r^{-\alpha_0} S_{\text{INV}}(\Lambda)$$

for $r > 0$ and Λ a Borel subset of $\aleph_{\text{INV}}$. Note for $d = 2$, $S_{\text{INV}}(\aleph_{\text{INV}}) > 0$ but for $d > 2$, it is possible that $S_{\text{INV}} \equiv 0$. See Example 5.2 below.

Recall from Section 1.3, that the domain of the polar coordinate transform is $[\mathbf{0},\boldsymbol{\infty})\setminus\{\mathbf{0}\}$. Define

$$T_{\text{INV}}:(0,\infty]^d\setminus\{\boldsymbol{\infty}\}\mapsto(0,\infty)\times\aleph_{\text{INV}};\quad T_{\text{INV}}(\boldsymbol{x})=\left(\|\boldsymbol{x}\|_{\text{INV}},\frac{\boldsymbol{x}}{\|\boldsymbol{x}\|_{\text{INV}}}\right),$$

so that $T_{\text{INV}}^{\leftarrow}:(0,\infty)\times\aleph_{\text{INV}}\mapsto(0,\infty]^d\setminus\{\boldsymbol{\infty}\}$. For a random vector $\boldsymbol{V}$ defined on $(0,\infty]^d$ define

$$T_{\text{INV}}(\boldsymbol{V})=(R_{\text{INV}},\boldsymbol{\Theta}_{\text{INV}}).$$

Regular variation on the subcone $(\mathbf{0},\boldsymbol{\infty}]$ can be characterized in terms of these coordinates.

Theorem 5.2. *For a random vector $\boldsymbol{V}$ on $\mathbb{E}$, $\boldsymbol{V}$ is regularly varying on $\mathbb{E}_{00}:=(\mathbf{0},\boldsymbol{\infty}]$ with scaling function $b_{00}(t)$, limit measure ν_{00}:*

$$t\,\mathrm{P}\left[\frac{\boldsymbol{V}}{b_{00}(t)}\in\cdot\right]\xrightarrow{v}\nu_{00},$$

in $M_+(\mathbb{E}_{00})$ iff

$$t\,\mathrm{P}\left[\left(\frac{R_{\text{INV}}}{b_{00}(t)},\boldsymbol{\Theta}_{\text{INV}}\right)\in\cdot\right]\xrightarrow{v}c_{00}\nu_{\alpha_{00}}\times S_{\text{INV}}$$

in $M_+((0,\infty]\times\aleph_{\text{INV}})$, with $c_{00}\in(0,\infty)$.

The proof is similar to the one used to express multivariate regular variation on $\mathbb{E}$ in terms of the usual polar coordinates and is hence omitted.

Remark 5.1. The characterization in Theorem 1 of Resnick (2002b) shows why in Theorem 5.2 we cannot get a characterization in terms of transformed coordinates on $\mathbb{E}_0$. Note, however, when $d=2$ that $\mathbb{E}_0=\mathbb{E}_{00}$. The following example gives further insight into why a characterization on $\mathbb{E}_0$ is not possible.

Example 5.1. Suppose $d=3$ and X_1,X_2,X_3 are i.i.d. Pareto random variables with parameter 1. Then one checks that $\alpha=1$, $\alpha_0=2$ and $\alpha_{00}=3$. Thus the regular variation behavior on $\mathbb{E}_{00}$ is essentially different from the regular variation on $\mathbb{E}_0$, as the latter is determined by the behavior on the two-dimensional faces.

Except for some clearly noted exceptions, we proceed assuming $d=2$ and assuming $\|\boldsymbol{x}\|_{\text{INV}}=x^{(1)}\wedge x^{(2)}$.

5.3. **Hidden regular variation in standard form.** Theoretical developments are often worked out for multivariate regular variation when a standard form is assumed (cf. Resnick, 1987, pp. 265, 277) which allows the scaling function to be linear. For hidden regular variation, there are two scaling functions and we have to choose which one to make linear. It is more convenient to transform so that the hidden scaling function $b_0(t)$ becomes linear.

Theorem 5.3. *The vector $\boldsymbol{V}$ possesses hidden regular variation and is regularly varying on $\mathbb{E}$ and $\mathbb{E}_0$ with limit measures ν,ν_0, indices α,α_0, and scaling functions $b(t),b_0(t)=(1/\,\mathrm{P}[\|\boldsymbol{V}\|_{\text{INV}}>\cdot])^{\leftarrow}(t)$ with $b(t)/b_0(t)\to\infty$ iff for*

$$U(x)=\frac{1}{P[\|\boldsymbol{V}\|_{\text{INV}}>x]},\tag{5.2}$$

we have $U(\boldsymbol{V}):=\big(U(V^{(1)}),U(V^{(2)})\big)$ regularly varying on $\mathbb{E},\mathbb{E}_0$ with limit measures ν_*,ν_{0*}, indices $(\alpha/\alpha_0,1)$ and scaling functions $b_*(t)=U\circ b(t)$ and $b_{0*}(t)=t$.

Proof. For regular variation on $\mathbb{E}_0$: We have

$$U\circ b_0(t)\sim t,$$

and therefore for $\boldsymbol{x}>\mathbf{0}$,

$$\begin{aligned}t\,\mathrm{P}\left[\frac{U(\boldsymbol{V})}{t}>\boldsymbol{x}\right]=&t\,\mathrm{P}\left[\boldsymbol{V}>\big(U^{\leftarrow}(tx^{(1)}),U^{\leftarrow}(tx^{(2)})\big)\right]\\=&t\,\mathrm{P}\left[\frac{\boldsymbol{V}}{U^{\leftarrow}(t)}>\left(\frac{U^{\leftarrow}(tx^{(1)})}{U^{\leftarrow}(t)},\frac{U^{\leftarrow}(tx^{(2)})}{U^{\leftarrow}(t)}\right)\right]\end{aligned}$$

$$\to \nu_0\left(\left(\left((x^{(1)})^{1/\alpha_0}, (x^{(2)})^{1/\alpha_0}\right), \boldsymbol{\infty}\right]\right) =: \nu_{0*}\big((\boldsymbol{x}, \boldsymbol{\infty}]\big).$$

For regular variation on $\mathbb{E}$: We have

$$\lim_{t\to\infty} t\,\mathrm{P}\left[\frac{U(V^{(i)})}{U(b(t))} > x\right] = \lim_{t\to\infty} t\,\mathrm{P}\left[\frac{V^{(i)}}{b(t)} > \frac{U^{\leftarrow}\big(xU(b(t))\big)}{b(t)}\right]$$
$$= \lim_{t\to\infty} t\,\mathrm{P}\left[\frac{V^{(i)}}{b(t)} > \frac{U^{\leftarrow}\big(xU(b(t))\big)}{U^{\leftarrow}(U(b(t)))}\right]$$

since $U^{\leftarrow} \circ U(t) \sim t$ and because $U(b(t)) \to \infty$, we get this limit to be

$$= \lim_{t\to\infty} \left(\frac{U^{\leftarrow}(tx)}{U^{\leftarrow}(t)}\right)^{-\alpha} = x^{-\alpha/\alpha_0},$$

where $\alpha/\alpha_0 \leqslant 1$. □

Take the inverse norm transform of the coordinates in standard form. Define

$$R_* = \|U(\boldsymbol{V})\|_{\text{INV}} = U(V^{(1)}) \wedge U(V^{(2)}); \quad \boldsymbol{\Theta}_* = \frac{U(\boldsymbol{V})}{R_*}.$$

Corollary 5.1. *Suppose the vector $\boldsymbol{V}$ possesses hidden regular variation and is regularly varying on $\mathbb{E}$ and $\mathbb{E}_0$ with limit measures ν, ν_0, indices (α, α_0), and scaling functions $b(t), b_0(t) = (1/\mathrm{P}[\|\boldsymbol{V}\|_{\text{INV}} > \cdot])^{\leftarrow}(t)$ with $b(t)/b_0(t) \to \infty$. In standard form, for $(R_*, \boldsymbol{\Theta}_*)$ we have*

$$t\,\mathrm{P}\left[\left(\frac{R_*}{t}, \boldsymbol{\Theta}_*\right) \in dr \times d\boldsymbol{\theta}\right] \xrightarrow{\text{v}} r^{-2}dr \times S_{\text{INV}*}(d\boldsymbol{\theta})$$

in $M_+((0,\infty] \times \aleph_{\text{INV}})$, where $S_{\text{INV}*}$ is a finite measure on $\aleph_{\text{INV}}$.

