Analytic computation schemes for the discrete-time bulk service queue

A.J.E.M. Janssen[a] & J.S.H. van Leeuwaarden[b]

[a] Digital Signal Processing Group
Philips Research, WO-02
5656 AA Eindhoven, The Netherlands
e-mail: a.j.e.m.janssen@philips.com

[b] EURANDOM
P.O. Box 513, 5600 MB Eindhoven, The Netherlands
e-mail: j.s.h.v.leeuwaarden@tue.nl

January 2004

Abstract

In commonly used approaches for the discrete-time bulk service queue, the stationary queue length distribution follows from the roots inside or outside the unit circle of a characteristic equation. We present analytic representations of these roots in the form of sample values of periodic functions with analytically given Fourier series coefficients, making these approaches more transparent and explicit. The resulting computational scheme is easy to implement and numerically stable. We also discuss a method to determine the roots by applying successive substitutions to a fixed point equation. We outline under which conditions this method works, and compare these conditions with those needed for the Fourier series representation. Finally, we present a solution for the stationary queue length distribution that does not depend on roots. This solution is explicit and well-suited for determining tail probabilities up to a high accuracy, as demonstrated by some numerical examples.

keywords: discrete-time bulk service, multi-server, roots, stationary distribution, Szegö curve, Spitzer's identity.

1 Introduction and motivation

During the last two decades, discrete-time queueing models have been applied to model digital communication systems such as multiplexers and packet switches. In this field, the multi-server or bulk service queue fulfills a key role due to its wide range of applications, among which ATM switching elements [7], data transmission over satellites [31], high performance serial busses [24], and cable access networks [16].

The discrete-time bulk service queue, to be referred to as the $D^A/D^s/1$ queue, is defined by the recursion

$$X_{n+1} = \max\{X_n - s, 0\} + A_n. \tag{1}$$

Here, time is assumed to be slotted, X_n denotes the queue length at the beginning of slot n, A_n denotes the number of new arriving customers during slot n, and s denotes the fixed number of customers that can be served during one slot. The sequence of A_n is assumed to be i.i.d. according to a discrete random variable A with probability generating function (pgf) $A(z)$. Without loss of generality we assume throughout that $\mathbb{P}(A = 0) > 0$. The pgf of the stationary queue length in the $D^A/D^s/1$ queue was first derived by Bruneel & Wuyts [8], although the same pgf occurs in earlier work on the $D/G/1$ queue by Servi [27] and on bulk queues by Powell [26]. The solution requires finding the roots of $z^s = A(z)$ within the unit circle. Zhao & Campbell [32] presented a full solution for the stationary queue length distribution in terms of the roots of $z^s = A(z)$ outside the unit circle, assuming that $A(z)$ is a polynomial. Chaudhry & Kim [10] used the same technique along with some numerical work.

The technique of finding roots to complete a transform has become a classic one in queueing theory. It started from the analysis of the $M/D/s$ queue by Crommelin [13], whose solution required finding the roots within the unit circle of $z^s = e^{\lambda(z-1)}$ for some value $\lambda < s$. Through the years, root-finding turned out to be particularly important in the theory of bulk queues, originating from the work of Bailey [5] and Downton [17], who consider a bulk service queue with Poisson arrivals. For an overview on bulk queues we refer to Powell [26] and Chaudhry & Templeton [11].

Initially, the potential difficulties of root-finding were considered to be a slur on the unblemished transforms, since the determination of the roots can be numerically hazardous and the roots themselves have no probabilistic interpretation. However, Chaudhry and others [9] have made every effort to dispel the scepticism towards root-finding in queueing theory. They emphasize that root-finding in queueing is well-structured, in the sense that the roots are distinct for most models and that their location is well-predictable, so that numerical problems are not likely to occur.

While in general this is true for the moments of the stationary queue length, for determining the distribution itself, dependence on the roots might cause some problems, in particular for tail probabilities. In Chaudhry & Kim [10] a comparison is made between using the roots of $z^s = A(z)$ either inside or outside the unit circle. The performance of both approaches, though, heavily depends on the model parameters. Since it is therefore difficult to give a fair comparison, we choose to stress their common weakness: their performance inherently depends on how precise the roots of $z^s = A(z)$ are determined. Any deviation of the numerically determined roots from their true values results in errors in the computed probabilities.

The main purpose of this paper is to present an analytical rather than a numerical framework for dealing with the $D^A/D^s/1$ queue. In particular, we will present explicit expressions for the roots of $z^s = A(z)$ and the stationary queue length distribution.

Under some mild conditions, we show that the roots of $z^s = A(z)$, both inside and outside the unit circle, can be represented as sample values of a periodic function with analytically given Fourier coefficients. In this way, the roots are no longer implicitly defined, and one can determine the roots as accurately as one wishes in a totally transparent way. We also show the Fourier series representation of the roots can also be applied in case of the discrete-time $G/G/1$ queue.

Another way to determine the roots while maintaining transparency, results from applying successive substitution to a fixed-point equation. This idea originates from the work of Harris et al. [20] on root-finding for the continuous-time $G/E_k/1$ queue, and was presented more formally by Adan & Zhao [4] who distinguished a class of continuous distributions for which the method works. In this paper we further investigate the method for finding the roots of

$z^s = A(z)$ for discrete distributions A. We present necessary conditions for the method to work and compare these to the conditions needed for the Fourier series representation of the roots.

In deriving explicit formulas for the stationary queue length distribution, we first note that the $D^A/D^s/1$ queue falls within the class of the $G/G/1$ queue. For the $G/G/1$ queue, the Laplace-Stieltjes transform of the stationary waiting time follows from Spitzer's identity (see e.g. [12]). We derive, using a Wiener-Hopf approach as in [12] to derive Spitzer's identity, a root-free expression for the pgf of the stationary queue length in the $D^A/D^s/1$ queue. The pgf explicitly involves an infinite series of convolutions of A, and can be easily inverted, yielding root-free expressions for the stationary queue length distribution. For the convolutions we present asymptotic expressions.

In conclusion, our goals in this paper are:

(i) To give a broad description of the discrete-time bulk service queue.

(ii) To present analytic formulas for the roots of $z^s = A(z)$.

(iii) To present a root-free representation of the stationary queue length distribution.

(iv) To demonstrate the numerical stability of the proposed method, in particular for tail probabilities.

2 The standard approach

In the above described $D^A/D^s/1$ queue, the stationary queue length X, defined as

$$x_j = \mathbb{P}(X = j) = \lim_{n\to\infty} \mathbb{P}(X_n = j), \quad j = 0, 1, 2, \ \dots \ , \tag{2}$$

exists under the assumption that $E(A) = \mu_A < s$. From the balance equations it then follows that the pgf of X is given by (see e.g. [7])

$$X(z) = \frac{A(z)\sum_{j=0}^{s-1} x_j(z^s - z^j)}{z^s - A(z)}, \tag{3}$$

which is assumed to be an analytic function in a disk $|z| \leq 1 + \epsilon$ with $\epsilon > 0$. The s unknowns $x_0, \dots, x_{s-1}$ in the numerator of (3) can be determined by consideration of the zeros of the denominator of (3) that lie in the closed unit disk (see e.g. [5, 32]). With Rouché's theorem (see [12]), it can be shown that there are exactly s of these zeros. Thus by analyticity, the numerator of $X(z)$ should vanish at each of the zeros, yielding s equations. One of the zeros equals 1, and leads to a trivial equation. The normalization condition $X(1) = 1$ provides an additional equation.

We can, however, eliminate $x_0, \dots, x_{s-1}$ from (3). Denoting the s roots of $z^s = A(z)$ in $|z| \leq 1$ by $z_0 = 1, z_1, \dots, z_{s-1}$, (3) can be written as (see e.g. [7, 27])

$$X(z) = \frac{A(z)(z-1)(s-\mu_A)}{z^s - A(z)} \prod_{k=1}^{s-1} \frac{z - z_k}{1 - z_k}. \tag{4}$$

When $A(z)$ is a polynomial of degree $n > s$, (3) can be written as

$$X(z) = A(z) \prod_{k=s}^{n-1} \frac{1 - z_k}{z - z_k}, \quad |z| \leq 1, \tag{5}$$

where $z_s, z_{s+1}, \ldots, z_{n-1}$ are the $n - s$ roots of $z^s = A(z)$ outside the unit circle. Expressing the pgf of the stationary queue length for the $D^A/D^s/1$ queue in terms of the roots outside the unit circle has been suggested by Zhao & Campbell [32], as already said, although the idea stems from much earlier work on bulk service queues by Bailey [5] and Downton [17].

2.1 Using roots inside the unit circle to compute x_j

We will now show how the stationary queue length distribution follows from (4). Let a_j denote the probability that A equals j, and recall that $a_0 > 0$.

From (4) we see that

$$X(z)(z^s - A(z)) =: cA(z)P(z), \tag{6}$$

where

$$c = \frac{s - \mu_A}{\prod_{k=1}^{s-1}(1 - z_k)}, \quad P(z) = \prod_{k=0}^{s-1}(z - z_k) = \sum_{j=0}^{s} p_j z^j. \tag{7}$$

Matching coefficients then gives for $j = 0, 1, \ldots$

$$x_j = \frac{1}{a_0} \sum_{n=1}^{j} (\delta_{n,s} - a_n) x_{j-n} - \frac{c}{a_0} \sum_{n=0}^{\min\{j,s\}} a_{j-n} p_n, \tag{8}$$

where $\delta_{n,s} = 1$ if $n = s$ and 0 otherwise.

