EXPONENTIAL BEHAVIOR IN THE PRESENCE OF DEPENDENCE IN RISK THEORY

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Abstract

We consider an insurance portfolio situation where there is possible dependence between the waiting time for a claim and its actual size. By employing the underlying random walk structure we obtain rather explicit exponential estimates for infinite and finite time ruin probabilities in the case of light-tailed claim sizes. The results are illustrated with several examples worked out for specific dependence structures.

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1. INTRODUCTION

Classical risk theory describing characteristics of the surplus process of a portfolio of insurance policies usually relies on the assumption of independence between claim sizes and claim inter-occurrence times. However, in many applications this assumption is too restrictive and generalizations to dependent scenarios are called for. In recent years, a number of results on ruin probabilities have been obtained for models that allow for specific types of dependence (see [2] for a survey on the subject).

One traditional technique to derive results in risk theory is to describe the surplus

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process as a random walk with independent increments between two claim instances. It is well-known that if the Laplace transform of the distribution of the increments exists in a left neighborhood of the origin, then the asymptotic behavior of ruin probabilities in infinite and finite time are determined by properties of the Laplace transform in that region.

In this paper we take up this random walk approach. However, we allow the interclaim time and its subsequent claim size to be dependent according to an arbitrary copula structure, thus separating the dependence behavior from the properties of the marginal distributions. The introduction of dependence modifies the shape of the Laplace transform, but the random walk structure is preserved and one can derive asymptotic results for the ruin function by studying properties of this Laplace transform. This approach seems to be new; the present paper is not meant to be an exhaustive treatment of the subject - it should rather be seen as a starting point.

Section 2 gives some preliminaries on random walk techniques and their connection with ruin theory. In Section 3 the Laplace transform of the increment distribution of the random walk is introduced and some of its properties are discussed. In Section 4 we rederive the Cramér-Lundberg approximation for the infinite-time ruin probability in terms of random walk quantities and discuss the behavior of the adjustment coefficient in the presence of dependence. Rather explicit exponential estimates of finite-time ruin probabilities for these dependent scenarios are given in Sections 5 and 6 and discussed in some detail for several specific dependence structures.

2. PRELIMINARIES

We start by introducing the main quantities from both ruin and random walk theory.

2.1. Portfolio Quantities

Let claim sizes arrive according to a renewal process with interclaim times $\{T_i, i = 1, 2, \ldots\}$ and $T_0 = 0$. The generic interclaim time T has distribution F_T. Let the claim sizes $\{U_i, i = 1, 2, \cdots\}$ form another renewal process generated by the random variable U with distribution F_U. We assume that there is constant payment of premiums at a rate c. We will normally assume that the bivariate process $\{(U_i, T_i); i = 1, 2, \cdots\}$

is a bivariate renewal process, generated by the pair (U, T). For then, the quantities $\{X_i := U_i - cT_i \ , i \geq 1\}$ are i.i.d. which is necessary for the random walk approach that we are following. The random variable $X = U - cT$ will be called the *generic variable.* As a special case we mention the famous Sparre Andersen model where the two processes are independent.

Denote by R_n the risk reserve immediately after payment of the $n-$th claim. Then obviously $R_0 = u$ is the initial reserve, while for $n \geq 0$

$$R_{n+1} = R_n + cT_{n+1} - U_{n+1} \ .$$

2.2. Random Walks

To introduce a random walk, define for all $n \geq 1$

$$X_n = U_n - cT_n$$

which can be interpreted as the loss between the $(n-1)-$th and the $n-$th claim. Then with $S_0 = 0$ and

$$S_{n+1} = S_n + X_{n+1} \quad , \quad n \geq 0$$

we can express $R_n = u - S_n$ in terms of the random walk $\{S_n\}$. Let

$$K(x) = P\{X \leq x\} = P\{U - cT \leq x\}$$

be the distribution of the generic variable X. Without further ado we assume that

$$EX = EU - cET < 0$$

as otherwise the company will be ruined with probability 1.

We use the terminology and the notation from general random walk theory (see e.g. [11]). At every instant, the random walk $\{S_n; n \geq 0\}$ itself is determined by the convolution

$$K^{*n}(x) = P\{S_n \leq x\}$$

where $K^{*0}(x)$ is the unit-step distribution at the origin. The characteristic function of X will be denoted by

$$\kappa(\zeta) := E\{\exp(i\zeta X)\} \quad , \quad \zeta \in \mathbb{R}.$$

The following quantities are among the prime study objects in random walk theory. The *first upgoing ladder index* is defined by $N := \inf\{n > 0 : S_n > 0\}$ and the corresponding *first upgoing ladder height* is then S_N. A famous result by Baxter [11] gives the hybrid transform of the pair (N, S_N). For $|s| < 1$ and $\theta \geq 0$,

$$E\{s^N\, e^{-\theta S_N}\} = 1 - \exp\left\{ -\sum_{n=1}^{\infty} \frac{s^n}{n} \int_{0+}^{\infty} e^{-\theta x} dK^{*n}(x) \right\} . \quad (1)$$

In the Feller notation [5], the above hybrid transform appears as the *right Wiener-Hopf factor* of the characteristic function $\kappa(.)$. By this we mean that

$$1 - s\kappa(\zeta) = (1 - \chi(s,\zeta))(1 - \tilde{\chi}(s,\zeta)) \quad (2)$$

where $\chi(s,\zeta) := E\{s^N\, e^{i\zeta S_N}\}$ and the quantity $\tilde{\chi}(.,.)$ similarly refers to the (weak) downgoing ladder index and ladder height.

The *maxima* of the random walk are defined by $M_0 = 0$ and for $n \geq 1$ by

$$M_n = max(0, S_1, \cdots, S_n).$$

We denote the distribution of M_n by $G_n(x) := P\{M_n \leq x\}$. The *supremum* of the random walk is defined by $M_\infty = \sup(0, S_1, S_2, \cdots)$ and further $G(x) := P\{M_\infty \leq x\}$.

A classification quantity that often appears is given by

$$B(s) := \sum_{n=1}^{\infty} \frac{s^n}{n}\, P(S_n > 0) .$$

Now, $B := B(1) < \infty$ iff $M_\infty < \infty$ a.s.; moreover then $\limsup S_n = -\infty$. In particular since $EX < 0$, automatically $B < \infty$.