If U given in (5.2) *is continuous and strictly increasing, R_* is Pareto with parameter 1.*

Proof. This follows from the fact that U is non-decreasing and

$$U\left(V^{(1)}\right) \wedge U\left(V^{(2)}\right) = U\left(V^{(1)} \wedge V^{(2)}\right).$$

□

5.3.1. *Identifying an infinite hidden angular measure.* We can identify when the hidden angular measure is infinite using $S_{\text{INV}*}$. Define the finite measures $G_i,\ i = 1, 2$ by

$$\bar{G}_1(s) = S_{\text{INV}*}\big(\{1\} \times (s, \infty]\big), \quad \bar{G}_2(s) = S_{\text{INV}*}\big((s, \infty \times \{1\}]\big), \quad s \geqslant 1.$$

Proposition 5.1. *The limit measure for regular variation on $\mathbb{E}_0$ has infinite angular measure, or equivalently,*

$$\nu_{0*}\{\boldsymbol{x} \in \mathbb{E}_0 : \|\boldsymbol{x}\| > 1\} = \infty,$$

iff

$$\max_{i=1,2} \int_1^\infty \bar{G}_i(s)ds = \infty. \tag{5.3}$$

Without standardization, the condition corresponding to (5.3) *becomes*

$$\max_{i=1,2} \int_1^\infty s^{\alpha_0 - 1}\bar{G}_i(s)ds = \infty,$$

where $\bar{G}_1(s) = \nu_0\left(\left\{\left(x^{(1)}, x^{(2)}\right) : \boldsymbol{x} \geqslant \mathbf{1}, x^{(1)}/x^{(2)} > s\right\}\right)$ and $\bar{G}_2(s)$ is defined similarly.

Proof. Let $0 < x^{(1)} < x^{(2)}$. We have

$$\begin{aligned}
\nu_{0*}\big((\boldsymbol{x},\boldsymbol{\infty}]\big) =& \nu_{0*}\big(\{\boldsymbol{y}\in\mathbb{E}_0 : \boldsymbol{y} > \boldsymbol{x}\}\big) = \nu_{0*}\Big(\{\boldsymbol{y} : \|\boldsymbol{y}\|_{\text{INV}}\cdot\frac{\boldsymbol{y}}{\|\boldsymbol{y}\|_{\text{INV}}} > \boldsymbol{x}\}\Big)\\
=& \iint\limits_{\substack{r\in(0,\infty)\\ \boldsymbol{\theta}\in\aleph_{\text{INV}}\\ r\boldsymbol{\theta}>\boldsymbol{x}}} \nu_{0*}(dr\times d\boldsymbol{\theta}) = \iint\limits_{\substack{r\in(0,\infty)\\ \boldsymbol{\theta}\in\aleph_{\text{INV}}\\ r\boldsymbol{\theta}>\boldsymbol{x}}} r^{-2}dr S_{\text{INV}*}(d\boldsymbol{\theta})\\
=& \int_{\boldsymbol{\theta}\in\aleph_{\text{INV}}}\int_{r>\frac{x^{(1)}}{\theta^{(1)}}\vee\frac{x^{(2)}}{\theta^{(2)}}} r^{-2}dr S_{\text{INV}*}(d\boldsymbol{\theta})\\
=& \int_{\boldsymbol{\theta}\in\aleph_{\text{INV}}}\left(\frac{x^{(1)}}{\theta^{(1)}}\vee\frac{x^{(2)}}{\theta^{(2)}}\right)^{-1} S_{\text{INV}*}(d\boldsymbol{\theta})
\end{aligned}$$

and splitting $\aleph_{\text{INV}} = \{1\}\times(1,\infty]\cup(1,\infty]\times\{1\}$ we get

$$= \int_1^\infty\Big(\frac{1}{x^{(1)}}\wedge\frac{s}{x^{(2)}}\Big)G_1(ds) + \int_1^\infty\Big(\frac{s}{x^{(1)}}\wedge\frac{1}{x^{(2)}}\Big)G_2(ds)$$

and thus

$$\nu_{0*}\big((\boldsymbol{x},\boldsymbol{\infty}]\big) = \Big(x^{(2)}\Big)^{-1}\left[\int_1^{x^{(2)}/x^{(1)}} sG_1(ds) + G_2((1,\infty])\right] + (x^{(1)})^{-1}\bar{G}_1(x^{(2)}/x^{(1)}). \tag{5.4}$$

Let $x^{(1)}\to 0$. Then $\nu_{0*}\big(((0,x^{(2)}),\boldsymbol{\infty}]\big) = \infty$ iff $\int_1^\infty \bar{G}_1(s)ds = \infty$.

Similarly, if we interchange the roles of $x^{(1)}$ and $x^{(2)}$, we get that $\nu_{0*}\big(((x^{(1)},0),\boldsymbol{\infty}]\big) = \infty$ iff $\int_1^\infty \bar{G}_2(s)ds = \infty$. □

If we suppose $S_{\text{INV}*}$ is a probability measure on $\aleph_{\text{INV}}$ and that $\boldsymbol{\Theta}_{\text{INV}*}$ is a random vector on $\aleph_{\text{INV}}$ with distribution $S_{\text{INV}*}$, then the previous result requires

$$\max_{i=1,2}\mathbb{E}\left[\boldsymbol{\Theta}^{(i)}_{\text{INV}*}\right] = \infty.$$

Though the above proposition has been given only for the case $d = 2$, it can be easily extended to $d > 2$. The extension follows from the following lemma, where we use the max-norm, namely $\|\boldsymbol{x}\| = \vee_{i=1}^d x^{(i)}$:

Lemma 5.1. *Suppose $\boldsymbol{X}$ is a random vector of dimension $d > 2$, which is multivariate regularly varying on $\mathbb{E}$ with hidden regular variation on $\mathbb{E}_0$, having limit measures ν, ν_0. Define $\tilde{\mathbb{E}}$ and $\tilde{\mathbb{E}}_0$ to be the spaces corresponding to $\mathbb{E}$ and $\mathbb{E}_0$ in two dimensions; that is*

$$\tilde{\mathbb{E}} = [0,\infty]^2\setminus\{(0,0)\};\quad \tilde{\mathbb{E}}_0 = (0,\infty]^2.$$

Then for some pair $1\leqslant i<j\leqslant d$, we must have

$$\nu_0(\{\boldsymbol{x}\in\mathbb{E}_0 : x^{(i)} > 0, x^{(j)} > 0\}) \neq 0 \tag{5.5}$$

and for all such pairs we must have $(X^{(i)}, X^{(j)})$ to be multivariate regularly varying on $\tilde{\mathbb{E}}$ with hidden regular variation on $\tilde{\mathbb{E}}_0$. Also the hidden angular measure of $\boldsymbol{X}$ is finite iff for all pairs $1\leqslant i<j\leqslant d$, satisfying the condition (5.5)*, the hidden angular measure of $(X^{(i)}, X^{(j)})$ is finite.*

Remark 5.2. Condition (5.5) is automatic when $d = 2$ but cannot be dropped for $d > 2$. The identical marginal distributions do not impose any restriction on the two-dimensional behavior as is illustrated in the following example.

Example 5.2. Suppose U_i, $i = 1,2,3$ are i.i.d. Pareto random variables on $[1,\infty)$ with parameter 1. Let B_i, $i = 1,2$ be i.i.d. Bernoulli random variables with

$$P[B_i = 0] = P[B_i = 1] = \frac{1}{2},$$

and $\{B_i\}$ and $\{U_i\}$ are independent. Define

$$W = (1 - B_1)U_3, \quad \boldsymbol{X} = B_2(U_1, 0, W) + (1 - B_2)(0, U_2, W).$$

Then

$$X^{(1)} = B_2 U_1, \quad X^{(2)} = (1 - B_2)U_2, \quad X^{(3)} = W$$

are identically distributed, having atom sized $\frac{1}{2}$ at 0 and having a Pareto density with parameter 1 and total mass $\frac{1}{2}$ on $[1, \infty)$. The distribution of $\boldsymbol{X}$ is supported on the planes where either of the first two coordinates vanish and furthermore $\|\boldsymbol{X}\|_{\text{INV}} = 0$. Thus $S_{\text{INV}} \equiv 0$. It is easy to see that $\boldsymbol{X}$ is multivariate regularly varying on $\mathbb{E}$ with $\alpha = 1$ and $b(t) = t$ and has hidden regular variation on $\mathbb{E}_0$ with $\alpha_0 = 2$ and $b_0(t) = \sqrt{t}$. However, $\nu_0(\{\boldsymbol{x} : x^{(1)} > 0, x^{(2)} > 0\}) = 0$. Also, this is an example, where we have hidden regular variation on $\mathbb{E}_0$, but *not* $\mathbb{E}_{00}$. This emphasizes the lesson learned in Example 5.1, that regular variation on $\mathbb{E}_0$ and $\mathbb{E}_{00}$ can be quite different.

