

1 Introduction and main results

Phase transitions are ubiquitous around us. The water-vapor transition is one of the most familiar examples in daily life. For mathematicians and theoretical physicists, self-avoiding walk, percolation, and the contact process are well-known models that exhibit phase transitions. For example, for percolation, there exists a percolation threshold p_c^{pe} such that there is almost surely no infinite cluster for $p < p_c^{\text{pe}}$, while for $p > p_c^{\text{pe}}$ there is almost surely a unique infinite cluster. The precise value of p_c^{pe} depends on the details of the model, and is only explicitly known in a few cases, such as for two-dimensional nearest-neighbor bond percolation [19].

In this paper, we will consider self-avoiding walk, percolation, oriented percolation and the contact process, where the interaction range L is taken to be large. When $L \gg 1$, the interaction in the considered models is relatively weak, and therefore the critical values are close to the critical value 1 of the respective mean-field models, i.e., random walk and branching random walk. We study the difference of the critical values for the above four models as $L \rightarrow \infty$. It turns out that, above the respective critical dimensions, we can write this difference to leading order as simple functions of the underlying random walk.

1.1 Models

First, we define the models. A *self-avoiding walk* is a path ω in the d -dimensional integer lattice \mathbb{Z}^d with $\omega(i) \neq \omega(j)$ for every distinct $i, j \in \{0, 1, \dots, |\omega|\}$. We define the weight of each path ω by

$$W_p(\omega) = p^{|\omega|} \prod_{i=1}^{|\omega|} D(\omega(i) - \omega(i-1)), \quad (1.1)$$

where D is a probability distribution on \mathbb{Z}^d . We suppose that D is symmetric with respect to the lattice symmetries and that $D(o) = 0$, where o is the origin in \mathbb{Z}^d . A more detailed definition will be given below. The self-avoiding walk two-point function is defined by

$$\tau_p^{\text{sa}}(x) = \sum_{\substack{\omega: o \rightarrow x \\ \text{saw}}} W_p(\omega), \quad (1.2)$$

where the sum is over all self-avoiding paths from o to x . It is known (see, e.g., [22]) that there is a critical value p_c^{sa} such that

$$\chi_p^{\text{sa}} = \sum_{x \in \mathbb{Z}^d} \tau_p^{\text{sa}}(x) \quad (1.3)$$

is finite if and only if $p < p_c^{\text{sa}}$ and diverges as $p \uparrow p_c^{\text{sa}}$.

For *percolation*, each bond $\{x, y\}$ is occupied with probability $p D(y-x)$ and vacant with probability $1-p D(y-x)$, independently of the other bonds, where $p \in [0, \|D\|_\infty^{-1}]$. Since $\sum_x D(x) = 1$, the percolation parameter p is the expected number of occupied bonds per site. We denote by

\mathbb{P}_p the probability distribution of the bond variables. We say that x is connected to y , and write $x \longleftrightarrow y$, if either $x = y$ or there is a path of occupied bonds between x and y . The percolation two-point function and its sum over $x \in \mathbb{Z}^d$ are denoted by

$$\tau_p^{\text{pe}}(x) = \mathbb{P}_p(o \longleftrightarrow x), \quad \chi_p^{\text{pe}} = \sum_{x \in \mathbb{Z}^d} \tau_p^{\text{pe}}(x). \quad (1.4)$$

Similarly to self-avoiding walk, there is a unique critical value p_c^{pe} such that χ_p^{pe} is finite if and only if $p < p_c^{\text{pe}}$ and diverges as $p \uparrow p_c^{\text{pe}}$ (see, e.g., [7]).

Oriented percolation is a time-directed version of percolation. Each bond $((x, t), (y, t + 1))$ is an ordered pair of sites in $\mathbb{Z}^d \times \mathbb{Z}_+$, and is occupied with probability $p D(y - x)$ and vacant with probability $1 - p D(y - x)$, independently of the other bonds, where $p \in [0, \|D\|_\infty^{-1}]$. We say that (x, s) is connected to (y, t) , and write $(x, s) \longrightarrow (y, t)$, if either $(x, s) = (y, t)$ or there is an oriented path of occupied bonds from (x, s) to (y, t) . Let \mathbb{P}_p be the probability distribution of the bond variables. The oriented percolation two-point function and its sum over $\mathbb{Z}^d \times \mathbb{Z}_+$ are denoted by

$$\tau_p^{\text{op}}(x, t) = \mathbb{P}_p((o, 0) \longrightarrow (x, t)), \quad \chi_p^{\text{op}} = \sum_{t \in \mathbb{Z}_+} \sum_{x \in \mathbb{Z}^d} \tau_p^{\text{op}}(x, t). \quad (1.5)$$

Also oriented percolation exhibits a phase transition such that $\chi_p^{\text{op}} < \infty$ if and only if p is less than the critical value p_c^{op} , and that $\chi_p^{\text{op}} \uparrow \infty$ as $p \uparrow p_c^{\text{op}}$ (see, e.g., [8]).

The *contact process* is a model of the spread of an infection in \mathbb{Z}^d , and is a continuous-time version of oriented percolation in $\mathbb{Z}^d \times \mathbb{R}_+$. We now describe a graphical representation for the contact process. Along each time line $\{x\} \times \mathbb{R}_+$, where $x \in \mathbb{Z}^d$, we place points according to a Poisson process with intensity 1, independently of the other time lines. For each ordered pair of distinct time lines from $\{x\} \times \mathbb{R}_+$ to $\{y\} \times \mathbb{R}_+$, we place oriented bonds $((x, t), (y, t))$, $t \geq 0$, according to a Poisson process with intensity $p D(y - x)$, independently of the other Poisson processes, where the parameter $p \geq 0$ is the infection rate. We say that (x, s) is connected to (y, t) , and write $(x, s) \longrightarrow (y, t)$, if either $(x, s) = (y, t)$ or there is an oriented path in $\mathbb{Z}^d \times \mathbb{R}_+$ from (x, s) to (y, t) using the Poisson bonds and time-line segments traversed in the increasing-time direction without traversing the Poisson points. Let \mathbb{P}_p be the corresponding probability distribution. We denote the contact process two-point function and its integro-sum over $\mathbb{Z}^d \times \mathbb{R}_+$ by

$$\tau_p^{\text{cp}}(x, t) = \mathbb{P}_p((o, 0) \longrightarrow (x, t)), \quad \chi_p^{\text{cp}} = \int_0^\infty dt \sum_{x \in \mathbb{Z}^d} \tau_p^{\text{cp}}(x, t). \quad (1.6)$$

Again there is a critical value p_c^{cp} such that χ_p^{cp} is finite if and only if $p < p_c^{\text{cp}}$ and diverges as $p \uparrow p_c^{\text{cp}}$ (see, e.g., [20]).

We will omit the superscript referring to the precise model, and write p_c , when referring to the critical values in all models simultaneously. The goal in this paper is to study p_c when the range L of D is sufficiently large. We first state the precise assumptions on D .

Definition 1.1. Let h be a probability distribution over $\mathbb{R}^d \setminus \{o\}$, which is piecewise continuous, and is invariant under rotations by $\pi/2$ and reflections in the coordination hyperplanes. We define

$$D(x) = \frac{h(x/L)}{\sum_{y \in \mathbb{Z}^d} h(y/L)}. \quad (1.7)$$

Since $\int_{\mathbb{R}^d} h(x) d^d x = 1$, the denominator is $L^d [1 + o(1)]$, where $o(1)$ tends to 0 as $L \rightarrow \infty$. We require that there exist finite positive constants c, C, η such that

$$\sup_{x \in \mathbb{Z}^d} D(x) \leq CL^{-d}, \quad \eta \wedge (cL^2|k|^2) \leq 1 - \hat{D}(k) \leq 2 - \eta, \quad (1.8)$$

where $\hat{D}(k) = \sum_{x \in \mathbb{Z}^d} D(x) e^{ik \cdot x}$. There are a few more minor requirements that depend on the precise model under investigation. For details, see [10] for percolation and [13, 14, 15, 16] for the other three models, for which the requirements are virtually identical.

A simple example of D is

$$D(x) = \frac{\mathbb{1}_{[0 < \|x\|_\infty \leq L]}}{(2L+1)^d - 1}. \quad (1.9)$$

We will frequently use the fact that the n -fold convolution D^{*n} in \mathbb{Z}^d satisfies

$$D^{*n}(x) \leq \delta_{0,n} \delta_{o,x} + \frac{O(\beta)}{(1+n)^{d/2}}, \quad (1.10)$$

where

$$\beta = L^{-d}. \quad (1.11)$$

Equation (1.10) is a simple consequence of the local central limit theorem for random walk.

1.2 Main results

Let d_c denote the respective critical dimensions, i.e., $d_c = 6$ for percolation and $d_c = 4$ for the other three models. In this paper, we investigate the respective critical values when $d > d_c$ and $L \gg 1$, in a unified fashion.

Theorem 1.1. *For each model with $d > d_c$, as $L \rightarrow \infty$,*

$$p_c^{\text{sa}}, p_c^{\text{cp}} = 1 + \sum_{n=2}^{\infty} D^{*n}(o) + O(\beta^2), \quad (1.12)$$

$$p_c^{\text{op}} = 1 + \frac{1}{2} \sum_{n=2}^{\infty} D^{*2n}(o) + O(\beta^2), \quad (1.13)$$

$$p_c^{\text{pe}} = 1 + (D * D)(o) + \frac{1}{2} \sum_{n=3}^{\infty} (n+1) D^{*n}(o) + O(\beta^2). \quad (1.14)$$

The universal term 1 is the critical value for the mean-field models (random walk and branching random walk). Note that, by (1.10), the model-dependent terms in (1.12)–(1.14) are $O(\beta)$.