For further reference it is necessary to include information on the distributions $\{G_n(.)\}$. We introduce the generating function for this sequence. Let $|s| < 1$ and define

$$G(s,x) := \sum_{n=0}^{\infty} G_n(x)\, s^n . \quad (3)$$

It follows from the Spitzer-Baxter identity [5, 11] that

$$\int_0^{\infty} e^{i\zeta x} G(s,dx) \;=\; \frac{e^{-B(s)}}{(1-s)(1-\chi(s,\zeta))} . \quad (4)$$

Part of the Wiener-Hopf factor in (2) appears in the expression for the Laplace transform of the supremum. Indeed, it follows from (4) that at least for $\theta \geq 0$

$$\int_0^{\infty} e^{-\theta x} dG(x) = \exp\left\{ -\sum_{n=1}^{\infty} \frac{1}{n} \int_0^{\infty} (1 - e^{-\theta x})\, dK^{*n}(x) \right\} \;=\; \frac{e^{-B}}{1 - E(e^{-\theta S_N})} . \quad (5)$$

2.3. Connections

The links between the above random walk concepts and the risk quantities introduced in the beginning are straightforward. Let us define the *time of ruin with initial reserve* u as

$$\tau(u) := \inf\{n : u < S_n\} .$$

Then ruin will occur at the $n-$th claim if the total loss expressed in terms of the random walk S_n has annihilated the initial surplus. In terms of the maximum we get the fundamental relation

$$\{\tau(u) > n\} = \{M_n \leq u\} . \tag{6}$$

This equation immediately implies that ruin will occur *in finite time* but after the $n-$th claim if and only if M_n is not overshooting u but M_∞ will. Hence

$$P\{n < \tau(u) < \infty\} = P\{M_n \leq u < M_\infty\} = G_n(u) - G(u) . \tag{7}$$

3. THE GENERIC VARIABLE

While not fully necessary we will assume from now on that the joint distribution function $F_{U,T}(u,t) = P(U \leq u, T \leq t)$ has a bivariate density $f_{U,T}$. We are interested in the distribution K of the generic variable $X = U - cT$. Recall that $EX = EU - cET < 0$. Obviously, the density of X exists and is given by

$$k(z) = \frac{1}{c} \int_0^\infty f_{U,T}\Big(u, \frac{u-z}{c}\Big)\, du . \tag{8}$$

The characteristic function $\kappa(\zeta)$ of K can be obtained from the joint characteristic function

$$E\{e^{i\zeta_1 U + i\zeta_2 T}\} = \int_0^\infty du \int_0^\infty dt\, e^{i\zeta_1 u + i\zeta_2 t} f_{U,T}(u,t)$$

by choosing $\zeta_1 = \zeta$ and $\zeta_2 = -c\zeta$. In general one cannot be sure that $\kappa(\zeta)$ exists for any non-real value of ζ.

3.1. Double Laplace Transform

As shown by Widder [16], the distribution K will have an exponentially bounded right tail if and only if the double Laplace transform $\hat{K}(\theta) := \kappa(i\theta)$ converges in a left neighborhood of the origin. We will therefore replace ζ by $i\theta$ to obtain the (two-sided)

Laplace transform of X, rather than the characteristic function. Alternatively, the *left abscissa of convergence* $-\sigma_K$ of $\hat{K}(\theta)$ should be strictly negative (in which case we call K to be super-exponential). We will generally write $-\sigma_Y = -\sigma_H$ for the left abscissa of convergence of the Laplace transform of a random variable Y with distribution H. So the main object of study is

$$\hat{K}(\theta) = \int_{u=0}^{\infty} \int_{t=0}^{\infty} e^{-\theta(u-c\,t)} f_{U,T}(u,t)\, du\, dt\ . \tag{9}$$

and we will restrict our analysis in this paper to cases where $\sigma_X > 0$. Notice that the balance condition tells us that $\hat{K}'(0) = -EX > 0$. Also note that since $T \geq 0$, $X \leq U$. Hence we always have $\sigma_X \geq \sigma_U \geq 0$ meaning that exponentially bounded claim sizes automatically lead to an exponentially bounded generic variable.

It is well-known that every joint distribution function can be expressed as a copula function of its marginal distributions (this copula representation being unique for continuous multivariate distribution functions), so that we have $F_{U,T}(u,t) = C(F_U(u), F_T(t))$ for some copula C. This approach allows to completely separate the dependence structure from the properties of the univariate marginals (for a survey on copulas we refer to Joe [8]). In what follows we try to formulate our results in terms of copulas. Using the identity $1 - F_{U,T}(x,\infty) - F_{U,T}(\infty, y) + F_{U,T}(u,t) = \int_x^\infty du \int_y^\infty dt\, f_{U,T}(u,t)$, we obtain for every $\theta > -\sigma_X$

$$\begin{aligned}\hat{K}(\theta) - \hat{F}_U(\theta) - \hat{F}_T(-c\,\theta) + 1 = \\ - c\,\theta^2 \int_0^1 e^{-\theta F_U^{-1}(a)} dF_U^{-1}(a) \int_0^1 e^{c\theta F_T^{-1}(b)} dF_T^{-1}(b)(1 - a - b + C(a,b))\end{aligned} \tag{10}$$

or equivalently

$$\begin{aligned}\hat{K}(\theta) = \hat{F}_U(\theta)\hat{F}_T(-c\,\theta) \\ - c\,\theta^2 \int_0^1 e^{-\theta F_U^{-1}(a)} dF_U^{-1}(a) \int_0^1 e^{c\theta F_T^{-1}(b)} dF_T^{-1}(b)\ (C(a,b) - ab).\end{aligned} \tag{11}$$

If the copula function is absolutely continuous, we can also write (9) as

$$\hat{K}(\theta) = \int_{u=0}^{\infty} \int_{t=0}^{\infty} e^{-\theta(u-c\,t)} f_U(u) f_T(t)\ c(F_U(u), F_T(t))\, du\, dt, \tag{12}$$

where $c(a,b) = \frac{\partial^2 C(a,b)}{\partial a \partial b}$.

