Gibbs under stochastic dynamics?

F. den Hollander

EURANDOM, P.O. Box 513
5600 MB Eindhoven, The Netherlands
denhollander@eurandom.tue.nl

28th January 2004

Abstract

This paper is a mini-overview of some recent results on the evolution of Ising-spin systems under Glauber spin-flip dynamics, in particular, the question whether Gibbsianness is preserved, lost or recovered during the dynamics. Examples of all three scenarios are given, with an explanation of what drives the behavior. Some open problems are formulated as well.

MSC 2000. Primary 60K35, 82C22; Secondary 60G60.
Key words and phrases. Ising-spin systems; Glauber spin-flip dynamics; Gibbs measure; bad configuration; preservation, loss or recovery of Gibbsianness.

$*$ This paper is based on a lecture presented at the conference "Gibbs versus non-Gibbs in Statistical Mechanics and Related Fields", 8–10 December 2003, EURANDOM.

1 Introduction

1.1 Main question

The question that we address in this paper is the following. Consider Ising spins on $\mathbb{Z}^d$ evolving under a Glauber spin-flip dynamics from an *initial* Gibbs measure μ towards a *final* Gibbs measure ν ($\neq \mu$). Is it possible that along the way the Gibbs property is

- preserved?
- lost?
- recovered?

The answer to this question turns out to be yes in all three cases. The goal of this paper is to give examples with explanation. We will see that these examples are natural and typical. The results to be described below are taken from van Enter, Fernández, den Hollander and Redig [3], and rely on the work by C. Maes and C. Netocný [8]. For proofs we refer the reader to these papers.

In statistical physics the above three scenarios correspond to preservation, loss or recovery of *temperature* in a *non-equilibrium* setting (where the dynamics can be viewed as a transformation acting on the probability law of the Ising spins). It is well known that such behavior occurs in an *equilibrium* setting (under renormalization-type transformations). For an extensive account of the latter up to 1993, we refer the reader to the review paper by van Enter, Fernández and Sokal [2]. Later developments are described in van Enter [1], Fernández [5], van Enter, Maes and Shlosman [4], and Maes [7].

1.2 Gibbs measures

Let $\Omega = \{-1,+1\}^{\mathbb{Z}^d}$ be the Ising-spin configuration space.

Definition 1.2.1 *A probability measure ρ on Ω is Gibbs if it has the DLR-prpperty (Dobrushin-Lanford-Ruelle), i.e.,*

$$\rho\left(\sigma_\Lambda | \eta_{\Lambda^c}\right) = \frac{1}{Z_{\eta_{\Lambda^c}}} e^{-H(\sigma_\Lambda \vee \eta_{\Lambda^c})} \qquad \forall\, \sigma, \eta \in \Omega,\ \forall\, \Lambda \subset\subset \mathbb{Z}^d, \tag{1.2.1}$$

with a Hamiltonian $H\colon \Omega \mapsto \mathbb{R}$ of the form

$$H(\omega) = \sum_{A \subset\subset \mathbb{Z}^d} U_A(\omega), \qquad \omega \in \Omega, \tag{1.2.2}$$

where $(U_A)_{A\subset\subset\mathbb{Z}^d}$ are interaction potentials satisfying

$$\sup_{\omega\in\Omega} \sum_{\substack{A\subset\subset\mathbb{Z}^d \\ A\ni x}} |U_A(\omega)| < \infty \qquad \forall\, x \in \mathbb{Z}^d. \tag{1.2.3}$$

Here, $\subset\subset$ stands for finite subset, σ_Λ and η_{Λ^c} are the rectrictions of σ to Λ and η to Λ^c, respectively, $\sigma_\Lambda \vee \eta_{\Lambda^c}$ denotes their joining, $Z_{\eta_{\Lambda^c}}$ is the partition sum in Λ given η outside Λ, while $U_A(\omega)$ depends on ω_A only. The uniform absolute summability condition in (1.2.3) implies the *uniform non-nullness* and the *quasi-locality* that are characteristic of Gibbs measures.