2.2 Using roots outside the unit circle to compute x_j

Starting from (5), the following partial fraction expansion can be applied:

$$W(z) := \prod_{k=s}^{n-1} \frac{1 - z_k}{z - z_k} = \sum_{i=s}^{n-1} \frac{r_i}{z - z_i}, \tag{9}$$

where

$$\begin{aligned} r_i &= \lim_{z \to z_i} (z - z_i) W(z) \\ &= \frac{\prod_{k=s}^{n-1}(1 - z_k)}{\prod_{k=s, k \neq i}^{n-1}(z_i - z_k)}, \quad i = s, \ldots, n-1. \end{aligned} \tag{10}$$

When we rewrite (9) as

$$W(z) \;\; = \;\; -\sum_{k=0}^{\infty} \sum_{i=s}^{n-1} \left(\frac{r_i}{z_i}\right) \left(\frac{1}{z_i}\right)^k z^k, \tag{11}$$

it can be easily seen that the stationary queue length distribution is given by

$$x_j = -\sum_{k=0}^{j} a_k \sum_{i=s}^{n-1} \left(\frac{r_i}{z_i}\right) \left(\frac{1}{z_i}\right)^{j-k}, \quad j = 0, 1, 2, \ldots. \tag{12}$$

3 Analytic methods for finding the roots

We now pay further attention to the roots of $z^s = A(z)$. We first present an explicit expression for each of the roots as a Fourier series. Next, we elaborate on finding the roots using a fixed point iteration. We also point out how the conditions needed for the Fourier series representation and the fixed point iteration are related.

3.1 Fourier series representation

The roots of $z^s = A(z)$ lie on, what is called in [22], the generalized Szegö curve, defined by

$$\mathcal{S}_{A,s} := \{z \in \mathbb{C} \mid |z| \leq 1,\ |A(z)| = |z|^s\}. \tag{13}$$

For the notions used below from complex function theory we refer to [18, 28]. We impose the following condition:

Condition 3.1. *$\mathcal{S}_{A,s}$ is a Jordan curve with 0 in its interior, and $A(z)$ is zero-free on and inside $\mathcal{S}_{A,s}$.*

Recall that $a_0 > 0$ so that we have $|A(z)| > |z|^s$ for z in the interior of $\mathcal{S}_{A,s}$. Condition 3.1 is geometric in nature, and can be visually checked using some standard software package. A useful geometric formulation equivalent with Condition 3.1 is as follows:

Lemma 3.2. *Condition 3.1 is satisfied if and only if there is a Jordan curve J with $\mathcal{S}_{A,s}$ in its interior such that $A(z)$ is zero-free on and inside J while $|A(z)| < |z|^s$ on J.*

The proof that Condition 3.1 implies the existence of a J as in Lemma 3.2 uses continuity of A on $\mathcal{S}_{A,s}$ and some basic considerations of Jordan curve theory. The proof of the reverse implication can be based on the considerations in the proof of Lemma 3.3. For brevity we omit the details.

To present an equivalent form of Condition 3.1 of more analytic nature, we introduce the short-hand notation $C_{z^j}[f(z)]$ for the coefficient of z^j in $f(z)$. We have the following result:

Lemma 3.3. *Condition 3.1 is satisfied if and only if the coefficients $C_{z^{l-1}}[A^{l/s}(z)]$ decay exponentially in l.*

Proof. Assume that Condition 3.1 holds. Letting J as in Lemma 3.2 we see that we can define an analytic root $A^{1/s}(z)$ for z on and inside J that is positive at $z = 0$. We thus have by Cauchy's theorem

$$C_{z^{l-1}}[A^{l/s}(z)] = \frac{1}{2\pi i} \int_{z\in J} \frac{A^{l/s}(z)}{z^l}\, dz, \quad l = 1, 2, \ \ldots\ . \tag{14}$$

Since $|A(z)| < |z|^s$ for $z \in J$, it follows that

$$|C_{z^{l-1}}[A^{l/s}(z)]| \leq \frac{1}{2\pi} \text{length}(J) \Big(\max_{z\in J} \Big| \frac{A(z)}{z^s} \Big|^{1/s} \Big)^l, \tag{15}$$

and this decays exponentially, as required.

Now assume that $C_{z^{l-1}}[A^{l/s}(z)]$ decays exponentially. We shall sketch the proof that Condition 3.1 is valid; full details can be found in [22], proof of Lemma 4.1. We consider for w in a neighbourhood of 0 the equation

$$zA^{-1/s}(z) = w, \tag{16}$$

where we have taken in a neighbourhood of $z = 0$ the root $A^{-1/s}$ of A that is positive at $z = 0$ (recall $a_0 > 0$). By the Lagrange inversion theorem (see e.g [30], p. 133), the solution $z_0(w)$ of (16) has the power series representation

$$z_0(w) = \sum_{l=1}^{\infty} c_l w^l, \tag{17}$$

for w in a neighbourhood of 0 in which

$$c_l = \frac{1}{l!}\Big(\frac{d}{dz}\Big)^{l-1}\Big(\frac{z}{zA^{-1/s}(z)}\Big)^l\Big|_{z=0} = \frac{1}{l}C_{z^{l-1}}[A^{l/s}(z)]. \tag{18}$$

By assumption, we have that $c_l \to 0$ exponentially, whence the power series in (17) for $z_0(w)$ has a radius of convergence $R > 1$. It follows then from basic considerations in analytic function theory that $A^{-1/s}$ extends analytically to the open set $\{\sum_{l=1}^{\infty} c_l w_k^l \mid |w| < R\}$ and that $z_0(w)$ extends according to (17) on the set $|w| < R$. The Szegö set $\mathcal{S}_{A,s}$ in (13) occurs as

$$\mathcal{S}_{A,s} = \{z_0(e^{i\alpha}) \mid \alpha \in [0, 2\pi]\}, \tag{19}$$

and it can be shown that the parametrization

$$\alpha \in [0, 2\pi] \to z_0(e^{i\alpha}) = \sum_{l=1}^{\infty} c_l e^{il\alpha} \in \mathcal{S}_{A,s} \tag{20}$$

has no double points while a homotopy between $\{0\}$ and $\mathcal{S}_{A,s}$ is obtained according to

$$r \in [0, 1] \to \{z_0(re^{i\alpha}) \mid \alpha \in [0, 2\pi]\}. \tag{21}$$

From the latter facts it follows that $\mathcal{S}_{A,s}$ is a Jordan curve with 0 in its interior, and this completes the sketch of the proof of the converse statement. □

Note. The $z_0(w)$ of (17) is a univalent function, of a special type on an open set containing the closed unit disk $|w| \leq 1$. Hence, the results of the theory of univalent functions, as presented for instance in [18], Chs. 2-3, and [28], Ch. 12 become available. We shall not elaborate this point here, except for a casual note in Subsec. 3.2.

We now turn to the representation of the s roots of $z^s = A(z)$ in $|z| \leq 1$. These roots all lie inside the Jordan curve J in Lemma 3.2 and are given by

$$z_k = w_k A^{1/s}(z_k), \quad k = 0, 1, \ \dots\ , s-1, \tag{22}$$

where $w_k = e^{2\pi k i/s}$. Hence, from (20) we have

$$z_k = \sum_{l=1}^{\infty} c_l w_k^l, \quad k = 0, 1, \ \dots\ , s-1, \tag{23}$$

where c_l are explicitly given in (18).

When $A(z)$ is a polynomial of degree $n > s$, an expression similar to (23) can be derived for the $n - s$ roots of $z^s = A(z)$ outside the unit circle. Substituting $1/v$ into $z^s = A(z)$ and multiplying by v^n, we get

$$v^{n-s} = B(v), \tag{24}$$

where $B(v) = v^n A(1/v)$ is a polynomial of degree n. Note that from $|A(z)| < |z|^s$ for $1 < |z| < 1 + \delta$ for some $\delta > 0$, we have that $|B(v)| < |v|^{n-s}$ for $(1+\delta)^{-1} < |v| < 1$. Therefore, by Rouché's theorem, there occur exactly $n - s$ roots $v_k,\ k = \{s, s+1, \dots, n-1\}$ of (24) in $|v| \leq (1+\delta)^{-1}$, obviously satisfying $v_k = 1/z_k$. When there exists a Jordan curve (within $|v| < 1$) such that $B(v)$ is zero-free on and inside this curve, while 0 lies inside this curve and $|B(v)| < |v|^{n-s}$ on this curve, we find as above that

$$v_k = \sum_{l=1}^{\infty} \frac{1}{l} C_{v^{l-1}}[B^{l/(n-s)}(v)]\, e^{2\pi(k-s)il/(n-s)}, \quad k = s, s+1, \dots, n-1. \tag{25}$$

The Condition 3.1 and its equivalent forms as given per Lemmas 3.2 and 3.3 are equally useful in deciding whether a given A satisfies it. We present now some instances where Condition 3.1 is satisfied.

i. $A(z)$ is zero-free in $|z| \leq 1$. An appropriate Jordan curve J is found as $|z| = 1 + \delta$ with sufficiently small $\delta > 0$. Indeed, the assumptions on A imply that there is a $\delta > 0$ such that $0 < |A(z)| < |z|^s$ for $1 < |z| \leq 1 + \delta$.

ii. $A(z)$ is zero-free in $|z| < 1$. There may occur now a finite number of zeros of A on $|z| = 1$, necessitating a modification of the Jordan curve J in (i). We indent this J around the zeros such that the zeros are outside the new J while $|A(z)| < |z|^s$ for all z on the new J. As one sees, this technique may also work in cases where there are zeros of A strictly inside $|z| = 1$. A class of examples as in (i), (ii), follows from Kakeya's theorem [23] as follows:

   - when $a_0 > a_1 > \dots$, we have that $A(z)$ is zero-free in $|z| \leq 1$,
   - when $a_0 \geq a_1 \geq \dots$, we have that $A(z)$ is zero-free in $|z| < 1$.

iii. The c_l in (18) are all non-negative. It follows from Pringsheim's theorem [29] and the fact that $z_0(w)$ is well-defined for $w \in [0, 1+\delta]$ with some $\delta > 0$, that the radius of convergence of the power series in (17) exceeds 1. Thus Lemma 3.3 applies and it follows that Condition 3.1 is satisfied.

Below we give two examples where one can compute the $c_l = C_{z^{l-1}}[A^{l/s}(z)]$ explicitly, so that the criterion in Lemma 3.3 can be verified.