We will now shortly discuss three simple copulas that can be viewed as extremal cases of dependence:

Example 3.1. *Independence copula*

The independence copula is given by $C_I(a,b) := a\,b$ and we will denote the corresponding distribution by K_I. If U and T are independent, we have $\hat{K}_I(\theta) = \hat{F}_U(\theta)\hat{F}_T(-c\theta)$. Clearly then $\hat{K}_I(\theta)$ exists for all $\theta \in (-\sigma_U\ ,\ \frac{1}{c}\sigma_T)$. Hence $\sigma_{K_I} = \sigma_U$.

Note that for an arbitrary copula we have $\hat{K}'(0) = -E(U - cT) = \hat{K}_I'(0)$, since this is a property of the marginal distributions U and T only. However, the difference of the second derivatives $\hat{K}''(0)$ and $\hat{K}_I''(0)$ already reflects the dependence structure through the covariance of U and T ($\hat{K}''(0) < \hat{K}_I''(0)$ for $Cov(U, I) > 0$ and conversely).

Example 3.2. *Comonotone copula*

The strongest possible positive dependence between U and T is attained for the comonotone copula $C_M(a,b) := \min(a,b)$, corresponding to the distribution K_M. This copula is singular and its Laplace transform is given by

$$\hat{K}_M(\theta) = \int_0^\infty e^{-\theta(u - cF_T^{-1}(F_U(u)))} f_U(u)\,du.$$

In the special case of exponential marginal distributions U and T (with parameters λ_1, λ_2, resp.), one obtains

$$\hat{K}_M(\theta) = \frac{\lambda_1}{\lambda_1 + \theta(1 - c\,\lambda_1/\lambda_2)}\,.$$

For the comonotone (and for some related) copulas, one can construct examples of heavy-tailed distributions F_U that still lead to $\sigma_K > 0$.

Example 3.3. *Countermonotone copula*

The strongest possible negative dependence between U and T is attained for the (singular) countermonotone copula $C_W(a,b) = \max(a + b - 1, 0)$ and linked to the distribution K_W . The corresponding Laplace transform can be derived as

$$\hat{K}_W(\theta) = \int_0^\infty e^{-\theta(u - cF_T^{-1}(1 - F_U(u)))} f_U(u)\,du = \int_0^1 e^{-\theta(F_U^{-1}(v) - cF_T^{-1}(1-v))}\,dv.$$

Trivially $F_U^{-1}(v) - cF_T^{-1}(1 - v)) \leq F_U^{-1}(v)$, so that for $\theta \leq 0$

$$\hat{K}_W(\theta) \leq \int_0^1 e^{-\theta F_U^{-1}(v)}\,dv = \hat{F}_U(\theta),$$

implying $\sigma_{K_W} \geq \sigma_U$.

In the special case of exponential marginal distributions U and T (with parameters λ_1, λ_2, resp.), one obtains in terms of a beta-function

$$\hat{K}_W(\theta) = B\left(1 + \frac{\theta}{\lambda_1}\,,\, 1 - \frac{c\theta}{\lambda_2}\right), \qquad -\lambda_1 < \theta < c\,\lambda_2\,.$$

Note that the above comonotone and the countermonotone copulas are those degenerate cases of bivariate dependence, where one random variable is a deterministic function of the other.

Remark 3.1. Since any copula $C(a,b)$ is itself a joint distribution function with uniform marginals, we have $C_W(a,b) \le C(a,b) \le C_M(a,b)$ for all $0 \le a, b \le 1$ (often referred to as the Fréchet-Hoeffding bounds). By virtue of (10), we thus obtain that for fixed marginals the Laplace transform $\hat{K}(\theta)$ is bounded by

$$\hat{K}_M(\theta) \le \hat{K}(\theta) \le \hat{K}_W(\theta)$$

for those values of θ, where the quantities are defined.

4. INFINITE TIME RUIN

Due to the connection between ruin and the random walk we have that $P(\tau(u) < \infty) = 1 - G(u)$ where $G(u) = P(M_\infty \le u)$ is given by (4). From the Wiener-Hopf factorization (2) at $s = 1$ we know that

$$1 - \hat{K}(\theta) = (1 - E(e^{-\theta S_N}))(1 - \tilde{\chi}(1, i\theta))\,.$$

But then the abscissa of convergence of $\hat{K}(\theta)$ is the same as that of $E(e^{-\theta S_N})$ and therefore also that of G. Hence $\sigma_K = \sigma_G$.

Now assume that there exists an *adjustment coefficient* $R > 0$ for which $E(e^{RS_N}) = 1$. The Wiener-Hopf factorization above then implies $\hat{K}(-R) = 1$. We put $\beta := \theta + R$ in (5) to get

$$\int_0^\infty e^{-\beta x} d\left(\int_0^x e^{Ry} dG(y)\right) = \frac{e^{-B}}{1 - E(e^{-\beta \tilde{S}_N})}$$

where

$$P(\tilde{S}_N \le x) := \int_0^x e^{Ry} dP(S_N \le y)\,. \tag{13}$$

It is clear that the function $H_1(x) := \int_0^x e^{Ry} dG(y)$ is then a renewal function. By Blackwell's renewal theorem (cf. [4]) we have that

$$H_1(x+y) - H_1(x) \stackrel{\mathcal{D}}{\to} \frac{e^{-B}}{E(\tilde{S}_N)}\, y =: c_1\, y$$

when $x \to \infty$. But since $dG(x) = e^{-Rx}\, dH_1(x)$ we have

$$e^{Ru}(1-G(u)) = e^{Ru} \int_u^\infty e^{-Rx} dH_1(x) = \int_0^\infty e^{-Rw}\, H_1(u+dw) \to c_1 \int_0^\infty e^{-Rw} dw = \frac{c_1}{R}\ .$$

If we return to the original quantities, we find that for initial capital $u \to \infty$

$$P(\tau(u) < \infty) \sim \frac{e^{-B}}{R\, E(S_N\, e^{RS_N})}\, e^{-R\,u}\ , \tag{14}$$

which completes a particularly transparent proof in the spirit of Feller [5] of the well-known Cramér-Lundberg approximation for the infinite time ruin probability. It has been derived in various other ways in the literature (see e.g. [12]). In the above version, the constant in the approximation is expressed as a function of quantities related to the underlying random walk.