1.3 Glauber spin-flip dynamics

We consider the following situation:

1. At time $t = 0$, start from a translation-invariant Gibbs measure μ with finite-range interaction and with inverse temperature β_μ.
2. At times $t > 0$, run a Glauber dynamics with spin-flip rates that are finite-range, translation-invariant and strictly positive. This dynamics has as equilibrium *at least one* translation-invariant (reversible) Gibbs measure ν with finite-range interaction and with inverse temperature β_ν.

Let μ_t be the measure evolved at time t. Then in good situations we have

$$\mu_0 = \mu \quad \text{and} \quad \mu_t \Rightarrow \nu \text{ as } t \to \infty. \tag{1.3.1}$$

A priori, ν may depend on μ. Below we will only consider *high-temperature dynamics*, i.e., $0 \leq \beta_\nu \ll 1$, in which case μ_t converges to a unique ν. We are interested in finding out under what conditions

$$\mu_t \text{ is Gibbs for all/some/no } t > 0. \tag{1.3.2}$$

Inverse temperature can be viewed as a norm for the interaction.

1.4 Gibbs versus non-Gibbs

A necessary and sufficient condition for a probability measure not to be Gibbs is the *existence* of a *bad configuration.*

Definition 1.4.1 *A configuration $\eta \in \Omega$ is called bad for a probability measure ρ on Ω if there exist $\epsilon > 0$ and $x \in \mathbb{Z}^d$ such that:*

$$\begin{gathered} \forall\, \Lambda \ni x,\ \Lambda \subset\subset \mathbb{Z}^d \quad \exists\, \Gamma \supset \Lambda,\ \Gamma \subset\subset \mathbb{Z}^d \quad \exists\, \xi, \zeta \in \Omega : \\ \left|\, \rho_\Gamma\left(\sigma(x)\,|\,\eta_{\Lambda\setminus\{x\}} \vee \xi_{\Gamma\setminus\Lambda}\right) - \rho_\Gamma\left(\sigma(x)\,|\,\eta_{\Lambda\setminus\{x\}} \vee \zeta_{\Gamma\setminus\Lambda}\right) \right| > \epsilon, \end{gathered} \tag{1.4.1}$$

where ρ_Γ is ρ restricted to Γ.

The inequality in (1.4.1) signals the failure of quasi-locality in η.

The set of bad configurations has ρ-measure 0 or 1 when ρ is ergodic.

2 Main theorems and a criterion for Gibbsianness

Below, when we write "for all ν" we mean "for all Glauber spin-flip dynamics whose invariant measure is ν".

2.1 Main theorems

Theorem 2.1.1 *For all μ, ν there exists $t_0 = t_0(\mu, \nu) > 0$ such that μ_t is Gibbs for all $t \in [0, t_0)$.*

This says that *Gibbsianness is preserved for small times.*

Intuition: The set of sites where a spin-flip has occurred consists of "small islands" that are far apart in a "sea" of sites where no spin-flip has occurred. Consequently, sites that are far apart have disjoint histories with a high probability, implying (in a percolation-type fashion) that there are no bad configurations.

Theorem 2.1.2 *For all μ, ν such that $0 \leq \beta_\mu, \beta_\nu \ll 1$: μ_t is Gibbs for all $t \geq 0$.*

This says that *Gibbsianness is preserved for all times when both μ, ν have a high temperature.*

Intuition: The time-evolved measure stays in the regime where no phase transition occurs, implying that there are no bad configurations.

