Limit theorem for maximum of the storage process with fractional Brownian motion as input.

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Abstract: The maximum M_T of the storage process $Y(t) = \sup_{s\geq t}(X(s) - X(t) - c(s-t))$ in the interval $[0,T]$ is dealt with, in particular for growing interval length T. Here $X(s)$ is a fractional Browninan motion with Hurst parameter, $0 < H < 1$. For fixed T the asymptotic behaviour of M_T was analysed by Piterbarg (2001) by determining an approximation for the probability $P\{M_T > u\}$ for $u \to \infty$. Using this expression the convergence $P\{M_T < u_T(x)\} \to G(x)$ as $T \to \infty$ is derived where $u_T(x) \to \infty$ is a suitable normalization and $G(x) = \exp(-\exp(-x))$ the Gumbel distribution. Also the relation to the maximum of the process on a dense grid is analysed.

Key words and phrases. storage process, maximum, limit distribution, fractional Brownian motion, dense grid.

1 Introduction

We consider the storage process

$$Y(t) = \sup_{s\geq t}(X(s) - X(t) - c(s-t))$$

where $X(t), t \geq 0$, is a Fractional Brownian Motion (FBM) with Hurst parameter H, $0 < H < 1$ and the constant $c > 0$ is the service rate. The FBM is a centered Gaussian process with stationary increments having a.s. continuous sample paths such that $\mathbf{E}(X(t) - X(s))^2 = |t-s|^{2H}$, hence with variance $\mathrm{Var}(X(t)) = |t|^{2H}$.

This storage process was considered in Piterbarg (2001) who derived results on the large deviations. The particular probability $\mathbf{P}\{Y(0) > u\} = \mathbf{P}\{\sup_{t\geq 0} X(t) - ct > u\}$ was studied by Duffield and O'Connel (1996), Norros (1997) and Nayaran (1998). In particular for $u \to \infty$ the asymptotic behaviour was derived in Hüsler and Piterbarg (1999) and Nayaran (1998). Albin and Samorodnitsky generalize the result of Piterbarg (2001) for infinitely divisible input processes.

Piterbarg (2001) analysed the supremum $M(T) = \sup_{t\in[0,T]} Y(t)$ of the process $Y(t)$ in a finite interval $[0, T]$: $\mathbf{P}\{M(T) > u\}$ for large u. His proofs showed that T can even depend on u, if T is contained in a certain interval depending on u, without changing the results (see Corollary 2). We continue in this paper to investigate the asymptotic behaviour of the supremum $M(T)$ where T is growing in relation to u, now growing faster, so that T is not included in that interval. However, we assume that $u = u_T$ depends on T, in the sense of a normalization, such that we get an asymptotic distribution for the supremum $M(T)$ (Theorem 1):

$$\mathbf{P}\{M(T) \leq u_T(x)\} \to G(x) = \exp(-e^{-x})$$

for any $x \in \mathbb{R}$ and some suitable normalization $u_T(x) = a(T)x + b(T)$ where $a(T)$ and $b(T)$ are given in (6). The derivation of this result reveals also the complete dependence of the maximum $M_T^{(\delta)}$ defined with respect to $X(i\delta)$, taken on a discrete grid with mesh $\delta = \delta(T) > 0$. This maximum depends on the observations $X(i\delta)$, only, hence $\tilde{Y}(i\delta) = \sup_{l\geq 0}(X((l+i)\delta) - X(i\delta) - cl\delta)$. We will note that if $H > 1/2$, then δ does not tend to 0, but tends to ∞. (Theorem 2).

The next section discusses some properties of the storage process needed for the derivation of the two main results treated in Section 3.

2 Preliminaries

We state here some needed relations which were derived in Piterbarg (2001). We begin with the relation

$$\mathbf{P}\left(\sup_{t\in[0,T]} Y(t) \leq u\right) = \mathbf{P}\left(\sup_{s\in[0,T/u],\ \tau\geq 0} Z(s,\tau) \leq u^{1-H}\right)$$

where

$$Z(s,\tau) = \frac{X(u(s+\tau)) - X(su)}{\tau^H u^H v(\tau)}$$

with $v(\tau) = \tau^{-H} + c\tau^{1-H}$. The variance of the field is $v^{-2}(\tau)$. Note that $Z(s,\tau)$ is not dependent on u, that means for any u the Gaussian field $Z(s,\tau)$ has the same distribution. Thus we do not use u as additional parameter in the notation of $Z(s,\tau)$. This is relation (3) of Piterbarg (2001). It is basic for the derivation of the limit distribution of $M(T)$.

The correlation function $r(s,\tau;s',\tau')$ of $Z(s,\tau)$ equals

$$\begin{aligned} r(s,\tau;s',\tau') &= \mathbf{E}Z(s,\tau)Z(s',\tau')v(\tau)v(\tau') \\ &= \frac{-|s-s'+\tau-\tau'|^{2H} + |s-s'+\tau|^{2H} + |s-s'-\tau'|^{2H} - |s-s'|^{2H}}{2\tau^H\tau'^H}. \end{aligned}$$

We note that $Z(s,\tau)$ is stationary in s, but not in τ. $\sigma_Z(\tau) = v^{-1}(\tau)$ has a single maximum point at $\tau_0 = H/(c(1-H))$. Taylor expansions show that

$$\sigma_Z(\tau) = v^{-1}(\tau) = \frac{1}{A} - \frac{B}{2A^2}(\tau-\tau_0)^2 + O((\tau-\tau_0)^3) \tag{1}$$

as $\tau \to \tau_0$, where

$$\begin{aligned} A &:= \frac{1}{1-H}\left(\frac{H}{c(1-H)}\right)^{-H} = v(\tau_0), \\ B &:= H\left(\frac{H}{c(1-H)}\right)^{-H-2} = v''(\tau_0). \end{aligned}$$

and also

$$r(s,\tau;s',\tau') = 1 - \frac{1+o(1)}{2\tau_0^{2H}}(|s-s'+\tau-\tau'|^{2H} + |s-s'|^{2H}) \tag{2}$$

as $s-s' \to 0$, $\tau \to \tau_0$, $\tau' \to \tau_0$. These relations are derived in Piterbarg (2001). We need in addition an expression of the correlation function for $|s-s'| \to \infty$. By series expansion we find for any τ, τ' with $0 < \tau_1 < \tau, \tau' < \tau_2 < \infty$, with fixed $\tau_1 < \tau_0 < \tau_2$

$$|r(s,\tau;s',\tau')| \le C|s-s'|^{2H-2}$$

for some constant $C > 0$ and all s, s' with $|s-s'|$ sufficiently large, since

$$\begin{aligned} |r(s,\tau;s',\tau')| &= \frac{|s-s'|^{2H}}{2(\tau\tau')^H}\left(-|1+\frac{(\tau-\tau')}{(s-s')}|^{2H} + |1+\frac{\tau}{(s-s')}|^{2H}\right. \\ &\qquad \left. +|1-\frac{\tau'}{(s-s')}|^{2H} - 1\right) \\ &\le \frac{|s-s'|^{2H}}{\tau_1^{2H}} 2H|2H-1||s-s'|^{-2}\tau_2^2 \le C|s-s'|^{2H-2} \end{aligned}$$

if $2H \neq 1$. For $2H = 1$, we have $r(s,\tau,s,\tau') = 0$ for large $|s - s'|$ since the increments of the Brownian motion on disjoint intervals are independent.