Proof of Lemma 5.1. First observe that $\mathbb{E}_0 = \cup_{1 \leqslant i < j \leqslant d} \{\boldsymbol{x} : x^{(i)} > 0, x^{(j)} > 0\}$ and since ν_0 is a non-zero measure on $\mathbb{E}_0$, the condition (5.5) holds for some pair $1 \leqslant i < j \leqslant d$.

As usual, let b and b_0 be the scaling functions for $\boldsymbol{X}$ with indices $1/\alpha$ and $1/\alpha_0$ respectively. Now, for $\tilde{\boldsymbol{x}} = (x^{(1)}, x^{(2)})$, the d-dimensional set $A_{ij} = \{\boldsymbol{y} : y^{(i)} > x^{(1)}, y^{(j)} > x^{(2)}\}$ is relatively compact in $\mathbb{E}_0$ and hence

$$t\,\mathrm{P}\left[\frac{X^{(i)}}{b_0(t)} > x^{(1)}, \frac{X^{(j)}}{b_0(t)} > x^{(2)}\right] = t\,\mathrm{P}\left[\frac{\boldsymbol{X}}{b_0(t)} \in A_{ij}\right] \to \nu_0(A_{ij}) =: \tilde{\nu}_{0,i,j}\left((x^{(1)}, \infty] \times (x^{(2)}, \infty]\right).$$

Note that whenever the pair (i, j) satisfies the condition (5.5), the limiting measure $\tilde{\nu}_{0,i,j}$ is non-zero. Also, for any $i = 1, \ldots, d$

$$t\,\mathrm{P}\left[\frac{X^{(i)}}{b(t)} > x\right] \to x^{-\alpha}.$$

So if (i, j) satisfies the condition (5.5), $(X^{(i)}, X^{(j)})$ is multivariate regularly varying on $\tilde{\mathbb{E}}$ and has hidden regular variation on $\tilde{\mathbb{E}}_0$. The scaling functions and indices remain the same. Also if the pair (i, j) does not satisfy the condition (5.5), then clearly $\tilde{\nu}_{0,i,j}$ is identically the zero measure.

Now for a pair (i, j) satisfying the condition (5.5), the hidden angular measure of $(X^{(i)}, X^{(j)})$ is finite iff

$$\tilde{\nu}_{0,i,j}\{(x^{(1)}, x^{(2)}) \in \tilde{\mathbb{E}}_0 : x^{(1)} \vee x^{(2)} > 1\} = \nu_0\{\boldsymbol{x} \in \mathbb{E}_0 : x^{(i)} \vee x^{(j)} > 1\} < \infty.$$

Suppose $\boldsymbol{X}$ has finite hidden angular measure. Then

$$\begin{aligned}
\infty > &\nu_0\left(\{\boldsymbol{x} \in \mathbb{E}_0 : \vee_{i=1}^d x^{(i)} > 1\}\right) = \nu_0\left(\bigcup_{i=1}^d \{\boldsymbol{x} \in \mathbb{E}_0 : x^{(i)} > 1\}\right) \\
\geqslant &\nu_0\left(\{\boldsymbol{x} \in \mathbb{E}_0 : x^{(i)} \geqslant 1\} \cup \{\boldsymbol{x} \in \mathbb{E} : x^{(j)} \geqslant 1\}\right) \\
= &\tilde{\nu}_{0,i,j}\left(\{(x^{(1)}, x^{(2)}) : x^{(1)} \vee x^{(2)} \geqslant 1\}\right).
\end{aligned}$$

Conversely, if for all $1 \leqslant i < j \leqslant d$ satisfying (5.5), the vector $(X^{(i)}, X^{(j)})$ has finite hidden angular measures, then

$$\begin{aligned}
\nu_0\left(\{\boldsymbol{x} \in \mathbb{E}_0 : \vee_{i=1}^d x^{(i)} > 1\}\right) = &\nu_0\left(\left\{\boldsymbol{x} \in \mathbb{E}_0 : \bigvee_{1 \leqslant i < j \leqslant d} (x^{(i)} \vee x^{(j)}) > 1\right\}\right) \\
\leqslant &\sum_{i \neq j} \nu_0\left(\{\boldsymbol{x} \in \mathbb{E}_0 : x^{(i)} \vee x^{(j)} > 1\}\right) \\
= &\sum_{1 \leqslant i < j \leqslant d} \tilde{\nu}_{0,i,j}\left(\{(x^{(1)}, x^{(2)}) : x^{(1)} \vee x^{(2)} > 1\}\right) < \infty,
\end{aligned}$$

since we are summing a finite number of finite terms and zeros. □

Combining Proposition 5.1 and Lemma 5.1, we have the following result:

Theorem 5.4. *Suppose $\boldsymbol{X}$ is a d-dimensional random vector defined on $[\mathbf{0},\boldsymbol{\infty})$ which is multivariate regularly varying on $\mathbb{E}$ and has hidden regular variation on $\mathbb{E}_0$ with limit measure ν_0. Then ν_0 has finite angular measure iff for all pairs (i,j), $i \neq j$ the function*

$$\bar{G}_{i,j}(s) := \nu_0\left(\left\{\boldsymbol{x} \in \mathbb{E} : \boldsymbol{x} \geqslant \mathbf{1}, \frac{x^{(i)}}{x^{(j)}} > s\right\}\right)$$

satisfies

$$\int_1^\infty s^{\alpha_0 - 1}\bar{G}_{i,j}(s)ds < \infty.$$

5.4. **A partial converse to a theorem of Breiman.** A theorem of Breiman (1965) discusses the tail behavior of a product of two independent, non-negative random variables, one of which has regularly varying tail probabilities and the other has a lighter tail. Our class of examples in the next section requires the following partial converse.

Proposition 5.2. *Suppose ξ and η are two independent, non-negative random variables and ξ has a Pareto distribution with parameter 1:*

$$\mathrm{P}[\xi > x] = x^{-1}, \quad x \geqslant 1.$$

(a) We have

$$\mathrm{P}[\xi\eta > x] \in RV_{-\alpha}, \quad \alpha < 1,$$

iff

$$\mathrm{P}[\eta > x] \in RV_{-\alpha},$$

and then

$$\frac{\mathrm{P}[\xi\eta > x]}{\mathrm{P}[\eta > x]} \to \frac{1}{1-\alpha}.$$

(b) If $\mathrm{P}[\xi\eta > x] \in RV_{-1}$ and $\xi\eta$ has a heavier tail than ξ meaning

$$\frac{\mathrm{P}[\xi\eta > x]}{\mathrm{P}[\xi > x]} = \int_0^x \mathrm{P}[\eta > y]dy \to \infty,$$

i.e., $\mathbb{E}[\eta] = \infty$, then

$$\int_0^x \mathrm{P}[\eta > s]ds =: L(x) \uparrow \infty$$

is slowly varying. If in addition $L(x) \in \Pi$, the de Haan function class Π (cf. Bingham et al., 1987, de Haan, 1970, Geluk and de Haan, 1987, Resnick, 1987), then

$$\mathrm{P}[\eta > x] \in RV_{-1}$$

and

$$\frac{L(x)}{x\,\mathrm{P}[\eta > x]} = \frac{\mathrm{P}[\xi\eta > x]}{\mathrm{P}[\eta > x]} \to \infty.$$

Remark 5.3. The assumption that the integral is Π-varying in (b) is implied by supposing that $\mathrm{P}[\eta > x] \in RV_{-1}$. We see from $[b]$, that the product has a heavier tail than either factor random variable.

As an example, consider

$$\mathrm{P}[\eta > x] = \frac{e \log x}{x}, \quad x > e.$$

An easy calculation shows that

$$\mathrm{P}[\xi\eta > x] \sim \frac{1}{2}ex^{-1}(\log x)^2, \quad x \to \infty.$$

Proof of Proposition 5.2. (a) If part: If $\mathrm{P}[\xi\eta > x] \in RV_{-\alpha}$, the result follows from Breiman (1965).

