Throughput analysis of two carousels

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November 7, 2003

Abstract: In this paper we consider a system with two carousels operated by one picker. The items to be picked are randomly located on the carousels and the pick times follow a phase-type distribution. The picker alternates between the two carousels, picking one item at a time. Important performance characteristics are the waiting time of the picker and the throughput of the two carousels. The waiting time of the picker satisfies an equation very similar to Lindley's equation for the waiting time in the $PH/U/1$ queue. Although the latter equation has no simple solution, it appears that the one for the waiting time of the picker can be solved explicitly. Furthermore, it is well known that the mean waiting time in the $PH/U/1$ queue strongly depends on the coefficient of variation of the inter-arrival time. Numerical results show that, for the carousel system, the mean waiting time and throughput are not very sensitive to the coefficient of variation of the pick-time.

AMS Subject Classification: Primary 60K25, Secondary 90B22.

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1 Introduction

A carousel is an automated storage and retrieval system consisting of a large number of shelves rotating in a circle. In this note we study a system consisting of two identical carousels and one picker. At each carousel there is an infinite supply of pick orders needing to be processed. The picker alternates between the two carousels, picking one order at a time. An important performance characteristic is the throughput, i.e. the number of orders processed per unit time. Park *et al.* [1] determine the throughput when the pick times are deterministic or exponentially distributed. We consider pick times following a phase-type distribution and derive explicit expressions for the throughput. Phase-type distributions may be used to approximate any pick-time distribution; see Schassberger [2].

Following Park *et al.* [1] we model a carousel as a circle of length 1 and assume it rotates in one direction at unit speed. Each pick order requires exactly one item. The picking process may be visualized as follows: When the picker is about to pick an item at one of the carousels, he may have to wait until the item is rotated in front of him. In the meantime, the other carousel rotates towards the position of the next item. After completion of the first pick, the picker turns to the other carousel, where he may have to wait again, and so on. Let the random variables P_n, R_n and W_n ($n \geqslant 1$) denote the pick time, rotation time and waiting time for the nth item. Clearly, the waiting times W_n satisfy the recursion

$$W_{n+1} = (R_{n+1} - P_n - W_n)^+, \qquad n = 0, 1, \ldots; \qquad P_0 = W_0 \stackrel{\mathsf{def}}{=} 0, \tag{1.1}$$

where $(x)^+ = \max\{0, x\}$. Note the striking similarity to Lindley's equation for the waiting times in a single-server queue. The only difference is the sign of W_n. We assume that both $\{P_n, n \geqslant 1\}$ and $\{R_n, n \geqslant 1\}$ are sequences of independent identically distributed random variables, also independent of each other. The pick times P_n have a phase-type distribution $G(\cdot)$ and the rotation times R_n are uniformly distributed on $[0, 1)$ (which means that the items are randomly located on the carousels). Then $\{W_n\}$ is a Markov chain, with state space $[0, 1)$. In equilibrium, equation (1.1) becomes

$$W \stackrel{\mathcal{D}}{=} (R - P - W)^+. \tag{1.2}$$

Let $\pi_0 = \mathbb{P}(W = 0)$ and $f(\cdot)$ denote the density of W on $(0, 1)$. From (1.2) it readily follows that (cf. equation (3) in Park *et al.* [1])

$$f(x) = \pi_0 G(1 - x) + \int_0^{1-x} G(1 - x - z) f(z) dz, \qquad 0 \leqslant x < 1, \tag{1.3}$$

and the normalisation equation

$$\pi_0 + \int_0^1 f(x) dx = 1. \tag{1.4}$$

Once the solution to equations (1.3) and (1.4) is known, we can compute $\mathbb{E}[W]$ and thus also the throughput τ from

$$\tau = \frac{1}{\mathbb{E}[W] + \mathbb{E}[P]}. \tag{1.5}$$

As pointed out before, equation (1.2) (with a plus sign instead of minus sign for W) is precisely Lindley's equation for the stationary waiting time in a $PH/U/1$ queue. This equation has no simple solution, but it will appear that the one for the waiting time of the picker can be solved explicitly. In the following we shall explore various methods to solve (1.2), or equivalently (1.3) and (1.4).