3 Asymptotic approximations

Lemma 2 of Piterbarg (2001) says that we can restrict the considered domain of (s,τ) to a domain with $|\tau - \tau_0| \le \log v/v$, since there exists a constant C such that for any v, T

$$\mathbf{P}\{\sup_{\substack{|\tau-\tau_0|\ge \log v/v\\ 0\le s\le T}} AZ(s,\tau) > v\} \le CTv^{2/H}\exp\left(-\frac{1}{2}v^2 - b\log^2 v\right) \qquad (3)$$

where $b = B/(2A)$. We will choose $v = Au_T^{1-H}$.

Then we need Lemma 5 from Piterbarg (2001) for the remaining domain (with a correction of a misprint). Pickands constants with repect to $\alpha = 2H$ in the case of FBM are denoted by $\mathcal{H}_{2H}$.

Lemma 1. (Lemma 5, Piterbarg, 2001). *For any $L > 0$, with $b = B/(2A)$ and $a = 1/(2\tau_0^{2H})$*

$$\mathbf{P}\{\sup_{\substack{|\tau-\tau_0|\le \log v/v\\ 0\le s\le L}} AZ(s,\tau) > v\} = \sqrt{\pi}a^{\frac{2}{H}}b^{-\frac{1}{2}}\mathcal{H}_{2H}^2 Lv^{\frac{2}{H}-1}\Psi(v)(1+o(1))$$

as $v \to \infty$. This holds also for $L = v^{-1/H'}$, with $1 > H' > H$.

Actually we need the slightly more general result mentioned above which readily follows from the proof of the Lemma:

Corollary 2. *The assertion of the Lemma 1 holds true for L, depending of v such that $v^{-1/H'} < L < \exp(cv^2)$, for any $H' \in (H,1)$ and $c \in (0,1/2)$.*

For any L such that L/u satisfies the restriction of Corollary 2 we have together with (3) and Lemma 1 where $v = Au^{1-H} \to \infty$, setting $\tau^*(u) = \log(Au^{1-H})/Au^{1-H}$, with $u \to \infty$

$$
\begin{aligned}
\mathbf{P}\left(\sup_{\substack{s\in[0,L/u]\\ \tau\geq 0}} AZ(s,\tau) > Au^{1-H}\right) &= \mathbf{P}\left(\sup_{\substack{s\in[0,L/u]\\ |\tau-\tau_0|\leq\tau^*(u)}} AZ(s,\tau) > Au^{1-H}\right)\\
&\quad + O\left(\mathbf{P}\left(\sup_{\substack{s\in[0,L/u]\\ |\tau-\tau_0|>\tau^*(u)}} AZ(s,\tau) > Au^{1-H}\right)\right)\\
&\sim c_1\,(L/u)\left(Au^{1-H}\right)^{2/H-1}\Psi\left(Au^{1-H}\right)\\
&\sim c_2 L u^h \exp\left(-\frac{1}{2}A^2u^{2-2H}\right), \qquad (4)
\end{aligned}
$$

with $h = \frac{2(1-H)^2}{H} - 1$ where

$$
c_1 = \sqrt{\pi}a^{2/H}b^{-1/2}\mathcal{H}_{2H}^2 \quad \text{and} \quad c_2 = a^{2/H}\,(2b)^{-1/2}\,\mathcal{H}_{2H}^2\,A^{2/H-2}
$$

are constants evaluated from Lemma 1.

We are going to apply (4) for subdomains $\{(s,\tau) : s \leq L/u, \tau > 0\}$ of the domain $\{(s,\tau) : s \leq T/u, \tau > 0\}$ with suitably chosen $L = L(T)$ such that L/u satisfies the restriction of Corollary 2. Obviously $u = u_T$ depends on T as mentioned. Then we will show that the exceedances in these subdomains are asymptotically independent. The product of these probabilities will reveal the asymptotic law for the supremum on the whole domain. This asymptotic expression is based on the summation of the probabilities (4) related to the subdomains. In the next step we derive $u_T = u_T(x) = a(T)x + b(T)$.

The normalizating functions $b(T)$ and $a(T)$ are such that the asymptotic equation, for $T \to \infty$, holds:

$$
c_2 T\Big[b(T) + xa(T)\Big]^h \exp\left(-\frac{1}{2}A^2(b(T)+xa(T))^{2-2H}\right) \to e^{-x}. \qquad (5)
$$

We get by a lengthy calculation that

$$
\begin{aligned}
b(T) &= (2A^{-2}\log T)^{1/(2(1-H))} + \left[\frac{h(2A^{-2})^{1/(2(1-H))}\,\log(2A^{-2}\log T)}{4(1-H)^2}\right.\\
&\quad \left. + \frac{(2A^{-2})^{1/(2(1-H))}\log c_2}{2(1-H)}\right](\log T)^{-(1-2H)/(2(1-H)}\\
a(T) &= \frac{(2A^{-2})^{1/(2(1-H))}}{2(1-H)}(\log T)^{-(1-2H)/(2(1-H))} \qquad (6)
\end{aligned}
$$

as $T \to \infty$. Note that $a(T)$ is a positive function with

$$a(T)/\,b(T) \to 0 \text{ as } T \to \infty, \tag{7}$$

for any $H < 1$ and that

$$b(T) \sim (2A^{-2}\log T)^{1/(2(1-H))}. \tag{8}$$

These normalizations are derived as follows. Observe that

$$\begin{aligned}\frac{1}{2}A^2(b(T)+xa(T))^{2(1-H)} &= \log T\left[1+\left(\frac{h\,\log(2A^{-2}\log T)}{4(1-H)^2}\right.\right.\\ &\qquad\left.\left.+\frac{\log c_2}{2(1-H)}+\frac{x}{2(1-H)}\right)(\log T)^{-1}\right]^{2(1-H)}\\ &= \log T+\left(\frac{h\log(2A^{-2}\log T)}{2(1-H)}+\log c_2+x+o(1)\right)\end{aligned}$$

With this expression in the exponential term, the left hand side of (5) is asymptotically equivalent to

$$c_2 T b^h(T) T^{-1}(2A^{-2}\log T)^{-h/2(1-H)} c_2^{-1}\exp(-x+o(1)) \to \exp(-x)$$

as $T \to \infty$. So we state the limit distribution of M_T.

