

# (Weak) Bernoullicity of Random Walk in Exponentially Mixing Random Scenery

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## Abstract

Consider an irreducible random walk and a stationary random scenery on  $\mathbb{Z}^d$ . The latter is assumed to be exponentially mixing, a property introduced in this paper. Exponentially mixing random fields include Gibbs fields at sufficiently high temperatures and the Ising field on  $\mathbb{Z}^d$ ,  $d \geq 2$ , at sufficiently low temperatures. We give conditions on the random walk such that the associated random walk in random scenery process is or is not Bernoulli or weak Bernoulli. Our conditions closely resemble the ones given by den Hollander and Steif (1997), where scenery values were assumed to be independent and identically distributed. In particular, the conditions coincide for simple random walk on  $\mathbb{Z}^d$ , respectively, for a 1-dimensional symmetric random walk taking steps of size  $x$  with probability proportional to  $(1 + |x|)^{-(1+\delta)}$ . For these examples, random walk in exponentially mixing random scenery is not Bernoulli if  $d = 1$  or 2 (resp.  $\delta \geq 1$ ), Bernoulli but not weak Bernoulli if  $d = 3$  or 4 (resp.  $\frac{1}{2} \leq \delta < 1$ ), and weak Bernoulli if  $d \geq 5$  (resp.  $0 < \delta < \frac{1}{2}$ ).

## 1 Background

### 1.1 Definition of the Problem

Let  $d \geq 1$  be an integer and let  $(X_n)_{n \in \mathbb{Z}}$  be independent random variables, taking values in  $\mathbb{Z}^d$  according to a common distribution function  $m$ . Let  $S = (S_n)_{n \in \mathbb{Z}}$  be the corresponding two-sided random walk, i.e.,  $S_0 = 0$  and  $S_n - S_{n-1} = X_n$ , and assume that  $S$  is irreducible, i.e., every point in  $\mathbb{Z}^d$  is visited with positive probability.

Next, let  $F$  be a finite set and let  $C = (C_x)_{x \in \mathbb{Z}^d}$  be a random scenery on  $\mathbb{Z}^d$  taking values in  $F^{\mathbb{Z}^d}$  according to a distribution  $\mu$ . We assume that  $\mu$  is stationary with respect to translations in  $\mathbb{Z}^d$ , and ergodic with respect to translations in the subgroup of  $\mathbb{Z}^d$  generated by  $\{x - y : m(x)m(y) > 0\}$ . The random walk and random scenery are assumed to be independent.

The joint process

$$Y = (Y_n)_{n \in \mathbb{Z}} \text{ with } Y_n = (X_n, C_{S_n})$$

is called the *random walk in random scenery* associated with  $m$  and  $\mu$ .

Under the aforementioned assumptions, Meilijson (1974) proved that  $Y$  has a trivial right tail. The general problem is to determine, given  $m$  and  $\mu$ , what other mixing properties  $Y$  possesses. In this paper, we will focus on two such properties: Bernoullicity and weak Bernoullicity.

A stationary process  $(Y_n)_{n \in \mathbb{Z}}$  is Bernoulli if it is conjugate to an i.i.d process, and it is weak Bernoulli if the past  $(Y_n)_{n \leq 0}$  and the far away future  $(Y_n)_{n > N}$  are asymptotically independent in the sense of total variation, i.e.,

$$\lim_{N \rightarrow \infty} \|P_{(-\infty, 0] \cup [N, \infty)} - P_{(-\infty, 0]} \times P_{[N, \infty)}\|_{\text{tv}} = 0,$$

where  $P_I$  is the distribution of  $Y$  restricted to  $I \cap \mathbb{Z}$  and  $\|\cdot\|_{\text{tv}}$  denotes the total variation norm.

In general, it is difficult to determine whether or not a process is Bernoulli. In our setting, however, we can exploit the fact that  $Y$  is Bernoulli if and only if it is very weak Bernoulli (Theorem 1.9 in den Hollander and Steif (1997)). A stationary process  $(Y_n)_{n \in \mathbb{Z}}$  is very weak Bernoulli, if for all  $\varepsilon > 0$  there exists a positive integer  $N = N(\varepsilon)$  such that the following holds. If  $n \geq N$  and  $J \subseteq (-\infty, 0] \cap \mathbb{Z}$  with  $J$  finite, then there exists a set  $H = H(J, n, \varepsilon)$  with  $P(Y_J \in H) > 1 - \varepsilon$  such that

$$\bar{d}(P(Y_{(0, n]} \in \cdot, P(Y_{(0, n]} \in \cdot | Y_J = \eta)) < \varepsilon \quad \text{for all } \eta \in H.$$

Here  $Y_J$  denotes the vector  $(Y_j)_{j \in J}$  and if  $\mu_1$  and  $\mu_2$  are probability measures on  $F^N$ , then the  $\bar{d}$ -distance between  $\mu_1$  and  $\mu_2$  is defined by

$$\bar{d}(\mu_1, \mu_2) = \inf \int \left( \frac{1}{N} \sum_{i=1}^N \mathbb{1}_{\{\eta_i \neq \xi_i\}} \right) \nu(d\eta, d\xi),$$

where the infimum extends over all couplings  $\nu$  of  $\mu_1$  and  $\mu_2$ .

Like Bernoullicity, weak Bernoullicity allows an alternative definition in terms of couplings, which for many processes is easier to establish (Theorem 4.4.7 in Berbee (1979)). A stationary process  $(Y_n)_{n \in \mathbb{Z}}$  is weak Bernoulli if and only if there is a joint process  $(Y'_n, Y''_n)_{n \in \mathbb{Z}}$  such that:

1.  $(Y'_n)_{n \in \mathbb{Z}}$  and  $(Y''_n)_{n \in \mathbb{Z}}$  are equal to  $(Y_n)_{n \in \mathbb{Z}}$  in distribution,
2.  $(Y'_n)_{n \in \mathbb{Z}}$  and  $(Y'_n)_{n \leq 0}$  are independent,
3. a.s. there exists a (random) non-negative integer  $N$  such that  $Y'_n = Y''_n$  for  $n > N$ .

In the present paper, we will give necessary and sufficient conditions for  $Y$  to be (very weak) Bernoulli, respectively, weak Bernoulli, under the assumption that the random scenery satisfies a certain mixing condition, called *exponentially mixing* (Definition 2.1). This mixing condition is known to hold for some classes of random fields:

1. Gibbs fields at sufficiently high temperatures,
2. the Ising model on  $\mathbb{Z}^d$ ,  $d \geq 2$ , at sufficiently low temperatures,
3. the 2-dimensional Ising model at all supercritical temperatures,
4. the 2-dimensional  $q$ -state Potts model at all supercritical temperatures, for sufficiently large  $q$ .

For classes 1, 3 and 4 see Alexander (1998), and for class 2 see Burton and Steif (1995).

## 1.2 History of the Problem

### 1.2.1 i.i.d. Random Sceneries

For the special case that the random scenery  $(C_x)_{x \in \mathbb{Z}^d}$  is i.i.d., den Hollander and Steif (1997) give conditions on the random walk such that  $Y$  is or is not (very weak) Bernoulli, respectively, weak Bernoulli. Their main results are summarized in the next theorem.

**Theorem 1.1.** (den Hollander and Steif (1997), Theorems 2.2, 2.6 and 2.8) *Let  $C = (C_x)_{x \in \mathbb{Z}^d}$  be a non-constant i.i.d. random scenery.*

1.  *$Y$  is weak Bernoulli if and only if  $|Z| < \infty$  a.s., where  $Z = \{S_n : n > 0\} \cap \{S_n : n \leq 0\}$ .*
2. *If  $S$  is a transient random walk, then  $Y$  is (very weak) Bernoulli.*
3. *If  $S$  is a recurrent random walk satisfying property  $\spadesuit$  (defined in den Hollander and Steif (1997), Section 2.3) and  $\sum_{x \in \mathbb{Z}^d} |x|^\delta m(x) < \infty$  for some  $\delta > 0$ , then  $Y$  is not (very weak) Bernoulli.*

Property  $\spadesuit$  in the theorem above bounds the density of self-intersections of the random walk. It can be checked for large classes of random walks and is conjectured to hold for arbitrary random walk.