Since equation (1.3) is a Fredholm type equation, a natural way to proceed is by successive substitutions. This yields the formal solution

$$f(x) = \pi_0 \sum_{j=1}^{\infty} G^{j\,*}(1-x), \qquad 0 \leqslant x < 1, \tag{1.6}$$

where

$$G^{1\,*}(1-x) \stackrel{\mathsf{def}}{=} G(1-x); \qquad G^{n\,*}(1-x) \stackrel{\mathsf{def}}{=} \int_0^{1-x} G(1-x-z)\, G^{(n-1)\,*}(1-z)\, dz, \qquad n \geqslant 2.$$

Since $G(\cdot)$ is a distribution, from the last relation we have for $n \geqslant 1$ that

$$G^{(n+2)\,*}(x) \leqslant \int_0^x G^{(n+1)\,*}(1-z)\, dz \leqslant \int_0^x \int_0^{1-z} G^{n\,*}(1-y)\, dydz = \int_0^x \int_z^1 G^{n\,*}(y)\, dydz,$$

which implies that $G^{3\,*}(x) \leqslant 1/2$. Now, by induction, it can be easily shown that for $n \geqslant 1$

$$G^{(2n+1)\,*}(x) \leqslant \frac{1}{2^n}, \qquad 0 \leqslant x < 1,$$

and thus also

$$G^{2(n+1)\,*}(x) \leqslant \int_0^x G^{(2n+1)\,*}(1-z)\, dz \leqslant \frac{1}{2^n}, \qquad 0 \leqslant x < 1.$$

This means that the infinite sum (1.6) converges (uniformly) for $0 \leqslant x < 1$.

However, for a non-trivial distribution $G(\cdot)$, one cannot easily compute $f(\cdot)$ using (1.6). The difficulty lies in the fact that $G^{n\,*}(\cdot)$ is not the n-fold convolution of the distribution function $G(\cdot)$. Therefore, we continue by applying transforms to solve equation (1.3). This approach yields explicit and computable expressions for the density $f(\cdot)$ and the throughput τ, involving roots of a certain equation. An alternative approach that is presented, is by means of differentiation. Although the two methods superficially appear to be different, they are strongly related and we shall try to highlight the analogies between them.

In Section 2 we will first consider pick times with an Erlang distribution and prove that the density of the waiting time can be expressed as a sum of exponentials. In the next section we extend this result to pick times with a phase-type distribution. In Section 4 we proceed with some numerical results demonstrating that the throughput is fairly insensitive to the squared coefficient of variation of the pick times; the dominant factor is just the mean. We conclude with a brief summary and further research plans in Section 5.

2 Erlang pick times

In this section we assume that the pick times follow an Erlang distribution $\mathrm{Erl}(\mu, n)$ with scale parameter μ and n stages, that is

$$G(x) = 1 - e^{-\mu x} \sum_{j=0}^{n-1} \frac{(\mu x)^j}{j!}, \qquad x \geqslant 0.$$

In the first part we obtain the density of the waiting time through Laplace transforms. In the second part we present an alternative approach to the problem. Specifically we construct a differential equation that together with some initial conditions gives the solution to (1.3).

2.1 Laplace transform approach

Let $\phi(\cdot)$ denote the Laplace transform of $f(\cdot)$ over the interval $(0, 1)$, which means

$$\phi(s) = \int_0^1 e^{-sx} f(x) dx.$$

We should emphasise that for the Laplace transform over a bounded interval, the standard properties are, unfortunately, no longer valid. For example, the Laplace transform of a convolution is not the product of the transforms of the functions involved. Note that $\phi(\cdot)$ is analytic in the whole complex plane. It is convenient to replace x by $(1 - x)$ in (1.3), yielding

$$f(1 - x) = \pi_0 G(x) + \int_0^x G(x - z) f(z) dz, \qquad 0 \leqslant x < 1. \tag{2.1}$$

By taking the Laplace transform of (2.1) and using (1.4) we get

$$\begin{aligned} e^{-s}\phi(-s) = \pi_0 &\left(\frac{1 - e^{-s}}{s} - \sum_{j=0}^{n-1} \frac{\mu^j}{(\mu + s)^{j+1}} + \sum_{j=0}^{n-1} \sum_{i=0}^{j} \frac{\mu^j}{i!(\mu + s)^{j+1-i}} e^{-(\mu+s)} \right) \\ &- \frac{e^{-s}}{s}(1 - \pi_0) + \frac{1}{s}\phi(s) - \sum_{j=0}^{n-1} \frac{\mu^j}{(\mu + s)^{j+1}} \phi(s) \\ &+ e^{-(\mu+s)} \sum_{j=0}^{n-1} \sum_{i=0}^{j} \sum_{\ell=0}^{i} \binom{i}{\ell} \frac{\mu^j}{i!(\mu + s)^{j+1-i}} \phi^{(\ell)}(-\mu), \end{aligned}$$

which, by rearranging terms and using the identity

$$\sum_{j=0}^{n-1} \frac{\mu^j}{(\mu + s)^{j+1}} = \frac{(\mu + s)^n - \mu^n}{s(\mu + s)^n},$$