Example 3.4. Consider the Poisson case, $a_j = e^{-\lambda}\lambda^j/j!,\ j = 0, 1, \dots$, and $A(z) = \exp(\lambda(z-1))$ with $0 \leq \theta := \lambda/s < 1$. In this case, Condition 3.1 is always satisfied. Furthermore, there holds that

$$c_l = e^{-l\theta}\, \frac{(l\theta)^{l-1}}{l!}. \tag{26}$$

In Fig. 1 we have pictured $\mathcal{S}_{A,s}$ for $\theta = 0.1, 0.5, 1.0$. The dots on the curves indicate the roots z_k for the case $s = 20$, obtained by calculating the sum in (23) up to $l = 50$.

Example 3.5. Consider the binomial case, $a_j = \binom{n}{j} q^j (1-q)^{n-j},\ j = 0, \dots, n$, and $A(z) = (p + qz)^n$ where $p,\ q \geq 0,\ p + q = 1$ and $A'(1) = nq < s$. We compute in this case

$$c_l = \frac{1}{l}\, p^{l\beta - l + 1}\, q^{l-1} \binom{l\beta}{l-1}, \quad l = 1, 2, \dots \tag{27}$$

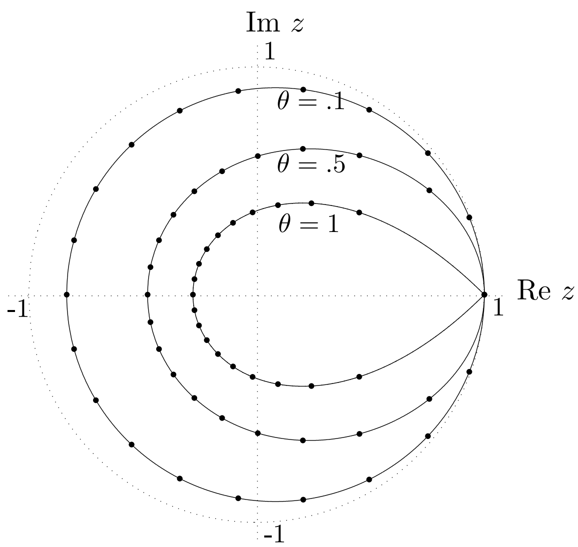

Figure 1: $\mathcal{S}_{A,s}$ for Poisson case, $\theta = .1, .5, 1$. The dots indicate $z_0, \ldots, z_{19}$ for $s = 20$, obtained by calculating the sum in (23) up to $l = 50$.

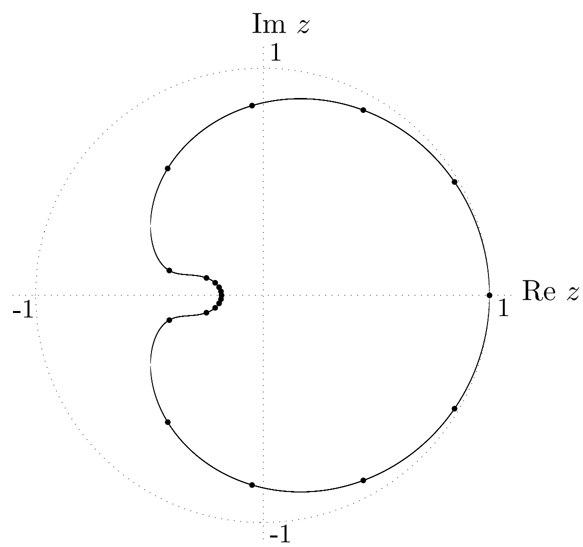

Figure 2: $\mathcal{S}_{A,s}$ for binomial case, $\beta = 0.5$, $q = .82$. The dots indicate $z_0, \ldots, z_{19}$ for $s = 20$, obtained by calculating the sum in (23) up to $l = 50$.

where $\beta := n/s$. In [22] the c_l are shown to have exponential decay for $\beta \geq 1$ (which covers in fact all practically relevant instances). It is further shown that for $0 \leq \beta < 1$ the c_l have exponential decay if and only if

$$p^{\beta-1} q (1-\beta)^{1-\beta} \beta^{\beta} < 1. \tag{28}$$

For $\beta = 1/2$, $s = 20$, constraint (28) requires q to be less than $2(\sqrt{2}-1)$. In Fig. 2 we plotted the $\mathcal{S}_{A,s}$ for $q = 0.82 < 2(\sqrt{2}-1)$, and the dots indicate the roots z_k obtained by calculating the sum in (23) up to $l = 50$. When q is increased, such that $q > 2(\sqrt{2}-1)$, $\mathcal{S}_{A,s}$ turns from a smooth Jordan curve containing zero into two separate closed curves (see [22]), and (23) no longer holds.

For the Poisson and binomial distribution we have (26) and (27), respectively, to determine the c_l. In general, the values of the c_l can be determined using the following property:

Property 3.6. *For $A(z) = \sum_{j=0}^{\infty} a_j z^j$ and $\alpha \in \mathbb{R}$, and $A^{\alpha}(z) = \sum_{j=0}^{\infty} b_j z^j$, the coefficients b_j follow from the coefficients a_j according to $b_0 = a_0^{\alpha}$ and*

$$b_{j+1} = \alpha a_0^{\alpha-1} a_{j+1} + \frac{1}{(j+1)a_0} \sum_{n=0}^{j-1} [\alpha(n+1) - (j-n)] a_{n+1} b_{j-n}, \quad j = 0, 1, \ldots. \tag{29}$$

The proof of Property 3.6 consists of computing the b_j's successively by equating coefficients in $A(z)(A^{\alpha})'(z) = \alpha A'(z) A^{\alpha}(z)$.

In [9] it is shown that the condition that A is infinitely-divisible, or the somewhat weaker condition that $A(z)$ has no zeros inside the unit circle, are sufficient for the roots of $z^s = A(z)$ on and within the unit circle to be distinct. However, examples exist of $A(z)$ having zeros inside the unit circle and at the same time having distinct roots (see e.g. Example 3.5). It is therefore that in both [9] and [20] the urge of finding a necessary condition for distinctness is expressed. In this respect, we have the following result:

Lemma 3.7. *When Condition 3.1 is satisfied, the roots of $z^s = A(z)$ on and within the unit circle are distinct.*

Proof. The roots lie inside J, and satisfy (22). Since $|A(z)|^{1/s} < |z|$ for all $z \in J$, it follows from Rouché's theorem that for each w_k, the function $z - w_k A^{1/s}(z)$ has as many zeros inside J as z. $\square$

Although Condition 3.1 is not necessary for the roots to be distinct (as appears to be the case in Example 3.5 with $\beta = 1/2$ and $q = 0.83$), it covers a far larger class of distributions A than those for which $A(z)$ has no zeros within the unit circle.

3.2 Fixed point iteration

We now discuss a way to determine the roots by applying successive substitution to a fixed-point equation. This idea originates from the work of Harris et al. [20] on root-finding for the continuous-time $G/E_k/1$ queue, and was presented more formally by Adan & Zhao [4] who distinguished a class of continuous distributions for which the method works. We further investigate the method for discrete distributions A. We present necessary conditions for the method to work and compare these to the conditions needed for the Fourier series representation of the roots introduced in the previous section.

When $A(z)$ is assumed to have no zeros for $|z| \leq 1$, we know that the s roots of $z^s = A(z)$ in $|z| \leq 1$ satisfy

$$z = wG(z), \tag{30}$$

with $G(z) = A^{1/s}(z)$ and $w^s = 1$. For each feasible w, Equation (30) can be shown as in Lemma 3.7 to have one unique root in $|z| \leq 1$. One could try to solve the equations by successive substitutions (see [4, 20]) as

$$z_k^{(n+1)} = w_k G(z_k^{(n)}), \quad k = 0, 1, ..., s-1, \tag{31}$$

with starting values $z_k^{(0)} = 0$.

Lemma 3.8. *When for $|z| \leq 1$, $A(z)$ is zero-free and $|G'(z)| < 1$, the fixed point equations (31) converge to the desired roots.*

Proof. For $|z| \leq 1$, $|w| \leq 1$,

$$|wG(z)| \leq G(|z|) \leq G(1) = 1, \tag{32}$$

so $wG(z)$ maps $|z| \leq 1$ into itself. For $|\tilde{z}|, |\hat{z}| \leq 1$ we have that

$$|wG(\tilde{z}) - wG(\hat{z})| \leq |\tilde{z} - \hat{z}| \max_{0 \leq t \leq 1} |G'(\hat{z} + t(\tilde{z} - \hat{z}))|. \tag{33}$$

Hence, from (33) and $|G'(z)| < 1$ for all $|z| \leq 1$, we conclude that $wG(z)$ is a contraction on $|z| \leq 1$. $\square$

For the Poisson distribution with $\lambda < s$, it is readily seen that $A(z) \neq 0$ and $|G'(z)| < 1$ for $|z| \leq 1$, so that the iteration (31) works. We want to consider, however, also distributions for which $A(z)$ has zeros within the unit circle (see e.g. Example 3.5). We restrict here naturally to $A(z)$ that allow a root $G(z) = A^{1/s}(z)$ that is analytic around $\mathcal{S}_{A,s}$ and positive at 0. Hence we introduce the following condition:

Condition 3.9. *Condition 3.1 should be satisfied and for all points $z \in \mathcal{S}_{A,s}$ there should hold that $|G'(z)| < 1$.*

According to the maximum principle we have that Condition 3.9 implies that $|G'(z)| < 1$ holds for all points inside $\mathcal{S}_{A,s}$ as well. Condition 3.9 thus ensures that for $\alpha \in [0, 2\pi]$ the point z_k is an attractor for the iteration (31).