Only if part: We have

$$\mathrm{P}[\xi\eta > x] = \int_1^\infty \mathrm{P}\left[\eta > \frac{x}{s}\right] s^{-2} ds = x^{-1} \int_0^x \mathrm{P}[\eta > y] dy.$$

So

$$\mathrm{P}[\xi\eta > x] \in RV_{-\alpha},\ \alpha < 1 \quad \text{iff} \quad \int_0^x \mathrm{P}[\eta > y] dy \in RV_{1-\alpha}.$$

By the monotone density theorem and Karamata's theorem (Bingham et al., 1987, Geluk and de Haan, 1987, Resnick, 1987), we have $\mathrm{P}[\eta > y] \in RV_{-\alpha}$ and then by Karamata's theorem

$$\frac{\mathrm{P}[\xi\eta > x]}{\mathrm{P}[\eta > x]} = \frac{\int_0^x \mathrm{P}[\eta > y] dy}{x\,\mathrm{P}[\eta > x]} \to \frac{1}{1-\alpha}.$$

(b) This follows from the fact that the derivative of a Π-varying function U with a non-increasing derivative U' is -1-varying with auxiliary function $xU'(x)$ and that $U(x)/(xU'(x)) \to \infty$. See Bingham et al. (1987), de Haan (1976), Geluk and de Haan (1987), Resnick (1987). □

Remark 5.4. The result is not true without supposing ξ has a Pareto tail. The following very clever example devised by Daren Cline is an example where $\xi\eta$ has a Pareto tail, the tail of $\xi\eta$ is heavier than that of ξ, but η does not have a regularly varying tail.

Example 5.3 (Cline)**.** Let F_1 be any distribution on $(0, \infty)$ with finite first moment and set $F_2(x) = F_1(e^{-\pi/\alpha} x)$. Define

$$C_i = \int_0^\infty x \cos(\log x) F_i(dx) \text{ and } S_i = \int_0^\infty x \sin(\log x) F_i(dx).$$

Then $C_2 = -e^\pi C_1$ and $S_2 = -e^\pi S_1$. Define

$$F(x) = \frac{e^\pi F_1(x) + F_2(x)}{e^\pi + 1},$$

so that

$$\int_0^\infty x \cos(\log x) F(dx) = \int_0^\infty x \sin(\log x) F(dx) = 0.$$

Now let

$$\bar{G}(y) = (1 + .5 \sin(\log y)) y^{-1},$$

for $y > 1$. Note that its derivative is

$$-\left(1 + .5\sin(\log y) + .5\cos(\log y)\right) y^{-2} < 0.$$

Suppose ξ has distribution F and η has distribution G. Then

$$\begin{aligned}
\mathrm{P}[\xi\eta > t] &= \int_0^\infty \bar{G}(t/x) F(dx) = \int_0^\infty (1 + .5\sin(\log(t/x)))(t/x)^{-1} F(dx) \\
&= t^{-1} \Bigg(\int_0^\infty x F(dx) + .5 \sin(\log t) \int_0^\infty x \cos(\log x) F(dx) \\
&\qquad - .5 \cos(\log t) \int_0^\infty x \sin(\log x) F(dx) \Bigg) \\
&= t^{-1} \int_0^\infty x F(dx).
\end{aligned}$$

Thus, $\mathrm{P}[\xi\eta > t]$ is -1-varying, $\bar{G}$ is not -1-varying and $\bar{F}(t) = o(\mathrm{P}[\xi\eta > t])$.

5.5. **A class of examples of hidden regular variation with infinite hidden angular measure.** Suppose R_{EG} is a Pareto random variable on $[1,\infty)$ with parameter 1 and $\boldsymbol{\Theta}_{\text{EG}}$ is a random vector living on $\aleph_{\text{INV}}$ and for some probability distribution G concentrating on $[1,\infty)$, we have

$$\mathrm{P}[\boldsymbol{\Theta}_{\text{EG}} \in \{1\}\times(s,\infty]] = \mathrm{P}[\boldsymbol{\Theta}_{\text{EG}} \in (s,\infty]\times\{1\}] = \frac{1}{2}\bar{G}(s), \quad s \geqslant 1. \tag{5.6}$$

Suppose R_{EG} and $\boldsymbol{\Theta}_{\text{EG}}$ are independent. Define

$$\boldsymbol{V}_{\text{EG}} = R_{\text{EG}} \cdot \boldsymbol{\Theta}_{\text{EG}}. \tag{5.7}$$

The assumption (5.6) is the same as assuming that

$$V_{\text{EG}}^{(1)} \stackrel{d}{=} V_{\text{EG}}^{(2)} \tag{5.8}$$

which is consistent with our standing assumption that marginal distributions are equal. Observe that by (5.6), we have for $s > 1$,

$$\mathrm{P}[\Theta_{\text{EG}}^{(1)} > s] = \mathrm{P}[\boldsymbol{\Theta}_{\text{EG}} \in (s,\infty]\times\{1\}] = \frac{1}{2}\bar{G}(s) = \mathrm{P}[\boldsymbol{\Theta}_{\text{EG}} \in \{1\}\times(s,\infty]] = \mathrm{P}[\Theta_{\text{EG}}^{(2)} > s],$$

so that (5.6) implies (5.8). Conversely, if (5.8) holds, then $R_{\text{EG}}\Theta_{\text{EG}}^{(1)} \stackrel{d}{=} R_{\text{EG}}\Theta_{\text{EG}}^{(2)}$ and therefore

$$\int_1^\infty \mathrm{P}\left[\Theta_{\text{EG}}^{(1)} > \frac{x}{y}\right] y^{-2}dy = \int_1^\infty \mathrm{P}\left[\Theta_{\text{EG}}^{(2)} > \frac{x}{y}\right] y^{-2}dy.$$

This is equivalent to

$$\int_0^x \mathrm{P}\left[\Theta_{\text{EG}}^{(1)} > s\right] ds = \int_0^x \mathrm{P}\left[\Theta_{\text{EG}}^{(2)} > s\right] ds$$

and it follows that

$$\mathrm{P}\left[\Theta_{\text{EG}}^{(1)} > s\right] = \mathrm{P}\left[\Theta_{\text{EG}}^{(2)} > s\right], \quad \forall s > 1.$$

Proposition 5.3. *With $\boldsymbol{V}_{\text{EG}}$ as specified, $\boldsymbol{V}_{\text{EG}}$ is regularly varying on $\mathbb{E}_0$ with index $\alpha_0 = 1$. The limit measure $\nu_{\text{EG},0}$ has infinite angular measure iff*

$$\int_1^\infty \bar{G}(s)ds = \infty. \tag{5.9}$$

Furthermore, if (5.9) *holds:*

(i) *$\boldsymbol{V}_{\text{EG}}$ is regularly varying on $\mathbb{E}$ with index $\alpha < 1$ (and hence possesses hidden regular variation with infinite hidden angular measure) iff*

$$1 - G \in RV_{-\alpha}.$$

In this case, for $i = 1,2$, we have

$$\mathrm{P}\left[V_{\text{EG}}^{(i)} > x\right] \sim \frac{1}{2(1-\alpha)}\bar{G}(x), \quad x \to \infty.$$

(ii) *$\boldsymbol{V}_{\text{EG}}$ is regularly varying on $\mathbb{E}$ with index $\alpha = 1$ (and hence possesses hidden regular variation with infinite hidden angular measure) iff*

$$L(x) := \int_0^x \bar{G}(s)ds \in RV_0 \quad \text{and } L(x) \uparrow \infty.$$

A sufficient condition is $\bar{G} \in RV_{-1}$ with $\int_0^\infty \bar{G}(s)ds = \infty$ in which case

$$\lim_{x\to\infty} \frac{\mathrm{P}\left[V_{\text{EG}}^{(i)} > x\right]}{\mathrm{P}\left[R_{\text{EG}} > x\right]} = \lim_{x\to\infty} \frac{\mathrm{P}\left[V_{\text{EG}}^{(i)} > x\right]}{\mathrm{P}\left[\Theta_{\text{EG}}^{(i)} > x\right]} = \infty,$$

for $i = 1,2$.