We now summarize previous results on the critical values. We start by self-avoiding walk. Penrose’s result in [24] implies that the critical value for self-avoiding walk defined by (1.9) with $L \gg 1$ satisfies

$$1 + c\beta^{2/7} \log \beta^{-1} \geq p_c^{\text{sa}} \geq \begin{cases} 1, & \text{if } d \geq 3, \\ 1 + c'\beta \log \beta^{-1}, & \text{if } d = 2, \\ 1 + c''\beta^{4/5}, & \text{if } d = 1, \end{cases} \quad (1.15)$$

for some positive constants c, c', c'' . A similar result for *spread-out lattice trees*, which is another phase transition model, was also obtained in [24]. For $d > 4$, Madras and Slade [22, Corollary 6.2.7] improved (1.15) to $p_c^{\text{sa}} = 1 + O(\beta)$. In [14, 16], this result was extended to more general D as defined in Definition 1.1. We will rely on the results in [14, 16], whose proof is based on the *lace expansion* and a generalized inductive approach. We will also use the lace expansion to derive the expression of the $O(\beta)$ term in (1.12).

For percolation, the best previous result is $p_c^{\text{pe}} = 1 + O(\beta^{2/d-\kappa})$ for $d > 6$ and $L \gg 1$, where $\kappa > 0$ is an arbitrarily small number [9]. However, if we combine Lemma 3.1 proved below and the estimates for the lace expansion in [10], then we obtain the better estimate $p_c^{\text{pe}} = 1 + O(\beta)$. The result in (1.14), which is also obtained by an application of the lace expansion, identifies the expression of this $O(\beta)$ term.

When $d > 4$ and $L \gg 1$, both p_c^{op} and p_c^{cp} were proved to be $1 + O(\beta)$ [13, 14, 15]. Similarly to self-avoiding walk, the proofs of these results rely on the lace expansion and an adaptation of the inductive approach. For the contact process, Durrett and Perkins [6] proved that for the example in (1.9),

$$\lim_{L \rightarrow \infty} \frac{p_c^{\text{cp}} - 1}{\beta} = \sum_{n=2}^{\infty} U^{*n}(o), \quad (1.16)$$

holds for $d > 2$, where U^{*n} is the n -fold convolution of the uniform probability density over $[-1, 1]^d \subset \mathbb{R}^d$. The Green’s function exhibits a logarithmic divergence in $d = 2$. By compensating the logarithmic divergence, Durrett and Perkins also proved that $(p_c - 1)/(\beta \log \beta^{-1})$ converges to $3/(2\pi)$ for $d = 2$. The limit (1.16) can be derived from (1.12) when $d > 4$. In [13], we also obtained the lace expansion results for the contact process in $d \leq 4$, and expect that these results could be used to prove a stronger version of (1.16) for $d = 3, 4$, but this will need serious work using block constructions as used in [6].

We expect that (1.12)–(1.14) remain valid for $d = d_c - 1$ and d_c when we change $O(\beta^2)$ to $o(\beta)$. As mentioned above, this is the case for the contact process [6]. When $d \leq d_c - 2$, the second terms in (1.12)–(1.14) become divergent, so that Theorem 1.1 cannot hold. However, we expect that the asymptotics of the critical point will, as for the contact process, again be described by the divergence of the sums in (1.12)–(1.14).

When $d > d_c$, we expect that the $O(\beta^2)$ terms can be identified in terms of D as well, but to do so will require a serious amount of work.

A related problem is to obtain the asymptotics of the critical points for the nearest-neighbor models, when $D(x) = (2d)^{-1} \mathbb{1}_{[|x|=1]}$ and $d \rightarrow \infty$. In [11], p_c^{sa} was proved to have an asymptotic expansion into powers of $(2d)^{-1}$, and the first six coefficients were obtained. For unoriented percolation, the first three coefficients were also computed in [11], but the proof of the asymptotic expansion only appeared in [17, 18]. The proofs of these results are again based on the lace expansion. For nearest-neighbour oriented percolation and the nearest-neighbour contact process, it is proved that $p_c^{\text{op}} = 1 + O(d^{-2})$ (see [5]) and $p_c^{\text{cp}} = 1 + O(d^{-1})$ (see, e.g., [21]), using different methods.

1.3 Overview of the proof

To prove Theorem 1.1, we will apply the lace expansion (see, e.g., [10, 13, 16, 22, 23]). For example, the lace expansion for self-avoiding walk gives the recurrence relation

$$\tau_p^{\text{sa}}(x) = \delta_{o,x} + \sum_v [p D(v) + \Pi_p^{\text{sa}}(v)] \tau_p^{\text{sa}}(x - v), \quad (1.17)$$

where $\Pi_p^{\text{sa}}(x)$ is a certain expansion coefficient. It was proved in [14, 16] that $\hat{\Pi}_p^{\text{sa}} \equiv \sum_x \Pi_p^{\text{sa}}(x) = O(\beta)$ for $p \leq p_c^{\text{sa}}$, if $d > 4$ and $L \gg 1$ (see Section 2). Summing both sides of (1.17) over $x \in \mathbb{Z}^d$ and solving the resulting equation in terms of χ_p^{sa} , we obtain

$$\chi_p^{\text{sa}} = (1 - p - \hat{\Pi}_p^{\text{sa}})^{-1}, \quad (1.18)$$

and thus

$$p_c^{\text{sa}} = 1 - \hat{\Pi}_{p_c^{\text{sa}}}^{\text{sa}}. \quad (1.19)$$

To estimate p_c^{sa} , we thus need to investigate $\hat{\Pi}_{p_c^{\text{sa}}}^{\text{sa}}$. We will prove that, since $p_c^{\text{sa}} = 1 + O(\beta)$, we can replace $\hat{\Pi}_{p_c^{\text{sa}}}^{\text{sa}}$ by $\hat{\Pi}_1^{\text{sa}}$ up to an error of order $O(\beta^2)$. When $p = 1$, then the exponential growth is taken out (see, e.g., (1.1)), and the main ingredient to the proof is to show that the remaining problem is close to a random walk problem.

We organize the rest of this paper as follows. We begin with self-avoiding walk in Section 2, and explain the key steps to estimate p_c^{sa} . Following the same steps, we estimate p_c^{op} and p_c^{cp} in Section 3.1, and p_c^{pe} in Section 3.2.

2 Critical point for self-avoiding walk

In this section, we prove (1.12), using (1.19). Throughout this section, we will omit the superscript ‘‘sa’’ and write, e.g., $p_c = p_c^{\text{sa}}$ and $\hat{\Pi}_p = \hat{\Pi}_p^{\text{sa}}$.

Before computing the asymptotics of $\hat{\Pi}_{p_c}$ in (1.19), we first note that $p_c \geq 1$. This is because removal of the self-avoidance constraint in (1.2) results in $\sum_{\omega: o \rightarrow x} W_p(\omega)$, whose sum over $x \in \mathbb{Z}^d$ equals $(1 - p)^{-1}$ for any $p \leq 1$. For self-avoiding walk,

$$\Pi_p(x) = \sum_{N=1}^{\infty} (-1)^N \pi_p^{(N)}(x), \quad (2.1)$$

where, e.g., $\pi_p^{(1)}(x)$ is a “1-loop diagram” at the origin [22]:

$$\pi_p^{(1)}(x) = \delta_{o,x} (pD * \tau_p)(o) = \delta_{o,x} \sum_{\substack{\omega: o \rightarrow o \\ |\omega| \geq 1}} W_p(\omega) I(\omega), \quad (2.2)$$

and where $I(\omega) = 1$ if there are no self-intersection points except for $\omega(0) = \omega(|\omega|)$, and $I(\omega) = 0$ otherwise.

For $d > 4$ and $L \gg 1$, it was proved in [16] that, for $\hat{\pi}_p^{(N)} = \sum_x \pi_p^{(N)}(x)$, we have

$$\hat{\pi}_p^{(N)} \leq O(\beta)^N, \quad p \partial_p \hat{\Pi}_p \leq O(\beta), \quad (2.3)$$

for all $p \leq p_c$ and $N \geq 1$. Together with (1.19) and (2.1), we immediately obtain that $p_c = 1 + O(\beta)$. Moreover, by the mean-value theorem, there is a $p \in (1, p_c)$ such that

$$p_c = 1 - \hat{\Pi}_1 - (\hat{\Pi}_{p_c} - \hat{\Pi}_1) = 1 - \hat{\Pi}_1 - (p_c - 1) \partial_p \hat{\Pi}_p = 1 + \hat{\pi}_1^{(1)} + O(\beta^2), \quad (2.4)$$

where

$$\hat{\pi}_1^{(1)} = \sum_{\substack{\omega: o \rightarrow o \\ |\omega| \geq 1}} W_1(\omega) I(\omega) = \sum_{\substack{\omega: o \rightarrow o \\ |\omega| \geq 1}} W_1(\omega) - \sum_{\substack{\omega: o \rightarrow o \\ |\omega| \geq 1}} W_1(\omega) [1 - I(\omega)]. \quad (2.5)$$

Since the first term equals $\sum_{n=2}^{\infty} D^{*n}(o)$, to complete the proof of (1.12), it suffices to prove that the second term is $O(\beta^2)$. We will do so now.

We first note that $I(\omega)$ can take the value 0 or 1. If $I(\omega) = 0$, so that $1 - I(\omega) = 1$, then there must be a pair $\{s, t\} \neq \{0, |\omega|\}$ with $0 \leq s < t \leq |\omega|$ such that $\omega(s) = \omega(t)$. Denoting the parts of ω corresponding to these three time intervals by ω_i , $i = 1, 2, 3$, respectively, we obtain

$$\sum_{\substack{\omega: o \rightarrow o \\ |\omega| \geq 1}} W_1(\omega) [1 - I(\omega)] \leq \sum_{x \in \mathbb{Z}^d} \sum_{\substack{\omega_1, \omega_3: o \rightarrow x \\ |\omega_1| + |\omega_3| \geq 1}} \sum_{\substack{\omega_2: x \rightarrow x \\ |\omega_2| \geq 1}} \prod_{i=1}^3 W_1(\omega_i) = (D * G^{*2})(o) (D^{*2} * G)(o), \quad (2.6)$$

where $G(x) = \sum_{n=0}^{\infty} D^{*n}(x)$ is the random walk Green’s function. By (1.10), both $(D * G^{*2})(o)$ and $(D^{*2} * G)(o)$ are $O(\beta)$. This completes the proof of (1.12) for self-avoiding walk. \square

3 Critical points for percolation models

In this section, we compute the asymptotics of the critical values for the other three models, and thus complete the proof of Theorem 1.1.