## 1.2.2 Dependent Random Sceneries

In den Hollander et al. (2003), a first attempt was undertaken to generalize part 1 of Theorem 1.1 to dependent random sceneries. The generalization of the left to right implication is essentially complete.

**Theorem 1.2.** (den Hollander et al. (2003), Theorem 1) *If  $|Z| = \infty$  a.s. and  $\mu$  is non-atomic, then  $Y$  is not weak Bernoulli.*

Generalizing the right to left implication requires more than just a mild condition on the random scenery, as is indicated by Theorem 2 in den Hollander et al. (2003). For Markov random sceneries, they develop two conditions, a high and a low noise condition, that imply weak Bernoullicity of  $Y$ .

The low noise condition is defined in terms of disagreement paths. For  $x \in \mathbb{Z}^d$ , let  $E_{\neq}(x)$  be the set of pairs of sceneries in  $F^{\mathbb{Z}^d} \times F^{\mathbb{Z}^d}$  such that there is a nearest-neighbor path between 0 and  $x$  on which the sceneries disagree everywhere, and put  $\phi_\mu(x) = (\mu \times \mu)(E_{\neq}(x))$ . Let  $S_+ = \{S_n : n > 0\}$  and  $S_- = \{S_n : n \leq 0\}$ .

**Theorem 1.3.** (den Hollander et al. (2003), Theorem 3) *Assume that*

1.  *$\mu$  is Markov,*
2.  *$\sum_{x, y \in \mathbb{Z}^d} \phi_\mu(x) < \infty$ ,*
3.  *$\sum_{x \in S_+, y \in S_-} \phi_\mu(x - y) < \infty$  a.s.*

*Then  $Y$  is weak Bernoulli.*

Condition 3 in the theorem above implies that  $|Z| < \infty$  a.s. Unfortunately, the conditions 2 and 3 are generally hard to check. If  $\mu$  is the plus or minus phase of the nearest-neighbor Ising model in  $\mathbb{Z}^d$  ( $d \geq 2$ ) at sufficiently low temperature, then it follows from Burton and Steif (1995), Proposition 2.4, that  $\phi_\mu(x) \leq Ce^{-\lambda|x|}$  for some  $C, \lambda > 0$ . In that case, Condition 2 is satisfied and the validity of Condition 3 follows from the next theorem.

**Theorem 1.4.** (den Hollander et al. (2003), Theorem 5) *Let  $f : \mathbb{Z}^d \rightarrow [0, 1]$  and  $F = \sum_{x \in S_+, y \in S_-} f(x - y)$ . Then  $E(F) < \infty$ , with  $E$  expectation over  $S$ , in either of the following cases:*

1.  *$d \geq 5$ ,  $\sum_{x \in \mathbb{Z}^d} f(x) < \infty$ ,*
2.  *$1 \leq d \leq 4$ ,  $\sum_{x \in \mathbb{Z}^d} |x|f(x) < \infty$ ,  $\sum_{x \in \mathbb{Z}^d} xm(x) \neq 0$ ,  $\sum_{x \in \mathbb{Z}^d} |x|^6 m(x) < \infty$ .*

The high noise condition is defined in terms of standard site percolation, very similar to the low noise condition.

## 2 Main Results

In Section 2.1, we give the definition and some examples of exponentially mixing random fields. In Sections 2.2, 2.3 and 2.4, we give sufficient conditions for a random walk in an exponentially mixing random scenery to be, respectively, weak Bernoulli, very weak Bernoulli and not very weak Bernoulli. Sufficient conditions for not weak Bernoullicity have already been given in Theorem 1.2.

### 2.1 Exponentially Mixing Random Sceneries

We start by introducing the exponentially mixing condition.

**Definition 2.1.** For a function  $f : \mathbb{Z}^d \rightarrow [0, \infty)$ , the random scenery  $C$  (or its distribution  $\mu$ ) is said to be *mixing with respect to  $f$* , if for all finite and disjoint sets  $A, B \subset \mathbb{Z}^d$ ,

$$\|\mu|_{A \cup B} - \mu|_A \times \mu|_B\|_{\text{tv}} \leq \sum_{x \in A, y \in B} f(x - y),$$

where  $\mu|_A$  denotes the restriction of  $\mu$  to  $A$ . We say that the random scenery is *exponentially mixing*, if it is mixing with respect to an exponential function  $f(x) = Ce^{-\lambda|x|}$ , for some  $C, \lambda > 0$ .

*Remark 1.* Let  $f : \mathbb{Z}^d \rightarrow [0, \infty)$  be such that  $\lim_{n \rightarrow \infty} f(x_n) = 0$  for all sequences  $(x_n)_{n \in \mathbb{N}}$  with  $\lim_{n \rightarrow \infty} |x_n| = \infty$ . Assume that  $\mu$  is a stationary distribution on  $F^{\mathbb{Z}^d}$  that is mixing with respect to  $f$ . Then  $\mu$  is totally ergodic, i.e., ergodic with respect to translations in any non-zero subgroup of  $\mathbb{Z}^d$ . If the random scenery is non-constant, then it is non-atomic and hence by Theorem 1.2, if  $|Z| = \infty$ , then  $Y$  is not weak Bernoulli.

Recall from Section 1.2.2 that  $E_{\neq}(x)$  is the set of pairs of sceneries such that there is a nearest neighbor path between 0 and  $x$  (including 0 and  $x$  itself) on which the sceneries disagree everywhere, and that  $\phi_{\mu}(x) = (\mu \times \mu)(E_{\neq}(x))$ .

**Lemma 2.1.** *If  $\mu$  is a Markov random field such that  $\sum_{x, y \in \mathbb{Z}} \phi_{\mu}(x) < \infty$ , then  $\mu$  is mixing with respect to  $2\phi_{\mu}(x)$ .*

*Remark 2.* Burton and Steif (1995), Proposition 2.4, show that  $\phi_{\mu}(x) \leq Ce^{-\lambda|x|}$  for some  $C, \lambda > 0$ , when  $\mu$  is the plus-phase or the minus-phase of the nearest-neighbor Ising model on  $\mathbb{Z}^d$ ,  $d \geq 2$ , at sufficiently low temperatures.

The following definition is taken from Martinelli and Olivieri (1994). The random scenery is said to be *conditionally exponentially mixing* (Martinelli and Olivieri use the loaded term *weakly mixing*), if there exist  $C, \lambda > 0$  such that for all finite sets  $A, B \subset \mathbb{Z}^d$  with  $A \subseteq B$ ,

$$\left\| \mu^{B^c, \eta}|_A - \mu^{B^c, \eta'}|_A \right\|_{\text{tv}} \leq C \sum_{x \in A, y \in B^c} e^{-\lambda|x-y|} \quad \text{for all } \eta, \eta' \in F^{B^c},$$

where  $B^c$  denotes the complement of  $B$  in  $\mathbb{Z}^d$  and  $\mu^{B^c, \eta}$  denotes the conditional measure given that the random scenery on  $B^c$  is  $\eta$ .

