Downside Risk Portfolio Diversification Effects

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Abstract

Risk managers use portfolios to diversify away the unpriced risk of individual securities. In this paper we compare the benefits of portfolio diversification for downside risk with the variance. The risk of a security is decomposed into a part which is attributable to the market risk and an orthogonal risk factor. The orthogonal part consists of an idiosyncratic part and a part which is attributable to dependency between assets, but which cannot be explained by the market risk. We find that the idiosyncratic downside risk evaporates more rapidly than the idiosyncratic variance risk. For this we offer a theoretical explanation on basis of the heavy tail properties of the asset return distributions. We also find that the non-diversifiable non-market factors are more important for the downside risk than for the global risk measure.

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1 Introduction

Risk managers use portfolios to diversify away the unpriced risk of individual securities. This topic has been well studied for global risk measures like the variance, see e.g. the textbook by Elton and Gruber (1995, ch.4). In this paper we study the benefits of portfolio diversification with respect to extreme downside risk known as the zero-th lower partial moment (i.e. the inverse of the VaR -Value at Risk- risk measure). The risk of a security is decomposed into a part which is attributable to the market risk and an orthogonal risk factor. The orthogonal part consists of an idiosyncratic part and a part which is attributable to dependency between assets but which cannot be explained by the market risk, i.e. there are other factors. Let the "diversification speed" denote the inverse of the number of assets that have to be included into the portfolio in order to eliminate the idiosyncratic parts of the orthogonal risk factors. We compare the diversification speed of the downside risk measure with the diversification speed regarding the variance. We find that the idiosyncratic downside risk evaporates at double the speed by which the idiosyncratic variance part disappears. For this we offer a theoretical explanation on basis of the heavy tail properties of the asset return distributions. We also find that the non-diversifiable non-market factors are more important for the downside risk than for the global risk measure. Furthermore, we provide predictions for the downside risk diversification benefits beyond the range of the empirical distribution function.

The diversification speed of the variance risk measure for the idiosyncratic risk part is equal to $1/n$, where n is the number of securities (assuming the idiosyncratic parts of the different securities have comparable variances). With independent risks, the variance of the sum is just the sum of the variances, regardless the distribution of the risks (provided the variance exists). The intuition for the higher diversification speed of the downside risk measure is as follows. For downside risk measures like the lower partial moments the tail shape of the distribution is important, since the tail shape governs the upper boundary of the integral. It is a well known fact that security returns are heavy tail distributed, see e.g. Jansen and De Vries (1991). Thus the diversification speed is dictated by the heavy tail property of the sum (by comparison, this would be an exponential rate in case of the normal distribution). For a global risk measure like the variance, the tail properties only play a role in the determination of the size of the variance of the individual random variable. But for partial moments the sum operator enters the boundary of the integral. A sine qua non for the existence of the variance is that the tail of the distribution must go down at least at a quadratic speed. Thus the tail shape, even of the heavy tail distributions, must be such that it beats the quadratic rate and hence the higher diversification speed of the downside risk measure. For example, in case of independent Student-t distributions with $\nu > 2$ degrees of freedom, ensuring the existence of the variance, the diversification speed of the downside risk measure equals $1/n^{1/(\nu-1)}$ (and hence exceeds the speed of the variance in all cases for which the variance exists, i.e. if $\nu > 2$). This intuition is made rigorous below by means of the celebrated Feller convolution theorem for heavy tailed (i.e. regularly varying) distributions.

For the empirical counterpart of this analysis, we briefly review the semi-parametric approach to estimating the (extreme) downside risk. The heavy tail feature is captured by a Pareto distribution like term, of which one needs to estimate the tail index and scale coefficient. We consider estimation by means of a pooled data set on basis of the assumption that the tail indices of the different securities and risk components are equal. We do allow for heterogeneity of the scale coefficients, though. Most securities' distributions display equal hyperbolic tail coefficients, but do differ considerably in terms of their scale coefficients, see Hyung and De Vries (2002). Within this framework it is possible to calculate the diversification effects beyond the sample range and for hypothetically larger portfolios, if we make some assumptions regarding the market model betas and scale coefficients of the orthogonal risk factors.

We start our essay by reviewing the Feller's convolution theorem for distributions with heavy tails. Subsequently, we study the diversification problem in more detail by adding the market factor. The relevance of the theoretical results for the downside risk portfolio diversification question is demonstrated by an application to S&P stock returns.