To discuss oriented percolation and the contact process simultaneously, it is convenient to introduce the following oriented percolation on $\mathbb{Z}^d \times \varepsilon\mathbb{Z}_+$, which is the time-discretized contact process with a discretization parameter $\varepsilon \in (0, 1]$. A bond is a directed pair $((x, t), (y, t + \varepsilon))$ of sites in $\mathbb{Z}^d \times \varepsilon\mathbb{Z}_+$. Each bond is either occupied or vacant, independently of the other bonds, and a bond $((x, t), (y, t + \varepsilon))$ is occupied with probability

$$q(y - x) = \begin{cases} 1 - \varepsilon, & \text{if } x = y, \\ p\varepsilon D(y - x), & \text{otherwise,} \end{cases} \quad (3.1)$$

provided that $\sup_x q(x) \leq 1$. In this notation, the model with $\varepsilon = 1$ is the ordinary oriented percolation, and the weak limit as $\varepsilon \downarrow 0$ is the ordinary contact process [3]. Similarly to ordinary oriented percolation, there is a critical value $p_c^{(\varepsilon)}$ for every $\varepsilon \in (0, 1]$, such that $p_c^{(1)} = p_c^{\text{op}}$ and $\lim_{\varepsilon \downarrow 0} p_c^{(\varepsilon)} = p_c^{\text{cp}}$ [25]. We will call the model with $\varepsilon \in (0, 1]$ the *time-discretized contact process*.

To summarise notation for percolation and the time-discretized contact process, we will write $\Lambda = \mathbb{Z}^d$ for percolation and $\Lambda = \mathbb{Z}^d \times \varepsilon\mathbb{Z}_+$ for oriented percolation. We will also use bold letters to represent elements of Λ . For example, $\mathbf{o} = o$, $\mathbf{x} = x$ for percolation, and $\mathbf{o} = (o, 0)$, $\mathbf{x} = (x, t)$ for the time-discretized contact process. For a bond $b = (\mathbf{u}, \mathbf{v})$, we write $\underline{b} = \mathbf{u}$ and $\bar{b} = \mathbf{v}$. We also omit the superscripts ε , pe, op and cp, if no confusion can arise.

As mentioned in Section 1, the lace expansion for percolation models takes a similar form as in (1.17), and reads (see, e.g., [10, 13])

$$\tau_p(\mathbf{x}) = [\delta_{\mathbf{o}, \mathbf{x}} + \Pi_p(\mathbf{x})] + \sum_{\mathbf{u}, \mathbf{v} \in \Lambda} [\delta_{\mathbf{o}, \mathbf{u}} + \Pi_p(\mathbf{u})] q(\mathbf{v} - \mathbf{u}) \tau_p(\mathbf{x} - \mathbf{v}). \quad (3.2)$$

In particular, $q(\mathbf{v} - \mathbf{u}) = pD(v - u)$ for percolation and oriented percolation for which $\varepsilon = 1$. To unify notation, we will also regard unoriented percolation as a model with $\varepsilon = 1$. The lace expansion coefficient $\Pi_p(\mathbf{x})$ equals

$$\Pi_p(\mathbf{x}) = \sum_{N=0}^{\infty} (-1)^N \pi_p^{(N)}(\mathbf{x}), \quad (3.3)$$

where $\pi_p^{(N)}(\mathbf{x})$, $N \geq 0$, are model-dependent diagram functions. The result of the lace expansion will be explained in Sections 3.1–3.2. For the time-discretized contact process with $\varepsilon \in (0, 1]$, $d > d_c$ and $L \gg 1$, it has been proved [13, 15] that $\hat{\Pi}_p \equiv \varepsilon \sum_{\mathbf{x} \in \Lambda} \Pi_p(\mathbf{x})$ is $O(\beta) \varepsilon^2$ for all $p \leq p_c$. The same estimate is proved to hold for unoriented percolation (with $\varepsilon = 1$), using the lace expansion in [10] and Lemma 3.1 proved below in Section 3.2.

As in the derivation of (1.18), solving (3.2) in terms of $\chi_p = \varepsilon \sum_{\mathbf{x} \in \Lambda} \tau_p(\mathbf{x})$ gives

$$\chi_p = \frac{1 + \frac{1}{\varepsilon} \hat{\Pi}_p}{1 - p - (1 - \varepsilon + p\varepsilon) \frac{1}{\varepsilon^2} \hat{\Pi}_p}, \quad (3.4)$$

and thus, equating the denominator to zero,

$$p_c = 1 - \frac{1}{\varepsilon^2} \hat{\Pi}_{p_c} - (p_c - 1) \frac{1}{\varepsilon} \hat{\Pi}_{p_c}. \quad (3.5)$$

This expression holds uniformly in ε . We will use it to compute p_c^{cp} and p_c^{pe} by taking $\varepsilon = 1$ and p_c^{cp} by taking the limit when $\varepsilon \downarrow 0$ [25], respectively. In particular, the third term is $O(\beta^2)$ when $\varepsilon = 1$, and it has no contribution in the limit $\varepsilon \downarrow 0$. Therefore, we are left to prove that, apart from an error term of order $O(\beta^2)$, the second term in (3.5) equals the second term in (1.12) when $\varepsilon \downarrow 0$, and equals the second term in (1.13) for oriented percolation and that in (1.14) for (unoriented) percolation when $\varepsilon = 1$. We again note that $p_c^{(\varepsilon)} \geq 1$, because

$$\chi_p \leq \varepsilon \sum_{n=0}^{\infty} \sum_{x \in \mathbb{Z}^d} q^{*n}(x) = (1 - p)^{-1}$$

for $p \leq 1$. In addition, similarly to (1.10), the n -fold convolution q^{*n} with $p = 1$ and $\varepsilon < 1$ satisfies [13]

$$q^{*n}(x) \leq (1 - \varepsilon)^n \delta_{o,x} + \frac{O(\beta)}{(1 + n\varepsilon)^{d/2}}. \quad (3.6)$$

Note that when $\varepsilon = 1$, (3.6) reduces to (1.10).

To complete the proof of Theorem 1.1, we investigate $\hat{\Pi}_{p_c}$ for oriented percolation and the contact process in Section 3.1, and for unoriented percolation in Section 3.2.

3.1 Asymptotics of p_c^{op} and p_c^{cp}

In this section, we investigate $\hat{\Pi}_{p_c}$ for the discretized contact process, and derive (1.13) for oriented percolation (i.e., $\varepsilon = 1$) and (1.12) for the contact process (i.e., $\varepsilon \downarrow 0$).

To describe the diagram functions $\pi_p^{(N)}(\mathbf{x})$, $N \geq 0$, we need some definitions. We say that \mathbf{x} is *doubly connected to* \mathbf{y} , if either $\mathbf{x} = \mathbf{y}$ or there are at least two nonzero bond-disjoint occupied paths from \mathbf{x} to \mathbf{y} . Following the notation in [15] as closely as possible, we denote this event by $\mathbf{x} \Longrightarrow \mathbf{y}$, and define

$$\pi_p^{(0)}(\mathbf{x}) = \mathbb{P}_p(\mathbf{o} \Longrightarrow \mathbf{x}) - \delta_{\mathbf{o},\mathbf{x}}. \quad (3.7)$$

If \mathbf{o} is connected but not doubly connected to \mathbf{x} , there is a *pivotal bond* $b = (\underline{b}, \bar{b})$ for $\mathbf{o} \longrightarrow \mathbf{x}$ such that both $\mathbf{o} \longrightarrow \underline{b}$ and $\bar{b} \longrightarrow \mathbf{x}$ occur, and that $\mathbf{o} \longrightarrow \mathbf{x}$ occurs if and only if b is set occupied. For $A \subset \Lambda$, we write $\mathbf{y} \longrightarrow \mathbf{x}$ *via* A if there is a vertex $\mathbf{z} \in A$ such that both $\mathbf{y} \longrightarrow \mathbf{z}$ and $\mathbf{z} \longrightarrow \mathbf{x}$ occur. We define $E[b, \mathbf{x}; A]$ to be the event that b is occupied, that $\bar{b} \longrightarrow \mathbf{x}$ via A , and that there are no pivotal bonds b' for $\bar{b} \longrightarrow \mathbf{x}$ such that $\bar{b} \longrightarrow \underline{b}'$ via A . Let $\tilde{C}^b(\mathbf{o})$ be the set of vertices in Λ connected from \mathbf{o} without using b . Then,

$$\pi_p^{(1)}(\mathbf{x}) = \sum_b \mathbb{P}_p(\mathbf{o} \Longrightarrow \underline{b}; E[b, \mathbf{x}; \tilde{C}^b(\mathbf{o})]). \quad (3.8)$$

The higher order diagram functions $\pi_p^{(N)}(\mathbf{x})$, $N \geq 2$, are defined in the same way, but are irrelevant in this paper (see [13, Section 3] for a complete definition, with slightly different notation).