Theorem 1. *Let $M_T = \sup_{0\le t\le T} Y(t)$ be the supremum of the storage process $Y(t)$ with FBM as input, with Hurst parameter $H<1$. Then with the normalizations $a(T)$ and $b(T)$ we have*

$$\mathbf{P}\{M_T \le b(t) + x\,a(T)\} \to \exp(-e^{-x})$$

as $T \to \infty$.

By (4) and (5) we find also for any fixed x and suitably large $L(T)$ which defines the subdomain $\{(s,\tau) : s \le L(T), \tau \ge 0\}$

$$\begin{aligned}&\mathbf{P}\left(\sup_{s\in[0,L(T)],\,\tau\ge 0} Z(s,\tau) > (b(T)+xa(T))^{1-H}\right)\\ &\quad\sim\ c_2 L(T)(b(T)+xa(T))^{h+1}\exp\left(-\frac{A^2}{2}(b(T)+xa(T))^{2-2H}\right)\\ &\quad\sim\ (L(T)b(T)/T)\exp(-x).\end{aligned}$$

if $L(T)$ satisfies $A^{1/H'}(b(T)+xa(T))^{-(1-H)/H'} \le L(T) \le \exp(cA^2(b(T)+xa(T))^{2(1-H)})$ for some $1 > H' > H$ and $c < 1/2$. The condition of Corollary 2 holds for $L(T)$, if

$$(1+o(1))A^{1/H'}[b(T)]^{-(1-H)/H'} \le L(T) \le \exp((2+o(1))c\log T) = T^{(2+o(1))c}$$

for some $c < 1/2$, by using (8). We choose a slowly increasing $L(T)$: $L(T) = v_T = Au_T^{1-H} \sim A(b(T))^{1-H} \sim (2\log T)^{1/2}$ which satisfies the condition of Corollary 2. Hence, we will use

$$\begin{aligned} &\mathbf{P}\left(\sup_{s\in[0,L(T)],\,\tau\ge 0} Z(s,\tau) > (b(T)+xa(T))^{1-H}\right) \\ \sim\ & c_2\,L(T)\,(b(T))^{h+1}\exp\left(-\frac{A^2}{2}(b(T)+xa(T))^{2-2H}\right) \end{aligned} \tag{9}$$

as $T\to\infty$.

Now we work in the following tedious, but known way (cf. Leadbetter et al. (1983)). For $L(T)$ and $0<\delta<L(T)$ define the two-dimensional intervals $I_k = [(k-1)L(T), kL(T)-\delta)\times J(\tau_0)$ and $I_k^* = [kL(T)-\delta, kL(T))\times J(\tau_0)$ for any $k\ge 1$, where $J(\tau_0) = \{\tau : |\tau-\tau_0| \le \tau^*(u)\}$. These are in the first components 'long' and 'short' intervals, respectively. They depend on T which we do not denote. Then

$$[0, T/u_T]\times J(\tau_0) = \cup_{k=1}^{K_T}(I_k\cup I_k^*)\cup I_{K_T+1}$$

where $I_{K_T+1} = [K_T L(T), T/u_T]\times J(\tau_0)$ with $K_T = [T/(u_T L(T))]\in\mathbb{N}$. Hence $|I_{K_T+1}| \le 2L(T)\times\tau^*(u)$. Thus with the chosen $L(T)$ we get $K_T = [T/u_T L(T)] \sim T/(Au_T^{2-H})$.

Lemma 3. *With the definitions of I_k, $k\ge 1$, and some $\delta>0$, we get for $T\to\infty$*

$$\mathbf{P}\{\sup_{t\le T} Y(t) > u_T\} \sim \mathbf{P}\{\sup_{(s,\tau)\in\cup_{k\le K_T} I_k} AZ(s,\tau) > Au_T^{1-H}\}$$

Proof: With $v = Au_T^{1-H} = A(b_T + xa(T))^{1-H}$, any x, we have for large T

$$\begin{aligned} \mathbf{P}\{\sup_{t\le T} Y(t) > u_T\} &\sim \mathbf{P}\{\sup_{\substack{|\tau-\tau_0|\le\tau^*(u_T)\\ 0\le s\le T/u_T}} AZ(s,\tau) > Au_T^{1-H}\} \\ &\ge \mathbf{P}\{\sup_{(s,\tau)\in\cup_k I_k} AZ(s,\tau) > Au_T^{1-H}\} \end{aligned}$$

as lower bound, and with the Bonferroni inequality the upper bound

$$\mathbf{P}\{\sup_{\substack{|\tau-\tau_0|\le\tau^*(u_T)\\ 0\le s\le T/u_T}} AZ(s,\tau) > Au_T^{1-H}\} \le \mathbf{P}\{\sup_{(s,\tau)\in\cup_k I_k} AZ(s,\tau) > Au_T^{1-H}\}$$
$$+\mathbf{P}\{\sup_{(s,\tau)\in I_{K_T+1}} AZ(s,\tau) > Au_T^{1-H}\} + \mathbf{P}\{\sup_{(s,\tau)\in\cup_k I_k^*} AZ(s,\tau) > Au_T^{1-H}\}$$

We show that the last two probabilities of the upper bound are asymptotically negligible. For $\delta > 0$ by Corollary 2

$$\begin{aligned}
\mathbf{P}\{\sup_{(s,\tau)\in\cup_k I_k^*} AZ(s,\tau) > Au_T^{1-H}\} &\le \sum_{k\le K_T}\mathbf{P}\{\sup_{(s,\tau)\in I_k^*} AZ(s,\tau) > Au_T^{1-H}\}\\
&\le CK_T\delta\, u_T^{(1-H)(\frac{2}{H}-1)}\Psi(Au_T^{1-H})\\
&\sim C\delta T/(u_T L(T))u_T^{h+1}\exp(-(1/2)A^2u_T^{2(1-H)})\\
&= O\left(\delta/L(T)\right) = o(1)
\end{aligned}$$

since $L(T)\to\infty$ where C and in the following also $\tilde{C}$ denote generic positive constants. We used that the term in (5) tends to a constant by the choice of u_T. In the same way the probability that an exceedance of u_T happens in the interval I_{K_T+1}, is asymptotically negligible, for

$$\mathbf{P}\{\sup_{(s,\tau)\in I_{K_T+1}} AZ(s,\tau) > Au_T^{1-H}\} \le CL(T)u_T^{(1-H)(\frac{2}{H}-1)}\Psi(Au_T^{1-H})$$

$$= O(L(T)u_T/T) = o(1)$$

since $L(T) = o(T/u_T)$. □

It means we deal now only with the intervals I_k and show in a following step that the suprema of these intervals are asymptotically independent. To establish this claim we apply Berman's inequality which holds only for sequences of Gaussian r.v.'s. Therefore we define a family of grid points (s,τ) in our domain of interest, depending on T.