Remark 4.1. Note that the classical form of the Cramér-Lundberg approximation for the compound Poisson model, where U and $T \sim \text{Exp}(1/\lambda)$ are independent, can be retained from (14) by using the corresponding Wiener-Hopf factorization

$$1 - \frac{\lambda/c}{\lambda/c - i\zeta}\, E[e^{i\zeta U}] = \left(1 - \frac{\lambda/c}{\lambda/c - i\zeta}\right)\left(1 - \frac{\lambda}{c}\frac{1 - E[e^{i\zeta U}]}{i\zeta}\right),$$

from which it follows that

$$E[e^{-\theta S_N}] = 1 + \frac{\lambda}{c\,\theta}(1 - E[e^{-\theta U}])$$

and thus

$$E(S_N\, e^{RS_N}) = \frac{\lambda\, E(Ue^{RU}) - c}{cR}.$$

This together with $e^{-B} = \mathbb{P}(N = \infty) = 1 - \lambda\, E(U)/c$ leads to the well-known expression

$$P(\tau(u) < \infty) \sim \frac{c - \lambda\, E(U)}{\lambda E(Ue^{RU}) - c}\, e^{-R\,u}, \quad u \to \infty.$$

In the general case, it follows from (14) that the asymptotic behavior of the ruin probability is determined by the value of the adjustment coefficient R defined by

$\hat{K}(-R) = 1$. Let us fix the marginal distributions of U and T, and define R_I to be the adjustment coefficient in the case of independence of U and T, i.e. $\hat{F}_U(-R_I)\hat{F}_T(c\,R_I) = 1$. If now U and T are positively quadrant dependent (that is $P(U > u, T > t) \geq P(U > u)P(T > t)$ for all $0 \leq u, t < \infty$), then we have $C(a,b) \geq a\,b$ for all $0 \leq a, b \leq 1$ for its copula and thus it follows from (11) that

$$\hat{K}(\theta) \leq \hat{K}_I(\theta) \quad \text{for all } \theta \in (-\sigma_K, 0), \tag{15}$$

so that $R > R_I$. Conversely, for negatively quadrant dependent variables U and T we get

$$\hat{K}(\theta) \geq \hat{K}_I(\theta) \quad \text{for all } \theta \in (-\sigma_K, 0), \tag{16}$$

implying $R < R_I$. In order to quantify the difference of R and R_I, one can use the Lagrange expansion, by which the value of R can be expressed in terms of properties of the Laplace transform at the value of the adjustment coefficient of the independence case. In that way we obtain

$$\begin{aligned} -R &= -R_I + \sum_{n=1}^{\infty} \frac{d^{n-1}}{dw^{n-1}} \left(\frac{w + R_I}{\hat{K}(w) - \hat{K}(-R_I)} \right)^n \Bigg|_{w=-R_I} \cdot \frac{(1 - \hat{K}(-R_I))^n}{n!} \\ &= -R_I + \frac{1 - \hat{K}(-R_I)}{\hat{K}'(-R_I)} - \frac{1}{2} \frac{\hat{K}''(-R_I)}{\hat{K}'(-R_I)^3} (1 - \hat{K}(-R_I))^2 + \ldots, \end{aligned}$$

the series being convergent as long as the inverse of $\hat{K}(\theta)$ is analytic in the domain under consideration and $\hat{K}'(-R_I) \neq 0$. This formula is particularly useful for investigating the sensitivity of the adjustment coefficient on the presence of dependency between U and T. Some specific examples, where R can even be expressed explicitly as a function of a dependence measure will be given in the next section.

Remark 4.2. Although quadrant dependence is one of the weakest dependence concepts, due to (11) it turns out to be sufficient for deriving inequalities for the adjustment coefficient. Other dependence concepts such as association, tail monotonicity, stochastic monotonicity and likelihood ratio dependence all imply quadrant dependence and thus inequalities (15) and (16) follow accordingly for these concepts.
In general, whenever there is a concordance ordering among two copulas $C_1(a,b)$ and $C_2(a,b)$ (i.e. $C_1(a,b) \geq C_2(a,b)\ \forall\, 0 \leq a, b \leq 1$), then by (11) we have that $R_1 \geq R_2$ (for a survey of dependence concepts and orderings we refer to Joe [8] or Nelsen [10]).

5. FINITE TIME RUIN

We adapt a result from the literature on random walk theory [13, 14]. The exponential speed of convergence of a random walk towards its upper limit immediately translates into the following finite time ruin estimate for our risk process.

Theorem 1. *Assume that*

(i) $-\infty \le EX < 0$;

(ii) $\hat{K}(\theta)$ *converges for* $-\sigma_K < \theta \le 0$ *where* $\sigma_K > 0$*;*

(iii) for some $\omega \in (0\,,\,\sigma_K)$, $\hat{K}(\theta)$ *attains a minimum* $\hat{K}(-\omega) := \gamma < 1$.

Then for all finite $u \ge 0$ *as* $n \to \infty$

$$P\{n < \tau(u) < \infty\} \sim c\,H(u)\,\gamma^n\,n^{-3/2} \tag{17}$$

where c *is a known constant and* H *a function solely depending on* u.

The quantity $c = \frac{\gamma}{1-\gamma}\,c_1$ where $c_1^2 = \gamma(2\pi\omega^2\hat{K}''(-\omega))^{-1}$ is given explicitly. For more information about the function H, see Section 6.
Let us have a closer look at the conditions of the theorem. Condition (i) is of course the balance condition, necessary for the eventual survival of the portfolio. The second condition has already been discussed in Section 3. Clearly, the existence of an adjustment coefficient R is sufficient for both (ii) and (iii).

Example 3.1 continued: *Independent Case*
Assume that $\sigma_{K_I} = \sigma_U > 0$. Then $-\omega$ is the solution of

$$\psi_U(\theta) = c\,\psi_T(-c\theta),$$

where $\psi_U(\theta) := -\frac{\hat{F}_U'(\theta)}{\hat{F}_U(\theta)}$ denotes the logarithmic derivative of $\hat{F}_U(\theta)$ (and $\psi_T(\theta)$ is defined analogously). Since U and T are nonnegative random variables, $\psi_U(\theta)$ is monotonically decreasing in $\theta \in (-\sigma_U, \infty)$ and $\psi_T(-c\theta)$ is monotonically increasing in θ over $(-\infty, 0]$. Since at the origin $\psi_U(0) = E(U) < c, E(T) = c\psi_T(0)$, the existence and uniqueness of $-\omega$ is guaranteed.