Note that Condition 3.9 is what is minimally needed to ensure (31) to converge locally. However, under Condition 3.9 the iterates are by no means guaranteed to stay in $\mathcal{S}_{A,s}$ and its interior. This is already seen for the binomial case with $\beta < 1$, s even, and the iteration (31) for $k = s/2$, i.e

$$z_{s/2}^{(n+1)} = -1(p + q z_{s/2}^{(n)})^{\beta}. \tag{34}$$

For this iteration, the $z_{s/2}^{(n)}$, $n = 0, 1, \ldots$, are alternatingly inside and outside $\mathcal{S}_{A,s}$. The iteration, though, converges to the correct point when q is not too large. It is difficult, in general, to give guarantees for convergence; nevertheless, convergence seems to occur in most cases where Condition 3.9 holds.

We shall now present a consequence of assuming Condition 3.9. While Condition 3.1 implies $\mathcal{S}_{A,s}$ being a closed curve without double points, Condition 3.9 apparently does not hold for all such curves. To get more to the point, we recall the notion of starshapedness (see [25], p.125, Exercise 109).

Definition 3.10. *A closed curve without double points is called starshaped with respect to a point in its interior if any ray from this point intersects the curve at exactly one point (all the points from this curve can be seen from this point).*

We have the following result:

Lemma 3.11. *When Condition 3.9 is satisfied, the curve $\mathcal{S}_{A,s}$ is starshaped with respect to 0.*

Proof. See Appendix A.1. □

Note that the notion of starshapedness is much weaker than convexity. We have observed, both by visual inspection of $\mathcal{S}_{A,s}$ and analytically, that convexity of $\mathcal{S}_{A,s}$ implies that Condition 3.9 is satisfied in the case of the binomial distribution. It might well be that this holds for a broader class of distributions as well. Hence, Condition 3.9 seems to be in between requiring starshapedness and convexity.

Example 3.5 gives a nice demonstration of Lemma 3.11. From an inspection of $\mathcal{S}_{A,s}$ in Fig. 2 one sees that $\mathcal{S}_{A,s}$ is not starshaped with respect to 0, and one can thus immediately conclude that Condition 3.9 is not satisfied and hence the iteration (31) cannot be applied to determine the roots.

Example 3.12. Consider the binomial case, $A(z) = (p + qz)^n$ where p, $q \geq 0$, $p + q = 1$ and $\beta = n/s$. When $\beta \geq 1$, we have that

$$\max_{z \in \mathcal{S}_{A,s}} |G'(z)| = \max_{z \in \mathcal{S}_{A,s}} |\beta q (p + qz)^{\beta - 1}|, \tag{35}$$

occurs at $z = 1$ and equals $\beta q < 1$. For $\beta \in (0, 1)$, it can be shown that Condition 3.9 holds if and only if

$$p^{\beta - 1} q (1 + \beta)^{1 - \beta} \beta^{\beta} < 1. \tag{36}$$

Now denote by $q_1(\beta)$ and $q_2(\beta)$ the values of q for which $<$ can be replaced by $=$ in (28) and (36), respectively. These values, that can be shown to be unique, are plotted in Fig. 3. Observe that the set of values q for which the Fourier series representation holds ($q < q_1(\beta)$)

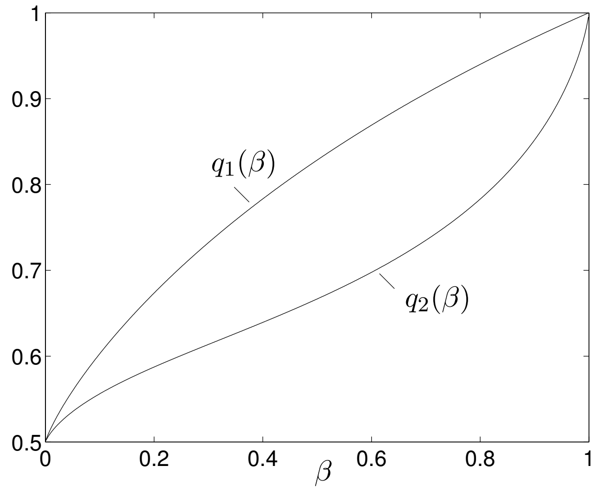

Figure 3: $q_1(\beta)$ and $q_2(\beta)$ for $\beta \in (0, 1)$.

is much larger than the set for which Condition 3.9 holds ($q < q_2(\beta)$). We have numerical evidence that whenever $q < q_2(\beta)$ the iteration (31) works. Finally note that the roots in Fig. 2 ($\beta = 1/2, q = 0.82$), which are computed using the Fourier series representation, cannot be obtained using the fixed point iteration.

Although the fixed point iteration is a very efficient method, the class of distributions A for which it can be applied is clearly smaller than the class of distributions A for which the Fourier series representation holds. That is, Condition 3.1 is much weaker than Condition 3.9.

3.3 Numerical results

We will now present two examples for which the roots can be determined through (31). For each root we stop the iteration when

$$|z_k^{(n+1)} - z_k^{(n)}| < 10^{-14}, \tag{37}$$

and we denote the resulting values by $\hat{z}_k$.

Denote by $z_k(L)$ the estimated root value that results when we truncate l at L in (23). We would like to have some more insight in how fast $z_k(L)$ converges to z_k, where the $\hat{z}_k$ determined above are considered to be sufficiently accurate approximations of the z_k to serve as references.

For the Poisson distribution with $\lambda = 8$, $s = 10$, Table 1 displays the roots $\hat{z}_k$, along with the distance between $\hat{z}_k$ and $z_k(L)$ for $L = 10, 20, 50$. As it appears, the Fourier series (23) converge quite rapidly. We further note that the most slowly convergent series is $z_0(L)$. This can be explained from the following result:

Lemma 3.13. *The truncation error* $|z_k(L) - z_k|$ *is largest for* $k = 0$ *among all* $k = 0, 1, \ldots, s-1$*, when the coefficients* $c_l \geq 0$*,* $l = 1, 2, \ldots$*.*

Proof. Follows directly from (23). □

For the Poisson case the c_l are greater or equal to 0, indeed. This is also the case for e.g. the geometric distribution $a_j = (1-p)p^j$ with $0 \leq p < 1$, and for the binomial distribution in Example 3.5 with $\beta \geq 1$, but it fails to hold for the latter distribution with $0 < \beta < 1$.

In general, if one applies (23) to a distribution A for which $c_l \geq 0$, then $|z_0(L) - z_0| = |z_0(L) - 1|$ being small is a good test for convergence, since it reflects the maximum distance between the estimated and true values of the roots.

Table 2 displays $\hat{z}_k$ and $|z_k(L) - \hat{z}_k|$, $L = 10, 20, 50$, for the binomial distribution with $n = 16$, $q = 0.5$ and $s = 10$, so that all $c_l \geq 0$.

Table 1: Poisson distribution, $\lambda = 8$, $s = 10$. The roots of $z^s = A(z)$ for $|z| \leq 1$ determined with (31) (denoted as $\hat{z}_k$), along with the distance between $\hat{z}_k$ and $z_k(L)$ for $L = 10, 20, 50$.

k	Re $\hat{z}_k$	Im $\hat{z}_k$	$\lvert z_k(10) - \hat{z}_k\rvert$	$\lvert z_k(20) - \hat{z}_k\rvert$	$\lvert z_k(50) - \hat{z}_k\rvert$
0	1.000	0.000	0.110194	0.048179	0.009637
1	0.300	0.486	0.017461	0.005283	0.000694
2	−0.017	0.442	0.009539	0.002817	0.000366
3	−0.206	0.321	0.006988	0.002052	0.000266
4	−0.308	0.166	0.005961	0.001747	0.000226
5	−0.342	0.000	0.005673	0.001662	0.000215

Table 2: Binomial distribution, $n = 16$, $q = 0.5$, $s = 10$. The roots of $z^s = A(z)$ for $|z| \leq 1$ determined with (31) (denoted as $\hat{z}_k$), along with the distance between $\hat{z}_k$ and $z_k(L)$ for $L = 10, 20, 50$.

k	Re $\hat{z}_k$	Im $\hat{z}_k$	$\lvert z_k(10) - \hat{z}_k\rvert$	$\lvert z_k(20) - \hat{z}_k\rvert$	$\lvert z_k(50) - \hat{z}_k\rvert$
0	1.000	0.000	0.118685	0.037943	0.003067
1	0.169	0.439	0.024368	0.006010	0.000364
2	−0.066	0.315	0.013329	0.003216	0.000192
3	−0.164	0.199	0.009766	0.002344	0.000140
4	−0.208	0.096	0.008330	0.001996	0.000119
5	−0.221	0.000	0.007928	0.001899	0.000113

We stress that there are many distributions A for which the iteration (31) fails to work, while (23) still holds, i.e. Condition 3.1 is satisfied (see e.g Example 3.5 and 5.7). We simply chose the above examples so that we could obtain precise estimates of the real roots without invoking some other, less transparent, numerical method than (31).