2 Diversification Effects and the Feller Convolution Theorem

In this section we only consider securities which are independent and identically distributed. In the next section this counterfactual assumption, as least as far as equities are concerned, is relaxed by allowing for common factors. Let R_i denote the logarithmic return of the $i-$th security. Suppose the $\{R_i\}$ are generated by a distribution with heavy tails in the sense of regular variation at infinity. Thus far from the origin the Pareto term dominates:

$$\Pr\{R_i \leq -x\} = A_i x^{-\alpha}[1 + o(1)],\ \alpha > 0,\ A_i > 0, \tag{1}$$

as $x \to \infty$. The Pareto term implies that only moments up to α are bounded and hence the informal terminology of heavy tails. Per contrast the normal distribution has all moments bounded thanks to the exponential tail shape. Distributions like the Student-t, Pareto, non-normal sum-stable distributions all have regularly varying tails. Downside risk measures like the VaR directly pick up differences in tail behavior. In this paper we focus on the loss probability $\Pr\{R_i \leq -x\}$ at high loss levels x as our measure of downside risk, rather than at the loss level x as in case of the VaR measure.

An implication of the regular variation property is the simplicity of the tail probabilities for convoluted data. Suppose the $\{R_i\}$ are generated by a heavy-tailed distribution which satisfies (1). From the Feller's Theorem (1971, VIII.8), the distribution of the $k-$sum satisfies

$$\Pr\left\{\sum_{i=1}^{k} R_i \leq -x\right\} = kAx^{-\alpha}[1 + o(1)], \text{ as } x \to \infty.$$

From this one can derive the diversification effect for the equally weighted portfolio $\overline{R} = \frac{1}{k}\sum_{i=1}^{k} R_i$, see Dacorogna et al. (2001). The following first order approximation for the diversification effect regarding the downside risk obtains

$$\Pr\left\{\frac{1}{k}\sum_{i=1}^{k} R_i \leq -x\right\} \approx k^{1-\alpha} A x^{-\alpha}. \tag{2}$$

By comparison, if the variance would be used as the risk measure (assuming $\alpha > 2$), the portfolio has a variance which declines at a rate k only.

Under the heterogeneity of the scale coefficients A_i, the equivalent of equation (2) reads

$$\Pr\left\{\frac{1}{k}\sum_{i=1}^{k} R_i \leq -x\right\} \approx k^{-\alpha}\left(\sum_{i=1}^{k} A_i\right) x^{-\alpha}. \tag{3}$$

At a constant VaR level x, increasing the number k of included securities decreases the probability of loss by $k^{1-\alpha}$. If $\alpha > 2$, then downside diversification goes at a higher speed than if the returns would be normally distributed. Hyung and De Vries (2002) investigate these diversification effects by means of a simulation study and an application to portfolios with equities. This study did not distinguish between different risk components and their contribution to the downside risk. The following section extends the results to the case where there are common and idiosyncratic risk factors.

3 Diversification Effects in Factor Models

We relax the assumption of independence between security returns and allow for non-diversifiable market risk. The market risk reduces the benefits from diversification to the elimination of the idiosyncratic component of the risk. First consider the single index model in which all risk which is orthogonal to the market risk R is security specific

$$R_i = \beta_i R + Q_i, \tag{4}$$

where R, is the return on the market portfolio, β_i is the amount of market risk and Q_i is the idiosyncratic risk of the return on asset i. The idiosyncratic risk may be diversified away fully in arbitrarily large portfolios and hence is not priced. But the cross-sectional dependence induced by common market risk factor has to be held in any portfolio.

We now apply Feller's theorem again for deriving the benefits from cross-sectional portfolio diversification in the single index model. Consider an equally weighted portfolio of k assets. Let $\overline{\beta} = \frac{1}{k}\sum_{i=1}^{k}\beta_i$. The case of unequally weighted portfolios is but a minor extension left to the reader for consideration of space. In the single index model the Q_i are cross-sectionally independent and are independent from the market risk factor R. Suppose in addition that the Q_i satisfy $\Pr\{Q_i \leq -x\} \approx A_i x^{-\alpha}$ for all i, and that $\Pr\{R \leq -x\} \approx A_r x^{-\alpha}$. The diversification benefits from the equally weighted portfolio regarding the downside risk measure for the case of homogenous scale coefficients $A_i = A$ then follow as