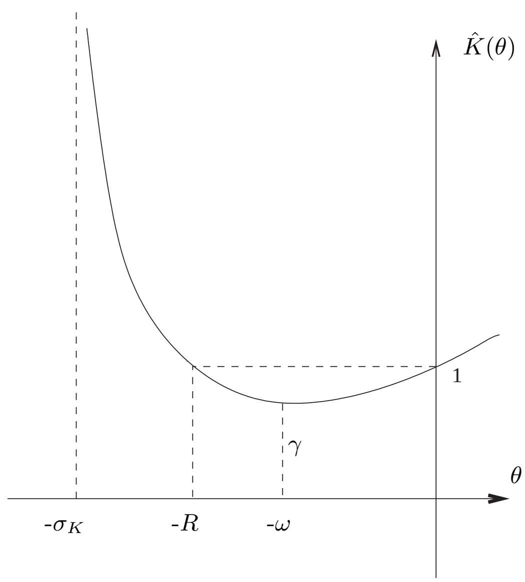

FIGURE 1: The Laplace transform $\hat{K}(\theta)$

Let us now fix the marginals U and T again and consider the behavior of the crucial quantities γ and ω in the presence of dependence. From (11) it follows that $\gamma < \gamma_I$ ($\gamma > \gamma_I$) for positively (negatively, respectively) quadrant dependent U and T, where γ_I corresponds to the case of independence (and Remark 4.2 on other dependence concepts applies here accordingly). However, one cannot establish such inequalities for ω and ω_I (see e.g. Example 5.6, where ω is insensitive to the degree of dependence). $\hat{K}'(\theta)$ is analytic at $-\omega_I$ and thus, if $\hat{K}''(-\omega_I) \neq 0$, we obtain through Lagrange expansion

$$\begin{aligned} -\omega &= -\omega_I + \sum_{n=1}^{\infty} \frac{d^{n-1}}{dw^{n-1}} \left(\frac{w+\omega_I}{\hat{K}'(w) - \hat{K}'(-\omega_I)} \right)^n \Bigg|_{w=-\omega_I} \cdot \frac{(-\hat{K}'(-\omega_I))^n}{n!} \\ &= -\omega_I - \frac{\hat{K}'(-\omega_I)}{\hat{K}''(-\omega_I)} + \frac{1}{2} \frac{\hat{K}'''(-\omega_I)\, \hat{K}'(-\omega_I)^2}{\hat{K}''(-\omega_I)^3} + \ldots \end{aligned}$$

This series converges, if $\omega - \omega_I$ is sufficiently small. By means of Bürmann's theorem (cf. [15]), we can get information on the value of γ directly in terms of properties of $\hat{K}$ at $-\omega_I$, namely

$$\gamma = \hat{K}(-\omega_I) + \sum_{n=1}^{m-1} \frac{d^{n-1}}{dw^{n-1}} \left(\hat{K}'(w)\, \frac{w+\omega_I}{\hat{K}'(w) - \hat{K}'(-\omega_I)} \right)^n \Bigg|_{w=-\omega_I} \cdot \frac{(-\hat{K}'(-\omega_I))^n}{n!} + R_m \tag{18}$$

where the remainder term is given by

$$R_m = \frac{1}{2\pi i}\int_{-\omega_I}^{-\omega}\int_D \left(\frac{\hat{K}'(-\omega)-\hat{K}'(-\omega_I)}{\hat{K}'(t)-\hat{K}'(-\omega_I)}\right)^{m-1} \frac{\hat{K}'(t)\hat{K}''(-\omega)\,dt\,d\omega}{\hat{K}'(t)-\hat{K}'(-\omega)}$$

and D is a contour in the t-plane enclosing the points $-\omega_i$ and $-\omega$ such that the equation $\hat{K}'(t)=\hat{K}'(\zeta)$ has no roots inside or on D except $t=\zeta$, where ζ is any point inside D. The first few terms of (18) are thus given by

$$\gamma = \hat{K}(-\omega_I) - \frac{1}{2}\frac{\hat{K}'(-\omega_I)^2}{\hat{K}''(-\omega_I)} - \frac{\hat{K}'(-\omega_I)^3\hat{K}'''(-\omega_I)}{2\hat{K}''(-\omega_I)^3} + \dots$$

The above expansions provide an approach to obtain sensitivity results on the degree of dependence of the quantities determining the asymptotic behavior of the risk process, if the Laplace transform $\hat{K}(\theta)$ is given for the dependent case. In some cases it might be possible to obtain an empirical Laplace transform from data sets of U and T. In what follows we will illustrate the above result on several examples.

5.1. Some General Cases

In quite a number of cases, a copula C can be decomposed into a convex combination of two more fundamental copulas. Suppose that for some quantity $\alpha \in (0\ ,\ 1)$, the distribution K has copula C given by $C(a,b)=\alpha C_1(a,b)+(1-\alpha)C_2(a,b)$ where for $i=1,2$, C_i is a copula linked to the distribution K_i through the expression (10). If σ_i refers to the abscissa for $K_i, i=1,2$, then the corresponding abscissa σ for K is given by $\sigma=\min(\sigma_1,\sigma_2)$. Moreover, on the interval $(-\sigma\ ,\ 0]$ all three functions $\hat{K}(\theta),\hat{K}_1(\theta)$ and $\hat{K}_2(\theta)$ are positive and convex. In particular, if for $i=1,2$, $\hat{K}_i(\theta)$ has a minimum γ_i at $-\omega_i$, then the minimum γ of $\hat{K}(\theta)$ is attained at a value $-\omega$ satisfying $\min(\omega_1,\omega_2)\le\omega\le\max(\omega_1,\omega_2)$. Moreover, $\gamma \ge \alpha\,\gamma_1+(1-\alpha)\,\gamma_2$.