4 The general approach

To find explicit expressions for the stationary queue length distribution, we start from the observation that the queue length at the beginning of slot n in (1) can be viewed as being the sojourn time of the n-th customer in a queue for which s equals the deterministic and integer-valued interarrival time between customer n and $n+1$, and A_n is the service time of customer $n+1$. This model is also referred to as the $D/G/1$ queue (see e.g. Servi [27]), and falls within the class of the $G/G/1$ queue. For the $G/G/1$ queue, the Laplace transform of the stationary waiting time follows from Spitzer's identity (see e.g. [12]). Using similar arguments as used for the derivation of Spitzer's identity, we can prove the following result:

Theorem 4.1. *The pgf of the stationary queue length distribution is given by*

$$X(z) = A(z) \exp\Big\{ -\sum_{l=1}^{\infty} \frac{1}{l} P(S_l > 0)\Big\} \exp\Big\{ \sum_{l=1}^{\infty} \frac{1}{l} E(z^{S_l}\mathbf{1}\{S_l > 0\})\Big\}, \tag{38}$$

where $S_l = \sum_{i=1}^{l} (A_i - s)$*, and* $\mathbf{1}\{B\} = 1$ *if* B *holds and 0 otherwise.*

Proof. This proof is based on Wiener-Hopf decomposition, analogously to [12] p. 338 for the continuous-time case. From recursion (1) we have

$$\begin{aligned} E(z^{X_{t+1}}) &= E(z^{A_t}\mathbf{1}\{X_t \le s\}) + E(z^{X_t+A_t-s}\mathbf{1}\{X_t > s\}) \\ &= \mathbb{P}(X_t \le s)E(z^{A_t}) + E(z^{X_t+A_t-s}) - E(z^{X_t+A_t-s}\mathbf{1}\{X_t \le s\}). \end{aligned} \tag{39}$$

Letting $t \to \infty$ and observing that X_t and A_t are independent then yields

$$\frac{X(z)}{A(z)}(1 - z^{-s}A(z)) = \mathbb{P}(X \le s) - E(z^{X-s}\mathbf{1}\{X \le s\}). \tag{40}$$

We denote the right-hand side of (40) as $X^*(z)$. Using

$$\frac{1}{1-z} = \exp\{-\ln(1-z)\} = \exp\Big\{\sum_{l=1}^{\infty} \frac{z^l}{l}\Big\}, \quad |z| < 1, \tag{41}$$

we have that

$$\begin{aligned} (1 - z^{-s}A(z))^{-1} &= \exp\Big\{ \sum_{l=1}^{\infty} \frac{1}{l} (z^{-s}A(z))^l \Big\} \\ &= \exp\Big\{ \sum_{l=1}^{\infty} \frac{1}{l} E(z^{S_l}\mathbf{1}\{S_l > 0\})\Big\} \cdot \exp\Big\{ \sum_{l=1}^{\infty} \frac{1}{l} E(z^{S_l}\mathbf{1}\{S_l \le 0\})\Big\}. \end{aligned} \tag{42}$$

Substituting (42) into (40) yields

$$\frac{X(z)}{A(z)} \exp\Big\{ -\sum_{l=1}^{\infty} \frac{1}{l} E(z^{S_l}\mathbf{1}\{S_l > 0\})\Big\} = X^*(z) \exp\Big\{ \sum_{l=1}^{\infty} \frac{1}{l} E(z^{S_l}\mathbf{1}\{S_l \le 0\})\Big\}. \tag{43}$$

The left-hand side and right-hand side of (43) are analytic in $|z| < 1$ and $|z| > 1$, respectively, and continuous up to $|z| = 1$. Also, the left-hand side and right-hand side of (43) are bounded (see [12] p. 338) and analytic in $|z| < 1$ and $|z| > 1$, respectively. Therefore, their analytic continuation contains no singularities in the entire complex plane, whence upon using Liouville's theorem (see e.g. [30]) the left-hand side of (43) is constant, i.e.

$$\frac{X(z)}{A(z)} = K \exp\Big\{ \sum_{l=1}^{\infty} \frac{1}{l} E(z^{S_l}\mathbf{1}\{S_l > 0\})\Big\}. \tag{44}$$

The constant K follows from $X(1)/A(1) = 1$ yielding

$$K = \exp\Big\{ -\sum_{l=1}^{\infty} \frac{1}{l} \mathbb{P}(S_l > 0)\Big\}, \tag{45}$$

which completes the proof. $\square$

In principle, the mean and variance of the stationary queue length can be determined from the x_j. However, this can be done more directly from (38), due to $\mu_X = X'(1)$ and $\sigma_X^2 = X''(1) + X'(1) - X'(1)^2$. Denoting by A^{*l} the l-fold convolution of A, and by $c_{i,l}$ the probability that $A^{*l} = i$, i.e. $c_{i,l} = C_{z^i}[A^l(z)] = \mathbb{P}(A^{*l} = i)$, this gives

$$\mu_X = \mu_A + \sum_{l=1}^{\infty} \frac{1}{l} \sum_{i=ls}^{\infty} (i - ls)\, c_{i,l}, \tag{46}$$

$$\sigma_X^2 = \sigma_A^2 + \sum_{l=1}^{\infty} \frac{1}{l} \sum_{i=ls}^{\infty} (i - ls)^2\, c_{i,l}, \tag{47}$$

which are root-free expressions for μ_X and σ_X^2. For comparison, we mention that taking derivatives of (4) instead of (38) yields, upon a lengthy calculation,

$$\mu_X = \frac{\sigma_A^2}{2(s-\mu_A)} + \frac{1}{2}\mu_A - \frac{1}{2}(s-1) + \sum_{k=1}^{s-1} \frac{1}{1-z_k}, \tag{48}$$

$$\begin{aligned}\sigma_X^2 = \sigma_A^2 \;&+\; \frac{A'''(1) - s(s-1)(s-2)}{3(s-\mu_A)} + \frac{A''(1) - s(s-1)}{2(s-\mu_A)} \\ &+\; \Big(\frac{A''(1) - s(s-1)}{2(s-\mu_A)}\Big)^2 - \sum_{k=1}^{s-1} \frac{z_k}{(1-z_k)^2}.\end{aligned} \tag{49}$$

4.1 Stationary queue length distribution

From (38) the following is readily seen:

Lemma 4.2. *The stationary queue length distribution is given by*

$$x_j = d \sum_{k=0}^{j} a_k\, C_{z^{j-k}} \Big[\exp\Big\{\sum_{l=1}^{\infty}\sum_{i=1}^{\infty} \frac{1}{l}\, c_{ls+i,l} z^i\Big\}\Big], \quad j = 0, 1, \ldots, \tag{50}$$

and

$$d = \exp\Big\{-\sum_{l=1}^{\infty} \sum_{i=ls+1}^{\infty} \frac{1}{l}\, c_{i,l}\Big\}. \tag{51}$$

We showed in [22] that (50) can be alternatively derived using Fourier sampling. We note here that He and Sohraby [21] also derived a root-free expression of a similar type for the stationary queue length distribution, using combinatorial arguments and Ballot theorems.

Expression (50) provides for each x_j a representation that does not depend on the roots of $z^s = A(z)$. However, the series contain two infinite sums. So, in working with (50) we should have some feeling for the speed of convergence of these series in relation with choosing appropriate truncation levels for the sums.

For determining the coefficients $c_{i,l}$ in (50) and (51) we can use the following property:

Property 4.3. *For $A(z) = \sum_{j=0}^{\infty} a_j z^j$ and $B(z) = \sum_{j=0}^{\infty} b_j z^j$ for which it holds that $A(z) = \exp\{B(z)\}$, the coefficients a_j follow recursively from the coefficients b_j (and vice versa) according to*

$$a_0 = \exp(b_0); \quad a_j = \frac{1}{j}\sum_{n=1}^{j} n b_n a_{j-n}, \quad j = 1, 2, \ldots. \tag{52}$$

The proof of Property 4.3 consists of computing the a_j's successively by equating coefficients in $A'(z) = B'(z)A(z)$.

4.2 Extension to the discrete-time G/G/1 queue

The $D^A/D^s/1$ queue can be generalized to a $D^A/D^B/1$ queue, where the service capacity s is no longer fixed, i.e.,

$$X_{n+1} = \max\{X_n - B_n, 0\} + A_n, \tag{53}$$

where X_n denotes the queue length at the beginning of slot n, $X_0 = 0$, A_n denotes the number of newly arriving customers during slot n, and B_n denotes the number of customers that can be served during one slot. The B_n, $n = 1, 2, \ldots$ are assumed to be i.i.d. according to a discrete random variable B with $b_j = \mathbb{P}(B = j)$ that can take values in $\{0, 1, \ldots, m\}$. Its pgf is given by

$$B(z) = \sum_{j=0}^{m} b_j z^j, \tag{54}$$

which is certainly analytic in an open set containing the closed unit disk $|z| \leq 1$. We will show that (53) fits into the framework of the $D^A/D^s/1$ queue, which implies that the methods as developed in this section can be applied for the $D^A/D^B/1$ queue.

Let X denote the random variable following the stationary distribution of the Markov chain defined by (53), with $x_j = \mathbb{P}(X = j)$, which exists under the stability condition

$$\mu_A = A'(1) < \mu_B = B'(1). \tag{55}$$

It that case, the pgf of X is given by (see Bruneel [6])

$$X(z) = \frac{A(z)(\mu_B - \mu_A)(z-1)}{z^m - z^m A(z) B(1/z)} \prod_{k=1}^{m-1} \frac{z - z_k}{1 - z_k}, \tag{56}$$

where $z_0 = 1, z_1, \ldots, z_{m-1}$ denote the m roots of $z^m = z^m A(z)B(1/z)$ on and inside the unit circle. Introducing a random variable $U = m + A - B$ with $U(z) = z^m A(z) B(1/z)$, we have

$$\begin{aligned} X(z) &= \frac{A(z)(m - (m + \mu_A - \mu_B))(z-1)}{z^m - z^m A(z)B(1/z)} \prod_{k=1}^{m-1} \frac{z - z_k}{1 - z_k} \\ &= \frac{A(z)(m - \mu_U)(z-1)}{z^m - U(z)} \prod_{k=1}^{m-1} \frac{z - z_k}{1 - z_k}, \end{aligned} \tag{57}$$

which fits completely into the framework of the discrete-time bulk service queue.

Another interesting aspect of (53) is that it represents Lindley's recursion for the sojourn time of the $n+1$-th customer in a discrete-time $G/G/1$ queue. The discrete-time $G/G/1$

queue thus also fits into the framework of the $D^A/D^s/1$ queue, which enlarges the range of application of the method presented in Sec. 3, where the roots are represented as Fourier series. The expressions (46), (47) and (50) also hold for the $D^A/D^B/1$ queue with A and s replaced by U and m, respectively.

5 Guidelines for numerical work

In this section we present some guidelines for calculating the stationary queue length characteristics as given by (46), (47) and (50). We discuss on how to choose and appropriate truncation level and how to calculate the convolutions involved. In particular, we present asymptotic expression for the convolutions.

5.1 Choosing the truncation level

First note that when $A(z)$ is a polynomial of degree n, the series in (46), (47) and (50) over i is finite, running from $i = 1$ to $i = ln$. For non-polynomial $A(z)$, we should truncate A at n such that $\mathbb{P}(A > n)$ is negligible.