Proof. Note, that since $\mathbf{\Theta}_{\text{EG}}$ lives on $\aleph_{\text{INV}}$, we have $\|\mathbf{\Theta}_{\text{EG}}\|_{\text{INV}} = 1$ and thus $R_{\text{INV}} = R_{\text{EG}}$. Similarly, since

$$\|R_{\text{EG}}\mathbf{\Theta}_{\text{EG}}\|_{\text{INV}} = R_{\text{EG}}\|\mathbf{\Theta}_{\text{EG}}\|_{\text{INV}}$$

we have

$$\frac{\boldsymbol{V}_{\text{EG}}}{\|\boldsymbol{V}_{\text{EG}}\|_{\text{INV}}} = \mathbf{\Theta}_{\text{EG}}.$$

Hence, for this example,

$$t\,\mathrm{P}\left[\left(\frac{\|\boldsymbol{V}_{\text{EG}}\|_{\text{INV}}}{t}, \frac{\boldsymbol{V}_{\text{EG}}}{\|\boldsymbol{V}_{\text{EG}}\|_{\text{INV}}}\right) \in dr \times d\boldsymbol{\theta}\right] = t\,\mathrm{P}\left[\frac{R_{\text{EG}}}{t} \in dr\right] \mathrm{P}[\mathbf{\Theta}_{\text{EG}} \in d\boldsymbol{\theta}] \to r^{-2}dr \times \mathrm{P}[\mathbf{\Theta}_{\text{EG}} \in d\boldsymbol{\theta}].$$

Regular variation on $\mathbb{E}_0$ follows from Theorem 5.2. The statement about when the angular measure is infinite follows from Proposition 5.1. The statement about regular variation on $\mathbb{E}$ comes from Proposition 5.2 and Theorem 5.1. The asymptotic form for the marginal tail of $\boldsymbol{V}_{\text{EG}}$ comes from the fact that for $x > 1$,

$$\begin{aligned}\mathrm{P}\left[V_{\text{EG}}^{(i)} > x\right] =& \mathrm{P}\left[R_{\text{EG}}\Theta_{\text{EG}}^{(i)} > x\right] = \int_1^\infty \mathrm{P}\left[\Theta_{\text{EG}}^{(i)} > \frac{x}{r}\right] r^{-2}dr \\ =& x^{-1}\int_0^x \mathrm{P}[\Theta_{\text{EG}}^{(i)} > y]dy \\ =& x^{-1}\int_0^x \frac{1}{2}(1+\bar{G}(y))dy\end{aligned}$$

and the result follows from application of Karamata's theorem. ☐

This result shows that the class of multivariate distributions which are regularly varying on $\mathbb{E}_0$ with infinite angular measure is at least as large as the class of distributions on $[1,\infty)$ which have infinite mean. The class of multivariate distributions on $\mathbb{E}$ which are regularly varying and possess hidden regular variation is at least as large as the class of distributions on $[1,\infty)$ which have regularly varying tails of index less than 1.

For this class of examples, we have $\boldsymbol{V}_{\text{EG}} = R_{\text{EG}} \cdot \mathbf{\Theta}_{\text{EG}}$, and thus

$$\frac{V_{\text{EG}}^{(1)}}{V_{\text{EG}}^{(2)}} = \frac{\Theta_{\text{EG}}^{(1)}}{\Theta_{\text{EG}}^{(2)}}.$$

Let Θ be a random variable concentrating on $[1,\infty)$ with distribution G and suppose B is a Bernoulli random variable with equally likely values 0 and 1, independent of Θ and R_{EG} and then set

$$\mathbf{\Theta}_{\text{EG}} = B(\Theta, 1) + (1-B)(1,\Theta),$$

so that $\mathbf{\Theta}_{\text{EG}}$ concentrates on $\aleph_{\text{INV}}$ and has distribution given in (5.6). Then

$$\frac{\Theta_{\text{EG}}^{(1)}}{\Theta_{\text{EG}}^{(2)}} = B\Theta + (1-B)\frac{1}{\Theta}.$$

Since $1/\Theta \leqslant 1$ it does not affect tail behavior in the mixture and we can write, as $x \to \infty$,

$$\begin{aligned}\mathrm{P}\left[\frac{\Theta_{\text{EG}}^{(1)}}{\Theta_{\text{EG}}^{(2)}} > x\right] =& \mathrm{P}\left[\frac{\Theta_{\text{EG}}^{(2)}}{\Theta_{\text{EG}}^{(1)}} > x\right] = \frac{1}{2}\,\mathrm{P}[\Theta > x] + \frac{1}{2}\,\mathrm{P}[\Theta^{-1} > x] \\ \sim& \frac{1}{2}\,\mathrm{P}[\Theta > x] = \frac{1}{2}\bar{G}(x).\end{aligned}$$

We conclude that for this class of examples, $\boldsymbol{V}_{\text{EG}}$ is regularly varying on both $\mathbb{E}$ and $\mathbb{E}_0$ with indices $(\alpha, 1)$, with $\alpha < 1$, iff $\frac{V_{\text{EG}}^{(1)}}{V_{\text{EG}}^{(2)}} \in RV_{-\alpha}$.

However, more general cases of regular variation on $\mathbb{E}$ and $\mathbb{E}_0$ will not necessarily follow this pattern. Consider the following example.

Example 5.4. Suppose $\beta < \alpha < 1$ and that $U_i,\ i = 1,2$ are i.i.d. random variables concentrating on $[1,\infty)$ having Pareto distributions with parameter β. Let M be a multinomial random variable with values $\{1,2,3\}$ and equal probabilities $1/3$. Suppose M is independent of $U_i,\ i = 1,2$ and $\boldsymbol{V}_{\text{EG}}$ constructed above. Define

$$\boldsymbol{X} = 1_{[M=1]}\boldsymbol{V}_{\text{EG}} + 1_{[M=2]}\left(1, \frac{1}{U_1}\right) + 1_{[M=3]}\left(\frac{1}{U_2}, 1\right).$$

Since $1/U_i \leqslant 1$, these terms cannot affect tail behavior:

$$t\,\mathrm{P}\left[\frac{\boldsymbol{X}}{t} > \boldsymbol{x}\right] \sim \frac{1}{3} t\,\mathrm{P}\left[\frac{\boldsymbol{V}_{\text{EG}}}{t} > \boldsymbol{x}\right], \quad \boldsymbol{x} > \boldsymbol{0},$$
$$tP\left(\left[\frac{\boldsymbol{X}}{b(t)} \leqslant \boldsymbol{x}\right]^c\right) \sim \frac{1}{3} tP\left(\left[\frac{\boldsymbol{V}_{\text{EG}}}{b(t)} \leqslant \boldsymbol{x}\right]^c\right),$$

showing that $\boldsymbol{X}$ is regularly varying on $\mathbb{E}$ and $\mathbb{E}_0$. However,

$$\frac{X^{(1)}}{X^{(2)}} = 1_{[M=1]}\frac{V^{(1)}_{\text{EG}}}{V^{(2)}_{\text{EG}}} + 1_{[M=2]}U_1 + 1_{[M=3]}\frac{1}{U_2}$$

so that

$$\mathrm{P}\left[\frac{X^{(1)}}{X^{(2)}} > x\right] \in RV_{-\beta}.$$

Example 5.5. Suppose $\boldsymbol{Z} = (Z^{(1)}, Z^{(2)})$ are i.i.d. Pareto random variables on $[1,\infty)$ with parameter $1/2$. Then

$$t\,\mathrm{P}\left[\frac{\boldsymbol{Z}}{t} \in \cdot\right] \xrightarrow{v} \nu_0$$

in $\mathbb{E}_0$ where

$$\nu_0(\boldsymbol{x},\infty] = \frac{1}{\sqrt{x^{(1)}x^{(2)}}}, \quad \boldsymbol{x} > \boldsymbol{0},$$

which has density

$$\nu_0'(\boldsymbol{x}) = \frac{1}{4}(x^{(1)}x^{(2)})^{-3/2}.$$

An easy calculation shows that

$$\nu_0\left\{\boldsymbol{x} : \boldsymbol{x} \geqslant 1; \frac{x^{(1)}}{x^{(2)}} > \theta\right\} = \theta^{-1/2}(\text{const}),$$

which is consistent with the class of examples discussed.

The following gives conditions for a distribution having hidden regular variation to be tail equivalent on $\mathbb{E}$ and $\mathbb{E}_0$ to $\boldsymbol{V}_{\text{EG}}$.