can be simplified to

$$e^{-s}\phi(-s) - \frac{\mu^n}{s(\mu+s)^n}\,\phi(s) = \pi_0\left(\frac{\mu^n}{s(\mu+s)^n} + e^{-(\mu+s)}\sum_{j=0}^{n-1}\sum_{i=0}^{j}\frac{\mu^j}{i!(\mu+s)^{j+1-i}}\right.$$
$$\left. - \frac{e^{-s}}{s}\right) + e^{-(\mu+s)}\sum_{j=0}^{n-1}\sum_{i=0}^{j}\sum_{\ell=0}^{i}\binom{i}{\ell}\frac{\mu^j}{i!(\mu+s)^{j+1-i}}\phi^{(\ell)}(-\mu). \quad (2.2)$$

In the above expression, $\phi^{(\ell)}(\cdot)$ denotes the derivative of order ℓ of $\phi(\cdot)$. Note that both $\phi(-s)$ and $\phi(s)$ appear in (2.2). To obtain an additional equation we replace s by $-s$ in (2.2) and form a system from which $\phi(s)$ can be solved, yielding:

Theorem 1. *For all s, the transform $\phi(s)$ satisfies*

$$\phi(s)R(s) = -e^{-s}s(\mu+s)^n A(-s) - \mu^n A(s), \quad (2.3)$$

where

$$\begin{aligned} R(s) &= s^2(\mu^2-s^2)^n + \mu^{2n}, \\ A(s) &= \pi_0\left(\mu^n + e^{-(\mu+s)}\sum_{j=0}^{n-1}\sum_{i=0}^{j}\frac{s\mu^j(\mu+s)^{n-j-1+i}}{i!}\right) - e^{-s}(\mu+s)^n \\ &\quad + e^{-(\mu+s)}\sum_{j=0}^{n-1}\sum_{i=0}^{j}\sum_{\ell=0}^{i}\binom{i}{\ell}\frac{s\mu^j(\mu+s)^{n-j-1+i}}{i!}\phi^{(\ell)}(-\mu). \end{aligned}$$

In (2.3) we still need to determine the $n+1$ unknowns π_0 and $\phi^{(\ell)}(-\mu)$ for $\ell = 0, \ldots, n-1$. Note that for any zero of the polynomial $R(\cdot)$, the left-hand side of (2.3) vanishes (since $\phi(\cdot)$ is analytic everywhere). This implies that the right-hand side should also vanish. Hence, the zeros of $R(\cdot)$ provide the equations necessary to determine the unknowns.

Lemma 1. *The polynomial $R(\cdot)$ has exactly $2n+2$ simple zeros $r_1, \ldots, r_{2n+2}$ satisfying $r_{2n+3-i} = -r_i$ for $i = 1, \ldots, n+1$.*

Proof: Since $R(s)$ is a polynomial of degree $n+1$ of s^2, it follows that $R(s)$ has exactly $2n+2$ zeros, with the property that each zero s has a companion zero $-s$. Furthermore, it is easily verified that $\gcd[R(s), R'(s)] = 1$. This means that the polynomials $R(s)$ and $R'(s)$ have no common factor of degree greater than zero or that $R(s)$ has only simple zeros. □

In the following lemma we prove that the $2n+2$ zeros of $R(\cdot)$ produce $n+1$ independent linear equations for the unknowns.

Lemma 2. *The probability π_0 and the parameters $\phi^{(\ell)}(-\mu)$ for $\ell = 0, \ldots, n-1$ are the unique solution to the $n+1$ linear equations,*

$$e^{-r_i}r_i(\mu+r_i)^n A(-r_i) + \mu^n A(r_i) = 0, \qquad i = 1, \ldots, n+1.$$

Proof: For any zero of $R(\cdot)$ the right-hand side of (2.3) should vanish. Hence, for two companion zeros r_i and $r_{2n+3-i} = -r_i$, $i = 1, \ldots, n+1$, we get

$$
\begin{aligned}
e^{-r_i} r_i(\mu + r_i)^n A(-r_i) + \mu^n A(r_i) &= 0, && (2.4)\\
-e^{r_i} r_i(\mu - r_i)^n A(r_i) + \mu^n A(-r_i) &= 0. && (2.5)
\end{aligned}
$$

The determinant of (2.4) and (2.5), treated as equations for $A(-r_i)$ and $A(r_i)$, is equal to $R(r_i) = 0$. Hence, (2.4) and (2.5) are dependent, so we may omit one of them. This leaves a system of $n+1$ linear equations for the unknowns π_0 and $\phi^{(\ell)}(-\mu)$ for $\ell = 0, \ldots, n-1$. The uniqueness of the solution follows from the general theory of Markov chains that implies that there is a unique equilibrium distribution and thus also a unique solution to (2.2). □