For some small $d > 0$ and any T, let

$$q = q(T) = du_T^{-(1-H)/H}$$

and define the grid points

$$s_{k,l} = (k-1)L(T) + lq \quad\text{and}\quad \tau_j = \tau_0 + jq$$

with $(s_{k,l}, \tau_j) \in I_k$ for integers $j \in \mathbb{Z}$, $l \geq 0, k \geq 1$. These grid points are denoted simpler by $(s, \tau) \in I_k \cap \mathcal{R}$ for fixed k, without mentioning the dependence on T. We need later to select $d = d(T) \to 0$ slowly. We select $d = d(T) = 1/\log\log T$.

For any k the index l of points $s_{k,l}$ is bounded by $L^* = [L(T)/q] \sim Au_T^{(1-H^2)/H}/d \to \infty$ as $T \to \infty$. In the whole for $s_{k,l} \leq T/u_T$ we have less than $L(T)K_T/q \sim d^{-1}Tu_T^{(1-2H)/H}$ number of points $s_{k,l}$ in the first component. Since $|\tau - \tau_0| \leq \tau^*(u_T)$ we have also $|j| \leq [(\tau^*(u_T)/q)] \sim \frac{1-H}{Ad}(\log u_T)u_T^{(1-H)^2/H} \to \infty$ for any $H < 1$. intreval $(\tau_0 - (\log u_T)/u_T, \tau_0 + (\log u_T)/u_T)$. For the other cases $H > 1/2$, there is only one point in this interval, namely $\tau_j = \tau_0$.

The steps of proof are as follows. We show that with $w = u_T^{1-H}$

$$
\begin{aligned}
P\{\sup_{t\leq T} Y_t \leq u_T\} &\sim P\{\sup_{(s,\tau)\in\cup_k I_k} AZ(s,\tau) \leq Au_T^{1-H}\} \\
&\sim \; P\{\sup_{(s,\tau)\in\cup_k I_k\cap\mathcal{R}} Z(s,\tau) \leq w\} \qquad (10)\\
&\sim \; \prod_{k=1}^{K_T} P\{\sup_{(s,\tau)\in I_k\cap\mathcal{R}} Z(s,\tau) \leq w\} \qquad (11)\\
&\sim \; \prod_{k=1}^{K_T} P\{\sup_{(s,\tau)\in I_k} Z(s,\tau) \leq w\} \qquad (12)\\
&\sim \; \exp\left(-K_T P\{\sup_{(s,\tau)\in I_1} Z(s,\tau) > w\}\right) \qquad (13)\\
&\to \; \exp(-e^{-x}). \qquad (14)
\end{aligned}
$$

Note that $P\{\sup_{(s,\tau)\in I_k} Z(s,\tau) \leq w\}$ is the same for each k, since the FBM $X(t)$ has stationary increments, implying the mentioned stationarity in the first component. Hence (13) is immediate. We have shown already the convergence (14) by the proper choice of u_T. (10) and (12) hold by the same reasoning in Lemma 6 and (11) will be shown by Berman's inequality in Lemma 8.

To prove (10) and (12) we investigate now the exceedances in a small domain $\{(s,\tau) \in [s_{k,l}, s_{k,l+1}) \times [\tau_j, \tau_{j+1})\}$ by conditioning on the value $Z(s_{k,l}, \tau_j)$. We define for fixed k, l, j the Gaussian field

$$\tilde{Z}^{(u)}(t,\xi) = \tilde{Z}^{(u)}_{k,l,j}(t,\xi) = w(Z(s_{k,l} + tq, \tau_j + \xi q) - w)$$

with $0 \leq t, \xi \leq 1$ where

$$\begin{aligned} E(\tilde{Z}^{(u)}(t,\xi)) &= -w^2 \\ \mathrm{Var}(\tilde{Z}^{(u)}(t,\xi)) &= w^2 v^2(\tau_j + \xi q) \end{aligned}$$

and also with $r(s,\tau,s',\tau')$ given in Section 2

$$\mathrm{Corr}(\tilde{Z}^{(u)}(t,\xi), \tilde{Z}^{(u)}(t',\xi')) = r^{(u)}(t,\xi,t',\xi')$$

$$= \frac{-|q(t-t'+\xi-\xi')|^{2H} + |q(t-t')+\tau_j+\xi q|^{2H} + |q(t-t')-\tau_j-\xi' q|^{2H} - |q(t-t')|^{2H}}{\tau_0^{2H}(1+(j+\xi)q/\tau_0)^H(1+(j+\xi')q/\tau_0)^H}$$

The conditional mean, variance and covariance and their approximations are as follows. For the conditional mean we get with $0 \leq t, \xi \leq 1$

$$\begin{aligned} &E(\tilde{Z}^{(u)}(t,\xi)|\tilde{Z}^{(u)}(0,0) = y) = -w^2 + r^{(u)}(t,\xi,0,0)\frac{v^{-1}(\tilde{\xi})}{v^{-1}(\tau_j)}(y+v^2) \\ &= \quad y + (y+w^2)\left(\frac{v(\tau_j)}{v(\tilde{\xi})} - 1\right) - (1 - r^{(u)}(t,\xi,0,0))\frac{v(\tau_j)}{v(\tilde{\xi})}(y+v^2) \end{aligned}$$

where $\tilde{\xi} = \tau_j + \xi q$. Since the lags tq and ξq tend to 0, using the Taylor expansion for $v(\tau)$, we get an approximation for $v(\tau_j)/v(\tilde{\xi})$, and using (2) an approximation for the correlation function. Thus the conditional mean is for fixed y

$$= y - \frac{1+o(1)}{2\tau_0^2} d^{2H}((t+\xi)^{2H} + t^{2H}) =: \mu(t,\xi,y) \ . \tag{15}$$

However, for all $y \leq -\gamma$ we derive with the same expansions that $\mu(t,\xi,y) = y(1 + O(d^{2H})/\gamma))$, uniformly in y. We have to choose $\gamma \to 0$ also, so let $\gamma = \gamma(T) = d^H \to 0$. For the selected $d = d(T)$ and γ, the term $O(d^{2H}/\gamma)$ tends to 0. This bound is sufficient for our approximations.

