

The Chaotic Chameleon

Richard D. Gill

Mathematical Institute, University of Utrecht, Netherlands, &
EURANDOM, Eindhoven, Netherlands

<http://www.math.uu.nl/people/gill>

August 7, 2003

Abstract

Various supposedly local hidden variables models for the singlet correlations exploit the detection loophole, or other loopholes connected with post-selection on coincident arrival times. I consider the connection with a probabilistic simulation technique called rejection-sampling, and pose some natural questions concerning what can be achieved and what cannot be achieved with local (or distributed) rejection sampling.

Mathematics Subject Classification: 65C50, 81P68, 81P05, 60-08, 62P35.

<http://arxiv.org/abs/quant-ph/0307217>

1 Introduction

It has been well known since Pearle (1970) that local realistic models can explain the singlet correlations when these are determined on the basis of post-selected coincidences rather than on pre-selected event pairs. These models are usually felt to be unphysical and conspiratorial. However, Accardi, Imafuku and Regoli (2002), Thompson and Holstein (2002), and others have argued that their models could make physical sense. Further examples are provided by Hess and Philipp (2001a,b), Kracklauer (2002), Sanctuary (2003), in many cases unwittingly! Already, Gisin and Gisin (1999) show

that these models can be simple and elegant, and should not be thought of as being artificial.

In this paper I do not want to enter into philosophical debate, nor address questions of physical legitimacy of these models. Instead I would like to extract a mathematical kernel from this literature, exposing some natural, probably open, problems concerning properties of these models. I would like to pose these problems to experts in probability theory (though possibly some answers are already known to experts on Bell-type experiments and on distributed quantum computation).

I will therefore use the language of Applied Probability: simulation, rejection-sampling, and so on; and avoid reference to physics or philosophy.

2 The Problem

Suppose we want to simulate two random variables X, Y from a joint probability distribution depending on two parameters a, b . To fix ideas, let me give two key examples:

Case 1 (The Singlet Correlations). *X, Y are binary, taking the values ± 1 . The parameters a, b are two directions in real, three dimensional space. We will represent them with two unit vectors in \mathbb{R}^3 (two points on the unit sphere \mathbb{S}^2). The joint density of X, Y (their joint probability mass function) is*

$$\Pr_{a,b}\{X = x, Y = y\} = p(x, y; a, b) = \frac{1}{4}(1 - xy a \cdot b), \quad (1)$$

where $a \cdot b$ stands for the inner product of the unit vectors a and b and $x, y = \pm 1$. Note that the marginal laws of X and Y are both Bernoulli ($\frac{1}{2}$) on $\{-1, +1\}$, and their covariance equals their correlation equals $-a \cdot b$. In particular, the marginal law of X does not depend on b nor that of Y on a .

Case 2 (The Singlet Correlations Restricted). *This is identical to the previous example except that we are only interested in a and b taking values in two particular, possibly different, finite sets of points on \mathbb{S}^2 .*

Next I describe two different protocols for “distributed Monte-Carlo simulation experiments”; the difference is that one allows rejection sampling, the other does not. The idea is that the random variables X and Y are going to be generated on two different computers, and the inputs a, b are only given to each computer separately. The two computers are to generate dependent random variables, so they will start with having some shared randomness between them. The programmer is allowed to start with any number of random

variables, distributed just how he likes, for this purpose. Cognoscenti will realize that it suffices to have just one random variable, uniformly distributed on the interval $[0, 1]$, or equivalently, an infinite sequence of fair independent coin tosses. There is no need for the two computers to have access to further randomness—they may as well share everything they might ever need, separately or together, from the start.

The difference between the two protocols, or two tasks, is that the first has to get it right first time, or if you prefer, with probability one. The second protocol is allowed to make mistakes, as long as the mistakes are also “distributed”. Another way to say this, is that we allow “distributed rejection sampling”. Moreover, we allow the second protocol not to be completely accurate. It might be, that the second protocol can be made more and more accurate at the expense of a smaller and smaller acceptance (success) probability. This is precisely what we want to study. Success probability and accuracy can both depend on the parameters a and b so one will probably demand *uniformly* high success probability, and *uniformly* good accuracy.