For $d > 4$ and $L \gg 1$, it was proved in [13] that, for $\hat{\pi}_p^{(N)} = \varepsilon \sum_{\mathbf{x} \in \Lambda} \pi_p^{(N)}(\mathbf{x})$, we have

$$\hat{\pi}_p^{(N)} \leq O(\beta)^{N+1} \varepsilon^2, \quad p \partial_p \hat{\Pi}_p \leq O(\beta) \varepsilon^2, \quad (3.9)$$

for all $p \leq p_c$ and $N \geq 0$. Together with (3.3) and (3.5), we obtain $p_c = 1 + O(\beta)$. Moreover, by the mean-value theorem, there is a $p \in (1, p_c)$ such that

$$\begin{aligned} p_c &= 1 - \frac{1}{\varepsilon^2} \hat{\Pi}_{p_c} - (p_c - 1) \frac{1}{\varepsilon} \hat{\Pi}_{p_c} = 1 - \frac{1}{\varepsilon^2} \hat{\Pi}_1 - (p_c - 1) \frac{1}{\varepsilon^2} \partial_p \hat{\Pi}_p + O(\beta^2) \varepsilon \\ &= 1 - \frac{1}{\varepsilon^2} \hat{\pi}_1^{(0)} + \frac{1}{\varepsilon^2} \hat{\pi}_1^{(1)} + O(\beta^2). \end{aligned} \quad (3.10)$$

To prove (1.12)–(1.13), it thus suffices to investigate $\hat{\pi}_1^{(0)}$ and $\hat{\pi}_1^{(1)}$.

Analysis of $\hat{\pi}_1^{(0)}$. We prove

$$\frac{1}{\varepsilon^2} \hat{\pi}_1^{(0)} \begin{cases} = \frac{1}{2} \sum_{n=2}^{\infty} D^{*2n}(o) + O(\beta^2), & \text{for } \varepsilon = 1, \\ \rightarrow \sum_{n=2}^{\infty} D^{*n}(o) + O(\beta^2), & \text{when } \varepsilon \downarrow 0. \end{cases} \quad (3.11)$$

Recall (3.7). To describe a double connection by a pair of two random walk paths, we order the support of D in an arbitrary but fixed manner. For x, y in the support of D , we write $x \prec y$ if x is *lower* than y in that order. For a pair of paths consisting of bonds in Λ , $\omega = (b_1, \dots, b_N)$ and $\omega' = (b'_1, \dots, b'_N)$ with $b_1 = b'_1$ and $b_N = b'_N$, we say that ω is *lower* than ω' , denoted by $\omega \prec \omega'$, if at the first time $n \in \{1, \dots, N\}$ when ω is incompatible with ω' (therefore $b_i = b'_i$ for all $i < n$) we have $b_n \prec b'_n$. We also say that ω_2 is *higher* than ω_1 .

A path $\omega = (b_1, \dots, b_{|\omega|})$ is said to be *occupied* if all bonds along ω are occupied. We define $E_{\succ}(\omega)$ to be the event that ω is the lowest occupied path among all occupied paths from b_1 to $b_{|\omega|}$, and that there is another occupied path ω' from b_1 to $b_{|\omega|}$ which is bond-disjoint from ω (denoted by $\omega \cap \omega' = \emptyset$). Given a path ω , we also define $E_{\prec}(\omega'; \omega)$ to be the event that ω' is the highest occupied path among all occupied paths from b_1 to $b_{|\omega|}$ that are bond-disjoint from ω . Such an occupied path ω' exists on $\{b_1 \Rightarrow b_{|\omega|}\} \cap E_{\succ}(\omega)$ by definition.

Using the above notation, we have, for $\mathbf{x} \neq \mathbf{o}$,

$$\{\mathbf{o} \Rightarrow \mathbf{x}\} = \bigcup_{\substack{\omega_1, \omega_2: \mathbf{o} \rightarrow \mathbf{x} \\ \omega_1 \cap \omega_2 = \emptyset \\ \omega_1 \prec \omega_2}} \{\omega_1, \omega_2 \text{ occupied}; E_{\succ}(\omega_1) \cap E_{\prec}(\omega_2; \omega_1)\}. \quad (3.12)$$

We define the right-hand side to be empty if $\mathbf{x} = \mathbf{o}$. Then,

$$\hat{\pi}_1^{(0)} = \varepsilon \sum_{\mathbf{x} \in \Lambda} \sum_{\substack{\omega_1, \omega_2: \mathbf{o} \rightarrow \mathbf{x} \\ \omega_1 \cap \omega_2 = \emptyset \\ \omega_1 \prec \omega_2}} \mathbb{P}_1(\omega_1, \omega_2 \text{ occupied}; E_{\succ}(\omega_1) \cap E_{\prec}(\omega_2; \omega_1)). \quad (3.13)$$

Since \mathbb{P}_1 is a product measure, if we ignore $E_{\succ}(\omega_1) \cap E_{\prec}(\omega_2; \omega_1)$, then we obtain

$$\begin{aligned} & \sum_{\substack{\omega_1, \omega_2: \mathbf{o} \rightarrow \mathbf{x} \\ \omega_1 \cap \omega_2 = \emptyset \\ \omega_1 \prec \omega_2}} \mathbb{P}_1(\omega_1, \omega_2 \text{ occupied}) \\ &= \sum_{\substack{u, v: u \prec v \\ y, z: y \neq z}} q(u) q(v) q(x - y) q(x - z) \sum_{\substack{\omega_1: \mathbf{u} \rightarrow \mathbf{y} \\ \omega_2: \mathbf{v} \rightarrow \mathbf{z} \\ \omega_1 \cap \omega_2 = \emptyset}} \mathbb{P}_1(\omega_1 \text{ occupied}) \mathbb{P}_1(\omega_2 \text{ occupied}), \end{aligned} \quad (3.14)$$

where $\mathbf{u} = (u, \varepsilon)$, $\mathbf{v} = (v, \varepsilon)$, $\mathbf{y} = (y, t - \varepsilon)$ and $\mathbf{z} = (z, t - \varepsilon)$. By an inclusion-exclusion relation, the correction is bounded by

$$\sum_{\substack{\omega_1, \omega_2: \mathbf{o} \rightarrow \mathbf{x} \\ \omega_1 \cap \omega_2 = \emptyset \\ \omega_1 \prec \omega_2}} \left[\mathbb{P}_1(\omega_1, \omega_2 \text{ occupied}; E_{\succ}(\omega_1)^c) + \mathbb{P}_1(\omega_1, \omega_2 \text{ occupied}; E_{\prec}(\omega_2; \omega_1)^c) \right].$$

We will prove below that, for E equal to $E_{\succ}(\omega_1)$ or $E_{\prec}(\omega_2; \omega_1)$,

$$\varepsilon \sum_{\mathbf{x} \in \Lambda} \sum_{\substack{\omega_1, \omega_2: \mathbf{o} \rightarrow \mathbf{x} \\ \omega_1 \cap \omega_2 = \emptyset}} \mathbb{P}_1(\omega_1, \omega_2 \text{ occupied}; E^c) = O(\beta^2) \varepsilon^2. \quad (3.15)$$

We investigate (3.14) to obtain the expression of $O(\beta)$ from (3.13). If we ignore the restriction $\omega_1 \cap \omega_2 = \emptyset$, then we obtain

$$\sum_{\substack{u,v:u \prec v \\ y,z:y \neq z}} q(u) q(v) q(x-y) q(x-z) q^{*(t/\varepsilon-2)}(y-u) q^{*(t/\varepsilon-2)}(z-v), \quad (3.16)$$

where $t \in [2, \infty) \cap \varepsilon\mathbb{Z}_+$. We will prove below that the correction satisfies

$$\varepsilon \sum_{\mathbf{x} \in \Lambda} \sum_{\substack{u,v:u \prec v \\ y,z:y \neq z}} q(u) q(v) q(x-y) q(x-z) \sum_{\substack{\omega_1: \mathbf{u} \rightarrow \mathbf{y} \\ \omega_2: \mathbf{v} \rightarrow \mathbf{z} \\ \omega_1 \cap \omega_2 \neq \emptyset}} \mathbb{P}_1(\omega_1 \text{ occupied}) \mathbb{P}_1(\omega_2 \text{ occupied}) = O(\beta^2) \varepsilon^2. \quad (3.17)$$

Therefore, we only need to consider the contribution to (3.13) from (3.16). By changing variables as $y' = x - y$ and $z' = x - z$ and using symmetry between $u \prec v$ and $u \succ v$, the sum of (3.16) over $x \in \mathbb{Z}^d$ equals

$$\begin{aligned} & \sum_{\substack{u,v:u \prec v \\ y',z':y' \neq z'}} q(u) q(v) q(y') q(z') \sum_x q^{*(t/\varepsilon-2)}(x-y'-u) q^{*(t/\varepsilon-2)}(x-z'-v) \\ &= \frac{1}{2} \sum_{\substack{u,v:u \neq v \\ y,z:y \neq z}} q(u) q(v) q(y) q(z) q^{*(2t/\varepsilon-4)}(v+z-y-u). \end{aligned} \quad (3.18)$$

Recall (3.1). Since there is at most one temporal (or vertical) bond growing out of every site in Λ , we must have $q(u) = \varepsilon D(u)$ or $q(v) = \varepsilon D(v)$, so that we obtain at least one factor of ε . By the same reason, we should have $q(y) = \varepsilon D(y)$ or $q(z) = \varepsilon D(z)$, so that we obtain a second factor of ε . Therefore, the number of combinations for the product of four factors of q in (3.18) is nine: one combination is proportional to ε^4 , four others are proportional to $(1-\varepsilon)\varepsilon^3$, and the remaining four are proportional to $(1-\varepsilon)^2\varepsilon^2$. Only the first cases arises for oriented percolation for which $\varepsilon = 1$, while only the third case arises for the contact process for which $\varepsilon \downarrow 0$, respectively.

We first complete the proof of (3.11) for oriented percolation. When $\varepsilon = 1$, and using inclusion-exclusion on the restrictions $u \neq v$ and $y \neq z$, the sum of (3.18) over $t \geq 2$ equals

$$\frac{1}{2} \sum_{u,v,y,z} D(u) D(v) D(y) D(z) \sum_{t=2}^{\infty} D^{*(2t-4)}(v+z-y-u) + O(\beta^2) = \frac{1}{2} \sum_{t=2}^{\infty} D^{*2t}(o) + O(\beta^2), \quad (3.19)$$

where we use (1.8) to obtain an error of order $O(\beta^2)$ that comes from contributions where $u = v$ or $y = z$.