**Lemma 2.2.** *If  $\mu$  is conditionally exponentially mixing with parameters  $C$  and  $\lambda$ , then  $\mu$  is exponentially mixing with the same parameters.*

*Remark 3.* For some particular cases, the conditionally exponentially mixing property is known to hold (see Alexander (1998)):

1. In arbitrary dimension, for Gibbs fields satisfying Dobrushin's uniqueness condition, i.e., for Gibbs potentials  $\Phi = \{\Phi_A : A \subset \mathbb{Z}^d \text{ finite}\}$  satisfying

$$\sup_{x \in \mathbb{Z}^d} \sum_{A: x \in A} (|A| - 1) \text{osc}(\Phi_A) < 2,$$

where  $\text{osc}(\Phi_A) = \sup\{|\Phi_A(\eta) - \Phi_A(\zeta)| : \eta, \zeta \in \mathbb{F}^{\mathbb{Z}^d}\}$  is the oscillation of  $\Phi_A$ .

2. In the uniqueness region of the 2-dimensional Ising model.
3. In the uniqueness region of the 2-dimensional  $q$ -state Potts model, for sufficiently large  $q$ .

## 2.2 Sufficient Conditions for Weak Bernoullicity

In the following theorem we give a sufficient condition for weak Bernoullicity of  $Y$ . This condition generalizes the low noise condition (Theorem 1.3 of this paper) and the high noise condition (den Hollander et al. (2003), Theorem 4) for Markov random fields.

**Theorem 2.1.** *Let  $f : \mathbb{Z}^d \rightarrow [0, \infty)$  be such that  $\sum_{x \in S_-, y \in S_+} f(x - y) < \infty$  a.s. and  $f(x) > 0$  for some  $x \in \mathbb{Z}^d$ . If  $\mu$  is mixing w.r.t.  $f$ , then  $Y$  is weak Bernoulli.*

The condition that  $\sum_{x \in S_-, y \in S_+} f(x - y) < \infty$  implies that  $|Z| < \infty$  a.s.

**Lemma 2.3.** *If  $f : \mathbb{Z}^d \rightarrow [0, \infty)$  is a function with  $\sum_{x \in S_-, y \in S_+} f(x - y) < \infty$  a.s. and  $f(x) > 0$  for some  $x \in \mathbb{Z}^d$ , then  $|Z| < \infty$  a.s.*

Theorem 1.4 gives conditions under which  $\sum_{x \in S_-, y \in S_+} f(x - y) < \infty$  a.s. The following lemma extends this result for a specific class of 1-dimensional random walks.

**Lemma 2.4.** *Let  $S$  be a 1-dimensional random walk satisfying  $m(x) \asymp (1 + |x|)^{-(1+\delta)}$  for some  $\delta > 0$ , where  $\asymp$  means that the ratio of the two sides is bounded between two constants independent of  $x$ . Then  $E(\sum_{x \in S_-, y \in S_+} f(x - y)) < \infty$ , whenever  $0 < \delta < \frac{1}{2}$  and  $\sum_{x \in \mathbb{Z}} f(x) < \infty$ .*

## 2.3 Sufficient Conditions for Very Weak Bernoullicity

In this section, we will generalize part 2 of Theorem 1.1 to exponentially mixing sceneries. Let  $G$  denote the Green's function associated with the random walk. For  $r \in \mathbb{R}$ , let  $B(r)$  denote the box  $[-r, r]^d \cap \mathbb{Z}^d$ .

**Theorem 2.2.** *Suppose that at least one of the following holds:*

1.  $\sum_{x \in \mathbb{Z}^d} xm(x) \neq 0$ ,
2. There are  $\alpha, \gamma > 0$  and  $C < \infty$  such that  $\lim_{n \rightarrow \infty} P(S_n \in B(n^\alpha)) = 0$  and  $G(x) \leq C(1 + |x|)^{-\gamma}$  for all  $x \in \mathbb{Z}^d$ .

*If  $\mu$  is exponentially mixing, then  $Y$  is very weak Bernoulli.*

*Remark 4.* The second condition in Theorem 2.2 is satisfied in the following cases.

1. If  $m$  satisfies  $\sum_{x \in \mathbb{Z}^d} |x|^2 m(x) < \infty$ , then it follows from the central limit theorem that  $\lim_{n \rightarrow \infty} P(S_n \in B(n^\alpha)) = 0$  for all  $0 \leq \alpha < \frac{1}{2}$ . Let  $d \geq 3$ , suppose that  $m$  satisfies

$$\begin{aligned} \sum_{x \in \mathbb{Z}^d} |x|^2 m(x) &< \infty && \text{if } d = 3, \\ \sum_{x \in \mathbb{Z}^d} |x|^2 \log(1 + |x|) m(x) &< \infty && \text{if } d = 4, \\ \sum_{x \in \mathbb{Z}^d} |x|^{d-2} m(x) &< \infty && \text{if } d \geq 5, \end{aligned}$$

and  $\sum_{x \in \mathbb{Z}^d} xm(x) = 0$ . Then it follows from Theorem 2 in Uchiyama (1998) that  $G(x) \leq C(1 + |x|)^{2-d}$  for all  $x \in \mathbb{Z}^d$  and some constant  $C < \infty$ .

2. Consider a 1-dimensional symmetric random walk with  $m(x) \asymp (1 + |x|)^{-(1+\delta)}$  and  $0 < \delta < 1$ . According to Theorem 7.7 in Durrett (1996), the sequence  $(\frac{S_n}{n^{1/\delta}})_{n \geq 0}$  converges in distribution to a non-degenerate random variable  $Z$ . Hence,  $\lim_{n \rightarrow \infty} P(|S_n| < n^\alpha) = 0$  for all  $\alpha < 1/\delta$ . From Theorem 5.5 in den Hollander and Steif (1997) it follows that  $G(x) \asymp (1 + |x|)^{\delta-1}$ .

## 2.4 Sufficient Conditions for Not Very Weak Bernoullicity

Finally, we generalize part 3 of Theorem 1.1 to exponentially mixing sceneries.

The following property is a slightly stronger version of property  $\spadesuit$  in den Hollander and Steif (1997).

**Definition 2.2.** A random walk  $S = (S_n)_{n \in \mathbb{Z}}$  has property  $\clubsuit$  if there exist constants  $C, \gamma, \lambda > 0$  such that for all integers  $M, N \geq 1$  and all  $0 < r < 1$

$$P(E_{N,r}^M) \leq CN^{-\gamma}r^{-2},$$

where  $E_{N,r}^M$  is the event

$$\left\{ \exists \mathcal{I} \subseteq \{0, \dots, N\}, |\mathcal{I}| \geq rN : \text{dist}(S[(i-1)M, iM], S[(j-1)M, jM]) \leq M^\lambda \forall i, j \in \mathcal{I} \right\}$$

and  $S[a, b]$  denotes the set  $\{S_n : a \leq n \leq b\}$ .

As was done in den Hollander and Steif (1997) with property  $\spadesuit$ , we conjecture that arbitrary random walk has property  $\clubsuit$ . Observe that if some coordinate of the random walk satisfies property  $\clubsuit$ , then so does the random walk itself.

**Theorem 2.3.** *Let  $S$  be a recurrent random walk satisfying property  $\clubsuit$  and  $\sum_{s \in \mathbb{Z}^d} |x|^\delta m(x) < \infty$  for some  $\delta > 0$ . If the random scenery is non-constant and exponentially mixing, then  $Y$  is not very weak Bernoulli.*

The following theorem is a generalization of Theorem 2.11 in den Hollander and Steif (1997) and gives conditions for a random walk to satisfy the  $\clubsuit$  property.

**Theorem 2.4.** *Let  $S$  be a 1-dimensional random walk for which there exists a sequence  $(a_n)_{n \in \mathbb{N}}$  with  $a_n > 0$  such that  $S_n/a_n$  converges in distribution to a stable law with index  $\alpha \geq 1$  and/or skewness parameter  $\theta \in (0, 1)$ . Then  $S$  satisfies property  $\clubsuit$ .*

We conclude this section by looking back on the claims made in the abstract concerning simple random walk (SRW) on  $\mathbb{Z}^d$  and 1-dimensional symmetric long-range random walk (LRW) with  $m(x) \asymp (1 + |x|)^{-(1+\delta)}$ ,  $\delta > 0$ . The claims follow from the results presented in Section 1.2.2 and Section 2 and the following observations.