The performance characteristics can be obtained by calculating the sum over l up to a certain level, say until convergence is obtained to within an appropriate criterion. A good overall check is provided by the fact that determining d is the bottleneck, which can be seen in the following way. Denote by $x_j(L)$ and $d(L)$ the estimated value of x_j and d that result from truncating the series over l at $l = L$ in (50) and (51), respectively. The relative error made then equals

$$\begin{aligned} \frac{d(L)-d}{d} &= \exp\Big\{\sum_{l=L+1}^{\infty}\sum_{i=ls+1}^{\infty}\frac{1}{l}\,c_{i,l}\Big\}-1 \\ &\approx \sum_{l=L+1}^{\infty}\sum_{i=ls+1}^{\infty}\frac{1}{l}\,c_{i,l} = \sum_{l=L+1}^{\infty}\sum_{i=1}^{\infty}\frac{1}{l}\,c_{ls+i,l}, \end{aligned} \tag{58}$$

where the far right-hand side of (58) sums all truncation errors $\sum_{l=L+1}^{\infty}\frac{1}{l}\,c_{ls+i,l}$ that appear in (50) when computing x_j.

A good check for overall accuracy is provided by the relation (see [22])

$$d = \sum_{j=0}^{s} x_j, \tag{59}$$

from which it follows that

$$\sum_{j=0}^{s}\sum_{k=0}^{j} a_k\, C_{z^{j-k}}\Big[\exp\Big\{\sum_{l=1}^{L}\sum_{i=1}^{\infty}\frac{1}{l}\,c_{ls+i,l}z^i\Big\}\Big] \uparrow 1, \quad L\to\infty. \tag{60}$$

Hence, when the quantity at the left-hand side of (60) is close enough to 1, the accuracy of the estimated values seems guaranteed.

5.2 Asymptotic behaviour of $\mathbb{P}(A^{*l} = n)$

We now examine the asymptotic behaviour of

$$C_{z^n}[A^l(z)] = \mathbb{P}(A^{*l} = n), \tag{61}$$

for l large, for reasons described below. We obtain asymptotic expansions using the saddle point method (or method of steepest descent), see e.g. De Bruijn [14].

The applicability of the expressions (46), (47) and (50) indisputably depends on the ability of computing the discrete convolutions involved. A straightforward way would be to iteratively determine the distribution of the l-fold convolution A from the $(l-1)$-fold convolution of A. A problem could be the computational expense involved. The discrete convolution at each iteration is a computationally expensive operation with sequences that might contain many points. Particularly in case of a high occupation rate (i.e. μ_A/s), many iterations could be required.

As suggested by Ackroyd [2, 3], one can apply a fast Fourier transform algorithm for computing discrete convolutions. In that way, given the pgf $A(z)$, the probability distribution of the l-fold convolution can be obtained directly from its pgf $A^l(z)$. Ackroyd [2] shows that the computational speed gained is considerable. For a description of the fast Fourier transform approach to invert a pgf we refer to Abate & Whitt [1].

Irrespective of the method used to compute the convolutions, the issue of truncating the infinite sum remains a matter of choice, likely to depend on the precision criterion required. We, however, aim at steering a middle course, in the sense that we will take into account the asymptotic behaviour of the l-fold convolution (i.e. for l large). For a certain value of l the difference between the real probability distribution and its asymptotic approximation will be negligible. From that value on, one could replace the true convolutional distributions by its asymptotic expression.

We have

$$c_{nl} := C_{z^n}[A^l(z)] = \frac{1}{2\pi i}\oint_{\mathcal{C}} \frac{A^l(z)}{z^{n+1}}\mathrm{d}z, \tag{62}$$

where $\mathcal{C}$ is any contour around 0 within the analyticity region of $A(z)$. We observe that only $n \geq ls$ are required. We have assumed that $A'(1) < s$ and so there is a δ such that

$$|A(z)| \leq A(|z|) < |z|^s\ , \quad 1 < |z| < 1 + \delta. \tag{63}$$

We apply the saddle point method to the integral in (62), so as to derive the asymptotic behaviour of c_{nl} for l large. To that end, we introduce the function

$$h(z) := l \ln A(z) - n \ln z\ , \quad z \geq 1, \tag{64}$$

so that

$$c_{nl} = \frac{1}{2\pi i}\oint_{\mathcal{C}} \frac{1}{z} e^{h(z)}\mathrm{d}z. \tag{65}$$

Lemma 5.1. *There is at most one $z \geq 1$ such that $h'(z) = 0$.*

Proof. See Appendix A.2. □

In case there is no $z \geq 1$ such that $h'(z) = 0$, we have $A^l(z) \leq z^n$, $z \geq 1$, so $A^l(z)$ is a polynomial of degree smaller or equal than n, and such that $A^l(z)/z^n$ decreases in $z \geq 1$. Then

$$c_{nl} = \lim_{z\to\infty} \frac{A^l(z)}{z^n} \ = \begin{cases} 0, & lj(A) < n, \\ (a_{j(A)})^l, & lj(A) = n, \end{cases} \tag{66}$$

where $j(A)$ is the degree of A.

We assume henceforth that there is exactly one $z \geq 1$ such that $h'(z) = 0$, and call that point z_0. We next calculate

$$h''(z) = l\frac{A''(z)}{A(z)} - l\Big(\frac{A'(z)}{A(z)}\Big)^2 + \frac{n}{z^2}. \tag{67}$$

At $z = z_0$ we have $A'(z)/A(z) = n/(lz)$, so that

$$h''(z_0) = \frac{l^2 z_0^2 A''(z_0) - n(n-l)A(z_0)}{l z_0^2 A(z_0)}, \tag{68}$$

for which we have the following result.

Lemma 5.2. *There holds that $h''(z_0) > 0$.*

Proof. See Appendix A.3. □

Combining the above results provides us with the following asymptotic expression for the coefficients c_{nl}.

Theorem 5.3. *For the c_{nl} the following approximation can be obtained from asymptotic analysis:*

$$c_{nl} \approx \frac{1}{z_0\sqrt{2\pi h''(z_0)}}\frac{A^l(z_0)}{z_0^n}. \tag{69}$$

Proof. From Lemma 5.1 and 5.2 we conclude that z_0 is a saddle point for the integral in (65), and by the saddle point method (see [14], p. 87), we thus obtain the approximation

$$\begin{aligned} c_{nl} &\approx \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{1}{z_0 + it} e^{h(z_0) - \frac{1}{2}t^2 h''(z_0)} i\mathrm{d}t \\ &= \frac{1}{2\pi z_0} e^{h(z_0)} \sqrt{\frac{\pi}{\frac{1}{2}h''(z_0)}}, \end{aligned} \tag{70}$$

which can be rewritten as (69). This completes the proof. □

Example 5.4. In the Poisson case, $A(z) = \exp(\lambda(z-1))$, we have

$$h(z) = l\lambda(z-1) - n\ln z, \tag{71}$$

so that

$$h'(z) = l\lambda - \frac{n}{z}, \quad h''(z) = \frac{n}{z^2}. \tag{72}$$

Therefore

$$z_0 = \frac{n}{l\lambda}, \quad h''(z_0) = \frac{1}{n}(l\lambda)^2, \tag{73}$$

and hence

$$c_{nl} \approx \frac{1}{n/(l\lambda)} \cdot \frac{1}{\sqrt{2\pi\frac{1}{n}(l\lambda)^2}} \cdot \frac{\exp(l\lambda(\frac{n}{l\lambda}) - 1)}{(\frac{n}{l\lambda})^n} = \frac{1}{\sqrt{2\pi n}}\Big(\frac{l\lambda}{n} \cdot e^{1 - \frac{l\lambda}{n}}\Big)^n. \tag{74}$$

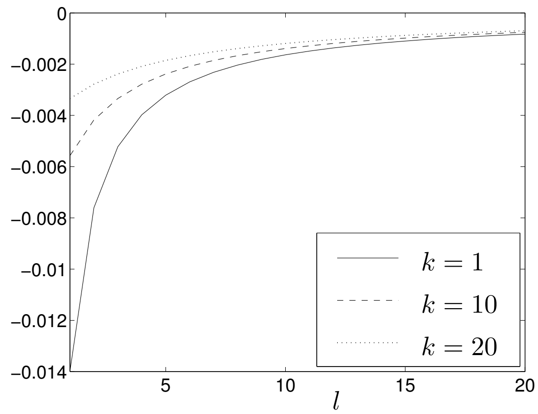

Figure 4: Relative error $100(c_{nl} - \hat{c}_{nl})/c_{nl}$, with $\hat{c}_{nl}$ the approximation (74) for the Poisson case with $s = 5$, $\lambda = 4$, and $c_{ls+k,l}$ for $l = 1, 2, \ldots, 20$ and $k = 1, 10, 20$.

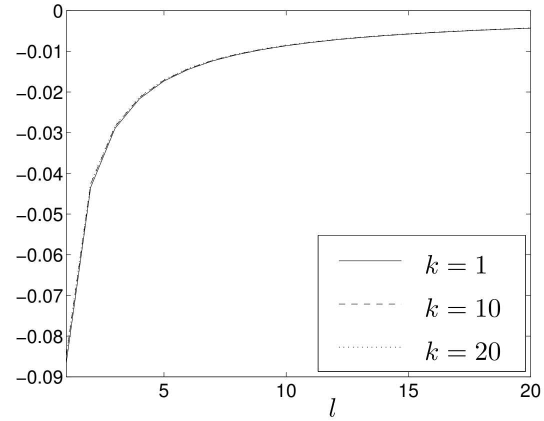

Figure 5: Relative error $100(c_{nl} - \hat{c}_{nl})/c_{nl}$, with $\hat{c}_{nl}$ the approximation (81) for the geometric case with $s = 5$, $p = 4/5$, and $c_{ls+k,l}$ for $l = 1, 2, \ldots, 20$ and $k = 1, 10, 20$.