For the contact process, for which $\varepsilon \downarrow 0$, the leading contribution is due to the four combinations of order $(1-\varepsilon)^2\varepsilon^2$ mentioned above, where either u or v is o , and either y or z is o . Therefore, the coefficient of $(1-\varepsilon)^2\varepsilon^2$ in (3.18) is

$$\begin{aligned} & \frac{1}{2} \left[\sum_{u,y} D(u) D(y) q^{*(2t/\varepsilon-4)}(-y-u) + \sum_{u,z} D(u) D(z) q^{*(2t/\varepsilon-4)}(z-u) \right. \\ & \left. + \sum_{v,y} D(v) D(y) q^{*(2t/\varepsilon-4)}(v-y) + \sum_{v,z} D(v) D(z) q^{*(2t/\varepsilon-4)}(v+z) \right] = 2(D^{*2} * q^{*(2t/\varepsilon-4)})(o). \end{aligned}$$

Summing this expression (multiplied by ε) over $t \in [2, \infty) \cap \varepsilon\mathbb{Z}_+$ gives

$$2 \int_{[-\pi, \pi]^d} \frac{d^d k}{(2\pi)^d} \hat{D}(k)^2 \varepsilon \sum_{n=0}^{\infty} [1 - \varepsilon + \varepsilon \hat{D}(k)]^{2n} = \int_{[-\pi, \pi]^d} \frac{d^d k}{(2\pi)^d} \frac{2\hat{D}(k)^2}{[1 - \hat{D}(k)][2 - \varepsilon + \varepsilon \hat{D}(k)]}$$

$$\xrightarrow{\varepsilon \downarrow 0} \int_{[-\pi, \pi]^d} \frac{d^d k}{(2\pi)^d} \frac{\hat{D}(k)^2}{1 - \hat{D}(k)} = \sum_{n=2}^{\infty} D^{*n}(o). \quad (3.20)$$

This completes the proof of (3.11). \square

Analysis of $\hat{\pi}_1^{(1)}$. We prove that $\frac{1}{\varepsilon^2} \hat{\pi}_1^{(1)}$ is asymptotically twice as large as the right-hand side of (3.11):

$$\frac{1}{\varepsilon^2} \hat{\pi}_1^{(1)} \begin{cases} = \sum_{n=2}^{\infty} D^{*2n}(o) + O(\beta^2), & \text{for } \varepsilon = 1, \\ \rightarrow 2 \sum_{n=2}^{\infty} D^{*n}(o) + O(\beta^2), & \text{when } \varepsilon \downarrow 0. \end{cases} \quad (3.21)$$

For a bond b , let $\{b \Rightarrow \mathbf{x}\}$ be the event that b is occupied and $\bar{b} \Rightarrow \mathbf{x}$. We define $\{\mathbf{u} \rightarrow b\}$ and a joint event $\{\mathbf{u} \rightarrow b \Rightarrow \mathbf{x}\}$ similarly. For events E_1 and E_2 , we denote by $E_1 \circ E_2$ the event that E_1 and E_2 occur *disjointly*, i.e., on disjoint bond sets (see e.g., [7, Section 2.3]). Recalling (3.8) and distinguishing between $\underline{b} = \mathbf{o}$ and $\underline{b} \neq \mathbf{o}$, we can rewrite $\hat{\pi}_1^{(1)}$ as

$$\hat{\pi}_1^{(1)} = \varepsilon \sum_{\mathbf{u}, \mathbf{x} \in \Lambda} \mathbb{P}_1(\{\{\mathbf{o}, \mathbf{u}\} \rightarrow \mathbf{x}\} \circ \{\mathbf{o} \rightarrow \mathbf{x}\}) + \varepsilon \sum_{\mathbf{x} \in \Lambda} \sum_{b: \underline{b} \neq \mathbf{o}} \mathbb{P}_1(\mathbf{o} \Rightarrow \underline{b}; E[b, \mathbf{x}; \tilde{C}^b(\mathbf{o})])$$

$$- \varepsilon \sum_{\mathbf{u}, \mathbf{x} \in \Lambda} \mathbb{P}_1(\{\{\{\mathbf{o}, \mathbf{u}\} \rightarrow \mathbf{x}\} \circ \{\mathbf{o} \rightarrow \mathbf{x}\}\} \setminus E[(\mathbf{o}, \mathbf{u}), \mathbf{x}; \tilde{C}^{(\mathbf{o}, \mathbf{u})}(\mathbf{o})]). \quad (3.22)$$

We will extract the leading contribution from the first term. Note that $\{\{\mathbf{o}, \mathbf{u}\} \rightarrow \mathbf{x}\} \circ \{\mathbf{o} \rightarrow \mathbf{x}\}$ is almost identical to $\{\mathbf{o} \Rightarrow \mathbf{x}\} = \{\mathbf{o} \rightarrow \mathbf{x}\} \circ \{\mathbf{o} \rightarrow \mathbf{x}\}$. However, the commutative symmetry between the two connections from \mathbf{o} to \mathbf{x} is lost in the former event, due to the bond (\mathbf{o}, \mathbf{u}) . Therefore (cf., (3.12)),

$$\{\{\mathbf{o}, \mathbf{u}\} \rightarrow \mathbf{x}\} \circ \{\mathbf{o} \rightarrow \mathbf{x}\} = \bigcup_{\substack{\omega_1: (\mathbf{o}, \mathbf{u}) \rightarrow \mathbf{x} \\ \omega_2: \mathbf{o} \rightarrow \mathbf{x} \\ \omega_1 \cap \omega_2 = \emptyset}} \left\{ \{\omega_1, \omega_2 \text{ occupied}; E_{\succ}(\omega_1) \cap E_{\prec}(\omega_2; \omega_1)\} \right.$$

$$\left. \dot{\cup} \{\omega_1, \omega_2 \text{ occupied}; E_{\succ}(\omega_2) \cap E_{\prec}(\omega_1; \omega_2)\} \right\}, \quad (3.23)$$

and its contribution to (3.22) is

$$\varepsilon \sum_{\mathbf{x} \in \Lambda} \sum_{\substack{\omega_1: (\mathbf{o}, \mathbf{u}) \rightarrow \mathbf{x} \\ \omega_2: \mathbf{o} \rightarrow \mathbf{x} \\ \omega_1 \cap \omega_2 = \emptyset}} \left[\mathbb{P}_1(\omega_1, \omega_2 \text{ occupied}; E_{\succ}(\omega_1) \cap E_{\prec}(\omega_2; \omega_1)) \right.$$

$$\left. + \mathbb{P}_1(\omega_1, \omega_2 \text{ occupied}; E_{\succ}(\omega_2) \cap E_{\prec}(\omega_1; \omega_2)) \right], \quad (3.24)$$

where $\omega_1 : (\mathbf{o}, \mathbf{u}) \rightarrow \mathbf{x}$ is a path from \mathbf{o} to \mathbf{x} starting by the bond (\mathbf{o}, \mathbf{u}) . Ignoring the conditions $E_{\succ}(\omega_1) \cap E_{\prec}(\omega_2; \omega_1)$ and $E_{\succ}(\omega_2) \cap E_{\prec}(\omega_1; \omega_2)$ as in (3.14) and following the same strategy as in

estimating $\hat{\pi}_1^{(0)}$, we obtain the main contribution to (3.21). The leading term of $\frac{1}{\varepsilon^2}\hat{\pi}_1^{(1)}$ is twice as large as that of $\frac{1}{\varepsilon^2}\hat{\pi}_1^{(0)}$, because both $\omega_1 \prec \omega_2$ and $\omega_1 \succ \omega_2$ give equal contributions to (3.24) (cf., (3.13)).

To complete the proof of (3.21), it suffices to show that the second and third terms in (3.22) are both $O(\beta^2)\varepsilon^2$. The event in the second term of (3.22) implies existence of $\mathbf{y} \in \Lambda$ such that $\{\mathbf{o} \rightarrow \mathbf{y} \rightarrow \underline{b}\} \circ \{\mathbf{o} \rightarrow \underline{b}\}$ and $\{\mathbf{y} \rightarrow \mathbf{x}\} \circ \{b \rightarrow \mathbf{x}\}$ occur disjointly. Let ω_1 denote a path from \mathbf{o} to \mathbf{x} through \mathbf{y} , ω_2 denote another path from \mathbf{o} to \mathbf{x} via the the bond b with $\underline{b} = \mathbf{z}$, and ω_3 denote another path from \mathbf{y} to \mathbf{z} . Then, the second term in (3.22) is bounded by

$$\varepsilon \sum_{\substack{\mathbf{x}, \mathbf{y}, \mathbf{z} \in \Lambda \\ \mathbf{z} \neq \mathbf{o}, \mathbf{x}}} \sum_{\substack{\omega_1: \mathbf{o} \rightarrow \mathbf{y} \rightarrow \mathbf{x} \\ \omega_2: \mathbf{o} \rightarrow \mathbf{z} \rightarrow \mathbf{x} \\ \omega_3: \mathbf{y} \rightarrow \mathbf{z} \\ \omega_i \cap \omega_j = \emptyset, i \neq j}} \prod_{i=1}^3 \mathbb{P}_1(\omega_i \text{ occupied}), \quad (3.25)$$

since \mathbb{P}_1 is a product measure. The third term in (3.22) is also bounded by the above expression. This is because the event in the third term in (3.22) implies existence of $\mathbf{y} \in \Lambda$ and a pivotal bond $b = (\mathbf{z}, \cdot)$ for $\mathbf{u} \rightarrow \mathbf{x}$ such that $\{\mathbf{o} \rightarrow \mathbf{y} \rightarrow \mathbf{x}\}$, $\{(\mathbf{o}, \mathbf{u}) \rightarrow b \rightarrow \mathbf{x}\}$ and $\{\mathbf{y} \rightarrow \mathbf{z}\}$ occur disjointly. We thus obtain (3.25) by the same random walk representation.