1. SRW is recurrent if and only if  $d \leq 2$ , and  $|Z| = \infty$  a.s. if and only if  $d \leq 4$  (Lawler, 1991).
2. LRW is recurrent if and only if  $\delta \geq 1$  (Spitzer (1976), Example 8.2), and  $|Z| = \infty$  a.s. if and only if  $\delta \geq \frac{1}{2}$  (den Hollander and Steif (1997), Corollary 5.6, Theorem 2.5).
3. If  $(S_n)_{n \in \mathbb{Z}}$  is LRW, then  $\frac{S_n}{n^{1/\delta}}$  converges in distribution to a stable law with index  $\delta$  and skewness parameter equal to 0 (Durrett (1996), Theorem 7.7).

### 3 Exponentially Mixing Sceneries, Proofs

*Proof of Lemma 2.1.* For  $x, y \in \mathbb{Z}^d$ , let  $E_{\neq}(x, y)$  be the set of pairs of sceneries such that there is a nearest-neighbor path between  $x$  and  $y$  (including  $x$  and  $y$  itself) on which the sceneries disagree everywhere. Let  $A, B$  be finite and disjoint subsets of  $\mathbb{Z}^d$ , and let  $E_{\neq}(A, B)$  denote the set  $\bigcup_{x \in A, y \in B} E_{\neq}(x, y)$ . In the last paragraph of Step 2 in the proof of Theorem 3, den Hollander et al. (2003) show that

$$\|\mu^\eta|_A - \mu|_A\|_{\text{tv}} \leq 2(\mu^\eta \times \mu)(E_{\neq}(A, B)),$$

for all  $\eta \in F^B$ , where  $\mu^\eta$  denotes the conditional measure given that the random scenery on  $B$  is  $\eta$ . Here the assumption  $\sum_{x, y \in \mathbb{Z}} \phi_\mu(x) < \infty$  was used. Hence,

$$\begin{aligned} \|\mu|_{A \cup B} - \mu|_A \times \mu|_B\|_{\text{tv}} &\leq \int_{\eta \in F^B} \mu|_B(d\eta) \|\mu^\eta|_A - \mu|_A\|_{\text{tv}} \\ &\leq 2 \int_{\eta \in F^B} \mu|_B(d\eta) (\mu^\eta \times \mu)(E_{\neq}(A, B)) \\ &= 2(\mu \times \mu)(E_{\neq}(A, B)) \\ &\leq 2 \sum_{x \in A, y \in B} (\mu \times \mu)(E_{\neq}(x, y)) \\ &= 2 \sum_{x \in A, y \in B} (\mu \times \mu)(E_{\neq}(x - y)). \end{aligned}$$

□

*Proof of Lemma 2.2.* Assume  $\mu$  is conditionally exponentially mixing with parameters  $C$  and  $\lambda$  and let  $A, B$  be finite and disjoint subsets of  $\mathbb{Z}^d$ . Let  $B(k) = [-k, k]^d \cap \mathbb{Z}^d$  be such that  $A, B \subseteq B(k)$ , and let  $C_k = B \cup B(k)^c$ . Then, for any  $k$ ,

$$\begin{aligned} \|\mu|_{A \cup B} - \mu|_A \times \mu|_B\|_{\text{tv}} &\leq \int_{\eta \in F^{C_k}} \mu|_{C_k}(d\eta) \int_{\eta' \in F^{C_k}} \mu|_{C_k}(d\eta') \left\| \mu^{C_k, \eta}|_A - \mu^{C_k, \eta'}|_A \right\|_{\text{tv}} \\ &\leq C \sum_{x \in A, y \in C_k} e^{-\lambda|x-y|} \\ &= C \sum_{x \in A, y \in B} e^{-\lambda|x-y|} + C \sum_{x \in A, y \notin B(k)} e^{-\lambda|x-y|}. \end{aligned}$$

Since  $C \sum_{x \in A, y \notin B(k)} e^{-\lambda|x-y|}$  tends to 0 as  $k \rightarrow \infty$ , the result follows. □

### 4 Sufficient Conditions for Weak Bernoullicity, Proofs

We start by proving Lemma 2.3.

*Proof of Lemma 2.3.* By exchangeability,  $|Z| < \infty$  occurs with probability 0 or 1. Assume that  $|Z| = \infty$  a.s. and let  $x_0 \in \mathbb{Z}^d$  be such that  $f(x_0) > 0$ . By irreducibility, there is an integer  $N > 0$  and a  $\delta > 0$  such that  $P(-x_0 \in S[0, N]) > \delta$ . Let  $T_0 = 0$  and define inductively, for  $n \geq 0$ ,

$$\begin{aligned} A_n &= S_{T_n} \cap (S[T_n, T_n + N] + x_0) \\ T_{n+1} &= \inf \left\{ m > T_n + N : S_m \in S_- \setminus \left( \bigcup_{k=0}^n A_k \right) \right\}. \end{aligned}$$

Since  $|Z| = \infty$  a.s., the  $T_n$ 's are well defined. The sets  $(A_n)_{n \geq 0}$  are disjoint, each  $A_n$  is empty or consists of one point from  $S_-$ , and if  $x \in A_n$ , then  $x - x_0 \in S_+$ . Hence a.s.,

$$\sum_{x \in S_-, y \in S_+} f(x - y) \geq \sum_{x \in \bigcup_{n=0}^{\infty} A_n} f(x_0) = f(x_0) \sum_{n=0}^{\infty} \mathbb{1}_{\{A_n \neq \emptyset\}}.$$

Using that the events  $\{A_n \neq \emptyset\}$  are independent, we get from Borel-Cantelli that  $P(A_n \neq \emptyset \text{ i.o.}) = 1$ . As a consequence,  $\sum_{x \in S_-, y \in S_+} f(x - y) = \infty$ .  $\square$

The proof of Theorem 2.1 consists of two ingredients. The first is Lemma 2.3, the second is Step 1 in the proof of Theorem 3 in den Hollander et al. (2003), stating that if the random scenery is weak Bernoulli along the random walk, then  $Y$  is weak Bernoulli. The random scenery is said to be weak Bernoulli along the random walk if

$$\lim_{N \rightarrow \infty} \left\| \mu|_{S_- \cup S_+^N} - \mu|_{S_-} \times \mu|_{S_+^N} \right\|_{\text{tv}} = 0 \quad \text{for a.s. all } S,$$

where  $S_+^N = \{S_n : n \geq N\}$ . In fact, if we extend the notion of weak Bernoullicity to non-stationary sequences, then this is equivalent to saying that  $(C_{S_n})_{n \in \mathbb{Z}}$ , the scenery factor of  $Y$ , is weak Bernoulli for a.s. all  $S$ .

*Proof of Theorem 2.1.* Fix a random walk  $S$  with  $|Z| < \infty$ , let  $N > 0$  be an integer such that  $S_- \cap S_+^N = \emptyset$  and let  $A \subset S_-$  and  $B \subset S_+^N$  be finite. Since  $\mu$  is mixing w.r.t.  $f$ , it satisfies

$$\|\mu|_{A \cup B} - \mu|_A \times \mu|_B\|_{\text{tv}} \leq \sum_{x \in A, y \in B} f(x - y).$$

Taking suprema over all finite  $A \subset S_-$ ,  $B \subset S_+^N$ , and using that the number of intersections of past and future of the random walk is finite a.s. by Lemma 2.3, we obtain that

$$\lim_{N \rightarrow \infty} \left\| \mu|_{S_- \cup S_+^N} - \mu|_{S_-} \times \mu|_{S_+^N} \right\|_{\text{tv}} \leq \lim_{N \rightarrow \infty} \sum_{x \in S_-, y \in S_+^N} f(x - y),$$

for a.s. all random walks  $S$ .