Once π_0 and $\phi^{(\ell)}(-\mu)$ for $\ell = 0, \ldots, n-1$ are determined, the transform $\phi(\cdot)$ is known. It remains to invert the transform. By collecting the terms that include e^{-s} we can rewrite (2.3) in the form

$$
\phi(s) = \frac{P(s)}{R(s)} + e^{-s}\frac{Q(s)}{R(s)}, \tag{2.6}
$$

where $P(s)$ and $Q(s)$ are polynomials of degree $2n+1$ and $n+1$, respectively. Since deg$[R]$ is greater than deg$[P]$ and deg$[Q]$, expression (2.6) can be decomposed into distinctive irreducible fractions. This leads to

$$
\phi(s) = \frac{c_1}{s - r_1} + \cdots + \frac{c_{2n+2}}{s - r_{2n+2}} + e^{-s}\left[\frac{\hat{c}_1}{s - r_1} + \cdots + \frac{\hat{c}_{2n+2}}{s - r_{2n+2}}\right], \tag{2.7}
$$

where the coefficients c_i and $\hat{c}_i$ are given by

$$
c_i = \lim_{s \to r_i} \frac{P(s)}{R(s)}(s - r_i) = \frac{P(r_i)}{R'(r_i)}, \qquad \hat{c}_i = \lim_{s \to r_i} \frac{Q(s)}{R(s)}(s - r_i) = \frac{Q(r_i)}{R'(r_i)}. \tag{2.8}
$$

Note that the derivative $R'(r_i)$ is non-zero, since r_i is a simple zero. Since the numerator of the right-hand side of (2.6) vanishes at all points r_i, we have

$$
P(r_i) = -e^{-r_i} Q(r_i), \qquad i = 1, \ldots, 2n+2.
$$

Hence, from (2.8) it follows that

$$
c_i = -e^{-r_i}\hat{c}_i, \tag{2.9}
$$

and thus

$$
\phi(s) = \sum_{i=1}^{2n+2} \frac{c_i}{s - r_i}\left[1 - e^{r_i - s}\right],
$$

which is the transform of a mixture of $2n+2$ exponentials. These findings are summarised in the following theorem.

Theorem 2. *The density of the waiting time is given by*

$$
f(x) = \sum_{i=1}^{2n+2} c_i e^{r_i x}, \qquad 0 \leqslant x < 1. \tag{2.10}
$$

Corollary 1. *The throughput τ satisfies*

$$
\tau^{-1} = \mathbb{E}[P] + \mathbb{E}[W] = \frac{n}{\mu} + \sum_{i=1}^{2n+2} \frac{c_i}{r_i^2}[1 + (r_i - 1)e^{r_i}].
$$

2.2 Differential equations approach

An alternative method to calculate the density $f(\cdot)$ is by means of differentiation. More precisely, we differentiate (2.1) with respect to x obtaining

$$\frac{d}{dx}f(1-x) = \pi_0\mu^n e^{-\mu x}\frac{x^{n-1}}{(n-1)!} + \int_0^x \mu^n e^{-\mu(x-z)}\frac{(x-z)^{n-1}}{(n-1)!}f(z)\,dz. \tag{2.11}$$

Multiplying with $e^{\mu x}$ and differentiating m more times, $m = 1,\ldots,n-1$, with respect to x gives us

$$\frac{d^m}{dx^m}\left[e^{\mu x}\frac{d}{dx}f(1-x)\right] = \pi_0\mu^n\frac{x^{n-m-1}}{(n-m-1)!} + \int_0^x \mu^n e^{\mu z}\frac{(x-z)^{n-m-1}}{(n-m-1)!}f(z)\,dz. \tag{2.12}$$

Note that the integral at the right hand side does not vanish. We shall need this remark later in order to derive the initial conditions of the differential equation. Now, by differentiating the one for $m = n-1$ we have

$$\frac{d^n}{dx^n}\left[e^{\mu x}\frac{d}{dx}f(1-x)\right] = e^{\mu x}f(x)\mu^n \quad \text{or} \quad \sum_{k=0}^{n}\binom{n}{k}\mu^{-k}\frac{d^{k+1}}{dx^{k+1}}f(1-x) = f(x).$$