Proposition 5.4. *Suppose $\boldsymbol{X}$ has a distribution possessing hidden regular variation in standard form on $\mathbb{E}$ and $\mathbb{E}_0$, with limit measures ν, ν_0 with $\nu_0\{\boldsymbol{x} : \|\boldsymbol{x}\| > 1\} = \infty$, scaling functions $b(t) \in RV_{1/\alpha},\ \alpha < 1,\ b_0(t) = t$, and such that*

$$S_{\text{INV}}(\aleph_{\text{INV}}) = \nu_0\{\boldsymbol{x} \in \mathbb{E}_0 : \boldsymbol{x} \geqslant \boldsymbol{1}, \|\boldsymbol{x}\|_{\text{INV}} = 1\} = 1.$$

Define

$$\bar{G}(t) = S_{\text{INV}}\big(\{1\} \times (t,\infty]\big) = S_{\text{INV}}\big((t,\infty] \times \{1\}\big),$$

and $\boldsymbol{V}_{\text{EG}}$ as in (5.7). *Then*

$$\boldsymbol{X} \overset{te(\mathbb{E},\mathbb{E}_0)}{\sim} \boldsymbol{V}_{\text{EG}}$$

iff the following hold:

(i) *We have*

$$\lim_{t\to\infty} t\,\mathrm{P}\left[X^{(1)}\wedge X^{(2)} > t, \frac{X^{(2)}}{X^{(1)}} > \theta\right] = \frac{1}{2}\bar{G}(\theta),$$

$$\lim_{t\to\infty} t\,\mathrm{P}\left[X^{(1)}\wedge X^{(2)} > t, \frac{X^{(1)}}{X^{(2)}} > \theta\right] = \frac{1}{2}\bar{G}(\theta),$$

and

(ii) *we have*

$$\bar{G}(\cdot) \in RV_{-\alpha}, \text{ and } \frac{\mathrm{P}[X^{(i)} > x]}{\bar{G}(x)} \to c \in (0,\infty), \quad x\to\infty.$$

If $\alpha = 1$, $b(t)/t \to \infty$ as $t\to\infty$, the result continues to hold provided (ii) is replaced by

(ii′) *We have*

$$\int_0^x \bar{G}(s)ds \in RV_0 \text{ and } \int_0^x \bar{G}(s)ds \sim cx\,\mathrm{P}[X^{(i)} > x], \quad x\to\infty$$

for some $c > 0$.

Proof. Suppose $\boldsymbol{X}$ has a distribution which is regularly varying on $\mathbb{E}, E_0$ with parameters $(\alpha, 1)$ and which is tail equivalent to $\boldsymbol{V}_{\text{EG}}$ on $\mathbb{E}, \mathbb{E}_0$. Then from tail equivalence on $\mathbb{E}$, for $i = 1, 2$:

$$\mathrm{P}[X^{(i)} > t] \sim c\,\mathrm{P}[V_{\text{EG}}^{(i)} > t] = c\,\mathrm{P}[R_{\text{EG}}\Theta_{\text{EG}}^{(i)} > t]$$

which implies $\mathrm{P}[\Theta_{\text{EG}}^{(i)} > t] \in RV_{-\alpha}$, by Proposition 5.2. Therefore, $\bar{G}(t) \in RV_{-\alpha}$ which gives (ii).

Furthermore, tail equivalence on $\mathbb{E}_0$ gives

$$\begin{aligned}\lim_{t\to\infty} t\,\mathrm{P}\left[X^{(1)}\wedge X^{(2)} > t, \frac{\boldsymbol{X}}{X^{(1)}\wedge X^{(2)}} \in \{1\}\times(\theta,\infty]\right] &= \lim_{t\to\infty} t\,\mathrm{P}\left[X^{(1)} > t, \frac{X^{(2)}}{X^{(1)}} > \theta\right]\\ &= \lim_{t\to\infty} t\,\mathrm{P}[R_{\text{EG}} > t]\,\mathrm{P}[\boldsymbol{\Theta}_{\text{EG}} \in \{1\}\times(\theta,\infty]]\\ &= \frac{1}{2}\bar{G}(\theta),\end{aligned}$$

which gives (i).

Conversely, suppose $\boldsymbol{X}$ has a distribution which is regularly varying on $\mathbb{E}, \mathbb{E}_0$ and (i) and (ii) hold. Condition (i) implies

$$\boldsymbol{X} \overset{\text{te}(\mathbb{E}_0)}{\sim} \boldsymbol{V}_{\text{EG}}.$$

Condition (ii) guarantees (in light of Theorem 5.1) tail equivalence on $\mathbb{E}$. □

Referring to Example 5.5 we observe for $\theta > 1$,

$$\begin{aligned} t\,\mathrm{P}\left[Z^{(1)} > t, \frac{Z^{(2)}}{Z^{(1)}} > \theta\right] &= t\int_t^\infty \frac{1}{2}s^{-3/2}(\theta s)^{-1/2}ds\\ &= \theta^{-1/2}\frac{t}{2}\int_t^\infty s^{-2}ds = \frac{\theta^{-1/2}}{2} = \bar{G}(\theta).\end{aligned}$$

Full disclosure: While this result clarifies when a distribution possessing hidden regular variation with infinite angular measure has the structure of $\boldsymbol{V}_{\text{EG}}$, it does not provide a full characterization.

5.6. **Random vectors with prescribed infinite hidden angular measure.** The examples of random vectors with infinite hidden angular measure given in the earlier subsections are of dimensions 2 and 3. The construction given in Subsection 5.5 is for dimension 2. In this subsection, we again consider general dimensions. Given an infinite hidden angular measure S_0 and scaling function b satisfying $b(t)/t \to \infty$, we construct a random vector $\boldsymbol{Z}$ in standard form with scaling function b on $\mathbb{E}$ and hidden angular measure S_0. We use the standard form for notational simplicity. The case of a general scaling function b_0 for the subcone $\mathbb{E}_0$ can be easily obtained through the transformation outlined in Theorem 5.3.

As in Theorem 4.1, define the random variables $I, X_1, \dots, X_d$. Each X_i has the property that for $x > 0$,

$$tP[X_i > b(t)x] \to x^{-\alpha}.$$

Now we specify the random vector $\boldsymbol{V}$ which will give regular variation on $\mathbb{E}_0$. Define the non-decreasing, left continuous function

$$\Phi(t) = S_0\left(\left\{\boldsymbol{a} \in \aleph_0 : \wedge_{i=1}^d \|\boldsymbol{a} - \boldsymbol{e}_i\| > \frac{1}{t}\right\}\right).$$

Since the hidden angular measure S_0 is infinite, Φ increases to ∞. Also define a non-decreasing function ϕ as follows:

$$\phi(t) = \inf\left\{\log \frac{b(2^u)}{2^u} : u \geqslant t\right\}.$$

Since $b(t)/t \to \infty$, ϕ also increases to ∞. Further observe that

$$\phi(\log_2 t) = \inf\left\{\log \frac{b(2^u)}{2^u} : u \geqslant \log_2 t\right\} = \inf\left\{\log \frac{b(u)}{u} : u \geqslant t\right\} \leqslant \log \frac{b(t)}{t}$$

and hence we have

$$\lim_{t\to\infty} \frac{\phi(\log_2 t)t}{b(t)} = 0,$$

which further implies

$$\lim_{t\to\infty} \frac{b^{\leftarrow}(\phi(\log_2 t)t)}{t} = 0, \tag{5.10}$$

since b is regularly varying of index $\alpha > 0$. Now we define an increasing sequence of pre-compact open covers $\{G_n\}_{n=0}^{\infty}$ for $\aleph_0$, none of which has too large S_0-mass: $G_0 = \emptyset$ and for $n \geqslant 1$,

$$G_n = \left\{\boldsymbol{a} \in \aleph_0 : \wedge_{i=1}^d \|\boldsymbol{a} - \boldsymbol{e}_i\| > \frac{1}{\Phi^{\leftarrow}(\phi(n))}\right\}.$$

Thus

$$S_0(G_n \setminus G_{n-1}) \leqslant S_0(G_n) = \Phi\Big(\Phi^{\leftarrow}\big(\phi(n)\big)\Big) \leqslant \phi(n), \tag{5.11}$$

since Φ is left continuous. Define the function

$$f(\boldsymbol{a}) = \sum_{n=1}^{\infty} 2^n \phi(n) \mathbb{1}_{G_n \setminus G_{n-1}}(\boldsymbol{a})$$

on $\aleph_0$. Then f is bounded on compact subsets of $\aleph_0$ and also satisfies

$$c := \int_{\aleph_0} \big(f(\boldsymbol{a})^{-1} \wedge 1\big)\, S_0(d\boldsymbol{a}) \leqslant \int_{\aleph_0} f(\boldsymbol{a})^{-1} S_0(d\boldsymbol{a}) = \sum_{n=1}^{\infty} 2^{-n} \frac{S_0(G_n \setminus G_{n-1})}{\phi(n)} \leqslant \sum_{n=1}^{\infty} 2^{-n} = 1 < \infty, \tag{5.12}$$

since by (5.11), we have $S_0(G_n \setminus G_{n-1}) \leqslant \phi(n)$.