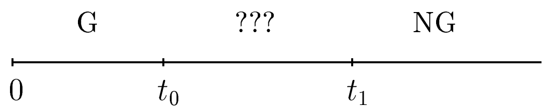
Theorem 2.1.3 *Assume that:*
(i) $1 \ll \beta_\mu < \infty$, with μ the plus-phase of the standard Ising Hamiltonian with magnetic field h.
(ii) $0 \leq \beta_\nu \ll 1$.
Under these assumptions:
I. If $h = 0$, then there exists $0 < t_1 = t_1(\mu, \nu) < \infty$ such that μ_t is not (!) Gibbs for all $t \in [t_1, \infty)$.
II. If $h > 0$, then there exists $0 < t_2 = t_2(\mu, \nu) < \infty$ such that μ_t is Gibbs for all $t \in [t_2, \infty)$.
III. Suppose that $d \geq 3$. If $0 < h \ll 1$, then there exist $0 < t_3 = t_3(\mu, \nu) < t_4 = t_4(\mu, \nu) < \infty$ such that μ_t is not (!) Gibbs for all $t \in [t_3, t_4)$.

This says that *Gibbsianness may get lost and may get recovered when the system is heated up from a low temperature to a high temperature.* Apparently, the magnetic field plays an important role in determining which scenario occurs.

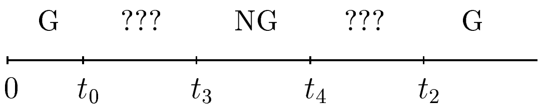
Intuition: Not immediate. See Section 3.

The result in Theorem 2.1.3 is quite remarkable, because the regime of exponentially fast convergence to a high-temperature Gibbs measure a priori seems unproblematic.

Figure 1 summarizes the statements in Theorem 2.1.3 (in combination with those in Theorems 2.1.1 and 2.1.2). We believe that picture (b) holds for all $d \geq 2$ and $h > 0$, but the proof requires the stated restrictions.



(a)



(b)

Fig. 1: (a) $h = 0$; (b) $d \geq 3$, $0 < h \ll 1$.

2.2 A criterion for Gibbsianness

Let σ_t be the spin configuration at time t. Consider the pair

$$(\sigma_0, \sigma_t), \tag{2.2.1}$$

and let $\hat{\mu}_t$ denote its joint distribution on $\Omega \times \Omega$. The left marginal is μ, the right marginal is μ_t.

Suppose that $\hat{\mu}_t$ is Gibbs (this is not obvious!). Then it has joint Hamiltonian $H_t(\sigma, \eta)$ given by

$$e^{-H_t(\sigma,\eta)} = e^{-H_\mu(\sigma)} \, p_t(\sigma, \eta), \tag{2.2.2}$$

or

$$H_t(\sigma, \eta) = H_\mu(\sigma) - \log p_t(\sigma, \eta), \tag{2.2.3}$$

where $p_t(\sigma, \eta)$ is the transition kernel of the spin-flip dynamics. Here, the last term has to be *properly interpreted* in the sense of a formal sum of t-dependent interaction potentials, like in (1.2.2) (this is not obvious!). For $\eta \in \Omega$, let

$$\mathcal{G}\,(H_t^\eta) \tag{2.2.4}$$

be the set of Gibbs measures associated with the Hamiltonian $H_t^\eta(\sigma) = H_t(\sigma, \eta)$, where η is fixed and σ is running. A key criterion in our analysis is the following:

Proposition 2.2.1 (Fernández and Pfister [6]) *Fix $t \geq 0$. Under the assumption that $\hat{\mu}_t$ is Gibbs:*
1. If $|\mathcal{G}(H_t^\eta)| = 1$ for all $\eta \in \Omega$, then μ_t is Gibbs.
2. For monotone interactions, if $|\mathcal{G}(H_t^\eta)| \geq 2$ for some $\eta \in \Omega$, then η is a bad configuration for μ_t, and hence μ_t is not Gibbs.

In part 2, the non-Gibbsianness comes from the presence of a phase transition in σ for fixed η. (If we look at the marginal μ_t at time t, then we are summing out over the marginal μ_0 at time 0.) The restriction to monotone interactions is believed to be redundant.

The idea is to use the above criterion for a *high-temperature dynamics* ($0 \leq \beta_\nu \ll 1$), for which it is possible to make sense of the *dynamical part* of H_t^η, i.e., the last term in the right-hand side of (2.2.3), as is shown in Maes and Netocný [8] with the help of a *space-time cluster expansion.*

For the proof of Theorems 2.1.2 and 2.1.3 we refer to van Enter, Fernández, den Hollander and Redig [3]. In Section 3 we consider the case $\beta_\nu = 0$, i.e., *infinite-temperature dynamics.* It turns out that this case already exhibits all the relevant features.