To see that  $\sum_{x \in S_-, y \in S_+^N} f(x - y)$  tends to 0 as  $N \rightarrow \infty$ , let  $W_N = S_+ \setminus S_+^N = \{x \in S_+ : S_n \neq x \text{ for all } n \geq N\}$ . By Lemma 2.3,  $|Z| < \infty$  a.s., which in turn implies that the random walk  $(S_n)_{n \in \mathbb{Z}}$  is transient (den Hollander and Steif (1997), Remark (d)), and consequently,  $\bigcup_{N=0}^{\infty} W_N = S_+$ . Since

$$\lim_{N \rightarrow \infty} \sum_{x \in S_-, y \in W_N} f(x - y) = \sum_{x \in S_-, y \in S_+} f(x - y)$$

is finite a.s. by assumption, we have that  $\sum_{x \in S_-, y \in S_+^N} f(x - y)$  tends to 0 as  $N \rightarrow \infty$ .

Hence, the random scenery is weak Bernoulli along the random walk. It follows from Step 1 in the proof of Theorem 3 in den Hollander et al. (2003) that  $Y$  is weak Bernoulli.  $\square$

*Proof of Lemma 2.4.* Observe that

$$E\left(\sum_{x \in S_-, y \in S_+} f(x - y)\right) = \sum_{x, y \in \mathbb{Z}} f(x - y) G(x) G(y) = \sum_{k \in \mathbb{Z}} f(k) \sum_{l \in \mathbb{Z}} G(k + l) G(l).$$



Since  $G(x) \asymp (|x| + 1)^{\delta-1}$  when  $0 < \delta < 1$  (den Hollander and Steif (1997), Theorem 5.5), there exists a constant  $C < \infty$  such that

$$\sup_{k \in \mathbb{Z}} \sum_{l \in \mathbb{Z}} G(k+l)G(l) \leq \sup_{k \in \mathbb{Z}} C \sum_{l \in \mathbb{Z}} (|k+l| + 1)^{\delta-1} (|l| + 1)^{\delta-1} \leq C \sum_{l \in \mathbb{Z}} (|l| + 1)^{2(\delta-1)},$$

where the last inequality follows from Cauchy-Schwarz. Hence,

$$E \left( \sum_{x \in S_-, y \in S_+} f(x-y) \right) \leq 2C \sum_{k \in \mathbb{Z}} f(k) \sum_{l \in \mathbb{Z}} (|l| + 1)^{2(\delta-1)},$$

which is clearly finite, whenever  $0 < \delta < \frac{1}{2}$  and  $\sum_{x \in \mathbb{Z}} f(x) < \infty$ .  $\square$

## 5 Sufficient Conditions for Very Weak Bernoullicity, Proofs

**Lemma 5.1.** *If at least one of the conditions of Theorem 2.2 is met, then there exists  $\beta > 0$  such that*

$$\lim_{N \rightarrow \infty} E \left( \frac{1}{N} \sum_{k=1}^N \mathbb{1}_{\{S_n \in S(-\infty, 0] + B(n^\beta)\}} \right) = 0.$$

*Proof.* By time homogeneity and time reversibility, it suffices to show that  $\lim_{n \rightarrow \infty} P(S[n, \infty) \cap B(n^\beta) \neq \emptyset) = 0$  for some  $\beta > 0$ . If  $\sum_{x \in \mathbb{Z}^d} xm(x) \neq 0$  exists, then this immediately follows from the strong law of large numbers for all  $0 < \beta < 1$ . So assume that  $\alpha, \gamma > 0$  and  $C < \infty$  are such that  $\lim_{n \rightarrow \infty} P(S_n \in B(n^\alpha)) = 0$  and  $G(x) < C(1 + |x|)^{-\gamma}$  for all  $x \in \mathbb{Z}^d$ . Then,

$$P(S[n, \infty) \cap B(n^\beta) \neq \emptyset) \leq P(S_n \in B(n^\alpha)) + P(S_n \notin B(n^\alpha), S[n, \infty) \cap B(n^\beta) \neq \emptyset),$$

for  $\beta < \alpha$ . The first term in the right-hand side tends to 0 by assumption. The second term can be estimated by

$$\begin{aligned} P(S_n \notin B(n^\alpha), S[n, \infty) \cap B(n^\beta) \neq \emptyset) &\leq \sup_{x \in B^c(n^\alpha), y \in B(n^\beta)} |B(n^\beta)| P((y-x) \in S[0, \infty)) \\ &\leq \sup_{x \in B^c(n^\alpha), y \in B(n^\beta)} (2n^\beta + 1)^d G(y-x). \end{aligned}$$

Using that  $G(x) \leq C(1 + |x|)^{-\gamma}$  for all  $x \in \mathbb{Z}^d$ , we obtain

$$\begin{aligned} P(S_n \notin B(n^\alpha), S[n, \infty) \cap B(n^\beta) \neq \emptyset) &\leq C(2n^\beta + 1)^d (n^\alpha - n^\beta)^{-\gamma} \\ &\leq C' n^{\beta d - \alpha \gamma}, \end{aligned}$$

for some constant  $C' < \infty$ . The last term tends to 0 as  $n \rightarrow \infty$  for all  $0 < \beta < \frac{\alpha \gamma}{d}$ .  $\square$

For the remainder of this section, fix a  $\beta > 0$  for which

$$\lim_{N \rightarrow \infty} E \left( \frac{1}{N} \sum_{k=1}^N \mathbb{1}_{\{S_n \in S(-\infty, 0] + B(n^\beta)\}} \right) = 0,$$

which is possible by the previous lemma, and for  $N > 0$ , define  $S_\beta^N$  by

$$S_\beta^N = \{S_n : n \geq N \text{ and } S_n \notin S(-\infty, 0] + B(n^\beta)\}.$$

**Lemma 5.2.** *If  $\mu$  is exponentially mixing, then*

$$\lim_{N \rightarrow \infty} \left\| \mu|_{S_- \cup S_\beta^N} - \mu|_{S_-} \times \mu|_{S_\beta^N} \right\|_{\text{tv}} = 0 \text{ for a.s. all } S.$$

*Proof.* Fix  $N$  and let  $A \subset S_\beta^N$  and  $B \subset S_-$  be finite sets. Since  $\mu$  is exponentially mixing, we have

$$\begin{aligned} \|\mu|_{A \cup B} - \mu|_A \times \mu|_B\|_{\text{tv}} &\leq C \sum_{x \in A} \sum_{y \in B} e^{-\lambda|x-y|} \\ &\leq C \sum_{x \in S_\beta^N} \sum_{y \in S_-} e^{-\lambda|x-y|} \\ &= C \sum_{n \in I(N)} \sum_{y \in S_-} e^{-\lambda|S_n-y|}, \end{aligned}$$

where  $I(N) = \{n \geq N : S_n \notin S(-\infty, 0] + B(n^\beta)\}$ . Hence,

$$\begin{aligned} \|\mu|_{A \cup B} - \mu|_A \times \mu|_B\|_{\text{tv}} &\leq C \sum_{n=N}^{\infty} \sum_{y \notin S_n + B(n^\beta)} e^{-\lambda|S_n-y|} \\ &= C \sum_{n=N}^{\infty} \sum_{z \notin B(n^\beta)} e^{-\lambda|z|} \\ &\leq C \sum_{n=N}^{\infty} \sum_{k > n^\beta} (2k+1)^{d-1} e^{-\lambda k}. \end{aligned}$$

Replacing the sums by integrals, it is straightforward to see that the last expression tends to 0 as  $N \rightarrow \infty$ .  $\square$

**Lemma 5.3.** *Let  $Y' = (Y'_n)_{n \in \mathbb{Z}}$  and  $Y'' = (Y''_n)_{n \in \mathbb{Z}}$  be copies of  $Y$  on the same probability space. Then there is a coupling of  $Y'$  and  $Y''$  with the following properties:*

1.  $Y, Y'$  and  $Y''$  are identically distributed,
2.  $(Y'_n)_{n \in \mathbb{Z}}$  and  $(Y''_n)_{n \leq 0}$  are independent,
3. almost surely, there is a (random) non-negative integer  $N$  such that  $Y'_n = Y''_n$  for all  $n \in I''(N)$ ,

where  $I''(N) = \{n \geq N : S''_n \notin S''_- + B(n^\beta)\}$  and  $S''$  is the random walk with steps  $(X''_n)_{n \in \mathbb{Z}}$  that form the first component of  $Y''$ .