Now define a pair of random variables $(R_0, \boldsymbol{\Theta}_0)$, taking values in $(0,\infty) \times \aleph_0$, independent of $I, X_1, \dots, X_d$, with joint distribution

$$\frac{1}{c} S_0(d\boldsymbol{\theta}) F_0(dr) 1_{\{(r',\boldsymbol{\theta}') : r' \geqslant f(\boldsymbol{\theta}')\}}(r, \boldsymbol{\theta}),$$

where F_0 is as defined in (4.1), with b_0 as the identity function. Hence F_0 is nothing but Pareto distribution function with parameter 1. The joint distribution is a probability distribution since

$$\frac{1}{c} \iint_{\{(r,\boldsymbol{\theta}) \in (0,\infty) \times \aleph_0 : r > f(\boldsymbol{\theta})\}} S_0(d\boldsymbol{\theta}) F_0(dr) = \frac{1}{c} \int_{\aleph_0} S_0(d\boldsymbol{\theta}) \overline{F_0}(f(\boldsymbol{\theta})) = \frac{1}{c} \int_{\aleph_0} \big(f(\boldsymbol{\theta})^{-1} \wedge 1\big)\, S_0(d\boldsymbol{\theta}) = 1,$$

by (5.12). Define $\boldsymbol{V} = R_0 \boldsymbol{\Theta}_0$. Finally define $\boldsymbol{Z}$ as in (4.2) by

$$\boldsymbol{Z} = \mathbb{1}_{[I=0]} \boldsymbol{V} + \sum_{i=1}^{d} \mathbb{1}_{[I=i]} X_i \boldsymbol{e}_i.$$

This is the required random vector $\boldsymbol{Z}$.

To verify the hidden regular variation of $\boldsymbol{Z}$ on the cone $\mathbb{E}_0$, first observe that given any compact subset Λ of $\aleph_0$ and $x>0$, we may choose N such that $\Lambda \subset G_N$. Then for all $t > 2^N \phi(N)/x$ and $\boldsymbol{\theta} \in \Lambda$, we have, $f(\boldsymbol{\theta}) < tx$ and hence

$$t\,\mathrm{P}\left[R_0 > tx, \boldsymbol{\Theta} \in \Lambda\right] = S_0(\Lambda) t\overline{F_0}(tx) \to \frac{1}{x} S_0(\Lambda), \tag{5.13}$$

which is the analog of (4.3) in the finite hidden angular measure case with $\alpha_0 = 1$. The decompositions (4.5) and (4.4) of R_Z and $\boldsymbol{\Theta}_Z$ respectively continue to hold. So we get the analogs of (4.6) in the infinite case as well.

Verifying regular variation on the bigger cone $\mathbb{E}$ is more difficult. We first check that scaling by b is too large for $\boldsymbol{V}$ on the cone $\mathbb{E}$ by checking that for any $x > 0$,

$$t\,\mathrm{P}[R_0 > b(t)x] \to 0.$$

It suffices to suppose $x = 1$. First define, for $t > 0$,

$$\kappa(t) = \sup\left\{u : 2^u \phi(u) \leqslant b(t)\right\}.$$

Then κ is increasing to ∞. Also we have

$$\phi(\kappa(t)) 2^{\kappa(t)} \leqslant b(t). \tag{5.14}$$

Furthermore, we have,

$$\phi(\kappa(t)+1) 2^{\kappa(t)+1} > b(t)$$

and hence

$$b^{\leftarrow}\left(\phi(\kappa(t)+1) 2^{\kappa(t)+1}\right) > t \tag{5.15}$$

Then observe that,

$$f(\boldsymbol{\theta}) \leqslant \phi(\lfloor\kappa(t)\rfloor) 2^{\lfloor\kappa(t)\rfloor} \leqslant b(t), \qquad \text{for } \boldsymbol{\theta} \in G_{\lfloor\kappa(t)\rfloor},$$

and

$$f(\boldsymbol{\theta}) \geqslant \phi(\lfloor\kappa(t)\rfloor + 1) 2^{\lfloor\kappa(t)\rfloor+1} > b(t), \qquad \text{for } \boldsymbol{\theta} \notin G_{\lfloor\kappa(t)\rfloor}.$$

Then we have

$$\begin{aligned}
ct\,\mathrm{P}[R_0 > b(t)] &= ct\,\mathrm{P}\left[R_0 > b(t), \boldsymbol{\Theta}_0 \in G_{\lfloor\kappa(t)\rfloor}\right] + t\,\mathrm{P}\left[R_0 > b(t), \boldsymbol{\Theta}_0 \notin G_{\lfloor\kappa(t)\rfloor}\right] \\
&= t \int_{G_{\lfloor\kappa(t)\rfloor}} \overline{F_0}(b(t)) S_0(d\boldsymbol{\theta}) + t \int_{G^c_{\lfloor\kappa(t)\rfloor}} \overline{F_0}(b_0(f(\boldsymbol{\theta}))) S_0(d\boldsymbol{\theta}) \\
&= \frac{t S_0(G_{\lfloor\kappa(t)\rfloor})}{b(t)} + t \int_{G^c_{\lfloor\kappa(t)\rfloor}} f(\boldsymbol{\theta})^{-1} S_0(d\boldsymbol{\theta}) \\
&\leqslant \frac{t\phi(\lfloor\kappa(t)\rfloor)}{b(t)} + t \sum_{n=\lfloor\kappa(t)\rfloor+1}^{\infty} 2^{-n} \frac{S_0(G_n \setminus G_{n-1})}{\phi(n)}, \quad \text{by (5.11)} \\
&\leqslant \frac{t\phi(\kappa(t))}{b(t)} + 2t 2^{-\kappa(t)}, \quad \text{since } \phi \text{ is increasing and by (5.11)} \\
&\leqslant 3t 2^{-\kappa(t)}, \quad \text{by (5.14)} \\
&< 6 \frac{b^{\leftarrow}\left(\phi(\kappa(t)+1) 2^{\kappa(t)+1}\right)}{2^{\kappa(t)+1}}, \quad \text{by (5.15)} \\
&= 6 \frac{b^{\leftarrow}(\phi(\log_2 u) u)}{u}, \quad \text{substituting } u = 2^{\kappa(t)+1} \\
&\sim 6 \frac{b^{\leftarrow}(\phi(\log_2 u) u)}{u}.
\end{aligned}$$

Now, using (5.10), we have this quantity going to zero, as required. Then, again as in Theorem 4.1, $\boldsymbol{Z}$ is regularly varying on $\mathbb{E}$.

So $\boldsymbol{Z}$ is a random vector having multivariate regular variation on $\mathbb{E}$ with scaling function b and having hidden regular variation on $\mathbb{E}_0$ with scaling by the identity function and having prescribed infinite hidden angular measure S_0. Thus, any random vector $\boldsymbol{Y}$ which is regularly varying on $\mathbb{E}$ and $\mathbb{E}_0$ with scaling functions b, b_0 and infinite hidden angular measure S_0 is tail equivalent to a mixture

$$1_{[I=0]}\mathbb{V} + 1_{[I\neq 0]}\boldsymbol{X}$$

where $\boldsymbol{X}$ is completely asymptotically independent and $\mathbb{V}$ is regularly varying on $\mathbb{E}_0$ with

$$tP[\mathbb{V} > b(t)\boldsymbol{x}] \to 0, \quad \boldsymbol{x} > \boldsymbol{0}.$$

6. Can hidden regular variation influence the tail behavior of the product of the components?

The tail behavior of distributions of products of heavy tailed variables arises fairly frequently. For the study of the heavy tailed sample correlation function, the tail behavior of X_1X_2 (where X_1, X_2 are iid with regularly varying tails of index $-\alpha$) was crucial and found to partly depend on whether $E(X_1^\alpha)$ was finite or infinite. See Cline (1983), Davis and Resnick (1985, 1986). In Internet studies of file downloads, download throughput or rate R and download duration L are heavy tailed and determine the size of the file F since $F = RL$ (Maulik et al., 2002).