Example 5.1. *The positive linear Spearman copula*

The positive linear Spearman copula has a particularly simple structure given by

$$C_{\rho_S}(a,b) = \begin{cases} (a+\rho_S(1-a))b, & b\le a \\ (b+\rho_S(1-b))a, & b>a \end{cases}$$

where we assume that $\rho_S \ge 0$. The name stems from the fact that the dependence parameter ρ_S coincides with Spearman's rank correlation coefficient, which is a measure of concordance. Note that there is also a simple relation to Kendall's τ, namely

$$\tau = \frac{1}{3}\rho_S(2 + \mathrm{sgn}(\rho_S)\rho_S).$$

The positive linear Spearman copula is a convex combination of the independent copula and the comonotone copula:

$$C_{\rho_S}(a, b) = (1 - \rho_S)C_\Pi(a, b) + \rho_S C_M(a, b).$$

This copula is an extreme value copula, since its asymptotic tail dependence coefficient $\lambda \in [0, 1]$ defined by $\lim_{\alpha \to 1^-} \frac{1}{1-\alpha}\,(1 - 2\alpha + C_{\rho_S}(\alpha, \alpha))$ (see e.g. [8]) is given by $\lambda = \rho_S$. If the dependence structure of U and T is governed by this copula, then we obtain

$$\hat{K}(\theta) = (1 - \rho_S)\hat{K}_I(\theta) + \rho_S \hat{K}_M(\theta). \tag{19}$$

From (19) it can immediately be seen that for $\rho_S < 1$, the marginal distribution U has to be super-exponential in order to satisfy condition (ii). Moreover, $\sigma_K = \sigma_U$ and ω is the solution of

$$\frac{\hat{K}'_M(-\omega)}{\hat{K}'_I(-\omega)} = -\frac{1 - \rho_S}{\rho_S} < 0.$$

From this it follows that

$$\omega_{\rho_S} > \omega_I \quad \text{and} \quad \gamma_{\rho_S} < \gamma_I.$$

If in addition we assume exponential marginal distributions $F_U(u) = 1 - \exp(-\lambda_1 u)$ and $F_T(t) = 1 - \exp(-\lambda_2 t)$, then, in order to satisfy condition (i) we have to have $c\lambda_1 > \lambda_2$. From (19) we obtain

$$\hat{K}(\theta) = (1 - \rho_S)\frac{\lambda_1}{\lambda_1 + \theta}\,\frac{\lambda_2}{\lambda_2 - c\theta} + \frac{\lambda_1\,\rho_S}{\theta\,(1 - \frac{c\lambda_1}{\lambda_2}) + \lambda_1}.$$

Example 5.2. *The negative linear Spearman copula*

We now assume that $\rho_S \le 0$. The copula is defined by

$$C_{\rho_S}(a, b) = \begin{cases} (1 + \rho_S)ab, & a + b \le 1 \\ ab + \rho_S(1 - a)(1 - b), & a + b > 1. \end{cases}$$

The simple relation to Kendall's τ again prevails. Also here, the negative linear Spearman copula is a convex combination, this time of $C_I(uv)$ and $C_W(a, b)$:

$$C_{\rho_S}(a, b) = (1 + \rho_S)\; C_I(a, b) - \rho_S\; C_W(a, b)$$

and accordingly

$$\hat{K}(\theta) = (1+\rho_S)\hat{K}_I(\theta) - \rho_S \hat{K}_W(\theta). \tag{20}$$

Thus, for $-1 < \rho_S < 0$, the marginal distribution U has to be super-exponential in order to satisfy condition (ii). Moreover, $\sigma_K \leq \sigma_U$ and ω is the solution of

$$\frac{\hat{K}'_W(-\omega)}{\hat{K}'_I(-\omega)} = \frac{1+\rho_S}{\rho_S} < 0$$

so that

$$\omega_{\rho_S} < \omega_I \quad \text{and} \quad \gamma_{\rho_S} > \gamma_I.$$

In case of exponential marginals we obtain in terms of a beta-function

$$\hat{K}(\theta) = (1-\rho_S)\frac{\lambda_1}{\lambda_1+\theta}\,\frac{\lambda_2}{\lambda_2 - c\theta} - \rho_S B\left(1+\frac{\theta}{\lambda_1}\,,\,1-\frac{c\theta}{\lambda_2}\right).$$

Example 5.3. *Farlie-Gumbel-Morgenstern copula*

This is an analytically simple and at the same time absolutely continuous copula given by

$$C(a,b) = ab(1+3\rho_S(1-a)(1-b)),$$

where $-1/3 \leq \rho_S \leq 1/3$ is again Spearman's rank correlation coefficient (and for Kendall's τ we have $\tau = 2\rho_S/3$). Thus this copula allows for weak dependence only. For exponential marginals with parameters as above one obtains

$$\hat{K}(\theta) = \frac{\lambda_1\lambda_2((\theta+2\lambda_1)(2\lambda_2 - c\theta) - 3c\rho_S\theta^2)}{(\theta+\lambda_1)(\theta+2\lambda_1)(\lambda_2 - c\theta)(2\lambda_2 - c\theta)}$$

and the determination of R and ω leads to polynomial equations of order 4 and 5, respectively.

Example 5.4. *Archimedean copulas*

Bivariate Archimedean copulas are an important subclass of copulas defined by

$$C(a,b) = \phi\left(\phi^{-1}(a) + \phi^{-1}(b)\right) \quad \text{for all } 0 \leq a,b \leq 1,$$

where the generator ϕ is the Laplace transform of a non-negative random variable. The concordance measure τ can easily be determined by the generator through

$$\tau = 1 - 4\int_0^\infty s\,(\phi'(s))^2\,ds.$$

Techniques for fitting these types of copulas to given bivariate data sets can be found in [6]. Here we will just state a general monotonicity result. Let us again assume that the marginal distributions of U and T are fixed. Since an Archimedean copula C_1 dominates another Archimedean copula C_2 in concordance order if and only if the function $\phi_1^{-1} \circ \phi_2$ is superadditive (cf. [8]), representation (10) allows to deduce $R_1 > R_2$ and $\gamma_1 < \gamma_2$, whenever the above superadditivity holds.

5.2. Specific Cases

We now deal with a few parametric bivariate distributions for which one can evaluate ω and γ explicitly as a function of the dependence parameter.