*Proof.* Let

$$X' = (X'_n)_{n \in \mathbb{Z}}, \quad C' = (C'_n)_{n \in \mathbb{Z}}, \quad X'' = (X''_n)_{n \in \mathbb{Z}}, \quad C'' = (C''_n)_{n \in \mathbb{Z}},$$

be independent copies of the random walk and the random scenery. Let  $S' = (S'_n)_{n \in \mathbb{Z}}$  and  $S'' = (S''_n)_{n \in \mathbb{Z}}$  be the associated random walks. Define

$$\begin{aligned} Y'_n &= (X'_n, C'_{S'_n}), & n \in \mathbb{Z}, \\ Y''_n &= (X''_n, C''_{S''_n}), & n \leq 0. \end{aligned}$$

For  $n > 0$ , let the first component of  $Y''_n$  be  $X''_n$ . To define the second component of  $Y''_n$  for  $n > 0$ , we follow the construction in Step 1 of the proof of Theorem 3 in den Hollander et al. (2003), with  $S''_+^N$  replaced by  $S''_+^N$ .

By Lemma 5.2,

$$\lim_{N \rightarrow \infty} \left\| \mu|_{S'' \cup S''_N} - \mu|_{S''} \times \mu|_{S''_N} \right\|_{\text{tv}} = 0 \quad \text{for a.s. all } S''.$$

Extending a result of Berbee (1979), Equation (4.4.2), to non-stationary processes, we obtain

$$\lim_{N \rightarrow \infty} \left\| \mu|_{S''_N|S''} - \mu|_{S''_N} \right\|_{\text{tv}} = 0,$$

for a.s. all random walks  $S''$  and a.s. all sceneries on  $S''$ . Here  $\mu|_{A|B}$  denotes the restriction of  $\mu$  to  $A$  conditional on the scenery in  $B$ . Note that  $\mu|_{A|B}$  is a random measure on  $F^A$ . By Goldstein (1979), Theorem 2.1, the measures  $\mu|_{S''_N|S''}$  and  $\mu|_{S''_N}$  can be successfully coupled after some random time a.s., where  $S''_N$  denotes  $S''^0$ . Let  $\nu$  denote the latter coupling. Let the second coordinate of  $Y''_n$  for  $n \in I'' = I''(0)$  be given by the first marginal of  $\nu$  conditioned on the second marginal of  $\nu$  being the scenery  $C'$  on  $S''_N \subset S'_N$ . Finally, choose the second coordinate of  $Y''_n$  for  $n \notin I''$  according to  $\mu$  conditionally on all previous choices.  $\square$

*Proof of Theorem 2.2.* For an interval  $J$ , let  $Y_J$  denote the vector  $(Y_j)_{j \in J \cap \mathbb{Z}}$ . We have to show that for all  $\varepsilon > 0$  there exists a positive integer  $N = N(\varepsilon)$  such that the following holds. If  $n \geq N$  and  $J \subseteq (-\infty, 0] \cap \mathbb{Z}$  with  $J$  finite, then there exists a set  $H = H(J, n, \varepsilon)$  with  $P(Y_J \in H) > 1 - \varepsilon$  such that

$$\bar{d}(P(Y_{(0,n]} \in \cdot), P(Y_{(0,n]} \in \cdot | Y_J = \eta)) < \varepsilon,$$

for all  $\eta \in H$ .

Fix  $\varepsilon > 0$ , let  $J \subseteq (-\infty, 0] \cap \mathbb{Z}$  be a finite set and let  $\hat{P}$  be the coupling from Lemma 5.3. It follows from Lemma 5.3 that

$$\begin{aligned} \hat{P}(Y' \in \cdot | Y'' = \eta) &= P(Y \in \cdot) \\ \hat{P}(Y'' \in \cdot | Y'' = \eta) &= P(Y \in \cdot | Y_J = \eta), \end{aligned}$$

for all  $\eta$  with  $P(Y_J = \eta) > 0$ . Suppose, for the moment, that we can find  $N$  such that for all  $n \geq N$ ,

$$\int \left( \frac{1}{n} \sum_{k=1}^n \mathbb{1}_{\{Y'_k \neq Y''_k\}} \right) d\hat{P} < \varepsilon^2.$$

It is not difficult to see that this implies that  $P(Y_J \in H) > 1 - \varepsilon$ , where  $H$  is defined as

$$H = \left\{ \eta : \int \left( \frac{1}{n} \sum_{k=1}^n \mathbb{1}_{\{Y'_k \neq Y''_k\}} \right) d\hat{P}(\cdot | Y'' = \eta) < \varepsilon \right\}.$$

Hence, for all  $n \geq N$  and  $\eta \in H$ ,

$$\begin{aligned} \bar{d}(P(Y_{(0,n]} \in \cdot), P(Y_{(0,n]} \in \cdot | Y_J = \eta)) &\leq \int \left( \frac{1}{n} \sum_{k=1}^n \mathbb{1}_{\{Y'_k \neq Y''_k\}} \right) d\hat{P}(\cdot | Y'' = \eta) \\ &< \varepsilon. \end{aligned}$$

To obtain the desired upper bound, we write

$$\begin{aligned} \int \left( \frac{1}{n} \sum_{k=1}^n \mathbb{1}_{\{Y'_k \neq Y''_k\}} \right) d\hat{P} &= \int \left( \frac{1}{n} \sum_{k \in \{1, \dots, n\} \cap I''(M)} \mathbb{1}_{\{Y'_k \neq Y''_k\}} \right) d\hat{P} \\ &\quad + \int \left( \frac{1}{n} \sum_{k \in \{1, \dots, n\} \setminus I''(M)} \mathbb{1}_{\{Y'_k \neq Y''_k\}} \right) d\hat{P}, \end{aligned}$$

where  $M$  is chosen such that  $\hat{P}(Y'_{I''(M)} \neq Y''_{I''(M)}) < \frac{\varepsilon^2}{2}$ , which is possible by Property 3 of Lemma 5.3. This directly implies that the first term in the last expression is smaller than  $\frac{\varepsilon^2}{2}$ . For  $N \geq M$ , the second term can be estimated by

$$\begin{aligned} \int \left( \frac{1}{n} \sum_{k \in \{1, \dots, n\} \setminus I''(M)} \mathbb{1}_{\{Y'_k \neq Y''_k\}} \right) d\hat{P} &\leq \int \left( \frac{1}{n} |\{1, \dots, n\} \setminus I''(M)| \right) d\hat{P} \\ &= \frac{M-1}{n} + \int \left( \frac{1}{n} \sum_{k=M}^n \mathbb{1}_{\{S''_k \in S''_- + B(k^\beta)\}} \right) d\hat{P}. \end{aligned}$$