$$\Pr\left\{\frac{1}{k}\sum_{i=1} R_i \leq -x\right\} \approx k^{1-\alpha} A x^{-\alpha}[1 + o(1)] + \overline{\beta}^{\alpha} A_r x^{-\alpha}[1 + o(1)], \tag{5}$$

as $x \to \infty$. If the scale coefficients are heterogenous, the equivalent of equation (5) reads

$$\Pr\left\{\frac{1}{k}\sum_{i=1} R_i \le -x\right\} \approx k^{-\alpha}\left(\sum_{i=1} A_i\right) x^{-\alpha} + \overline{\beta}^{\alpha} A_r x^{-\alpha}. \tag{6}$$

In arbitrarily large portfolios one should see that all downside risk is driven by the market factor, if $\alpha > 1$

$$\Pr\left\{\frac{1}{k}\sum_{i=1} R_i \le -x\right\} \approx \overline{\beta}^{\alpha} A_r x^{-\alpha}$$

for large k.

In general one finds the single index model does not hold exactly due to the fact that $\mathrm{Cov}[Q_i, Q_j]$ is typically non-zero for off diagonal elements as well. Thus though the Q_i are uncorrelated with the market risk factor R by construction, they are not cross sectionally independent from each other; moreover, one can imagine they may not even be independent from R. This case is usually referred to as the market model. For example, let there be one other common factor F. This factor is uncorrelated with R, but the $\mathrm{Cov}[Q_i, F]/\mathrm{Cov}[F, F] = \tau_i$ say. Let $\overline{\tau} = \frac{1}{k}\sum_{i=1}^{k} \tau_i$, and assume $\Pr\{F \le -x\} \approx A_f x^{-\alpha}$. Then, by analogy with the foregoing results

$$\Pr\left\{\frac{1}{k}\sum_{i=1} R_i \le -x\right\} \approx k^{-\alpha}\left(\sum_{i=1} A_i\right) x^{-\alpha} + \overline{\beta}^{\alpha} A_r x^{-\alpha} + \overline{\tau}^{\alpha} A_f x^{-\alpha}. \tag{7}$$

4 Estimation by Pooling

To investigate the relevance of the above downside risk diversification theory, we need to estimate the various downside risk components. To explain the details of the estimation procedure, consider again the simple setup in (3). To be able to calculate the downside risk measure, one needs estimates of the tail index α and the scale coefficients A_i. A popular estimator for the inverse of the tail index is Hill's (1975) estimator. If the only source of heterogeneity are the scale coefficients, one can pool all return series. Let $\{R_{11}, ..., R_{1n}, ..., R_{k1}, ..., R_{kn}\}$ be the sample of returns. Denote by $Z_{(i)}$ the i-th descending order statistic from $\{R_{11}, ..., R_{1n}, ..., R_{k1}, ..., R_{kn}\}$. If we estimate the left tail of the distribution, it is understood that we take the losses (reverse signs). The Hill estimator reads

$$\widehat{1/\alpha} = \frac{1}{m}\sum_{i=1}^{m} \ln\left(Z_{(i)}\right) - \ln\left(Z_{(m+1)}\right). \tag{8}$$

This estimator requires a choice of the number of the highest order statistics m to be included, i.e. one needs to locate the start of the tail area. We implemented the subsample bootstrap method proposed by Danielsson et al. (2000) to determine m. The estimator for the scale A when $A = A_i$ for all i is

$$\widehat{A} = \frac{m}{kn}(Z_{(m+1)})^{\widehat{\alpha}}.$$

Note that m/nk is the empirical probability associated with $Z_{(m+1)}$, and the estimator $\widehat{A}$ follows intuitively from (1). Under the heterogeneity of A_i one takes

$$\hat{A}_i = \frac{m_i}{n}(Z_{(m+1)})^{\hat{\alpha}}$$

where m_i is such that

$$R_{i(1)} \geq ... \geq R_{i(m_i)} \geq Z_{(m+1)} \geq R_{i(m_i+1)} \geq ... \geq R_{in}.$$