3 Sketch of proof for $\beta_\nu = 0$

3.1 Joint Hamiltonian

Consider the evolution of μ under a *product dynamics* where each spin flips independently at rate 1. Since under this dynamics the conditional probability of the event $\{\sigma_t(x) = \eta(x)\}$ given the event $\{\sigma_0(x) = \sigma(x)\}$ is

$$\begin{array}{ll} \frac{1}{2}(1+e^{-2t}) & \text{if } \sigma(x) = \eta(x), \\ \frac{1}{2}(1-e^{-2t}) & \text{if } \sigma(x) \neq \eta(x), \end{array} \tag{3.1.1}$$

the joint Hamiltonian in (2.2.3) is given by

$$H_t(\sigma,\eta) = H_\mu(\sigma) - \sum_x [h_t \eta(x)]\, \sigma(x) \tag{3.1.2}$$

with

$$h_t = \frac{1}{2} \log \frac{1+e^{-2t}}{1-e^{-2t}}. \tag{3.1.3}$$

(Note that constants do not matter in the Hamiltonian.) The *dynamical magnetic field* h_t is strictly decreasing in t with $h_0 = \infty$ and $h_\infty = 0$, corresponding to full correlation between σ and η at time $t = 0$, respectively, no correlation at time $t = \infty$.

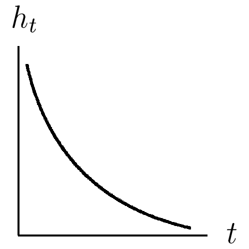


Fig. 2: $t \mapsto h_t$.

3.2 High- and low-temperature initial measure

• $0 < \beta_\mu \ll 1$:

H_μ has no phase transition. Since, for any η and t, H_t^η differs from H_μ only in the single-site interaction, it also has no phase transition. Hence Proposition 2.2.1, part 1, applies.

• $1 \ll \beta_\mu < \infty$:

We consider the plus-phase of the standard Ising Hamiltonian with magnetic field h, i.e.,

$$H_\mu(\sigma) = -\beta \sum_{x \sim y} \sigma(x)\sigma(y) - h \sum_x \sigma(x) \tag{3.2.1}$$

with $\beta = \beta_\mu$. Then the joint Hamiltonian in (3.1.2) reads

$$H_t^\eta(\sigma) = -\beta \sum_{x\sim y} \sigma(x)\sigma(y) - \sum_x [h + h_t\eta(x)]\,\sigma(x). \tag{3.2.2}$$

There are four subcases:

3.2.1 μ_t Gibbs for small t

For small t, h_t is large and, for given η, forces σ in the direction of η. Rewrite (3.2.2) as

$$H_t^\eta(\sigma) = \sqrt{h_t}\,\widetilde{H}_t^\eta(\sigma) \tag{3.2.3}$$

with

$$\widetilde{H}_t^\eta(\sigma) = -\frac{\beta}{\sqrt{h_t}} \sum_{x\sim y} \sigma(x)\sigma(y) - \sum_x \left[\frac{h}{\sqrt{h_t}} + \sqrt{h_t}\,\eta(x)\right] \sigma(x). \tag{3.2.4}$$

For small t, the last term in the right-hand side of (3.2.4) is the dominant interaction (independently of η, not of β, h). Therefore $\widetilde{H}_t^\eta$ has the unique ground state $\sigma = \eta$. Consequently, H_t^η in (3.2.3) satisfies the Dobrushin condition for large enough inverse temperature $\sqrt{h_t}$. Hence Proposition 2.2.1, part 1, applies.