Observe that $xe^{1-x} \in [0, 1)$ when x $(=l\lambda/n) \in [0, 1)$. In the present case we have, explicitly,

$$c_{nl} = \frac{e^{-l\lambda}}{n!}(l\lambda)^n \approx e^{-l\lambda}(n^n e^{-n}\sqrt{2\pi n})^{-1}(l\lambda)^n = \frac{1}{\sqrt{2\pi n}}\Big(\frac{l\lambda}{n} \cdot e^{1-\frac{l\lambda}{n}}\Big)^n, \tag{75}$$

where Stirling's formula $n! \approx n^{n+1/2}e^{-n}\sqrt{2\pi}$ has been used. It is thus seen that the approximation of c_{nl} as obtained per Theorem 5.3 amounts to replacing $n!$ in the exact expression for c_{nl} in (75) by its Stirling approximation. Accordingly, the approximation given by (74) has relative error independent of λ.

In Fig. 4 we have plotted the relative error $100(c_{nl} - \hat{c}_{nl})/c_{nl}$, with $\hat{c}_{nl}$ the approximation (74), for the Poisson case with $s = 5$, $\lambda = 4$, and $c_{ls+k,l}$ for $l = 1, 2, \ldots, 20$ and $k = 1, 10, 20$. The relative error decreases rapidly for larger values of l. Also, the relative error is smaller for larger values of k.

Example 5.5. In the case of the geometric distribution, $a_j = (1-p)p^j$, $j = 0, 1, \ldots$, so that

$$A(z) = \frac{1-p}{1-pz}; \quad A'(1) = \frac{p}{1-p} < s, \tag{76}$$

we have

$$h(z) = l\ln(1-p) - n\ln(1-pz) - n\ln z. \tag{77}$$

Hence

$$h'(z) = \frac{lp}{1-pz} - \frac{n}{z}, \quad h''(z) = \frac{1}{l}\Big(\frac{lp}{1-pz}\Big)^2 + \frac{n}{z^2}, \tag{78}$$

and

$$z_0 = \frac{n}{p(n+l)} = \frac{1}{p}\,\frac{n/l}{n/l+1} \in (1, 1/p), \tag{79}$$

since we assume $n/l > s$. Furthermore

$$h''(z_0) = \frac{p^2(n+l)^3}{nl}. \tag{80}$$

We thus get

$$
\begin{aligned}
c_{nl} &\approx \frac{p(n+l)}{n}\cdot\frac{1}{\sqrt{2\pi\frac{p^2(n+l)^3}{nl}}}\cdot\Big(\frac{1-p}{1-p\frac{n}{p(n+l)}}\Big)^l\Big/\Big(\frac{n}{p(n+l)}\Big)^n \\
&= \frac{1}{\sqrt{2\pi}}(1-p)^l p^n (n+l)^{n+l-1/2} n^{-n-1/2} l^{-l+1/2}. \qquad (81)
\end{aligned}
$$

By using the explicit representation

$$
\begin{aligned}
c_{nl} = (1-p)^l(-p)^n\binom{-l}{n} &= (1-p)^l(-p)^n\frac{-l(-l-1)\cdot\ldots\cdot(l-n+1)}{n!} \\
&= (1-p)^l p^n\frac{(n+l-1)!}{n!(l-1)!} \\
&= (1-p)^l p^n\frac{(n+l)!}{n!\ l!}\ \frac{l}{n+l}, \qquad (82)
\end{aligned}
$$

we get by Stirling's formula exactly (81). Accordingly, as in Example 5.4, the approximation given by (81) has relative error independent of p.

In Fig. 5 we have plotted the relative error $100(c_{nl}-\hat{c}_{nl})/c_{nl}$, with $\hat{c}_{nl}$ the approximation (81), for the geometric case with $s=5$, $p=4/5$, and $c_{ls+k,l}$ for $l=1,2,\ldots,20$ and $k=1,10,20$. Again, the relative error decreases rapidly for larger values of l. In contrast to Example 5.4, though, the relative error hardly depends on the value of k.

5.3 Numerical results

Example 5.6. Consider the Poisson case, $A(z)=\exp(\lambda(z-1))$, $\lambda<s$. For $s=10$ and $\lambda=5,8,9$, Table 3 displays the mean and variance of the stationary queue length. For the

Table 3: Mean and variance of X for the Poisson case with $s=10$, $\lambda=5,8,9$.

	μ_X		σ_X^2	
	(48)	(46)	(49)	(47)
λ=5	5.0237	5.0237	5.0519	5.0519
λ=8	8.8786	8.8786	11.6109	11.6104
λ=9	12.1012	11.9938	29.9067	28.4381

results obtained from (48) and (49) we have determined the roots using (31) with stopping criterion $|z_k^{(n+1)}-z_k^{(n)}|<10^{-14}$. For the results obtained from (46) and (47) we have truncated the sum over l at $l=30$ and the sum over i at $i=300$. We observe that the higher the load, the higher we should choose the level at which we truncate these sums. For $\lambda=5$ and $\lambda=8$ the truncation levels chosen are sufficient, while for $\lambda=9$ they should be taken somewhat higher.

Example 5.7. We take the example considered in [10], in which $A(z)=Y(z)^6$ where

$$Y(z)=0.1+0.15z+0.2z^2+0.2z^3+0.15z^4+0.1z^5+0.05z^6+0.01z^7+0.01z^8+0.03z^{10}. \qquad (83)$$

In [10] the stationary queue length distribution is determined from (12), for which the zeros outside the unit circle are determined numerically. The iteration (31) does not work for this example. We calculate the stationary queue length distribution from (8), (12) and (50). For (8) and (12) we calculate the roots of $z^s = A(z)$ inside and outside the unit circle using (23) and (25), respectively. For (23), (25) and (50) we truncate the sum over l at $l = 60$. The

Table 4: Stationary queue length distribution for $A(z) = Y(z)^6$, with $Y(z)$ given in (83), $s = 30$.

j	x_j from [10]	x_j from (8)-(23)	x_j from (12)-(25)	x_j from (50)
0	0.00000098	0.00000098	0.00000098	0.00000098
1	0.00000885	0.00000885	0.00000885	0.00000885
2	0.00004501	0.00004500	0.00004501	0.00004501
3	0.00016686	0.00016680	0.00016686	0.00016686
4	0.00049733	0.00049715	0.00049733	0.00049733
5	0.00125555	0.00125503	0.00125555	0.00125555
6	0.00277138	0.00277010	0.00277138	0.00277138
7	0.00546268	0.00545988	0.00546269	0.00546268
8	0.00976060	0.00975504	0.00976060	0.00976060
9	0.01598541	0.01597540	0.01598541	0.01598541
10	0.02420260	0.02418598	0.02420260	0.02420260
20	0.06498585	0.06487376	0.06498585	0.06498585
30	0.00728773	0.00661255	0.00728773	0.00728773
40	0.00015022	0.00049559	0.00015022	0.00015022
50	0.00000080	0.00072575	0.00000080	0.00000080

results are displayed in Table 4. We see that both (12) and (50) lead to similar results as obtained in [10]. Determining the probabilities from (8) gives problems when moving into the tail of the distribution. Although these problems might be resolved by truncating the sum over l in (23) at a higher level, (12) and (50) seem more stable. The truncation level of $l = 60$ is sufficient, although it is no problem to increase it from a numerical point of view.

Example 5.8. Consider the binomial case, $A(z) = (p + qz)^n$ where p, $q \geq 0$, $p + q = 1$, for which we take $n = 16$, $q = 0.5$, $s = 10$. Table 5 displays some of the x_j, calculated by $x_j(L)$ for $L = 10, 20, 30$. Additionally, the x_j have been determined from (12) where the roots of $z^s = A(z)$ outside the unit circle follow from (25) (with the sum over l truncated at $l = 60$). Note that for x_{50} and x_{100} we need some higher level of L to determine these small

Table 5: Stationary queue length distribution for the binomial case, $n = 16$, $q = 0.5$, $s = 10$.

j	$x_j(10)$	$x_j(20)$	$x_j(30)$	(12)
0	$0.13132067{\cdot}10^{-4}$	$0.13131227{\cdot}10^{-4}$	$0.13131225{\cdot}10^{-4}$	$0.13131228{\cdot}10^{-4}$
10	$0.12967227{\cdot}10^{-0}$	$0.12967413{\cdot}10^{-0}$	$0.12967413{\cdot}10^{-0}$	$0.12967413{\cdot}10^{-0}$
20	$0.10032061{\cdot}10^{-4}$	$0.10120244{\cdot}10^{-4}$	$0.10120578{\cdot}10^{-4}$	$0.10120543{\cdot}10^{-4}$
30	$0.25745593{\cdot}10^{-9}$	$0.29484912{\cdot}10^{-9}$	$0.29527164{\cdot}10^{-9}$	$0.29527217{\cdot}10^{-9}$
50	$0.07585901{\cdot}10^{-18}$	$0.22301750{\cdot}10^{-18}$	$0.25004662{\cdot}10^{-18}$	$0.25112237{\cdot}10^{-18}$
70	$0.01219941{\cdot}10^{-27}$	$0.11512297{\cdot}10^{-27}$	$0.19133604{\cdot}10^{-27}$	$0.21357399{\cdot}10^{-27}$
100	$0.00126202{\cdot}10^{-41}$	$0.09314437{\cdot}10^{-41}$	$0.28000048{\cdot}10^{-41}$	$0.52971556{\cdot}10^{-41}$

probabilities up to a reasonable accuracy. Increasing L would give no numerical difficulties, so that the accuracy is just a matter of choice. This makes this approach well-suited for calculating tail probabilities.

6 Conclusions

In the commonly used approaches to the discrete-time bulk service queue, the stationary queue length follows from the roots inside or outside the unit disk of a characteristic equation. We have presented representations of these roots as Fourier series, making the classic approach transparent and explicit. The Fourier series are easy to implement and numerically stable.

We further presented analytic formulas and asymptotic approximations for the stationary queue length distribution that do not depend on roots. The results are explicit and well-suited for determining tail probabilities up to a high accuracy.