Note that $\sum_{i=1}^{k} m_i = m$. This implies that by the pooling method we obtain exactly the same portfolio probabilities whether or not one assumes (counterfactually incorrect) identical or heterogenous scale coefficients, since

$$\begin{aligned} k^{-\hat{\alpha}}\left(\sum_{i=1}^{k}\hat{A}_i\right)x^{-\hat{\alpha}} &= k^{-\hat{\alpha}}\left(\sum_{i=1}^{k}\frac{m_i}{n}(Z_{(m+1)})^{\hat{\alpha}}\right)x^{-\hat{\alpha}} \\ &= k^{-\hat{\alpha}}\left(\frac{\sum_{i=1}^{k} m_i}{n}\right)(Z_{(m+1)})^{\hat{\alpha}}x^{-\hat{\alpha}} \\ &= k^{1-\hat{\alpha}}\hat{A}x^{-\hat{\alpha}}. \end{aligned}$$

We can adapt this pooling method to the market model with little modification. Pooling the series $\{R\}, \{Q_1\}, ... \{Q_k\}$, one can use the same procedure as in the case of cross-independence.[1] For the estimation of the tail index one uses again (8), where in this case $\{Z\} = \{R_{r1}, ..., R_{rn}, Q_{11}, ..., Q_{1n}, ..., Q_{k1}, ..., Q_{kn}\}$.

[1] The determination of the parameters β_i and the residuals Q_i entering in the definition of the market model is done by regressing the stock returns on the market return. The coefficient β_i is thus given by the ordinary least squares estimator, which is consistent as long as the residuals are white noise and have zero mean and finite variance. The idiosyncratic noise Q_i is obtained by subtracting β_i times the market return to the stock return.

Estimators for the scales are

$$\hat{A}_i = \frac{m_i}{n}(Z_{(m+1)})^{\hat{\alpha}}, i = 1, ..., k \text{ and } r,$$

where m_i is such that

$$X_{i(1)} \geq ... \geq X_{i(m_i)} \geq Z_{(m+1)} \geq X_{i(m_i+1)} \geq ... \geq X_{in},$$

where X_i can be R or Q_i.

In case the tail indices differ across securities and risk factors, the above can be easily adapted to estimation on individual series. There is however considerable evidence that the tail indices are comparable for equities from the S&P 500 index, see e.g. Jansen and De Vries (1991) and Hyung and De Vries (2002). Therefore we decided to proceed on basis of the assumption that the tail indices are equal.

5 Empirical Analysis of the Diversification Speed

We now apply our theoretical results to the daily returns of a set of stocks. In order to estimate the parameters of the market model we choose the Standard and Poor's 500 index as a representation of the market factor. This is certainly not the market portfolio as in the CAPM; nevertheless, the S&P 500 index represents about 80% of the total market capitalization. To see the effects of portfolio diversification, we choose arbitrarily 15 stocks from the S&P 100 index in March of 2001. We use the daily returns (close-to-close data), including cash dividends. The data were obtained from the Datastream. The data span runs from January 2, 1980, through March 6, 2001, giving a sample size of $n = 5{,}526$. Thus more than 20 years of daily data are considered, including the 1987 Crash. All results are in terms of the excess returns above the risk free interest rate (three month US Treasury bills).

The summary statistics for each stock return series and the market factor are given in Table 1. On an annual basis the excess returns hover around 7.5% and have comparable second moments. The excess returns all exhibit considerably higher than normal kurtosis. This latter feature is also captured by the estimates of the tail index α in Table 2. In this table we report tail index and scale estimates using the individual series, counter to the pooling method outlined above. This is done in order to show that the tail indices are indeed rather similar, while there is considerable variation in the scales. This motivates the single tail index, heterogenous scale model implemented in the other tables. Table 2 also gives the beta estimates for the market model.

In Table 3 computations proceed by using the pooling method, assuming identical tail indices for all risk components. We report the estimates of the scale parameter A, and the optimal number of order statistics m. Both are calculated for the series of excess returns and for the (constructed) orthogonal residuals from the market model (using the betas from Table 2). The tail index estimate using all excess returns is 3.163, while when we use all the residuals the tail index is 3.246. The scale parameter estimates, however, differ considerably since these range between 14.4 and 46.4 for the excess returns, and are between 4.3 and 42.2 for the market returns and residuals respectively. We note that the scale estimates for the excess returns using the pooling method are more homogeneous than when using the individual series approach from Table 2.