Next we derive a bound for the conditional variance. We have by (2)

$$\begin{aligned} &\mathrm{Var}(\tilde{Z}^{(u)}(t,\xi)|\tilde{Z}^{(u)}(0,0) = y) = \mathrm{Var}(\tilde{Z}^{(u)}(t,\xi))(1 - [r^{(u)}(t,\xi,0,0)]^2) \\ &= \quad \frac{w^2}{v^2(\tau_j+\xi q)} \frac{2+o(1)}{2\tau_0^{2H}}((t+\xi)^{2H} + t^{2H})\, q^{2H} \\ &\leq \quad C w^2 q^{2H} = C d^{2H} \end{aligned} \tag{16}$$

for all $t, \xi \leq 1$, with some constant $C > 0$.

We need also an upper bound for the variance of the conditional increments of $\tilde{Z}^{(u)}(t,\xi)$ which is

$$\mathrm{Var}(\tilde{Z}^{(u)}(t,\xi) - \tilde{Z}^{(u)}(t',\xi')|\tilde{Z}^{(u)}(0,0) = y) =$$

$$= \frac{\mathrm{Var}(\tilde{Z}^{(u)}(t,\xi) - \tilde{Z}^{(u)}(t',\xi')) - [\mathrm{Cov}(\tilde{Z}^{(u)}(t,\xi) - \tilde{Z}^{(u)}(t',\xi'), \tilde{Z}^{(u)}(0,0))]^2}{w^2 v^{-2}(\tau_j)}.$$

The variance of the increments is approximated first.

$$\begin{aligned}
&\mathrm{Var}(\tilde{Z}^{(u)}(t,\xi) - \tilde{Z}^{(u)}(t',\xi'))/w^2 = \\
&= \quad \frac{v^{-2}(\tau_j + \xi q) + v^{-2}(\tau_j + \xi' q) - 2r^{(u)}(t,\xi,t',\xi')}{v(\tau_j + \xi q)v(\tau_j + \xi' q)} \\
&\sim \quad A^{-4}[(v(\tau_j + \xi q) - v(\tau_j + \xi' q))^2 + 2(1 - r^{(u)}(t,\xi,t',\xi'))A^2(1 + o(1))]
\end{aligned}$$

The first term, the difference of the v-values, is of $o(q|\xi - \xi'|)$ because of the behaviour of v in the neighbourhood of τ_0, given in (1). The second term is approximated by (2) to get

$$\begin{aligned}
&A^2 \frac{(1 + o(1))}{\tau_0^{2H}} [|t - t' + \xi - \xi'|^{2H} + |t - t'|^{2H}] q^{2H} \\
\sim \quad & w^{-2} A^2 (d/\tau_0)^{2H} [|t - t' + \xi - \xi'|^{2H} + |t - t'|^{2H}]
\end{aligned}$$

Combining the two approximations, results in

$$\begin{aligned}
&\mathrm{Var}(\tilde{Z}^{(u)}(t,\xi) - \tilde{Z}^{(u)}(t',\xi')) \sim \\
&\sim \quad A^{-2}(d/\tau_0)^{2H}[|t - t' + \xi - \xi'|^{2H} + |t - t'|^{2H}] + o(|\xi - \xi'|^2) \\
&\leq \quad G(|t - t'|^{2H} + |\xi - \xi'|^{2H})
\end{aligned}$$

for some $G > 0$. The covariance of the increment and $\tilde{Z}^{(u)}(0,0)$ is a bit more tedious but straightforward with the same approximations.

$$\begin{aligned}
&\mathrm{Cov}(\tilde{Z}^{(u)}(t,\xi) - \tilde{Z}^{(u)}(t',\xi'), \tilde{Z}^{(u)}(0,0)) = \\
&= \quad \mathrm{Cov}(\tilde{Z}^{(u)}(t,\xi), \tilde{Z}^{(u)}(0,0)) - \mathrm{Cov}(\tilde{Z}^{(u)}(t',\xi'), \tilde{Z}^{(u)}(0,0)) \\
&\sim \quad \frac{w^2}{A} \frac{v(\tau_j + \xi' q)(r_1 - r_2) - r_2[v(\tau_j + \xi q) - v(\tau_j + \xi' q)]}{v(\tau_j + \xi q)v(\tau_j + \xi' q)}
\end{aligned}$$

with $r_1 = r^{(u)}(t,\xi,0,0)$ and $r_2 = r^{(u)}(t',\xi',0,0)$. By (2) the difference of $r_1 - r_2$ is bounded by

$$O(q^{2H}(|t - t' + \xi - \xi'|^{\alpha} + |t - t'|^{\alpha}))$$

with $\alpha = \min(2H, 1)$. The difference of the v-terms is again $O(q|\xi - \xi'|(\log w)/w)$. Together we have for

$$
\begin{aligned}
&[\mathrm{Cov}(\tilde{Z}^{(u)}(t,\xi) - \tilde{Z}^{(u)}(t',\xi'), \tilde{Z}^{(u)}(0,0))]/w^2 = \\
&= \; w^2\Big(O(q^{4H}(|t-t'+\xi-\xi'|^{\alpha} + |t-t'|^{\alpha})^2) + o(q^2(\log w)^2|\xi-\xi'|^2/w^2)\Big) \\
&= \; o(1)(|t-t'|^{2H} + |\xi-\xi'|^{2H}).
\end{aligned}
$$

Therefore the conditional variance of the increment, being the variance of the increments minus the above squared covariance term divided by the variance of $\tilde{Z}^{(u)}(0,0)$, is bounded by

$$
G(|t-t'|^{2H} + |\xi-\xi'|^{2H})
$$

for some $G > 0$. We are now ready to prove the following statement.

Lemma 4. *With the definition of* $\tilde{Z}^{(u)}(t,\xi)$ *we get*

$$
\mathbf{P}\{\sup_{0\le t,\xi\le 1} \tilde{Z}^{(u)}(t,\xi) > 0 | \tilde{Z}^{(u)}(0,0) = y\} \le C d^H |y|^{2/H-1}\phi(\tilde{C}|y|/d^H)
$$

for $y < -\gamma$ *and* $T \to \infty$, *with* $d = d(T) \to 0$ *and some constants* $C, \tilde{C} > 0$, *not depending on* γ.