Hidden regular variation condition on the cone $\mathfrak{C}_0 = \mathbb{E}_0$ may or may not influence the tail behavior of the distribution of the product of the components. For dimensions 3 or higher, problems may be more pathological due to the flexibility in the choice of the subcone $\mathfrak{C}_0$. (See Example 5.2.) The hidden regular variation definition does not prevent the limiting measure ν_0 from concentrating on the hyperfaces of the d-dimensional non-negative orthant. On the hyperfaces, at least one of the coordinates will be zero and so will be the product. This problem suggests we examine the convergence more carefully and seek the right scaling to understand the behavior on the subcone $\mathbb{E}_{00} = (\boldsymbol{0}, \boldsymbol{\infty}] = (0,\infty]^d$ instead of $\mathbb{E}_0$. However, the problem is deeper than the choice of the correct subcone. In case $d = 2$, the cone $(\boldsymbol{0}, \boldsymbol{\infty}]$ coincides with $\mathbb{E}_0$. As the following counterexample shows, even in that case hidden regular variation is not enough to characterize the behavior of the product of the components.

Example 6.1. Let us consider the two-dimensional random vector $\boldsymbol{Y}$, which is a mixture of extremally dependent random vector $\boldsymbol{U}$ with multivariate regularly varying tail and a random vector $\mathbb{V}$ supported on a curve to be described later. We assume that

$$\mathrm{P}\left[\frac{\boldsymbol{U}}{b(t)} \in \cdot\right] \xrightarrow{\mathrm{v}} \nu(\cdot) \text{ on } [\boldsymbol{0}, \boldsymbol{\infty}] \setminus \{\boldsymbol{0}\},$$

where $\nu((\boldsymbol{0}, \boldsymbol{\infty}]) \neq 0$ and satisfies the scaling property $\nu(cA) = c^{-\beta}\nu(A)$, for all $A \subset [\boldsymbol{0}, \boldsymbol{\infty}] \setminus \{\boldsymbol{0}\}$. We denote the distribution function of the polar coordinate representation of $\boldsymbol{U}$, (R_U, Θ_U) by F. (Since we are in dimension 2, we resort to the abuse of notation indicated in Subsection 1.3. Thus θ denotes the polar angle instead of the vector scaled to norm 1.) The other vector $\boldsymbol{V}$, given in the polar coordinates (R_V, Θ_V), has a distribution supported on the curve:

$$\{(r,\theta) \in (0,\infty) \times [0, \pi/2] : r\cos\theta \cdot r\sin\theta = r, \text{ or, } r\sin 2\theta = 2, \text{ for } \theta \in (0, \pi/2)\}. \tag{6.1}$$

Put another way, the distribution of $\mathbb{V}$ is supported on

$$\left\{\left(\frac{2}{\sin 2\theta}, \theta\right) : 0 < \theta < \frac{\pi}{2}\right\} = \left\{\left(r, \frac{1}{2}\arcsin\frac{2}{r}\right), \left(r, \frac{\pi}{2} - \frac{1}{2}\arcsin\frac{2}{r}\right) : r \geqslant 2\right\}. \tag{6.2}$$

Hence the product of the Cartesian coordinates of a point on this curve is equal to the norm of the point and $V_1V_2 = R_V$ almost surely. Assume that R_V has density $2^\alpha \alpha r^{-\alpha-1}$, for $r \geqslant 2$ and given $R_V = r$, Θ_V

is equally likely to take values $\frac{1}{2}\arcsin\frac{2}{r}$ and $\frac{\pi}{2}-\frac{1}{2}\arcsin\frac{2}{r}$. $\boldsymbol{Y}$ is $\boldsymbol{U}$ with probability $1-2^{-\alpha}$ and $\boldsymbol{V}$ with probability $2^{-\alpha}$. Then, the polar coordinate transform (R_Y,Θ_Y) of the vector $\boldsymbol{Y}$ has distribution

$$(1-2^{-\alpha})F(dr,d\theta)+\frac{1}{2}\alpha r^{-\alpha-1}\left(\delta_{\frac{1}{2}\arcsin\frac{2}{r}}(d\theta)-\delta_{\frac{\pi}{2}+\frac{1}{2}\arcsin\frac{2}{r}}(d\theta)\right)dr,$$

where δ_x is the Dirac's delta measure. Further assume that $\beta>\alpha$.

Next, we check the hidden regular variation property of the vector $\boldsymbol{Y}$. We use Corollary 1 of Resnick (2002b). First we consider the subcone $\mathbb{E}_0$. Let Λ be a compact sub-interval of $(0,\frac{\pi}{2})$. Observe that $\frac{1}{2}\arcsin\frac{2}{r}\to 0$ as $r\to\infty$. Thus, both $\frac{1}{2}\arcsin\frac{2}{r}$ and $\frac{\pi}{2}-\frac{1}{2}\arcsin\frac{2}{r}$ do not belong to Λ for all sufficiently large r. So, for sufficiently large t, we have

$$t\,\mathrm{P}\left[\frac{R_{\boldsymbol{Y}}}{b(t)}>r,\Theta_Y\in\Lambda\right]=(1-2^{-\alpha})t\,\mathrm{P}\left[\frac{R_{\boldsymbol{U}}}{b(t)}>r,\Theta_U\in\Lambda\right]\to(1-2^{-\alpha})\nu(\{\boldsymbol{x}:r_x>r,\theta_x\in\Lambda\})$$

and hence we have

$$t\,\mathrm{P}\left[\frac{\boldsymbol{Y}}{b(t)}\in\cdot\right]\xrightarrow{\mathrm{v}}(1-2^{-\alpha})\nu. \tag{6.3}$$

For the marginal distributions, observe that

$$t\,\mathrm{P}\left[\frac{Y^{(1)}}{t^{1/\alpha}}>x\right]=(1-2^{-\alpha})t\,\mathrm{P}\left[\frac{U^{(1)}}{t^{1/\alpha}}>x\right]+2^{-\alpha}t\,\mathrm{P}\left[\frac{R_{\mathbb{V}}\cos\Theta_{bV}}{t^{1/\alpha}}>x\right].$$

The first term on the right side converges to 0, since the correct scaling for $U^{(1)}$ is $b(t)=o(t^{1/\alpha})$. Also since

$$\cos(\frac{1}{2}\arcsin\frac{2}{r})\to 1,\ \text{and}\ \cos(\frac{\pi}{2}-\frac{1}{2}\arcsin\frac{2}{r})\to 0$$

as $r\to 1$, we have,

$$t\,\mathrm{P}\left[\frac{R_{\mathbb{V}}\cos\Theta_{\mathbb{V}}}{t^{1/\alpha}}>x\right]\sim\frac{1}{2}t\,\mathrm{P}[R_{\mathbb{V}}>t^{1/\alpha}x]=2^{\alpha-1}x^{-\alpha}.$$

So,

$$t\,\mathrm{P}\left[\frac{Y^{(1)}}{t^{1/\alpha}}>x\right]=\frac{1}{2}x^{-\alpha}. \tag{6.4}$$

Then, applying (6.3) and (6.4), we conclude that $\boldsymbol{Y}$ has hidden regular variation. Also, the limit measure on the subcone $\mathbb{E}_0$, $(1-2^{-\alpha})\nu$ is Radon on $\mathbb{E}$ and hence is finite on $\aleph_0$.

Now, observe that Y_1Y_2 is U_1U_2 with probability $1-2^{-\alpha}$ and R_V with probability $2^{-\alpha}$. Using Proposition 3.1 of Maulik et al. (2002), we have that $\mathrm{P}[U_1U_2>\cdot]$ is regularly varying of index $-\beta/2$. On the other hand, by assumption, R_V has a regularly varying tail of index $-\alpha$. Thus, Y_1Y_2 has a regularly varying tail of index $-\gamma$, where $\gamma=\alpha\wedge(\beta/2)$. We conclude that depending on the relationship of α,β, the hidden regular variation might or might not affect the tail behavior of the distribution of $Y^{(1)}Y^{(2)}$.

7. Concluding Remarks.

The inverse norm transformation of Subsection 5.2 seemed promising since choosing directions using a unit sphere which is a neighborhood of $\boldsymbol{\infty}$ rather than $\boldsymbol{0}$ allows one to ignore whether the hidden angular measure is finite or infinite. However, this technique did not lead to a complete characterization but only spawned a class of examples. The promise of this technique was not fully realized.

We were also frustrated that the mixture characterization in Subsection 5.6 did not lead to an entirely satisfying characterization for the infinite angular measure case. We cannot say with certainty that the hidden regular variation on $\mathbb{E}_0$ described by $\mathbb{V}$ can or cannot be extended to all of $\mathbb{E}$, possibly with a different index on $\mathbb{E}\setminus\mathbb{E}_0$.

The behavior of the distribution tail of products of regularly varying components seems rather intricate.

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