Therefore, it is sufficient to prove that (3.25) is bounded by $O(\beta^2)\varepsilon^2$. When $\varepsilon = 1$, we simply ignore the restriction $\omega_i \cap \omega_j = \emptyset, i \neq j$, and apply the Gaussian bound (1.10) to the part of ω_1 from \mathbf{y} to \mathbf{x} and to the part of ω_2 from \mathbf{o} to \mathbf{z} . Since $\mathbf{y} \neq \mathbf{x}$ and $\mathbf{z} \neq \mathbf{o}$, the term $\delta_{\mathbf{o}, \mathbf{x}}$ in (1.10) does not contribute, so that (3.25) is bounded by

$$\sum_{\substack{t, s, s' \in \mathbb{Z}_+ \\ 0 \leq s \leq s' \leq t}} \frac{O(\beta)}{(1+t-s)^{d/2}} \frac{O(\beta)}{(1+s')^{d/2}} \leq \sum_{t=0}^{\infty} \frac{O(\beta^2)}{(1+t)^{d/2}} \leq O(\beta^2), \quad (3.26)$$

where s, s' are the time variables of \mathbf{y} and \mathbf{z} , respectively. When $\varepsilon < 1$, we use the restriction $\omega_i \cap \omega_j = \emptyset, i \neq j$, to extract factors of q with pairwise different arguments, as in (3.14), out of the four intersection points $\mathbf{o}, \mathbf{y}, \mathbf{z}$ and \mathbf{x} . As explained above (3.19), each pair gives rise to a factor ε , and we obtain a total factor ε^4 . With the help of (3.6), (3.25) with $\varepsilon < 1$ is bounded by ε^{1+4} times the left-hand side of (3.26) with the region of summation being replaced by $\varepsilon\mathbb{Z}_+$. This is further bounded by $O(\beta^2)\varepsilon^2$, since the sum over $t, s, s' \in \varepsilon\mathbb{Z}_+$ eats up a factor ε^3 for the Riemann sum approximation. This completes the proof of (3.21). \square

Proof of (3.15). We only consider the case $E^c = E_{\succ}(\omega_1)^c$, which is the event that there is an $\eta \prec \omega_1$ from \mathbf{o} to \mathbf{x} , possibly sharing a first few steps with ω_1 , such that $E_{\succ}(\eta)$ occurs; the other case $E = E_{\prec}(\omega_2; \omega_1)$ can be estimated in a similar way. Let ω_3 be the part of η from the point, say \mathbf{y} , where η starts disagreeing from ω_1 until it hits ω_1 or ω_2 at \mathbf{z} . Since \mathbb{P}_1 is a product measure, (3.15) is bounded by

$$\varepsilon \sum_{\substack{\mathbf{x}, \mathbf{y}, \mathbf{z} \in \Lambda \\ \mathbf{z} \neq \mathbf{o}, \mathbf{x}}} \sum_{\substack{\omega_1: \mathbf{o} \rightarrow \mathbf{y} \rightarrow \mathbf{x} \\ \omega_2: \mathbf{o} \rightarrow \mathbf{x} \\ \omega_3: \mathbf{y} \rightarrow \mathbf{z} \\ \omega_i \cap \omega_j = \emptyset, i \neq j}} (\mathbb{1}_{[\mathbf{z} \in \omega_1 \setminus \{\mathbf{y}\}]} + \mathbb{1}_{[\mathbf{z} \in \omega_2]}) \prod_{i=1}^3 \mathbb{P}_1(\omega_i \text{ occupied}).$$

Since the contribution from $\mathbb{1}_{[\mathbf{z} \in \omega_2]}$ is equal to (3.25), we only need to investigate the contribution due to the other indicator $\mathbb{1}_{[\mathbf{z} \in \omega_1 \setminus \{\mathbf{y}\}]}$. We again discuss the case $\varepsilon = 1$ first, and then adapt the argument to the case $\varepsilon < 1$, as done below (3.26). When $\varepsilon = 1$, we ignore the restriction $\omega_i \cap \omega_j = \emptyset$, $i \neq j$, and apply (1.10) to the probability of ω_2 and ω_3 being occupied. By denoting the time variables of \mathbf{y} and \mathbf{z} by s and s' respectively, the contribution from $\mathbb{1}_{[\mathbf{z} \in \omega_1 \setminus \{\mathbf{y}\}]}$ is bounded by

$$\sum_{\substack{t, s, s' \in \mathbb{Z}_+ \\ 0 \leq s < s' \leq t}} \frac{O(\beta)}{(1+t)^{d/2}} \frac{O(\beta)}{(1+s'-s)^{d/2}} \leq \sum_{t=0}^{\infty} \frac{O(\beta^2)}{(1+t)^{(d-2)/2}} \leq O(\beta^2).$$

When $\varepsilon < 1$, we use the restriction $\omega_i \cap \omega_j = \emptyset$, $i \neq j$, along each of the four intersection points and obtain the eight factors of q with pairwise different arguments. Following the argument below (3.26), we obtain the desired bound $O(\beta^2) \varepsilon^2$. This completes the proof of (3.15). \square

Proof of (3.17). Since $\omega_1 \cap \omega_2 \neq \emptyset$, there is a sequence of bonds b_1, \dots, b_n such that ω_1 and ω_2 meet for the first time at \underline{b}_1 , share b_1, \dots, b_n , and split at \bar{b}_n (ω_1 and ω_2 may share a bond again after \bar{b}_n). This means that, together with $q(u)q(v)q(x-y)q(x-z)$ in (3.17), the left-hand side of (3.17) is bounded by the convolution of two non-vanishing bubbles and $\prod_{i=1}^n q(w_i)^2$, where each w_i is the spatial component of $\bar{b}_i - \underline{b}_i$. Using (3.6), we can bound (3.17) by

$$\varepsilon \sum_{\substack{t, s, s' \in \varepsilon \mathbb{Z}_+ \\ \varepsilon < s < s' < t - \varepsilon}} \frac{O(\beta) \varepsilon^2}{(1+s)^{d/2}} [(1-\varepsilon) \vee (\varepsilon \|D\|_\infty)]^{\frac{s'-s}{\varepsilon}} \frac{O(\beta) \varepsilon^2}{(1+t-s')^{d/2}} \leq O(\beta^2) \varepsilon^2,$$

where, as before, ε^3 is used up for the Riemann sum approximation. The above estimate can be improved to $O(\beta^3)$ for oriented percolation, using (1.8). This completes the proof of (3.17). \square

3.2 Asymptotics of p_c^{pe}

In this section, we compute the asymptotics of the critical point for (unoriented) percolation. We follow the strategy in Section 3.1 as closely as possible. However, there are a number of changes due to the fact that we have less control of the lace expansion coefficients. For example, the bounds on the derivative of $\hat{\Pi}_p$ with respect to p are not available in the literature, even though in the unpublished manuscript [12], this derivative is computed. To make this paper self-contained, we avoid the use of the derivative, which causes changes in the proof.

We start with some notation. Let

$$T_p = \sup_{x \in \mathbb{Z}^d} (pD * \tau_p^{*3})(x), \quad T'_p = \sup_{x \in \mathbb{Z}^d} \tau_p^{*3}(x). \quad (3.27)$$

We will use the following bounds:

Lemma 3.1. *Fix $d > 6$. For L sufficiently large, and all $p \leq p_c$,*

$$T_p \leq C\beta, \quad T'_p \leq 1 + C\beta. \quad (3.28)$$

We will defer the proof of Lemma 3.1 to the end of this section.

To compute the asymptotics of $\sum_x \Pi_p(x)$, we use (3.3), together with the bound (see e.g. [4, Proposition 4.1]) that

$$\hat{\pi}_p^{(N)} \leq T_p'(2T_p T_p')^{N \vee 1}. \quad (3.29)$$

Note that Lemma 3.1 together with (3.5) and (3.29) immediately imply that

$$p_c = 1 + O(\beta). \quad (3.30)$$

We now start the proof to improve (3.30) one term further. Together with Lemma 3.1, (3.29) proves that the contribution to $\sum_{N=2}^{\infty} \hat{\pi}_{p_c}^{(N)}$ is $O(\beta^2)$. Thus, we are left to compute $\hat{\pi}_{p_c}^{(0)}$ and $\hat{\pi}_{p_c}^{(1)}$. The goal of this section is to prove that

$$\hat{\pi}_{p_c}^{(0)} = \frac{1}{2} \sum_{n=3}^{\infty} (n-1) D^{*n}(o) + O(\beta^2), \quad \hat{\pi}_{p_c}^{(1)} = (D * D)(o) + \sum_{n=3}^{\infty} n D^{*n}(o) + O(\beta^2). \quad (3.31)$$

Using (3.5) and (3.31), we arrive at (1.14). Thus, we are left to prove (3.31).

We again compute the terms $\hat{\pi}_{p_c}^{(0)}$ and $\hat{\pi}_{p_c}^{(1)}$ separately. For percolation, we denote by $\{w \Leftrightarrow x\}$ the event that w is doubly connected to x . By definition [10], $\hat{\pi}_p^{(0)}(x)$ is given by

$$\hat{\pi}_p^{(0)}(x) = \mathbb{P}_p(o \Leftrightarrow x) - \delta_{o,x}. \quad (3.32)$$

We follow the argument in Section 3.1 as closely as possible. We first use *Russo's formula* (see, e.g., [7]), with the help of the fact that $\mathbb{P}_{p_c}(o \longleftrightarrow \infty) = 0$ under the situation we consider in this paper [2], to write

$$\hat{\pi}_{p_c}^{(0)} = \sum_{x \neq o} \mathbb{P}_{p_c}(o \Leftrightarrow x) = \hat{\pi}_1^{(0)} + (p_c - 1) \sum_{x \neq o} \sum_{(u,v)} D(v-u) \mathbb{P}_p((u,v) \text{ pivotal for } o \Leftrightarrow x), \quad (3.33)$$

for some $p \in (1, p_c)$. We will later identify the first term, and first show that the second term is an error term. Since $p_c = 1 + O(\beta)$, and since, by the *BK inequality* (see, e.g., [7]), we have

$$\begin{aligned} & \sum_{x \neq o} \sum_{(u,v)} D(v-u) \mathbb{P}_p((u,v) \text{ pivotal for } o \Leftrightarrow x) \\ & \leq \sum_{x \neq o} \sum_{(u,v)} D(v-u) \mathbb{P}_p(\{o \longleftrightarrow u\} \circ \{v \longleftrightarrow x\} \circ \{o \longleftrightarrow x\}) \\ & \leq \sum_{x \neq o} \sum_{(u,v)} D(v-u) \tau_p(u) \tau_p(x-v) \tau_p(x) \leq p^{-1} T_p \leq T_p, \end{aligned} \quad (3.34)$$

so that the second term in (3.33) is $O(\beta^2)$. We are left to analyse the first term. We follow the strategy around (3.12), but the details change somewhat.