Proof: Since the conditioned centered process $\tilde{Z}^{(u)}(t,\xi) - \mu(t,\xi,y)|\tilde{Z}^{(u)}(0,0)$ is a Gaussian process with variance of the increments given above, where $\mu(t,\xi,y)$ is derived in (15), we can apply Theorem 8.1 of Piterbarg (1996) for

$$
\begin{aligned}
&\mathbf{P}\{\sup_{0\le t,\xi\le 1} \tilde{Z}^{(u)}(t,\xi) - \mu(t,\xi) > -\mu(t,\xi,y)|\tilde{Z}^{(u)}(0,0) = y\} \le \\
&\le \; C\sigma^*|\mu(t,\xi,y)|^{2/H-1}\phi(|\mu(t,\xi,y)|/\sigma^*) \qquad (17)
\end{aligned}
$$

with $\sigma^{*2} = \sup_{t,\xi\le 1} \mathrm{Var}\,(\tilde{Z}^{(u)}(t,\xi)|\tilde{Z}^{(u)}(0,0))$ and C not depending on γ. Note that the conditional mean $|\mu(t,\xi,y)| = |y|(1+O(d^{2H}/\gamma)) > |y|(1-\epsilon)$, uniformly in $t,\xi \le 1, y \le -\gamma$, with d sufficiently small (T large), with the chosen $\gamma = d^H$. By (16) $\sigma^* \le d^H/\tilde{C}$. Hence we get as upper bound for (17)

$$
Cd^H|y|^{2/H-1}\phi(\tilde{C}|y|(1-\epsilon)/d^H) = Cd^H|y|^{2/H-1}\phi(\tilde{C}|y|/d^H)
$$

with suitable (generic) constants $C, \tilde{C} > 0$, not depending on t, ξ, y and γ. □

This allows now the approximation of the supremum of the process $Z(s,\tau)$ on the continuous points by the maximum on the grid in a small domain in the following way.

Lemma 5. *For the process $Z(s,\tau)$ we get for T large with $\gamma = d^H$*

$$P\{Z(s_{k,l},\tau_j) \le w - \gamma/w, \sup_{0\le t,\xi\le 1} Z(s_{k,l}+tq, \tau_j+\xi q) > w\}$$

$$= O(d^{H+2})\phi(wv(\tau_j))/w$$

uniformly in k,l,j, and for any $k \le K_T$

$$P\{\max_{(s,\tau)\in I_k\cap\mathcal{R}} Z(s,\tau) \le w - \gamma/w, \sup_{(s,\tau)\in I_k} Z(s,\tau) > w\}$$

$$= O(d^H L(T) w^{2(1-H)/H}\phi(Aw)) = O(d^H/K_T)$$

with T large and $d = d(T) > 0$ small.

Proof: For the process $\tilde{Z}^{(u)}(t,\xi)$ we apply Lemma 4

$$\begin{aligned}
&P\{Z(s_{k,l},\tau_j) \le w - \gamma/w, \sup_{0\le t,\xi\le 1} Z(s_{k,l}+tq, \tau_j+\xi q) > w\} = \\
&= \quad P\{\tilde{Z}^{(u)}(0,0) \le -\gamma, \sup_{0\le t,\xi\le 1} \tilde{Z}^{(u)}(t,\xi) > 0\} \\
&= \quad \int_{-\infty}^{-\gamma} \mathbf{P}\{\sup_{0\le t,\xi\le 1} \tilde{Z}^{(u)}(t,\xi) > 0 | \tilde{Z}^{(u)}(0,0) = y\} f_{\tilde{Z}^{(u)}(0,0)}(y)dy \\
&= \quad \int_{-\infty}^{-\gamma} \phi(v(\tau_j)(w + y/w))\, C d^H |y|^{2/H-1} \phi(\tilde{C}|y|/d^{2H}) dy v(\tau_j)/w \\
&\le \quad \frac{O(d^H)}{w}\phi(wv(\tau_j)) \int_{-\infty}^{-\gamma} |y|^{2/H-1} \exp\{-\frac{\tilde{C}y^2}{2d^{2H}} - yv^2(\tau_j) - \frac{y^2v^2(\tau_j)}{2w^2}\}dy \\
&\le \quad \frac{O(d^H)}{w}\phi(wv(\tau_j)) \int_{\gamma}^{\infty} y^{2/H-1} \exp\{-\frac{y^2}{2}(\frac{\tilde{C}}{d^{2H}} + o(1)) + yA^2(1+o(1))\}dy \\
&\le \quad C d^{H+2}\phi(wv(\tau_j))/w
\end{aligned}$$

since the integral can be bounded by d^2 for any $\gamma \ge 0$. The constant C does not depend on k,l,j.

The second claim follows by summing these bounds on l, j for fixed k. We use that $0 \leq v(\tau_j) - v(\tau_0) = (B + o(1))(jq)^2 \geq \tilde{B}(jq)^2$.

$$\begin{aligned}
\sum_{l,j} Cd^{H+2}\phi(wv(\tau_j))/w &\leq (L(T)/q)(Cd^{H+2}/w)\sum_j \phi(wv(\tau_j)) \\
&= (L(T)/q)(Cd^{H+2}/w)\phi(Aw)\sum_j e^{-\frac{1}{2}(v(\tau_j)-v(\tau_0))^2w^2 - w^2A(v(\tau_j)-v(\tau_0))} \\
&\leq O(d^{H+2})(L(T)/qw)\phi(Aw)\sum_j e^{-\tilde{B}Aw^2(jq)^2} \\
&\leq O(d^{H+2})(L(T)/qw)\phi(Aw)O(1/wq)\int_0^\infty e^{-\tilde{B}Ax^2}dx \\
&\leq O(d^{H+2}\,L(T)(d^{-2}w^{2(1-H)/H})\phi(Aw)) \\
&\leq O(d^H w^{2(1-H)/H}u^{-h-1}K_T^{-1})\,[K_T L(T)u^{h+1}\phi(Aw)] \\
&\leq O(d^H\,K_T^{-1})(Tu^h\phi(Aw)) \\
&\leq O(d^H/K_T)
\end{aligned}$$

using $Tu_T^h\phi(Aw) = O(1)$ with $w = u_T^{1-H}$. Because of the stationarity(homogeneity) of $Z(s,\tau)$ in the first component, this holds for any k, hence uniformly. □

We have by (9) and Lemma 1

$$P\{\max_{(s,\tau)\in I_k} Z(s,\tau) > w\} = (1+o(1))c_2L(T)\exp(-\frac{1}{2}A^2w^2)w^{2(1-H)/H}$$

for any k. We want now to show that for any k and $\gamma \to 0$ (slowly, chosen as $\gamma = d^H$ as $d = d(T) \to 0$)

$$P\{\max_{(s,\tau)\in I_k\cap\mathcal{R}} Z(s,\tau) > w\} = (1+o(1))c_2L(T)\exp(-\frac{1}{2}A^2w^2)w^{2(1-H)/H}$$