3.2.2 $h > 0$: μ_t Gibbs for large t

For large t, h_t is small and cannot overrule the effect of $h > 0$. Rewrite (3.2.2) as

$$H_t^\eta(\sigma) = \sqrt{\beta}\,\widetilde{H}_t^\eta(\sigma) \tag{3.2.5}$$

with

$$\widetilde{H}_t^\eta(\sigma) = -\sqrt{\beta} \sum_{x\sim y} \sigma(x)\sigma(y) - \sum_x \left[\frac{h}{\sqrt{\beta}} + \frac{h_t}{\sqrt{\beta}}\,\eta(x)\right] \sigma(x). \tag{3.2.6}$$

For large t, the middle term in the right-hand side of (3.2.6) is the dominant interaction (independently of η, not of β, h). Therefore $\widetilde{H}_t^\eta$ has the unique ground state $\sigma \equiv h/|h|$. Consequently, H_t^η in (3.2.5) satisfies the Dobrushin condition for large enough inverse temperature $\sqrt{\beta}$. Hence Proposition 2.2.1, part 1, applies.

3.2.3 $h = 0$: μ_t not Gibbs for large t

Pick $\eta = \eta_a$, the alternating configuration. For large t, h_t is small and

$$H_t^{\eta_a}(\sigma) = -\beta \sum_{x\sim y} \sigma(x)\sigma(y) - h_t \sum_x \eta_a(x)\sigma(x) \tag{3.2.7}$$

has two ground states, $\sigma \equiv +1$ and $\sigma \equiv -1$, because the last term in (3.2.7) is neutral in selecting them. By an application of Pirogov-Sinai theory, it follows that $H_t^{\eta_a}$ has a phase transition for large enough inverse temperature β, so η_a is a bad configuration. Hence Proposition 2.2.1, part 2, applies.

3.2.4 $d \geq 3$, $0 < h \ll 1$: μ_t not Gibbs for intermediate t

A rough argument goes as follows. For intermediate t, h and h_t are of the same order (both small). Therefore we can find a configuration η^* such that the external magnetic field in (3.2.2),

$$x \mapsto h + h_t \eta^*(x), \tag{3.2.8}$$

is "zero on average", i.e., its average over a large box tends to zero as the box tends to $\mathbb{Z}^d$ (for instance, $h_t/h = 2$ and η^* is a periodic repetition of $+1-1-1-1$). In that case $H_t^{\eta^*}$ has two ground states, $\sigma \equiv +1$ and $\sigma \equiv -1$, because the magnetic field is neutral in selecting them. By an application of Pirogov-Sinai theory, it follows that $H_t^{\eta^*}$ has a phase transition for large enough inverse temperature β, so η^* is a bad configuration. Hence Proposition 2.2.1, part 2, applies.

To make the above argument precise, we need the following. As shown by Zahradnik [10], in $d \geq 3$ the *random field Ising model* at large enough inverse temperature has a phase transition when the random field is small and zero on average. This phase transition occurs for a set of random fields with measure 1 under the Bernoulli measure. From this we conclude that η^* can be drawn from a set of configurations with measure 1 under the Bernoulli measure (not μ_t!). This in turn guarantees that η^* exists.

4 Open problems

Some challenges for the future are:

1. What is the physical origin of the transition from Gibbs to non-Gibbs? In van Enter, Fernández, den Hollander and Redig [3] it is suggested that a *nature versus nurture* transition may be responsible, namely, a crossover from a situation where fluctuations are dominated by the initial measure (small times) to a situation where fluctuations are dominated by the dynamics (large times). This suggestion has not yet been properly investigated.

2. In the case where μ_t is Gibbs for all $t \geq 0$, what can we say about the *trajectory* $t \mapsto H_{\mu_t}$? For instance, how does H_{μ_t} converge to H_ν? The space-time cluster expansion developed in Maes and Netocny [8] should serve as the starting point for such an analysis.

3. What about $1 \ll \beta_\nu < \infty$, i.e., *low-temperature dynamics*? Here, one of the main obstacles is to make sense of the dynamical part of H_t^η, i.e., the last term in the right-hand side of (2.2.3). Probably many different scenarios are possible, and *metastability phenomena* are to be expected.