Choose  $N > \frac{4(M-1)}{\varepsilon^2}$  such that

$$E \left( \frac{1}{n} \sum_{k=1}^n \mathbb{1}_{\{S_k \in S_- + B(k^\beta)\}} \right) < \frac{\varepsilon^2}{4} \text{ for all } n \geq N,$$

which is possible by our choice of  $\beta$ . Then for all  $n \geq N$ ,

$$\begin{aligned} \int \left( \frac{1}{n} \sum_{k \in \{1, \dots, n\} \setminus I''(M)} \mathbb{1}_{\{Y'_k \neq Y''_k\}} \right) d\hat{P} &< \frac{\varepsilon^2}{4} + \int \left( \frac{1}{n} \sum_{k=1}^n \mathbb{1}_{\{S''_k \in S''_- + B(k^\beta)\}} \right) d\hat{P} \\ &< \frac{\varepsilon^2}{2}. \end{aligned}$$

Hence,  $\int \left( \frac{1}{n} \sum_{k=1}^n \mathbb{1}_{\{Y'_k \neq Y''_k\}} \right) d\hat{P} < \varepsilon^2$ , for all  $n \geq N$ .  $\square$

## 6 Sufficient Conditions for Not Very Weak Bernoullicity, Proofs

This is the only section of this paper that cannot be read independently. The proofs in this section are modifications of the proofs of Theorem 2.3 and 2.4 in den Hollander and Steif (1997). The original proofs are technical and involve many parameters. In this section we adopt the same notation and for the definitions of symbols we refer to the original paper.

*Proof of Theorem 2.3.* We will sketch the adjustments to be made to the proof of Theorem 2.8 in den Hollander and Steif (1997). We begin by observing that the only parts of the proof that require independence of the random scenery are Lemma 6.3, which provides a recursive relation for  $f_k(p)$ , and the upper bound for  $f_0(p_0)$  in paragraph 6.4. Our first objective is to find an alternative recursive relation under the exponentially mixing assumption and property  $\clubsuit$ .

Assume that  $C, \gamma > 0$  are such that property  $\clubsuit$  is satisfied with parameters  $(C, \gamma, \gamma)$  and that  $\mu$  is exponentially mixing with parameters  $(C, \gamma)$ .

Redefine  $\theta'_{k+1}$  in paragraph 6.1 by

$$\begin{aligned} \theta'_{k+1} &= \{w \in \Omega : \text{for all } \mathcal{I} \subseteq \{0, \dots, \alpha_k\} \text{ with } |\mathcal{I}| \geq \alpha_k \beta_k \text{ there are } i, j \in \mathcal{I} \\ &\quad \text{such that } \text{dist}(S[(i-1)M, iM], S[(j-1)M, jM]) > M^\gamma\}. \end{aligned}$$

**Lemma 6.1.** *Let  $k \geq 0$  and suppose that  $p_{k+1}/p_k \leq 1 - 3\beta$ . Then*

$$f_{k+1}(p_{k+1}) \leq \alpha_k^2 |B_{m_{k+1}}|^2 \left( (f_k(p_k))^2 + C n_k^2 e^{-\gamma m_k^\gamma} \right).$$

*Proof.* For  $w \in \theta_{k+1}$ , let  $\mathcal{D}_w$  and  $\mathcal{E}_w$  be the index sets defined by

$$\begin{aligned}\mathcal{D}_w &= \{i \in \{1, 2, \dots, \alpha_k\} : w((i-1)n_k, in_k) \in \theta_k\}, \\ \mathcal{E}_w &= \{(i, j) \in \mathcal{D}_w \times \mathcal{D}_w : \text{dist}(S[(i-1)n_k, in_k](w), S[(j-1)n_k, jn_k](w)) > n_k^\gamma\}.\end{aligned}$$

Similarly as in the proof of Lemma 6.3 in den Hollander and Steif (1997), one can show that if  $w_1 \in \theta_{k+1}$  and  $c_2 \in F^{\mathbb{Z}^d}$ , then

$$\begin{aligned}A^{k+1, w_1, c_2, p_{k+1}} \subseteq & \bigcup_{(i, j) \in \mathcal{E}_{w_1}} \bigcup_{I, J \in B_{m_{k+1}}} \left\{ \left( \tau^{-S_{(i-1)n_k}(w_1)} A^{k, \sigma^{(i-1)n_k}(w_1), \tau^I(c_2), p_k} \right) \right. \\ & \left. \cap \left( \tau^{-S_{(j-1)n_k}(w_1)} A^{k, \sigma^{(j-1)n_k}(w_1), \tau^J(c_2), p_k} \right) \right\},\end{aligned}$$

where  $\sigma$  denotes the left-shift on  $(\mathbb{Z}^d)^{\mathbb{Z}}$  and  $\tau$  denotes the natural action of  $\mathbb{Z}^d$  on  $F^{\mathbb{Z}^d}$ . Consider the two events of which the intersection is taken between the braces in the expression above. Call the first one  $E(i, I)$ , the second one  $E(j, J)$ . Observe that  $E(i, I)$  is measurable with respect to the random scenery restricted to  $S[(i-1)n_k, in_k](w_1)$  and  $E(j, J)$  is measurable with respect to the random scenery restricted to  $S[(j-1)n_k, jn_k](w_1)$ . Since  $S[(i-1)n_k, in_k](w_1)$  and  $S[(j-1)n_k, jn_k](w_1)$  are at least distance  $n_k^\gamma$  apart for all  $(i, j) \in \mathcal{E}_{w_1}$ , it follows from the exponentially mixing property that

$$\begin{aligned}& |\mu(E(i, I) \cap E(j, J)) - \mu(E(i, I))\mu(E(j, J))| \\ & \leq \left\| \mu|_{S[(i-1)n_k, in_k](w_1) \cup S[(j-1)n_k, jn_k](w_1)} - \mu|_{S[(i-1)n_k, in_k](w_1)} \times \mu|_{S[(j-1)n_k, jn_k](w_1)} \right\|_{\text{TV}} \\ & \leq C |S[(i-1)n_k, in_k](w_1)| |S[(j-1)n_k, jn_k](w_1)| e^{-\gamma n_k^\gamma} \\ & \leq C n_k^2 e^{-\gamma n_k^\gamma}.\end{aligned}$$

Since the random scenery is stationary and since  $|\mathcal{E}_w| \leq |\mathcal{D}_w|^2 \leq \alpha_k^2$ , the lemma follows immediately.  $\square$

To get a non-recursive estimate for  $f_k(p_k)$ , let  $r_k = C n_k e^{-\frac{\gamma}{2} n_k^\gamma}$  and  $g_k = f_k(p_k) + r_k$ . If we assume that

$$\alpha_k^\gamma > 2 \quad \text{for all } k, \tag{1}$$

then  $r_{k+1} \leq \alpha_k r_k^2$ . Indeed, since  $n_{k+1} = \alpha_k n_k$  (den Hollander and Steif (1997), Equation 7), we have

$$\begin{aligned}r_{k+1} &= C \alpha_k n_k e^{-\frac{\gamma}{2} \alpha_k^\gamma n_k^\gamma} \\ &\leq \alpha_k C n_k e^{-\gamma n_k^\gamma} \\ &\leq \alpha_k r_k^2,\end{aligned}$$

where the second line follows from assumption 1. A recursive upper bound for  $g_k$  is easily obtained from the recursive upper bounds for  $f_k(p_k)$  in Lemma 6.1 and for  $r_k$ :

$$\begin{aligned}g_{k+1} &= f_{k+1} + r_{k+1} \\ &\leq \alpha_k^2 |B_{m_{k+1}}|^2 \left( (f_k(p_k))^2 + r_k^2 \right) + \alpha_k r_k^2 \\ &\leq (\alpha_k |B_{m_{k+1}}| + 1)^2 \left( (f_k(p_k))^2 + r_k^2 \right) \\ &\leq (\alpha_k |B_{m_{k+1}}| + 1)^2 g_k^2.\end{aligned}$$

Following paragraph 6.4, suppose that  $C_2$  is a constant satisfying

$$\alpha_k |B_{m_{k+1}}| + 1 \leq e^{C_2 2^k / (k+1)^2}. \quad (2)$$

Then we can estimate  $g_k$  by

$$g_k \leq g_0^{2^k} \exp\left[2^k C_2 \sum_{\ell=1}^k \ell^{-2}\right].$$

Hence  $\lim_{k \rightarrow \infty} f_k(p_k) = \lim_{k \rightarrow \infty} g_k = 0$  would hold as soon as  $g_0 < \exp(-C_2 \zeta(2))$  with  $\zeta(2) = \sum_{\ell=1}^{\infty} \ell^{-2}$ , for which it suffices that

$$C n_0 e^{-\frac{\gamma}{2} n_0^\gamma} < \frac{1}{2} e^{-C_2 \zeta(2)} \quad (3)$$

and

$$f_0(p_0) < \frac{1}{2} e^{-C_2 \zeta(2)}. \quad (4)$$

Our second objective is to find an upper bound for  $f_0(p_0)$ .