Up to this point we have differentiated with respect to x a total of $n+1$ times. In order to derive a homogeneous linear differential equation we replace x by $1-y$ and repeat the same procedure. This means that we shall differentiate (with respect to y now) a total of $n+1$ times more. We replace x by $1-y$ in the last relation and we obtain

$$\sum_{k=0}^{n}\binom{n}{k}\mu^{-k}(-1)^{k+1}\frac{d^{k+1}}{dy^{k+1}}f(y) = f(1-y). \tag{2.13}$$

The change of variables is practically equivalent to the replacement of s by $-s$ that we did in order to obtain equation (2.3). Differentiating once (2.13) with respect to y and combining the result with (2.11) yields

$$\sum_{k=0}^{n}\binom{n}{k}\mu^{-k}(-1)^{k+1}\frac{d^{k+2}}{dy^{k+2}}f(y) = \pi_0\mu^n e^{-\mu y}\frac{y^{n-1}}{(n-1)!} + \int_0^y \mu^n e^{-\mu(y-z)}\frac{(y-z)^{n-1}}{(n-1)!}f(z)\,dz.$$

As before, we multiply with $e^{\mu y}$ and we differentiate m more times with respect to y, for $m = 1,\ldots,n-1$. Furthermore, the remark we made before is still valid. Namely, all the intermediate steps have a right hand side of the same form as in (2.12). One more differentiation gives us

$$\sum_{k=0}^{n}\binom{n}{k}\mu^{-k}(-1)^{k+1}\frac{d^n}{dy^n}\left[e^{\mu y}\frac{d^{k+2}}{dy^{k+2}}f(y)\right] = e^{\mu y}f(y)\mu^n,$$

which after rewriting the derivatives and arranging the terms becomes

$$\sum_{k=0}^{n}\binom{n}{k}\mu^{-k}(-1)^{k+1}\sum_{j=0}^{n}\binom{n}{j}\mu^{-j}\frac{d^{j+k+2}}{dy^{j+k+2}}f(y) = f(y). \tag{2.14}$$

Equation (2.14) is a homogeneous linear differential equation of order $2n+2$. For the solution we need the roots of the characteristic function

$$\sum_{k=0}^{n}\binom{n}{k}\mu^{-k}(-1)^{k+1}\sum_{j=0}^{n}\binom{n}{j}\mu^{-j}r^{j+k+2}=1 \quad \text{or} \quad -r^2\left(1-\frac{r}{\mu}\right)^n\left(1+\frac{r}{\mu}\right)^n=1,$$

which agrees with $R(r)=0$. By Lemma 1 we know that the roots of this equation are simple, which means that the general solution of (2.14) is given by

$$f(x)=\sum_{i=1}^{2n+2}d_ie^{r_ix}, \qquad 0\leqslant x<1.$$

This proves Theorem 2, except that we still need to determine the coefficients d_i, $i=1,\dots,2n+2$.

For the solution we need as many initial conditions as the order of the differential equation. We are going to derive them from the intermediate steps of differentiation. From (1.3) we have that $f(1)=0$ and this will be the first condition. We derive the other $2n+1$ conditions from evaluating at zero each equation that we obtained from the intermediate steps of differentiation. We do not use the last differentiation with respect to y, because this yields equation (2.14), which is the differential equation. We summarise the above in the following relations: For $m=1,\dots,n-2$ we have

$$\left.\frac{d}{dx}f(1-x)\right|_{x=0}=0, \qquad \left.\frac{d^m}{dx^m}\left[e^{\mu x}\frac{d}{dx}f(1-x)\right]\right|_{x=0}=0,$$
$$\left.\frac{d^{n-1}}{dx^{n-1}}\left[e^{\mu x}\frac{d}{dx}f(1-x)\right]\right|_{x=0}=\pi_0\mu^n, \qquad \left.\frac{d^n}{dx^n}\left[e^{\mu x}\frac{d}{dx}f(1-x)\right]\right|_{x=0}=f(0)\mu^n,$$
$$\sum_{k=0}^{n}\binom{n}{k}\mu^{-k}(-1)^{k+1}\left.\frac{d^{k+2}}{dy^{k+2}}f(y)\right|_{y=0}=0, \qquad \sum_{k=0}^{n}\binom{n}{k}\mu^{-k}(-1)^{k+1}\left.\frac{d^m}{dy^m}\left[e^{\mu y}\frac{d^{k+2}}{dy^{k+2}}f(y)\right]\right|_{y=0}=0,$$

and

$$\sum_{k=0}^{n}\binom{n}{k}\mu^{-k}(-1)^{k+1}\left.\frac{d^{n-1}}{dy^{n-1}}\left[e^{\mu y}\frac{d^{k+2}}{dy^{k+2}}f(y)\right]\right|_{y=0}=\pi_0\mu^n.$$

Note that all these conditions define uniquely the coefficients d_i, but involve the unknown parameter π_0. We obtain this last parameter by using the normalisation equation (1.4) and this concludes the proof of Theorem 2. One observation is that by using this method, one needs to solve a system of linear equations twice as big as the one that appears in Lemma 2. Furthermore, we know that the coefficients d_i are equal to the coefficients c_i that appear in (2.10), thus they satisfy relation (2.9). However, these relations do not become immediately obvious from the analysis.