Task 1 (Perfect Distributed Monte-Carlo). *Construct a probability distribution of a random variable Z , and two transformations f and g of Z , each depending on one of the two parameters a and b , such that*

$$f(Z; a), g(Z; b) \sim X, Y \quad \text{for all } a, b. \quad (2)$$

The symbol ‘ \sim ’ means ‘is jointly distributed as’, and X, Y on the right hand side come from the prespecified (or target) joint law with the given values of the parameters a and b .

Task 2 (Imperfect Distributed Rejection Sampling). *As before, but there are two further transformations, let me call them $D = \delta(Z, a)$ and $E = \epsilon(Z; b)$, such that δ and ϵ take values 1 and 0 or if you like, ACCEPT and REJECT, and such that*

$$f(Z; a), g(Z; b) \mid D = 1 = E \sim X, Y. \quad (3)$$

The symbol ‘ \mid ’ stands for ‘conditional on’, and ‘ \sim ’ means ‘is approximately distributed as’. The quality of the approximation needs to be quantified; in our case, the supremum over a and b of the variation distance between the two probability laws could be convenient (a low score means high quality). Moreover, one would like to have a uniformly large chance of acceptance. Thus a further interesting score (high score means high quality) is $\inf_{a,b} \Pr\{D = 1 = E\}$.

3 The Solutions

By Bell (1964) there is no way to succeed in Task 1 for Case 1. Moreover, there is no way to succeed in Task 1 for Case 2 either, for certain suitably chosen two-point sets of values for a and b .

Consider now Task 2, and suppose first of all that there are only two possible different values of a and b each (Case 2). Let the random variable Z consist of independent coin tosses coding guesses for a and b , and a realization of the pair X, Y drawn from the *guessed* joint distribution. The transformations δ and ϵ check if each guess is correct. The transformations f and g simply deliver the already generated X, Y . One obtains perfect accuracy with success probability $1/4$. It is known that a much higher success probability is achievable at the expense of more complicated transformations.

Now consider Task 2 for Case 1. So there is a continuum of possible values of a and b . Note that the joint law of X, Y depends on the parameters a, b continuously, and the parameters vary in compact sets. So one can partition each of their ranges into a finite number of cells in such a way that the joint law of X, Y does not change much while each parameter varies within one cell of their respective partitions. Moreover, one can get less and less variation at the expense of more and more cells. Pick one representative parameter value in each cell.

Now, fix one of these pairs of partitions, and just play the obvious generalization of our guessing game, using the representative parameter values for the guessed cells. If each partition has k cells and the guesses are uniform and independent, our success probability is $1/k^2$, uniformly in a and b . We can achieve arbitrarily high accuracy, uniformly in a and b , at the cost of arbitrarily low success probability.

However, Gisin and Gisin (1999) show we can do much better in the case of the singlet correlations:

Theorem 1 (Perfect conditional simulation of the singlet correlations). *For Case 1 and Task 2, there exists a perfect simulation with success probability uniformly equal to $1/2$.*

See Gisin and Gisin (1999) for the very pretty details. Can we do better still? What is the maximum uniformly achievable success probability?

It would be interesting to study these problems in a wider context. Consider arbitrary biparameterized joint laws p ; extend from pairs to triples, ...

The joint laws coming from quantum mechanics always satisfy *no action at a distance* (“no Bell telephone”), i.e., the marginal of X does not depend on b nor that of Y on a . This should obviously be favourable to finding

solutions to our tasks. Does it indeed play a role in making these simulations spectacularly more easy for quantum mechanics, than in general? Does “no action at a distance” ensure that we can find a perfect solution to Task 2 with success probability uniformly bounded away from 0? Am I indeed correct in thinking that one find probability distributions p *with* action at a distance, depending smoothly on parameters a, b , for which one can only achieve perfection in the limit of zero success probability?