4. Is it true that μ_t is *weakly Gibbs* for all $t \geq 0$ always? Or even *almost Gibbs* for all $t \geq 0$ always? [1]

[1] Weakly Gibbs means that μ_t has an absolutely summable interaction potential for a set of configurations of measure 1 w.r.t. μ_t. Almost Gibbs means that the set of bad configurations for μ_t has measure 0 w.r.t. μ_t. This is stronger than weakly Gibbs.

5. What about other types of dynamics? It was shown by Le Ny and Redig [9] that Gibbsianness is preserved for short times under an *arbitrary reversible local dynamics*, i.e., Theorem 2.1.1 generalizes fully. Does a similar type of behavior as described in Theorems 2.1.2 and 2.1.3 hold for the lattice gas under Kawasaki dynamics at high temperature? Infinite-temperature Kawasaki dynamics corresponds to the exclusion process. Thus, we would first need to understand the evolution of Gibbs measures under the exclusion process and we would afterwards need to carry out an expansion for weak attraction on top of exclusion. However, the trouble is that exclusion is not a weak interaction itself. "Glassy dynamics" is an even greater challenge.

6. What about spins taking values in a *continuous* space? This question is addressed in the paper by Dereudre and Roelly appearing elsewhere in this volume, where for interacting diffusions a generalization of Theorems 2.1.1 and 2.1.2 is achieved. Phase transitions for continuous systems are typically hard to handle. A criterion like Proposition 2.2.1 is so far absent in the continuous setting. As is clear from Section 3, this criterion is the key tool in establishing the results described in Section 2.

References

[1] A.C.D. van Enter, The renormalization-group peculiarities of Griffiths and Pearce: What have we learned?, in: *Proceedings Mathematical Results in Statistical Mechanics, Marseille, 1998* (eds. S. Miracle-Sole, J. Ruiz and V. Zagrebnov), World Scientific, Singapore, 1999, pp. 509–526.

[2] A.C.D. van Enter, R. Fernández and A.D. Sokal, Regularity properties and pathologies of position-space renormalization-group transformations: Scope and limitations of Gibbsian theory, J. Stat. Phys. 72 (1993) 879–1167.

[3] A.C.D. van Enter, R. Fernández, F. den Hollander and F. Redig, Possible loss and recovery of Gibbsianness during the stochastic evolution of Gibbs measures, Commun. Math. Phys. 226 (2002) 101–130.

[4] A.C.D. van Enter, C. Maes and S.B. Shlosman, Dobrushin's program: Weakly Gibbs and almost Gibbs random fields, in: *On Dobrushin's way, From Probability to Statistical Mechanics* (eds. R. Minlos, Yu. Suhov and S.B. Shlosman), Am. Math. Transl. 198 (2000) 159–170.

[5] R. Fernández, Measures on lattice systems, in: *Proceedings IUPAP XX, Paris, 1998*, Physica A263 (1999) 117–130.

[6] R. Fernández and C.E. Pfister, Global specifications and non-quasilocality of projections of Gibbs measures, Ann. Probab. 25 (1997) 1284–1315.

[7] C. Maes, Weakly Gibbsian measures: How strong?, in: *Proceedings XIII-th International Congress on Mathematical Physics, London, 2000* (eds. A. Fokas, A. Grigoryan, T. Kibble and B. Zegarliński), International Press, 2001, pp. 235–242.

[8] C. Maes and C. Netocný, Space-time expansions for weakly interacting particle systems, J. Phys. A35 (2002) 3053–3077.

[9] A. Le Ny and F. Redig, Short time conservation of Gibbsianness under local stochastic evolutions, J. Stat. Phys. 109 (2002) 1073–1090.

[10] M. Zahradnik, On the structure of low-temperature phases in three-dimensional spin models with random impurities: A general Pirogov-Sinai approach, in: *Phase Transitions: Mathematics, Physics, Biology* (ed. R. Kotecký), World Scientific, Singapore, 1992, pp. 225–237.