3 Phase-Type pick times

Let us now assume that the pick times follow an $\mathrm{Erl}(\mu, n)$ with probability α_n, $n = 1, \ldots, N$, in other words,

$$G(x) = \sum_{n=1}^{N} \alpha_n \left(1 - e^{-\mu x} \sum_{j=0}^{n-1} \frac{(\mu x)^j}{j!} \right), \qquad x \geqslant 0. \tag{3.1}$$

The class of the phase-type distributions of the above form is dense in the space of distribution functions defined on $[0, \infty)$. This means that for any such distribution function $F(\cdot)$, there is a sequence $F_n(\cdot)$ of phase-type distributions of this class that converges weakly to $F(\cdot)$ as n goes to infinity; for details see Schassberger [2]. Below we show that the results of Subsection 2.1 can be extended to pick time distributions of the form (3.1).

Transforming now (2.1) gives us formulas analogous to the ones that appeared in the Laplace transform approach for Erlang pick times. So we have (cf. equation (2.2))

$$\begin{aligned}
e^{-s}\phi(-s) \quad = \quad & \pi_0 \sum_{n=1}^{N} \alpha_n \left(\frac{\mu^n}{s(\mu+s)^n} + e^{-(\mu+s)} \sum_{j=0}^{n-1} \sum_{i=0}^{j} \frac{\mu^j}{i!(\mu+s)^{j+1-i}} \right) \\
& + \sum_{n=1}^{N} \alpha_n \left(-\frac{e^{-s}}{s} + e^{-(\mu+s)} \sum_{j=0}^{n-1} \sum_{i=0}^{j} \sum_{\ell=0}^{i} \binom{i}{\ell} \frac{\mu^j}{i!(\mu+s)^{j+1-i}} \phi^{(\ell)}(-\mu) \right) \\
& + \phi(s) \sum_{n=1}^{N} \alpha_n \left(\frac{\mu^n}{s(\mu+s)^n} \right).
\end{aligned}$$

In order to obtain the transform $\phi(s)$ we form once more a 2×2 system of linear equations by replacing s with $-s$. This leads to the following result.

Theorem 3. *For all s, the transform $\phi(s)$ satisfies*

$$\phi(s)\widetilde{R}(s) = -e^{-s} s(\mu+s)^N \widetilde{A}(-s) - \sum_{n=1}^{N} \alpha_n \mu^n (\mu - s)^{N-n} \widetilde{A}(s), \tag{3.2}$$

where

$$\begin{aligned}
\widetilde{R}(s) \quad = \quad & s^2(\mu^2 - s^2)^N + \sum_{n=1}^{N} \sum_{m=1}^{N} \alpha_n \alpha_m \mu^n \mu^m (\mu - s)^{N-n} (\mu + s)^{N-m}, \\
\widetilde{A}(s) \quad = \quad & \pi_0 \sum_{n=1}^{N} \alpha_n \left(\mu^n (\mu+s)^{N-n} + e^{-(\mu+s)} \sum_{j=0}^{n-1} \sum_{i=0}^{j} \frac{s\mu^j (\mu+s)^{N-j-1+i}}{i!} \right) \\
& + \sum_{n=1}^{N} \alpha_n \left(-e^{-s}(\mu+s)^N + e^{-(\mu+s)} \sum_{j=0}^{n-1} \sum_{i=0}^{j} \sum_{\ell=0}^{i} \binom{i}{\ell} \frac{s\mu^j(\mu+s)^{N-j-1+i}}{i!} \phi^{(\ell)}(-\mu) \right).
\end{aligned}$$

The unknowns π_0 and $\phi^{(\ell)}(-\mu)$ for $\ell = 0, \ldots, n-1$ can be determined in the same way as in Subsection 2.1. The polynomial $\widetilde{R}(\cdot)$ has exactly $2N+2$ zeros, with the property that each zero s has a companion zero $-s$. We *assume* that all zeros are simple and label them $\widetilde{r}_1, \ldots, \widetilde{r}_{2N+2}$ such that $\widetilde{r}_{2N+3-i} = -\widetilde{r}_i$ for $i = 1, \ldots, N+1$. Then the following lemma can be readily established.