4 Variant 1: Coincidences

Instead of demanding that δ and ϵ in Task 2 are binary, one might allow them to take on arbitrary real values, and correspondingly allow a more rich acceptance rule. Suggestively changing the notation to suggest times, define now $S = \delta(Z; a)$ and $T = \epsilon(Z; b)$. Instead of conditioning on the separate events $D = 1$ and $E = 1$ condition on the event $|S - T| < c$ where c is some constant. Obviously the new variant contains the original, so Variant Task 2 has become easier. Accardi, Imafuku and Regoli (2002) tackle this variant task, claiming that it has nothing to do with detector efficiency, but on the contrary is intrinsic to quantum optics, that one will have to post-select on coincidences in arrival times of entangled photons. By Heisenberg uncertainty, photons will always have a chance to arrive (or to be measured) at different times. But then their joint state is not the singlet state, and if we collect data on all pairs (supposing 100% detector efficiency) we would not recover the singlet correlations.

Conjecture 1 (No improvement from coincidences). *There is no gain from Variant Task 2 over the original.*

There seems to be a connection with the work of Massar, Bacon, Cerf and Cleve (2001) on classical simulation of quantum entanglement using classical communication. After all, checking the inequality $|S - T| < c$ is a task which requires communication between the two observers.

5 Variant 2: Demanding More

Instead of making Task 2 easier, as in the previous section, we can try to make it harder by demanding further attractive properties of the simulated joint probability distribution of D, X, E, Y . For instance, Gisin and Gisin (1999) show how one can achieve nice symmetry and stochastic independence properties at the cost of an only slightly smaller success probability $4/9 = (2/3)^2$. In fact, this solution has even more nice properties, as follows.

One might like the simulated X to behave well, when $D = 1$, whether or not $E = 1$, and similarly for Y .

Suppose we start with a joint law of X, Y depending on a, b as before. Let η be a fixed probability. Modify Task 2 as follows: we require not only that given $D = 1 = E$, the simulated X, Y have the prespecified joint distribution, but also that conditional on $D = 1$ and $E = 0$, the simulated X has the prespecified marginal distribution, and also that, conditional on $D = 0$ and $E = 1$, the simulated Y has the prespecified marginal distribution, and also that D and E are independent Bernoulli(η). Another way to describe this is by saying that under the simulated joint probability distribution of X, D, Y, E , we have statistical independence between D, E , and (X, Y) , with (X, Y) distributed according to our target distribution and D and E Bernoulli(η), except that we don't care about X on $\{D = 0\}$ nor about Y on $\{E = 0\}$.

Gisin and Gisin (1999) show that this Variant Task 2 can be achieved for our main example Case 1, with $\eta = 2/3$. It is known from considerations of the Clauser and Horne (1974) inequality that it cannot be done with $\eta > 2/(1 + \sqrt{2}) \approx 0.828$. It seems that the precise boundary is unknown.

In fact, for some practical applications, achieving this task is more than necessary. A slightly more modest task is to simulate the joint probability distribution just described, conditionally on the complement of the event $\{D = 0 = E\}$, i.e. conditional on $D = 1$ or $E = 1$. This means to say that we also don't care what is the simulated probability of $\{D = 0 = E\}$. Gisin and Gisin (1999) show that this can be achieved with a variant of the same model, and with success probability 100% (i.e., the simulation never generates an event $\{D = 0 = E\}$), and $\eta = 2/3$.

6 Acknowledgments

I am grateful for the warm hospitality and support of the Quantum Probability group at the department of mathematics of the University of Greifswald, Germany, during my sabbatical there, Spring 2003. My research there was supported by European Commission grant HPRN-CT-2002-00279, RTN QP-Applications. This research has also been supported by project RESQ (IST-2001-37559) of the IST-FET programme of the European Commission.

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