Example 5.5. *Moran and Downton's bivariate exponential*
The joint density function is given by

$$f_{U,T}(u,t) = \frac{\lambda_1\lambda_2}{1-\rho} I_0\left(\frac{2\sqrt{\rho\lambda_1\lambda_2\, u\, t}}{1-\rho}\right) \exp\left(-\frac{\lambda_1 u + \lambda_2 t}{1-\rho}\right),$$

where $I_0(z) = \sum_{j=0}^{\infty} \frac{1}{j!^2}\left(\frac{z}{2}\right)^{2j}$ is the modified Bessel function of the first kind and order zero, $0 \le \rho \le 1$ is Pearson's correlation coefficient and $\lambda_1, \lambda_2, u, t > 0$ (cf. [9]). Here the marginal distributions U and T are exponential with parameters λ_1 and λ_2. From the particularly simple structure of the joint moment-generating function we obtain

$$\hat{K}(\theta) = \frac{\lambda_1\lambda_2}{c\theta^2(\rho-1) + \theta(\lambda_2 - c\lambda_1) + \lambda_1\lambda_2}, \tag{21}$$

and thus we have $\sigma_K = \frac{\lambda_2 - c\lambda_1 - \sqrt{(c\lambda_1-\lambda_2)^2 + 4c\lambda_1\lambda_2(1-\rho)}}{2c(1-\rho)}$. The adjustment coefficient is now given by

$$R = \frac{c\lambda_1 - \lambda_2}{c(1-\rho)},$$

which is positive, if $c\lambda_1 > \lambda_2$. But the latter is just the net balance condition (i) for the marginal distributions. From (21) it follows that

$$\omega = \frac{\lambda_2 - c\lambda_1}{2c(\rho-1)} = \frac{R}{2}\,.$$

Furthermore we have

$$\gamma = \hat{K}(-\omega) = \frac{\lambda_1\lambda_2}{\lambda_1\lambda_2 + (c\lambda_1 - \lambda_2)^2 \frac{1}{4c(1-\rho)}}\,.$$

Example 5.6. *Kibble and Moran's bivariate gamma*

This symmetric bivariate distribution with standard gamma marginals (shape parameter $\alpha > 0$) is defined through its joint moment-generating function

$$E(e^{t_1 U + t_2 T}) = \left(1 - \frac{\beta+1}{\beta}\, t_1 - \frac{\beta+1}{\beta}\, t_2 + \frac{\beta+1}{\beta}\, t_1\, t_2\right)^{-\alpha}.$$

Here $\beta > 0$ is the dependence parameter and Pearson's correlation coefficient is given by $\frac{1}{1+\beta}$. We thus have

$$\hat{K}(\theta) = \left(1 - \frac{\beta+1}{\beta}\Big((c-1)\,\theta + c\,\theta^2\Big)\right)^{-\alpha},$$

and $\sigma_K = \frac{(1-c)(1+\beta) - \sqrt{(1-c)^2(1+\beta)^2 + 4c\beta(1+\beta)}}{2c(1+\beta)}$. It follows easily that $R = \frac{c-1}{c}$ and

$$\omega = \frac{c-1}{2c} = \frac{R}{2},$$

which is positive, since condition (i) amounts to $c > 1$ in this case. Note that R and ω are independent of the dependence parameter β. The crucial quantity γ depends on β and is given by

$$\gamma = \left(\frac{4c\beta}{1 + 2c(\beta-1) + \beta + c^2(1+\beta)}\right)^{\alpha}.$$

Example 5.7. *Marshall and Olkin bivariate exponential*

The distribution is defined by

$$\mathbb{P}(U > u, T > t) = e^{-\lambda_1 u - \lambda_2 t - \lambda_3 \max(u,t)}, \quad u, t > 0.$$

In this example, the exponential marginal distributions with parameters $\lambda_1 + \lambda_3$ and $\lambda_2 + \lambda_3$, respectively, are a function of the degree of dependence. Pearson's correlation coefficient is determined through $\frac{\lambda_3}{\lambda_1+\lambda_2+\lambda_3}$ (cf. [9]). We obtain

$$\hat{K}(\theta) = \frac{(\lambda_1 + \lambda_2 + \lambda_3 + \theta(c-1))(\lambda_1 + \lambda_3)(\lambda_2 + \lambda_3) - c\lambda_3\theta^2}{(\lambda_1 + \lambda_3 + \theta)(\lambda_2 + \lambda_3 - c\theta)(\lambda_1 + \lambda_2 + \lambda_3 + \theta(1-c))}.$$

In this case both R and ω are solutions of polynomial equations of third order.

Other bivariate distributions that lead to polynomial equations of low order for ω include Freund's bivariate exponential distribution (order 3) and the bivariate gamma of Cheriyan and Ramabhadran (order 2) (see [9] for their definitions).

5.3. Conditioning on the Event of Ruin

If one conditions on the occurrence of ruin, then if the adjustment coefficient R exists, it is well-known that the asymptotic behavior of the random walk S_n can be studied in terms of its associated random walk $\tilde{S}_n$ defined by (13). For large u, we have

$$P(X_1 \leq x_1, \dots, X_n \leq x_n \,|\, \tau(u) < \infty) \sim P(\tilde{X}_1 \leq x_1, \dots, \tilde{X}_n \leq x_n)$$

(cf. Asmussen [1]), so that the properties of the surplus process conditioned on ruin are determined by the Laplace transform $\hat{K}(\theta)$ shifted by $-R$ to the right. For instance, $E(\tilde{X}_i) = -\hat{K}'(-R) > 0$ so that, conditioned on the occurrence of ruin in finite time, the random walk has a positive drift. Thus, by adapting Theorem B of [14] to our situation, we obtain for large u

$$P(\tau(u) > n \,|\, \tau(u) < \infty) \sim H_2(u)\, \gamma^n \, n^{-3/2} \quad \text{as } n \to \infty. \tag{22}$$

Here $H_2(u)$ is a function depending on u only which can be expressed in terms of quantities related to the random walk (cf. [14]). Hence by studying the behavior of γ for dependent U and T as in the previous sections, one can also derive rather sharp asymptotic results on the finite time ruin probability conditioned on the event of ruin.

6. THE FUNCTION H

This section is devoted to a closer look at the function H whose existence has been used in (17), but whose properties have not been revealed. First of all it follows from [13] that $H(0) = \exp -B(\frac{1}{\gamma})$. For $u > 0$, the function H has been given in a rather complicated form in [13]. However, if we use the Markovian structure of the random walk, then we are able to give a much neater interpretation of H in terms of its Laplace transform.