Lemma 3. *The probability π_0 and the parameters $\phi^{(\ell)}(-\mu)$ for $\ell = 0, \ldots, n-1$ are the unique solution to the $N+1$ linear equations,*

$$e^{-\widetilde{r}_i}\widetilde{r}_i(\mu+\widetilde{r}_i)^N\widetilde{A}(-\widetilde{r}_i) + \sum_{n=1}^{N}\alpha_n\mu^n(\mu-\widetilde{r}_i)^{N-n}\widetilde{A}(\widetilde{r}_i) = 0, \qquad i = 1, \ldots, N+1. \tag{3.3}$$

Given π_0 and $\phi^{(\ell)}(-\mu)$ for $\ell = 0, \ldots, n-1$, the transform $\phi(\cdot)$ is completely known. Partial fraction decomposition of the transform yields

$$\phi(s) = \sum_{i=1}^{2N+2}\frac{\widetilde{c}_i}{s-\widetilde{r}_i}\left[1 - e^{\widetilde{r}_i - s}\right],$$

from which we conclude that the density of the waiting time is a mixture of $2N+2$ exponentials. Hence, as it was the case for Erlang pick times, the density is given by

$$f(x) = \sum_{i=1}^{2N+2}\widetilde{c}_i e^{\widetilde{r}_i x}.$$

Remark 1. The analysis proceeds essentially in the same way when $R(\cdot)$ has multiple zeros. For example, if $\widetilde{r}_1 = \widetilde{r}_2$ (so $\widetilde{r}_1$ and thus also $\widetilde{r}_{2N+2}$ are double zeros), then equation (3.3) for $i = 1$ is identical to the one for $i = 2$. Nonetheless an additional equation can be obtained by requiring that also the derivative of the right-hand side of (3.2) at $s = r_1$ should vanish. The partial fraction decomposition of $\phi(\cdot)$ now becomes

$$\begin{aligned}\phi(s) \quad &= \quad \frac{\widetilde{c}_1}{(s-\widetilde{r}_1)^2}\left[1 - e^{\widetilde{r}_1 - s} - (s-\widetilde{r}_1)e^{\widetilde{r}_1 - s}\right] + \sum_{i=2}^{2N+1}\frac{\widetilde{c}_i}{s-\widetilde{r}_i}\left[1 - e^{\widetilde{r}_i - s}\right] \\ &\quad + \frac{\widetilde{c}_{2N+2}}{(s-\widetilde{r}_{2N+2})^2}\left[1 - e^{\widetilde{r}_{2N+2} - s} - (s-\widetilde{r}_{2N+2})e^{\widetilde{r}_{2N+2} - s}\right],\end{aligned}$$

the inverse of which is given by

$$f(x) = \widetilde{c}_1 x e^{\widetilde{r}_1 x} + \sum_{i=2}^{2N+1}\widetilde{c}_i e^{\widetilde{r}_i x} + \widetilde{c}_{2N+2} x e^{\widetilde{r}_{2N+2} x}.$$

Remark 2. In case the pick times follow a phase-type distribution of the form (3.1), we can still obtain the solution by similar methods as in the differential equations approach of Subsection 2.2. If $\widetilde{R}(\cdot)$ has only simple roots, then there are no differences in the analysis. If there are roots with multiplicity greater than 1, the differential equation is solved in a similar but not identical manner, involving exponentials multiplied with powers of x (cf. Remark 1). For each root r of multiplicity k we need to have k linearly independent solutions, which in this case will be of the form $x^i e^{rx}$, for $i = 0, \ldots, k-1$.

Remark 3. Hyper-exponential distributions form another useful class of phase-type distributions. These distributions may be used to model pick times with squared coefficient of variation greater than 1. The analysis for hyper-exponential pick times is very similar to the one presented in this section.

4 Numerical results

This section is devoted to some numerical results. For various values of the mean pick time $\mathbb{E}[P]$ we show in Figure 1 the throughput τ versus the squared coefficient of variation of the pick time, c_P^2. The mean pick time is chosen comparable to the mean rotation time, which is 1/2. For each setting we fitted a mixed Erlang or hyper-exponential distribution to $\mathbb{E}[P]$ and c_P^2, depending on whether the squared coefficient of variation is less or greater than 1 (see, for example, Tijms [3]). More specifically, if $1/n \leqslant c_P^2 \leqslant 1/(n-1)$ for some $n = 2, 3, \ldots$, then the mean and squared coefficient of variation of the mixed Erlang distribution

$$G(x) = p\left(1 - e^{-\mu x}\sum_{j=0}^{n-2}\frac{(\mu x)^j}{j!}\right) + (1-p)\left(1 - e^{-\mu x}\sum_{j=0}^{n-1}\frac{(\mu x)^j}{j!}\right), \qquad x \geqslant 0,$$

matches with $\mathbb{E}[P]$ and c_P^2, provided the parameters p and μ are chosen as

$$p = \frac{1}{1+c_P^2}[nc_P^2 - \{n(1+c_P^2) - n^2c_P^2\}^{1/2}], \qquad \mu = \frac{n-p}{\mathbb{E}[P]}.$$