**Lemma 6.2.** *Let  $r > 0$  and let  $A \subset \mathbb{Z}^d$  be a finite subset of  $\mathbb{Z}^d$  with the property that any two points in  $A$  have distance at least  $r$ . Then for any fixed scenery  $\xi$ ,*

$$\mu\{\eta : \eta_A = \xi_A\} \leq q^{|A|} + C|A|^2 e^{-\gamma r},$$

where  $q = \sup_{c \in F} \mu\{\eta : \eta_0 = c\}$ .

*Proof.* If  $x$  is a point in  $A$ , then

$$\begin{aligned} & |\mu\{\eta : \eta_x = \xi_x, \eta_{A \setminus x} = \xi_{A \setminus x}\} - \mu\{\eta : \eta_x = \xi_x\} \mu\{\eta : \eta_{A \setminus x} = \xi_{A \setminus x}\}| \\ & \leq \|\mu|_{x \cup (A \setminus x)} - \mu|_x \times \mu|_{A \setminus x}\|_{\text{TV}} \\ & \leq C(|A| - 1) e^{-\gamma r}, \end{aligned}$$

where the last inequality follows from the exponentially mixing property and from the fact that  $\text{dist}(x, A \setminus x) \geq r$ . Hence,

$$\begin{aligned} \mu\{\eta : \eta_A = \xi_A\} & \leq \mu\{\eta : \eta_x = \xi_x\} \mu\{\eta : \eta_{A \setminus x} = \xi_{A \setminus x}\} + C(|A| - 1) e^{-\gamma r} \\ & \leq q \mu\{\eta : \eta_{A \setminus x} = \xi_{A \setminus x}\} + C(|A| - 1) e^{-\gamma r}. \end{aligned}$$

Iterating this inequality  $|A| - 1$  times, we obtain

$$\mu\{\eta : \eta_A = \xi_A\} \leq q^{|A|} + C e^{-\gamma r} \sum_{k=1}^{|A|-1} k q^{|A|-k}.$$

Bounding  $q$  by 1 in the second term of the last expression, we get

$$\begin{aligned} \mu\{\eta : \eta_A = \xi_A\} & \leq q^{|A|} + C e^{-\gamma r} \frac{1}{2} |A| (|A| - 1) \\ & \leq q^{|A|} + C |A|^2 e^{-\gamma r}. \end{aligned}$$

□

Any finite set  $A \subset \mathbb{Z}^d$  has a subset of cardinality at least  $\frac{|A|}{(2r-1)^d}$  with the aforementioned property. Hence for  $p_0 < 1/n_0$  and  $r > 0$ ,

$$\begin{aligned} f_0(p_0) &\leq q^{\frac{L}{(2r-1)^d}} + C \frac{L^2}{(2r-1)^{2d}} e^{-\gamma r} \\ &\leq q^{\frac{L}{(2r)^d}} + C L^2 e^{-\gamma r}. \end{aligned}$$

Concerning the choice of parameters, we will keep the values given in paragraph 6.6, except for  $C_2$ , which we change into  $C_2 = L^\gamma$ . We fix the parameter  $r$  introduced above as  $r = L^{\frac{1}{2d}}$  and we will assume that  $\gamma < \frac{1}{2d}$ .

Since the  $\alpha_k$ 's are left unchanged, Condition 1 follows immediately. Following the last part of paragraph 6.6, Condition 2 is satisfied if for any positive number  $A$  we can find an  $L$  such that

$$A^k k^{A^k} L^A \leq e^{L^\gamma 2^k / (k+1)^2} \quad \text{for all } k \geq 0.$$

Clearly, this holds whenever  $L$  is sufficiently large. Recalling that  $n_0$  was chosen to be  $C_1 L^2$ , we have that

$$C n_0 e^{-\frac{\gamma}{2} n_0^\gamma} = C C_1 L^2 e^{-\frac{\gamma}{2} C_1^\gamma L^{2\gamma}},$$

which is smaller than  $\frac{1}{2} e^{-L^\gamma \zeta(2)}$  whenever  $L$  is sufficiently large. This settles Condition 3. Finally, regarding Condition 4, it follows that

$$f_0(p_0) \leq q^{\frac{\sqrt{L}}{2d}} + C L^2 e^{-\gamma L^{\frac{1}{2d}}},$$

which is smaller than  $\frac{1}{2} e^{-\zeta(2) L^\gamma}$  whenever  $L$  is sufficiently large, since we assumed that  $\gamma < \frac{1}{2d}$  and since  $q < 1$  by our assumption that the random scenery is non-constant.  $\square$

*Proof of Theorem 2.4.* We will make some small modifications to the proof of Theorem 2.11 in den Hollander and Steif (1997). Fix  $M, N \geq 1$ , let  $\sigma < \frac{1}{\alpha}$  and for  $i, j \geq 1$  let

$$Y_{ij} = \mathbb{1}\{\text{dist}(S[(i-1)M, iM], S[(j-1)M, jM]) < M^\sigma\}.$$

By Equation (12) in paragraph 7, it suffices to show that  $P(Y_{ij} = 1) \leq C(j-i)^{-\gamma}$  for some  $C < \infty$  and  $\gamma > 0$ . Following Equations (13–15), we have for arbitrary  $h > 0$  that

$$P(Y_{ij} = 1) \leq \frac{C_1 M^\sigma}{a_{(j-i)M}} + \frac{C_1(2h+1)}{a_{(j-i)M}} + C_1 \left(\frac{a_M}{h}\right)^\lambda,$$

for some  $C_1 < \infty$ . Choosing  $h = a_{(j-i)M}^{1/(1+\lambda)} a_M^{\lambda/(1+\lambda)}$ , we get

$$P(Y_{ij} = 1) \leq \frac{C_1 M^\sigma}{a_{(j-i)M}} + C_2 (j-i)^{-\gamma_2}$$

for some  $C_2 < \infty$  and  $\gamma_2 > 0$ . We can write  $a_{(j-i)M}$  as  $(j-i)^{1/\alpha} M^{1/\alpha} L((j-i)M)$  with  $L$  a slowly varying function. Choose  $0 < \delta < \frac{1}{\alpha} - \sigma$  and let  $D < \infty$  be such that  $D k^{-\delta} \leq L(k)$  for all  $k \geq 1$ . Then

$$\frac{C_1 M^\sigma}{a_{(j-i)M}} \leq \frac{C_1}{D} M^{-(1/\alpha - \sigma - \delta)} (j-i)^{-(1/\alpha - \delta)},$$

and hence

$$\begin{aligned} P(Y_{ij} = 1) &\leq \frac{C_1}{D} (j-i)^{-(1/\alpha - \delta)} + C_2 (j-i)^{-\gamma_2} \\ &\leq C (j-i)^{-\gamma} \end{aligned}$$

for some  $C < \infty$  and  $\gamma > 0$ .  $\square$

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