holds. This is true since by Lemma 5

$$\begin{aligned}
P\{\sup_{(s,\tau)\in I_k\cap\mathcal{R}} Z(s,\tau) > w\} &\leq P\{\sup_{(s,\tau)\in I_k} Z(s,\tau) > w\} \\
&\leq P\{\sup_{(s,\tau)\in I_k\cap\mathcal{R}} Z(s,\tau) > w - \gamma/w\} \\
&\quad + P\{\sup_{(s,\tau)\in I_k\cap\mathcal{R}} Z(s,\tau) \leq w - \gamma/w, \sup_{(s,\tau)\in I_k} Z(s,\tau) > w\} \\
&\leq (1+O(d^H))P\{\sup_{(s,\tau)\in I_k} Z(s,\tau) > w - \gamma/w\} \\
&= (1+O(d^H)+O(\gamma))P\{\sup_{(s,\tau)\in I_k} Z(s,\tau) > w\}
\end{aligned}$$

using $(w-\gamma/w)^2 = w^2 - 2\gamma + o(1)$ for small γ. With this result it is also straightforward to show that for small γ

$$P\{w-\gamma/w < \sup_{(s,\tau)\in I_k\cap\mathcal{R}} Z(s,\tau) \le w\} = O(\gamma L(T)\phi(Aw)w^{2(1-H)/H}) \qquad (18)$$

and

$$\begin{aligned} 0 \le P\{\sup_{(s,\tau)\in I_k\cap\mathcal{R}} Z(s,\tau)\le w\} - P\{\sup_{(s,\tau)\in I_k} Z(s,\tau)\le w\} \\ = \quad O(c_d L(T)\phi(Aw)w^{2(1-H)/H}) \qquad (19) \end{aligned}$$

where $c_d = \gamma + d^H = 2d^H \to 0$ as $d\to 0$. Hence we get the following statements.

Lemma 6. *For $d\to 0$ with $\gamma = d^H \to 0$*

$$0 \le \mathbf{P}\{\sup_{(s,\tau)\in\cup_k I_k\cap\mathcal{R}} Z(s,\tau)\le u_T^{1-H}\} - \mathbf{P}\{\sup_{(s,\tau)\in\cup_k I_k} Z(s,\tau)\le u_T^{1-H}\} \to 0$$

and also

$$0 \le \prod_{k=1}^{K_T}\mathbf{P}\{\sup_{(s,\tau)\in I_k\cap\mathcal{R}} Z(s,\tau)\le u_T^{1-H}\} - \prod_{k=1}^{K_T}\mathbf{P}\{\sup_{(s,\tau)\in I_k} Z(s,\tau)\le u_T^{1-H}\} \to 0.$$

Proof: We have

$$\begin{aligned} 0 &\le \mathbf{P}\{\sup_{(s,\tau)\in\cup_k I_k\cap\mathcal{R}} Z(s,\tau)\le w\} - \mathbf{P}\{\sup_{(s,\tau)\in\cup_k I_k} Z(s,\tau)\le w\} \\ &\le \sum_{k=1}^{K_T}\Big(\mathbf{P}\{\sup_{(s,\tau)\in I_k\cap\mathcal{R}} Z(s,\tau)\le w\} - \mathbf{P}\{\sup_{(s,\tau)\in I_k} Z(s,\tau)\le w\}\Big) \end{aligned}$$

Using (19) this term is bounded by

$$O\Big(K_T L(T) c_d \phi(Aw) w^{2(1-H)/H}\Big) = O\Big((T/u_T) c_d \phi(Aw) u^{2(1-H)^2/H}\Big)$$
$$= O\Big(c_d\, T\, u_T^h \exp(-\frac{1}{2}A^2 u_T^{2(1-H)})\Big) \to 0$$

as $d\to 0$. This shows the first claim. It implies also the second claim using the stationarity (homogeneity) of $Z(s,\tau)$ with respect to the first parameter s. □

Now we are considering the proof of (11). We begin with the approximation of the sum in Berman's comparison lemma.

Lemma 7. *Under the above definitions and properties of $Z(s,\tau)$ we have*

$$S_T = \sum_{k\neq k'} \sum_{\substack{(s_{k,l},\tau_j)\in I_k\times\tau_d \\ (s_{k',l'},\tau_j')\in I_{k'}\times\tau_d}} |r(s_{k,l},\tau_j,s_{k',l'},\tau_{j'})| \exp\{-\frac{A^2v^2}{1+r(s_{k,l},\tau_j,s_{k',l'},\tau_{j'})}\} \to 0.$$

Proof: Since $|s_{k,l}-s_{k',l'}|\geq\delta$ by definition, $r(s_{k,l},\tau_0,s_{k',l'},\tau_0)\leq\rho<1$. Furthermore we showed that

$$\sup_{|s_{k,l}-s_{k',l'}|\geq s} |r(s_{k,l},\tau_0,s_{k',l'},\tau_0)| \leq Cs^{\lambda}$$

for $\lambda=2H-2<0$ and some constant $C>0$, since also τ_j and $\tau_{j'}$ tend to τ_0. If $H=\frac{1}{2}$ we have $r(s_{k,l},\tau_j,s_{k',l'},\tau_{j'})=0$ if $|s_{k,l}-s_{k',l'}|$ is large. Set $\beta=(1-\rho)/(1+\rho)$ and split the sum into two sums $S_{T,1}$ and $S_{T,2}$ with $|s_{k,l}-s_{k',l'}|<\tilde{T}^{\beta}=(T/u_T)^{\beta}$ and $|s_{k,l}-s_{k',l'}|\geq\tilde{T}^{\beta}$, respectively. For the first sum there are $\tilde{T}^{1+\beta}/q^2$ many combinations of two points $s_{k,l},s_{k',l'}\in\cup_k I_k$. Together with the τ_j combinations there are $\tilde{T}^{1+\beta}(2\tau^*(u_T))^2/q^4$ terms in the sum $S_{T,1}$. Thus $S_{T,1}$ is bounded by

$$\begin{aligned}
&\rho\,\frac{\tilde{T}^{1+\beta}(2\tau^*(u_T))^2}{q^4}\exp\{-\frac{A^2w^2}{1+\rho}\} \\
\leq\ & 4\rho\exp\left\{(1+\beta)\log\tilde{T}+2\log(\tau^*(u_T)/q^2)-\frac{(2(1+o(1))\log T}{1+\rho}\right\} \\
\leq\ & 4\exp\left\{-(\log T)\left[\frac{2(1+o(1))}{1+\rho}-(1+\beta)(1-\frac{\log u_T}{\log T})-2\frac{\log(\tau^*(u_T)/q^2)}{\log T}\right]\right\} \\
\to\ & 0
\end{aligned}$$

since $1+\beta<2/(1+\rho)$ by the choice of β, using $\log(\tau^*(u_T)/q^2)=o(\log T)$ and $\log u_T=O(\log\log T)=o(\log T)$.
For the second sum $S_{T,2}$ with $|s_{k,l}-s_{k',l'}|\geq\tilde{T}^{\beta}$, we use that

$$\sup_{|s_{k,l}-s_{k',l'}|\geq\tilde{T}^{\beta}} |r(s_{k,l},\tau_0,s_{k',l'},\tau_0)| \leq C\tilde{T}^{\beta\lambda}$$