On the other hand, if $c_P^2 > 1$, then the mean and squared coefficient of variation of the hyper-exponential distribution

$$G(x) = p_1(1 - e^{-\mu_1 x}) + p_2(1 - e^{-\mu_2 x}), \qquad x \geqslant 0,$$

match with $\mathbb{E}[P]$ and c_P^2 provided the parameters μ_1, μ_2, p_1 and p_2 are chosen as

$$p_1 = \frac{1}{2}\left(1 + \sqrt{\frac{c_P^2 - 1}{c_P^2 + 1}}\right), \qquad p_2 = 1 - p_1,$$
$$\mu_1 = \frac{2p_1}{\mathbb{E}[P]} \qquad \text{and} \qquad \mu_2 = \frac{2p_2}{\mathbb{E}[P]}.$$

For single-server queuing models it is well-known that the mean waiting time depends (approximately linearly) on the squared coefficient of variation of the inter-arrival (and service) time. The results in Figure 1, however, show that for the carousel model, the mean waiting time is not very sensitive to the squared coefficient of variation of the pick time and thus neither is the throughput τ; it indeed decreases as c_P^2 increases, but very slowly. This phenomenon may be explained by the fact that the waiting time of the picker is bounded by 1, i.e. the time for a full rotation of the carousel.

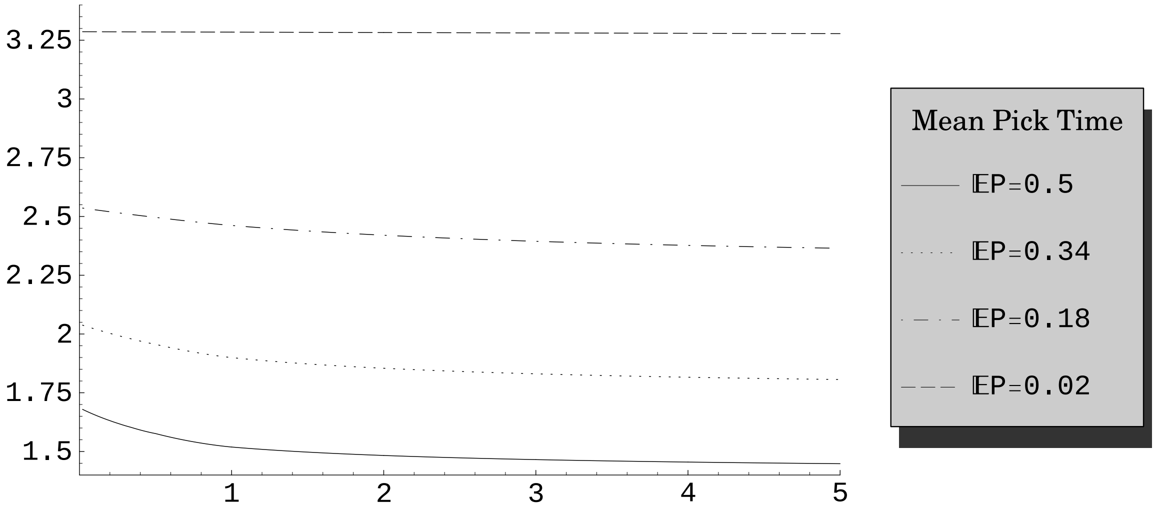

Figure 1: Plot of throughput vs. the squared coefficient of variation of the pick time.

5 Concluding remarks and further research

In this paper we have considered a system with two carousels operated by one picker. We have shown that if the pick time follows a phase-type distribution, then the density of the waiting time is a mixture of exponentials. Numerical results show that the squared coefficient of variation of the pick time does not influence the throughput significantly.

We have solved recursion (1.1) under specific assumptions for the random variables R_n and P_n. In particular, we assumed that R_n is uniformly distributed on $[0, 1)$ and P_n follows a phase-type distribution, for every n. This makes sense if one has a carousel application in mind. Nonetheless, it is mathematically interesting to try and solve this recursion under less restrictive assumptions. In further research we shall try to solve (1.1) allowing R_n and P_n to follow a more general distribution.

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