Let \mathcal{S}_x denote all self-avoiding walk paths from o to x , and order the elements in \mathcal{S}_x in an arbitrary way. Then we can write

$$\hat{\pi}_1^{(0)} = \sum_{x \neq o} \sum_{\substack{\omega_1, \omega_2 \in \mathcal{S}_x \\ \omega_1 \cap \omega_2 = \emptyset \\ \omega_1 \prec \omega_2}} \mathbb{P}_1(\omega_1, \omega_2 \text{ occupied}; E_{\succ}(\omega_1) \cap E_{\prec}(\omega_2; \omega_1)), \quad (3.35)$$

where $E_{\succ}(\omega_1)$ and $E_{\prec}(\omega_2; \omega_1)$ were defined between (3.11) and (3.12). In words, the event $E_{\succ}(\omega_1)$ holds when ω_1 is the lowest occupied self-avoiding walk path from o to x such that there is an occupied bond disjoint path from o to x . The event $E_{\prec}(\omega_2; \omega_1)$ holds when ω_2 is the highest occupied self-avoiding walk path from o to x that is bond disjoint from ω_1 .

Since \mathbb{P}_1 is a product measure, if we ignore $E_{\succ}(\omega_1) \cap E_{\prec}(\omega_2; \omega_1)$, we obtain

$$\sum_{\substack{\omega_1, \omega_2 \in \mathcal{S}_x \\ \omega_1 \cap \omega_2 = \emptyset \\ \omega_1 \prec \omega_2}} \mathbb{P}_1(\omega_1, \omega_2 \text{ occupied}) = \sum_{\substack{\omega_1, \omega_2 \in \mathcal{S}_x \\ \omega_1 \cap \omega_2 = \emptyset \\ \omega_1 \prec \omega_2}} \mathbb{P}_1(\omega_1 \text{ occupied}) \mathbb{P}_1(\omega_2 \text{ occupied}). \quad (3.36)$$

We can then follow the rest of the argument between (3.14) and (3.19) to arrive at the conclusion that

$$\hat{\pi}_1^{(0)} = \frac{1}{2} \sum_{n=3}^{\infty} (n-1) D^{*n}(o) + O(\beta^2). \quad (3.37)$$

Here the factor $1/2$ has the same origin as the one in (3.18), the sum starts at $n = 3$ since the smallest cycle in percolation has length 3, and the factor $n - 1$ comes from the number of places where $x \neq o$ can lie on a loop of size n . This completes the computation of the leading asymptotics of $\hat{\pi}_{p_c}^{(0)}$.

We next derive the asymptotics of $\hat{\pi}_{p_c}^{(1)}$, following the strategy in [17, 18], where the first three coefficients of the asymptotic expansion into powers of $(2d)^{-1}$ of the critical value p_c for nearest-neighbour percolation were computed. The details of the argument are changed considerably compared to [17, 18]. Indeed, since we are only interested in the leading order term, while in [17] the first three coefficients are computed, many terms that need explicit computation in [17, 18] will be error terms for us. On the other hand, since in [17, 18] the asymptotics in nearest-neighbour models for large dimensions are considered, long loops lead to error term in [17, 18], whereas they contribute to the leading asymptotics here. We follow the proof in [17, Section 4.2] as closely and as long as possible, and indicate where the argument diverges.

To define $\hat{\pi}_p^{(1)}$, we need the following definitions. Given a bond configuration and $A \subset \mathbb{Z}^d$, we say that x and y are *connected through* A , and write $x \xleftrightarrow{A} y$, if every occupied path connecting x to y has at least one bond with an endpoint in A . As defined below (3.7), the directed bond (u, v) is said to be *pivotal* for $x \xleftrightarrow{A} y$, if $x \xleftrightarrow{A} u$ and $v \xleftrightarrow{A} y$ occur, and if $x \xleftrightarrow{A} y$ occurs only when $\{u, v\}$ is set occupied. (Note that there is a distinction between the events $\{(u, v) \text{ is pivotal for } x \xleftrightarrow{A} y\}$ and $\{(v, u) \text{ is pivotal for } x \xleftrightarrow{A} y\} = \{(u, v) \text{ is pivotal for } y \xleftrightarrow{A} x\}$.) Let

$$E'(v, x; A) = \{v \xleftrightarrow{A} x\} \cap \{\#(u', v') \text{ occupied \& pivotal for } v \xleftrightarrow{A} x \text{ s.t. } v \xleftrightarrow{A} u'\}. \quad (3.38)$$

Then, by definition [10],

$$\hat{\pi}_p^{(1)} = \sum_x \sum_{(u, v)} p D(v - u) \mathbb{E}_0 \left[\mathbb{1}_{[o \leftrightarrow u]} \mathbb{P}_1(E'(v, x; \tilde{C}_0^{(u, v)}(o))) \right], \quad (3.39)$$

where the sum over (u, v) is a sum over directed bonds. On the right hand side, we use subscripts to identify the different expectations. Thus, the subscripts do *not* refer to the percolation threshold

p . The cluster $\tilde{C}_0^{(u,v)}(o)$ appearing on the right hand side of (3.39) is random with respect to the expectation \mathbb{E}_0 , but $\tilde{C}_0^{(u,v)}(o)$ should be regarded as a *fixed* set inside the probability \mathbb{P}_1 . The latter introduces a second percolation model which depends on the original percolation model via the set $\tilde{C}_0^{(u,v)}(o)$. We refer to the bond configuration corresponding to the j^{th} -expectation as the “level- j ” configuration.

By (3.29),

$$0 \leq \hat{\pi}_p^{(1)} \leq 2T'_p T_p T'_p. \quad (3.40)$$

We will use refinements of this bound in the following.

We first claim that the contribution to (3.39) due to $u \neq o$ is an error term of order $O(\beta^2)$. Indeed, if $u \neq o$ then at level-0 the origin is in a cycle of length at least 3. Standard diagrammatic estimates then allow for the replacement in (3.40) of a factor T'_p by a constant multiple of T_p . This improves the bound (3.40) from $O(\beta)$ to $O(\beta^2)$, by (3.28).

We are left with the contribution to (3.39) due to $u = o$, namely

$$\sum_{x,v} pD(v) \mathbb{E}_0 \left[\mathbb{P}_1(E'(v, x; \tilde{C}_0^{(o,v)}(o))) \right]. \quad (3.41)$$

If $x \notin \tilde{C}_0^{(o,v)}(o)$, then to obtain a non-zero contribution to $\mathbb{P}_1(E'(v, x; \tilde{C}_0^{(o,v)}(o)))$, x must be in an occupied cycle of length at least 3, in level-1 (in the language of [4, Section 3], the sausage containing x must consist of a cycle containing both x and an endpoint of the last pivotal bond for the connection from o to x). In this case, in (3.40), we may again replace a factor T'_p by a constant multiple of T_p , and again this contribution is $O(\beta^2)$. We are left to consider

$$\sum_{x,v} pD(v) \mathbb{E}_0 \left[\mathbb{1}_{[x \in \tilde{C}_0^{(o,v)}(o)]} \mathbb{P}_1(E'(v, x; \tilde{C}_0^{(o,v)}(o))) \right]. \quad (3.42)$$

This is as far as the analogy with the argument in [17, Section 4.2] goes. We now need to adapt the proof there to compute the asymptotics of $\hat{\pi}_{pc}^{(1)}$ when $L \rightarrow \infty$.

If $x \in \tilde{C}_0^{(o,v)}(o)$, and if $v \longleftrightarrow x$, then $v \xleftrightarrow{\tilde{C}_0^{(o,v)}(o)} x$. We next claim that the intersection with the second event in (3.38) leads to an error term. We write

$$\begin{aligned} & I_0[x \in \tilde{C}_0^{(o,v)}(o)] I_1[E'(v, x; \tilde{C}_0^{(o,v)}(o))] \\ &= I_0[x \in \tilde{C}_0^{(o,v)}(o)] I_1[v \longleftrightarrow x] (1 - I_1[\exists(u', v') \text{ occupied \& pivotal for } v \longleftrightarrow x \text{ s.t. } v \xleftrightarrow{\tilde{C}_0^{(o,v)}(o)} u']), \end{aligned}$$

where we write I_0 and I_1 for the indicator functions on levels 0 and 1, respectively. The latter term can be bounded by

$$\sum_{(u', v')} \sum_z I_0[z \in \tilde{C}_0^{(o,v)}(o)] I_1[\{v \longleftrightarrow z\} \circ \{z \longleftrightarrow u'\} \circ \{(u', v') \text{ occupied}\} \circ \{v' \longleftrightarrow x\}], \quad (3.43)$$

which, using the BK inequality, yields a bound of the form

$$\sum_{x,v,z} \sum_{(u', v')} pD(v) \mathbb{P}_0(o \longleftrightarrow x, o \longleftrightarrow z) \tau_p(z - v) \tau_p(u' - z) pD(v' - u') \tau_p(x - v'). \quad (3.44)$$

By the tree-graph inequality [1]

$$\mathbb{P}_o(o \longleftrightarrow x, o \longleftrightarrow z) \leq \sum_y \tau_p(y) \tau_p(x-y) \tau_p(z-y), \quad (3.45)$$

so that we end up with

$$\sum_{x,z,y} (pD * \tau_p)(y) \tau_p(x-y) \tau_p(z-y) \tau_p(z-v) (\tau_p * pD * \tau_p)(x-z) \leq T_p^2 = O(\beta^2), \quad (3.46)$$

which indeed is an error term. Thus, using the identity

$$\{x \in \tilde{C}_0^{(o,v)}(o)\} = \{o \longleftrightarrow x \text{ without using } (o,v)\},$$

we end up with

$$\hat{\pi}_p^{(1)} = \sum_{v,x} pD(v) \tau_p^{(o,v)}(x) \tau_p(x-v) + O(\beta^2), \quad (3.47)$$

where

$$\tau_p^{(o,v)}(x) = \mathbb{P}(o \longleftrightarrow x \text{ without using } (o,v)). \quad (3.48)$$

Note that we can think of $\tau_p^{(o,v)}(x)$ as the two-point function on \mathbb{Z}^d , where the bond (o,v) is removed. We will denote the resulting graph with vertex set \mathbb{Z}^d and edge set $\{\{x,y\} : x,y \in \mathbb{Z}^d, \{x,y\} \neq \{o,v\}\}$ by $\mathbb{Z}_{(o,v)}^d$, so that $\tau_p^{(o,v)}(x)$ is the two-point function on $\mathbb{Z}_{(o,v)}^d$. We will use this observation to compute $\tau_p^{(o,v)}(x)$.