with $\lambda=2H-2<0$. In this case there are $(\tilde{T}/q)^2$ many combinations of two points $s_{k,l}, s_{k',l'} \in \cup_k I_k$. Hence $S_{T,2}$ has the upper bound

$$\begin{aligned}
& C\tilde{T}^{\beta\lambda}\frac{(2\tilde{T}\tau^*(u_T))^2}{q^4}\exp\{-\frac{A^2w^2}{1+C\tilde{T}^{\beta\lambda}}\} \\
& \leq \quad C\exp\left\{\beta\lambda\log\tilde{T}+2\log\tilde{T}+2\log(\tau^*(u_T)/q^2)-\frac{2(1+o(1))\log T}{1+C\tilde{T}^{\beta\lambda}}\right\} \\
& \leq \quad C\exp\left\{(\log\tilde{T})\left[\beta\lambda+o(1)\right]\right\} \\
& \rightarrow \quad 0
\end{aligned}$$

since $\lambda < 0$. If $H = 1/2$, the sum $S_{T,2} = 0$ obviously. □

With Berman's comparison lemma we get finally

Lemma 8. *Under the above definitions and properties of $Z(s,\tau)$ we have*

$$P\{\sup_{(s,\tau)\in\cup_k I_k\cap\mathcal{R}} Z(s,\tau)\leq w\} - \prod_{k=1}^{K_T} P\{\sup_{(s,\tau)\in I_k\cap\mathcal{R}} Z(s,\tau)\leq w\} \rightarrow 0$$

as $T \rightarrow \infty$ with $d \rightarrow 0$.

Proof: To apply Berman's comparison lemma (cf. Hüsler 1983, or Leadbetter et al. 1983, for this general form) we have to standardize the Gaussian field yielding nonconstant boundaries $v(\tau)w$.

$$\begin{aligned}
& P\{\sup_{(s,\tau)\in\cup_k I_k\cap\mathcal{R}} Z(s,\tau)\leq w\} - \prod_{k=1}^{K_T} P\{\sup_{(s,\tau)\in I_k\cap\mathcal{R}} Z(s,\tau)\leq w\} \\
& = \quad P\{\sup_{(s,\tau)\in\cup_k I_k\cap\mathcal{R}} Z(s,\tau)v(\tau)\leq v(\tau)w\} - \prod_{k=1}^{K_T} P\{\sup_{(s,\tau)\in I_k\cap\mathcal{R}} Z(s,\tau)v(\tau)\leq v(\tau)w\} \\
& \leq \quad \sum_{k\neq k'}\sum_{\substack{(s_{k,l},\tau_j)\in I_k\cap\mathcal{R} \\ (s_{k',l'},\tau'_j)\in I'_k\cap\mathcal{R}}} |r(s_{k,l},\tau_j,s_{k',l'},\tau'_j)|\exp\left\{-\frac{(v^2(\tau_j)+v^2(\tau'_j))w^2}{2(1+r(s_{k,l},\tau_j,s_{k',l'},\tau'_j))}\right\} \\
& \leq \quad \sum_{k\neq k'}\sum_{\substack{(s_{k,l},\tau_j)\in I_k\cap\mathcal{R} \\ (s_{k',l'},\tau'_j)\in I'_k\cap\mathcal{R}}} |r(s_{k,l},\tau_j,s_{k',l'},\tau'_j)|\exp\left\{-\frac{v^2(\tau_0)w^2}{(1+r(s_{k,l},\tau_j,s_{k',l'},\tau'_j))}\right\}
\end{aligned}$$

which tends to 0 by Lemma 7. □

So we have proved every asymptotic equality (10) - (14) and thus the statement of the theorem, showing the limit distribution for M_T with the appropriate normalization $u_T = u_T(x)$.

The proof reveals a further result. We considered the maximum on the discrete process $\tilde{M}_T^{(\delta)} = \sup_{(s,\tau)\in\cup_k I_k \cap \mathcal{R}} Z(s,\tau)$ besides the maximum of the continuous process $\sup_{(s,\tau)\in\cup_k I_k} Z(s,\tau)$. The proof shows that they are asymptotically completely dependent. Obviously, this holds also for the maxima $M_T^{(\delta)}$ on the whole time domain not only on the $\cup_k I_k$ since the grid points are dense by the chosen $q(T)$ and $d(T)$. This statement holds for any $d \to 0$, not only for the chosen $d(T) = 1/\log\log T$. Note also that the assumption $q = du_T^{-(1-H)/H} \sim d(2A^{-2}\log T)^{-1/(2H)}$ does not depend really on the value x in the normalization $u_T = u_T(x)$. Therefore let $q = d(2A^{-2}\log T)^{-1/(2H)}$ for some $d \to 0$ for the following result.

Theorem 2 *Let $M_T = \sup_{0\le t\le T} Y(t)$ be the supremum of the storage process $Y(t)$ with FBM as input, with Hurst parameter $H < 1$. Then with the normalizations $a(T)$ and $b(T)$ we have*

$$P\{M_T^{(\delta)} \le b(T) + xa(T), M_T \le b(T) + ya(T)\} \to \exp(-\exp(-\min(x,y))).$$

Proof: Since $u_T(y) \le u_T(x)$ for all T and $y \le x$, we have for $y \le x$

$$\begin{aligned} P\{M_T^{(\delta)} \le u_T(x), M_T \le u_T(y)\} &= P\{M_T^{(\delta)} \le u_T(y), M_T \le u_T(y)\} \\ &= P\{M_T \le u_T(y)\} - P\{M_T^{(\delta)} \le u_T(y), M_T > u_T(y)\} \end{aligned}$$

and for $x \le y$

$$P\{M_T^{(\delta)} \le u_T(x), M_T \le u_T(y)\} = P\{M_T^{(\delta)} \le u_T(x)\} -$$

$$-P\{M_T^{(\delta)} \le u_T(x), M_T > u_T(y)\}.$$

The statement follows by using

$$P\{M_T^{(\delta)} \le u_T(x)\} \sim P\{M_T \le u_T(x)\}$$

and

$$P\{M_T^{(\delta)} \le u_T(x), M_T > u_T(x)\} = o(1)$$

by Lemma 6 for any dense grid with $d \to 0$. $\square$

Note that the grid is dense for the transformed storage process, for the Gaussian field. However, considering the grid for the storage process $Y(t)$ we have the grid points $uq = du_T^{1-(1-H)/H} = du_T^{(2H-1)/H}$ which tends to ∞, for $H > 1/2$. It means that we have to observe quite rarely the storage process to get the complete information on the maximum of the continuous storage process, assuming that d does not tend fast to 0.

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