A Appendix

A.1 Proof of Lemma 3.11

We have $z(\alpha) = z_0(e^{i\alpha})$ where $z_0(w)$ is the solution of

$$z_0(w) = wG(z_0(w)), \quad G(z) = A^{1/s}(z), \tag{84}$$

see proof of Lemma 3.3.

For the proof of Lemma 3.11 we should check, see [25], p.125, Exercise 109, whether the "angular velocity"

$$\varphi'(\alpha) := \frac{d}{d\alpha}[\arg z(\alpha)] = \mathrm{Re}\Big[w\frac{z_0'(w)}{z_0(w)}\Big], \quad w = e^{i\alpha}, \tag{85}$$

is positive for $\alpha \in [0, 2\pi]$. We compute from (84)

$$z_0'(w) = \frac{G(z_0(w))}{1-v}, \quad v = \frac{z_0(w)G'(z_0(w))}{G(z_0(w))}. \tag{86}$$

Using (84) once more, we find with v as in (86)

$$\varphi'(\alpha) = \mathrm{Re}\Big[\frac{1}{1-v}\Big]. \tag{87}$$

When $w = e^{i\alpha}$, $\alpha \in [0, 2\pi]$, there holds that $z_0(w) \in \mathcal{S}_{A,s}$ and by Condition 3.9 we have

$$\Big|\frac{z_0(w)}{G(z_0(w))}\Big| = 1, \quad |G'(z_0(w))| < 1. \tag{88}$$

Hence we have $|v| < 1$ for v in (86). Finally, for $|v| < 1$,

$$\mathrm{Re}\Big[\frac{1}{1-v}\Big] \geq \frac{1}{1+|v|} > \frac{1}{2}. \tag{89}$$

This implies that $\varphi'(\alpha) > 1/2$ so that the proof is complete. □

A.2 Proof of Lemma 5.1

Assume we have a $z \geq 1$ such that $h'(z) = 0$, and let z_0 be the smallest such z. We have

$$h'(z) = \frac{lA'(z)}{A(z)} - \frac{n}{z} = 0 \quad \Leftrightarrow \quad \sum_{j=0}^{\infty} a_j(lj - n)z^j = 0. \tag{90}$$

Let $z > z_0$, and let j_0 be such that

$$lj - n \leq 0, \quad j = 0, 1, \ldots, j_0 \ ; \quad lj - n > 0, \quad j = j_0 + 1, \ldots. \tag{91}$$

Then

$$\begin{aligned} \sum_{j=0}^{\infty} a_j(lj-n)z^j &= \sum_{j=0}^{j_0} a_j(lj-n)z_0^j\Big(\frac{z}{z_0}\Big)^j + \sum_{j=j_0+1}^{\infty} a_j(lj-n)z_0^j\Big(\frac{z}{z_0}\Big)^j \\ &\geq \sum_{j=0}^{j_0} a_j(lj-n)z_0^j\Big(\frac{z}{z_0}\Big)^{j_0} + \sum_{j=j_0+1}^{\infty} a_j(lj-n)z_0^j\Big(\frac{z}{z_0}\Big)^{j_0} \\ &= \Big(\frac{z}{z_0}\Big)^{j_0} \sum_{j=0}^{\infty} a_j(lj-n)z_0^j = 0, \end{aligned} \tag{92}$$

and there is equality if and only if $a_{j_0} = 1$, $a_j = 0$, $j \neq j_0$. The case of equality is excluded by the observation that $n \geq ls$ implies $j_0 \geq s$ which contradicts $A'(1) = j_0 < s$.

A.3 Proof of Lemma 5.2

Using $A(z_0) = lz_0A'(z_0)/n$, we have

$$\begin{aligned} l^2z_0^2A''(z_0) - n(n-l)A(z_0) &= l^2z_0^2A''(z_0) - l(n-l)z_0A'(z_0) \\ &= l\sum_{j=0}^{\infty}(lj(j-1) - (n-l)j)a_jz_0^j \\ &= l\sum_{j=0}^{\infty} j(lj-n)a_jz_0^j. \end{aligned} \tag{93}$$

As in the proof of Lemma 5.1 we let j_0 such that (91) holds. Then we see that

$$\begin{aligned} \sum_{j=0}^{\infty} j(lj-n)a_jz_0^j &= \sum_{j=0}^{j_0} j(lj-n)a_jz_0^j + \sum_{j=j_0}^{\infty} j(lj-n)a_jz_0^j \\ &\geq \sum_{j=0}^{j_0} j_0(lj-n)a_jz_0^j + \sum_{j=j_0}^{\infty} j_0(lj-n)a_jz_0^j \\ &= j_0\sum_{j=0}^{\infty}(lj-n)a_jz_0^j = 0, \end{aligned} \tag{94}$$

with equality if and only if $a_{j_0} = 1$, $a_j = 0$, $j \neq j_0$ (to be excluded since $j_0 \geq s$). This completes the proof.

Acknowledgement

The authors like to thank Onno J. Boxma for his help in proving Theorem 4.1.

References

[1] Abate, J., W. Whitt (1992). Numerical inversion of probability generating functions, *Operations Research Letters* **12**: 245-251.

[2] Ackroyd, M. (1980). Computing the waiting time distribution for the $G/G/1$ queue by signal processing methods, *IEEE Transactions on Communications* **28**: 52-58.

[3] Ackroyd, M. (1986). Approximate characterisation of nonstationary discrete time $G/G/1$ systems, *Performance Evaluation* : 52-58.

[4] Adan, I.J.B.F., Y.Q. Zhao (1996). Analyzing $GI/E_r/1$ queues. *Operations Research Letters* **19**: 183-190.

[5] Bailey, N.T.J. (1954). On queueing processes with bulk service. *Journal of the Royal Statistical Society* **16**: 80-87.

[6] Bruneel, H. (1984). A general model for the behaviour of infinite buffers with periodic service opportunities. *European Journal of Operational Research* **16**: 98-106.

[7] Bruneel, H., B.G. Kim (1993). *Discrete-Time Models for Communication Systems including ATM*, Kluwer Academic Publishers, Dordrecht.

[8] Bruneel, H., I. Wuyts (1994). Analysis of discrete-time multiserver queueing models with constant service times. *Operations Research Letters* **15**: 231-236.

[9] Chaudhry, M.L., C.M. Harris, W.G. Marchal (1990). Robustness of rootfinding in single-server queueing models, *ORSA Journal on Computing* **3**: 273-286.

[10] Chaudhry M.L., N.K. Kim (2003). A complete and simple solution for a discrete-time multi-server queue with bulk arrivals and deterministic service times. *Operations Research Letters* **31**: 101-107.

[11] Chaudhry, M.L., J.G.C. Templeton (1983). *A First Course in Bulk Queues*, John Wiley & Sons, New York.

[12] Cohen, J.W. (1982). *The Single Server Queue*, North-Holland, Amsterdam.

[13] Crommelin, C.D. (1932). Delay probability formulae when the holding times are constant. *Post Office Electrical Engineers Journal* **25**: 41-50.

[14] De Bruijn, N.G. (1981). *Asymptotic Methods in Analysis*, Dover Publications, New York.

[15] Denteneer, T.J.J., A.J.E.M. Janssen, J.S.H. van Leeuwaarden (2003). Moment series inequalities for the discrete-time multi-server queue. *Eurandom report series* 017; submitted for publication.

[16] Denteneer, T.J.J., J.S.H. van Leeuwaarden, J.A.C. Resing (2003). Bounds for a discrete-time multi-server queue with an application to cable networks. In *Proceedings of ITC 18*, Elsevier, Berlin, pp. 601-612.

[17] Downton, F. (1955). Waiting time in bulk service queues, *J. Roy. Stat. Soc. B*, **17**: 256-261.

[18] Duren, P.L. (1983). *Univalent Functions*, Springer, New York.

[19] Eenige, M.J.A. van (1996). *Queueing Systems with Periodic Service*, Ph.D. thesis, Technische Universiteit Eindhoven.

[20] Harris, C.M., W.G. Marchal, R.W. Tibbs (1992). An algorithm for finding characteristic roots of quasi-triangular Markov chains. In *Queueing and Related Models*, U.N. Bhat, Editor, Oxford University Press.

[21] He, J., K. Sohraby (2001). A new analysis framework for discrete time queueing systems with general stochastic sources. *IEEE Infocom 2001*, New York.

[22] Janssen, A.J.E.M., J.S.H. van Leeuwaarden (2003). A discrete queue, Fourier sampling on Szegö curves, and Spitzer's formula. *Eurandom report series* 018; submitted for publication.

[23] Kakeya, S. (1912). On the limits of roots of an algebraic equation with positive coefficients. *Tôhoku Mathematical Journal* **2**: 140-142.

[24] Norimatsu, T, H. Takagi, H.R. Gail (2002). Performance analysis of the IEEE 1394 serial bus. *Performance Evaluation* **50**: 1-26.

[25] Pólya, G., G. Szegö (1972). *Problems and Theorems in Analysis. Volume I*, Springer-Verlag, New York.

[26] Powell, W.B. (1985). Analysis of vehicle holding cancellation strategies in bulk arrival, bulk service queues. *Transportation Science* **19**: 352-377.

[27] Servi, L.D. (1986). $D/G/1$ queues with vacations. *Operations Research* **34**: 619-629.

[28] Silverman, H. (1975). *Complex Variables*, Houghton Mifflin, Boston.

[29] Titchmarsh, E.C. (1939). *The Theory of Functions*, 2^{nd} edition, Oxford University Press, New York.

[30] Whittaker, E.T. and Watson, G.N. (1963), *A Course of Modern Analysis*, 4^{th} edition, Cambridge University Press, Cambridge.

[31] Zhao, Y.Q., L.L. Campbell (1995). Performance analysis of a multibeam packet satellite system using random access techniques. *Performance Evaluation* **24**: 231-244.

[32] Zhao, Y.Q., L.L. Campbell (1996). Equilibrium probability calculations for a discrete-time bulk queue model. *Queueing Systems* **22**: 189-198.