We investigate the main term in the right-hand side of (3.47) further. Russo's formula, together with the BK inequality, yields that

$$\partial_p \tau_p(x) = \sum_{(y,z)} D(z-y) \mathbb{P}((y,z) \text{ pivotal for } o \longleftrightarrow x) \leq (\tau_p * D * \tau_p)(x), \quad (3.49)$$

$$\partial_p \tau_p^{(o,v)}(x) = \sum_{(y,z)} D(z-y) \mathbb{P}((y,z) \text{ pivotal for } o \longleftrightarrow x \text{ in the graph } \mathbb{Z}_{(o,v)}^d) \leq (\tau_p * D * \tau_p)(x). \quad (3.50)$$

Therefore, we obtain that for $p = p_c$,

$$\begin{aligned} \hat{\pi}_{p_c}^{(1)} &= \sum_{x,v} p_c D(v) \tau_{p_c}^{(o,v)}(x) \tau_{p_c}(x-v) + O(\beta^2) \\ &= \sum_{x,v} D(v) \tau_1^{(o,v)}(x) \tau_1(x-v) + O((p_c - 1)T_{p_c}) + O(\beta^2) \\ &= \sum_{x,v} D(v) \tau_1^{(o,v)}(x) \tau_1(x-v) + O(\beta^2), \end{aligned} \quad (3.51)$$

since $p_c = 1 + O(\beta)$. Furthermore, an argument similar to the one for $\hat{\pi}_p^{(0)}$ shows that

$$\tau_1(x) = G(x) + O((G * g * G)(x)), \quad (3.52)$$

$$\tau_1^{(o,v)}(x) = G(x)[1 - \delta_{v,x}] + \delta_{v,x}(D^{*2} * G)(x) + O((G * g * G)(x)) + O(D(v)(G(x-v) - \delta_{v,x})). \quad (3.53)$$

where

$$g(x) = G(x)(D * G)(x). \quad (3.54)$$

We will prove (3.52)–(3.53) in full detail below, and first complete the proof subject to (3.52)–(3.53). Using (3.52)–(3.53), together with the fact that for $u \neq o$, we have $G(u) = (D * G)(u)$, we end up with

$$\begin{aligned} \hat{\pi}_{pc}^{(1)} &= \sum_{x \neq v} G(x) D(v) G(x - v) + \sum_x (D^{*2} * G)(x) D(x) + O(\beta^2) + O((G^{*3} * g)(o)) \\ &= \sum_{n=2}^{\infty} (n-1) D^{*n}(o) + \sum_{n=3}^{\infty} D^{*n}(o) + O(\beta^2) = (D * D)(o) + \sum_{n=3}^{\infty} n D^{*n}(o) + O(\beta^2). \end{aligned} \quad (3.55)$$

by standard random walk estimates.

This completes the proof subject to (3.52)–(3.53) and Lemma 3.1. \square

Proof of (3.52)–(3.53). We start by proving (3.52), and then adapt the argument to prove (3.53). To see (3.52), we recall the arbitrary ordering of the elements in \mathcal{S}_x introduced above (3.35). Then we have that

$$\tau_1(x) = \sum_{\omega \in \mathcal{S}_x} \mathbb{P}_p(\omega \text{ occupied}; F_{\succ}(\omega)), \quad (3.56)$$

where $F_{\succ}(\omega)$ is the event that ω is the lowest occupied path in \mathcal{S}_x . Thus, we can write

$$\tau_1(x) = \sum_{\omega \in \mathcal{S}_x} \mathbb{P}_p(\omega \text{ occupied}) - \sum_{\omega \in \mathcal{S}_x} \mathbb{P}_p(\omega \text{ occupied}; F_{\succ}(\omega)^c). \quad (3.57)$$

The former term equals

$$\delta_{o,x} + (1 - \delta_{o,x}) \sum_{\omega \in \mathcal{S}_x} \prod_{i=0}^{|\omega|-1} D(\omega(i+1) - \omega(i)). \quad (3.58)$$

Clearly, by using inclusion-exclusion on the fact that ω is self-avoiding, as in (2.5), (3.58) equals

$$G(x) + O((G * G)(x)(G(o) - 1)), \quad (3.59)$$

which is a contribution to the error in (3.52) when we note that $G(o) - 1 = (D * G)(o)$. Similarly, the second term in (3.57) is bounded by $O((G * g * G)(x))$ using the fact that there must exist a $u \in \mathbb{Z}^d$ such that there exist bond disjoint occupied paths from o to u , two occupied paths from u to v (of which at least one is non-vanishing) and one from v to x . Thus, by the BK inequality, this term is bounded by

$$\sum_{u,v} G(u) G(v - u) (D * G)(v - u) G(x - v) = (G * g * G)(x).$$

The proof of (3.53) follows the same ideas. In (3.57) and (3.58), we only need to sum over self-avoiding walk paths that do not use the bond (o, v) . When $x = v$, this means that $|\omega| \geq 2$, so that we obtain

$$\tau_1^{(o,v)}(v) = (D^{*2} * G)(v) + O((G * g * G)(v)) + O(D(v)(G(o) - 1)). \quad (3.60)$$

When $x \neq v$, we can use inclusion-exclusion on the fact that the bond (o, v) is not used, and obtain

$$\tau_1^{(o,v)}(x) = \tau_1(x) + O(D(v)G(x-v)), \quad (3.61)$$

and then use (3.52). \square

Proof of Lemma 3.1. We use [10, (5.20)], which states that uniformly in $p \leq p_c$ and for L large enough

$$\hat{\tau}_p(k) \leq \frac{1 + o(1)}{1 - \hat{D}(k)}, \quad (3.62)$$

where $o(1)$ tends to 0 when $L \rightarrow \infty$. We also use the standard bound (see e.g. [4]) that for $x \neq 0$,

$$\tau_p(x) \leq (pD * \tau_p)(x). \quad (3.63)$$

We then follow the proof as in [4]. For T_p , we fix x and extract the term in (3.27) due to the case where every argument of τ_p is o , which is $pD(x) \leq pC\beta$ (see (1.8)). This gives

$$T_p(x) \leq pC\beta + p \sum_{(u,y,z) \neq (x,o,o)} \tau_p(y)\tau_p(z-y)D(u)\tau_p(x+z-u). \quad (3.64)$$

Therefore, by (3.63),

$$T_p \leq pC\beta + 3p^2 \sup_x (D^{*2} * \tau_p^{*3})(x), \quad (3.65)$$

where the factor 3 comes from the 3 factors τ_p whose argument can differ from o . In terms of the Fourier transform, this gives

$$T_p \leq pC\beta + 3p^2 \sup_x \int_{[-\pi,\pi]^d} \hat{D}(k)^2 \hat{\tau}_p(k)^3 e^{-ik \cdot x} \frac{d^d k}{(2\pi)^d} = pC\beta + 3p^2 \int_{[-\pi,\pi]^d} \hat{D}(k)^2 \hat{\tau}_p(k)^3 \frac{d^d k}{(2\pi)^d}, \quad (3.66)$$

where we use that $\hat{\tau}_p(k) \geq 0$ by [1]. We now substitute (3.62) to obtain that, for L large enough,

$$T_p \leq K \left[\beta + \int_{[-\pi,\pi]^d} \frac{\hat{D}(k)^2}{[1 - \hat{D}(k)]^3} \frac{d^d k}{(2\pi)^d} \right], \quad (3.67)$$

for some $K < \infty$. Using (1.8), we estimate the contribution to the integral in (3.67) from $k^2 > \eta/(cL^2)$ by

$$\int_{k \in [-\pi,\pi]^d : k^2 > \eta/(cL^2)} \frac{\hat{D}(k)^2}{[1 - \hat{D}(k)]^3} \frac{d^d k}{(2\pi)^d} \leq \eta^{-3} \int_{[-\pi,\pi]^d} \hat{D}(k)^2 \frac{d^d k}{(2\pi)^d} = O(\beta). \quad (3.68)$$

On the other hand, the contribution due to $k^2 \leq \eta/(cL^2)$ is, again using (1.8), bounded by

$$\int_{k^2 \leq \eta/(cL^2)} \frac{\hat{D}(k)^2}{[1 - \hat{D}(k)]^3} \frac{d^d k}{(2\pi)^d} \leq \int_{k^2 \leq \eta/(cL^2)} (cL^2 k^2)^{-3} \frac{d^d k}{(2\pi)^d} = O(\beta). \quad (3.69)$$

This proves the bound on T_p .

The bound on T'_p is a consequence of $T'_p \leq 1 + 3T_p$. Here the term 1 is due to the contribution to (3.27) where the arguments of the three factors of τ_p in T'_p in (3.27) are equal to o . If at least one of these arguments is nonzero, then we can use (3.63) for the corresponding two-point function. \square

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