Let us pin down the first time that the random walk hits its positive maximum. Introduce the auxiliary quantities

$$u_n(x) := P\{S_1 > 0, S_2 > 0, \cdots, S_{n-1} > 0, 0 < S_n \leq x\} \tag{23}$$

and $u(x) := \sum_{n=1}^{\infty} u_n(x)$. If we define

$$L_0 = 0\,,\; L_n = \min\{r \geq 0 : S_r = \max_{0 \leq k \leq n} S_k\}$$

then it is clear that

$$u_n(x) := P\{L_n = n\ ,\ S_n \le x\}\ . \tag{24}$$

If we link these portfolio variables with the random walk, then the Markovian character of the latter allows us to write that

$$P(n < \tau(u) < \infty) = \sum_{k=0}^{n} P(n-k < \tau(0) < \infty)\, P(L_k = k, S_k \le u)$$

(see e.g. [5]). Therefore

$$\frac{P(n < \tau(u) < \infty)}{P(n < \tau(0) < \infty)} = \sum_{k=0}^{n} \frac{P(n-k < \tau(0) < \infty)}{P(n < \tau(0) < \infty)}\ u_k(u)\ .$$

However, from (17) for $u = 0$ we immediately see that $\lim_{n\uparrow\infty} \frac{P(n-k<\tau(0)<\infty)}{P(n<\tau(0)<\infty)} = \gamma^{-k}$ for each fixed k. But it then follows that

$$\frac{H(u)}{H(0)} = \lim_{n\uparrow\infty} \frac{P(n < \tau(u) < \infty)}{P(n < \tau(0) < \infty)} = \sum_{k=0}^{\infty} \gamma^{-k}\ u_k(u)\ .$$

A fundamental relation is the following *Spitzer-Baxter identity* that gives the hybrid transform of the sequence $\{u_n(x)\}$. Let

$$U(s,x) := \sum_{n=0}^{\infty} u_n(x)\, s^n.$$

Then for $|s| < 1$ and $\zeta \in \mathbb{R}$,

$$\tilde{u}(s,\zeta) := \int_0^{\infty} e^{i\zeta x} U(s,dx) = \exp\left\{ \sum_{n=1}^{\infty} \frac{s^n}{n} \int_{0+}^{\infty} e^{i\zeta x} dK^{*n}(x) \right\}\ . \tag{25}$$

In view of (1) we thus have the remarkable formula for the Laplace transform of H

$$\hat{H}(\theta) = \frac{e^{-B(\frac{1}{\gamma})}}{1 - E\left(\gamma^{-N}\, e^{-\theta S_N}\right)}\ , \tag{26}$$

which resembles (5) closely.

To get a better look at the behavior of the function H we introduce *associated random walks.* Define for any $\delta \in (-\sigma_K\ ,\ 0]$ the distribution

$$K_\delta(x) = \frac{1}{\hat{K}(\delta)} \int_{-\infty}^{x} e^{-\delta u}\ dK(u)\ .$$

Then its bilateral Laplace transform is given by

$$\hat{K}_\delta(\theta) = \frac{\hat{K}(\theta+\delta)}{\hat{K}(\delta)}\ .$$

Recall the Laplace transform analogue of expression (2) for the random walk generated by K_δ

$$1 - s\hat{K}_\delta(\theta) = (1 - \chi_\delta(s, i\theta))(1 - \tilde{\chi}_\delta(s, i\theta)) \ .$$

But also $\hat{K}(\theta)$ has its own decomposition. Hence

$$1 - \frac{s}{\hat{K}(\delta)}\hat{K}(\theta+\delta) = \left(1 - \chi\Big(\frac{s}{\hat{K}(\delta)}, i(\theta+\delta)\Big)\right)\left(1 - \tilde{\chi}\Big(\frac{s}{\hat{K}(\delta)}, i(\theta+\delta)\Big)\right) \ .$$

By the uniqueness of the Wiener-Hopf decomposition, this means that

$$\chi_\delta(s, i\theta) = \chi\left(\frac{s}{\hat{K}(\delta)}, i(\theta+\delta)\right)$$

or in terms of ladder quantities

$$E\left(s^{N_\delta}\, e^{-\theta S_{N_\delta}}\right) = E\left(\left(\frac{s}{\hat{K}(\delta)}\right)^N e^{-(\theta+\delta)S_N}\right) \tag{27}$$

where N_δ is the first upgoing ladder index for the associated random walk generated by the distribution K_δ and S_{N_δ} is its corresponding ladder height.

If we now compare this formula with (26) then the substitution $\theta = \delta + \beta$ leads to the equality

$$\int_0^\infty e^{-\beta x} d\left(\int_0^x e^{-\delta u}\, dH(u)\right) = \frac{e^{-B(1/\gamma)}}{1 - E\left(\left(\frac{\hat{K}(\delta)}{\gamma}\right)^{N_\delta} e^{-\beta S_{N_\delta}}\right)}$$

which is valid for $-\sigma_K - \delta < \beta < -\delta$.

In view of (5) it then looks natural to choose δ in such a way that $\hat{K}(\delta) = \gamma$, or $\delta = -\omega$. For then

$$\int_0^\infty e^{-\beta x} d\left(\int_0^x e^{\omega u}\, dH(u)\right) = \frac{e^{-B(1/\gamma)}}{1 - E\left(e^{-\beta S_{N_\omega}}\right)} \ ,$$

valid for $-\sigma_K + \omega < \beta < \omega$.

But now we can repeat the procedure from Section 4. Using a similar application of the renewal theorem leads to the asymptotic expression for $u \to \infty$

$$1 - H(u) \sim \frac{e^{-B(1/\gamma)}}{\omega E(S_{N_\omega})}\, e^{-\omega u} \ . \tag{28}$$

7. CONCLUSION

The random walk approach presented in this paper allowed to extend several rather explicit asymptotic results for the independent risk process to a dependent framework. Moreover, the introduction of copula functions enables to study the dependence structures separated from the marginal behavior of the involved distributions. However, the present paper is just an attempt in trying to get a clearer picture on the impact of dependence in risk theory and a lot of open questions remain for further study. For instance, a similar study for heavy-tailed claims (possibly based on recent results of Baltrūnas [3]) is left for future research.

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