The effects of portfolio diversification are reported in Table 4. The downside risk measure is the probability of a loss in excess of the VaR level s; we report at four different loss levels (respectively s =7.10, 11.69, 13.33 and 15.97). Four different levels of portfolio aggregation are considered: one stock, 5 stocks, 10 stocks and 15 stocks. The numbers in row EMP are the probabilities from the empirical distribution function of the total return series. The normal law is often used as the workhorse distribution model in finance, even though it does not capture the characteristic tail feature of the data. Therefore in the rows labelled NOR we give the probabilities from the normal model based formula, using the mean and variance estimates from the averaged series. The estimated values in rows FAT were obtained by the heavy tail model using the averaged total excess returns $\sum_{i=1}^{k} R_i/k$. The rows CDp give the probability estimates from the pooled series on the basis of (6) assuming the heterogenous scale model. One notes that the normal model does well in the center, but performs miserable as one moves into the tail part. Per contrast, the averaged series in rows FAT is always quite close to the empirical distribution function in the tail area. This shows that the heavy tail model much better captures the tail properties. If we turn to the last rows , one notes that the model in (6) does capture a considerable part of the tail risk of the portfolio, but that there is a gap between the tail risk

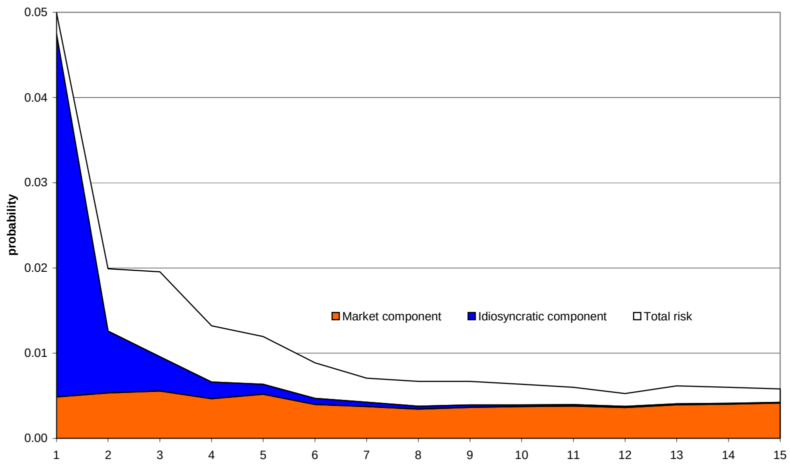

Figure 1: Downside Risk Decomposition at s=-7.10

which is explained by the model and which is left unexplained. This is further interpreted below.

To judge these results a graphical exposition is insightful. In Figures 1 and 2 we graph the downside risk (VaR probability levels) against the number of securities which are included in the portfolio. Figure 1 is for the 7.10 VaR level, and Figure 2 concerns the 15.97 VaR level. The top line gives the total amount of tail risk by means of the empirical distribution function. The grey area constitutes the market risk component, while the black area contains the idiosyncratic risk part of the residual risk from model (6). Note that the idiosyncratic risk is basically eliminated once the portfolio includes about seven stocks. To put this result into perspective, we also provide a graph for the speed of diversification concerning the variance, see Figure 3. As can be seen from this latter figure, it

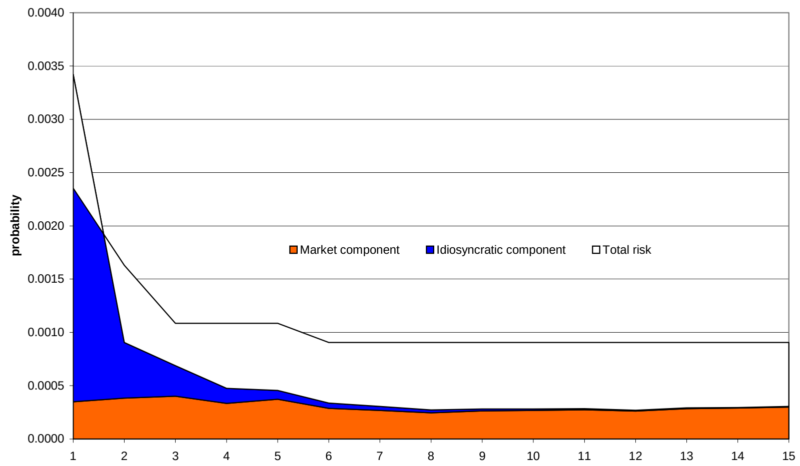

Figure 2: Downside Risk Decomposition at s=-15.97

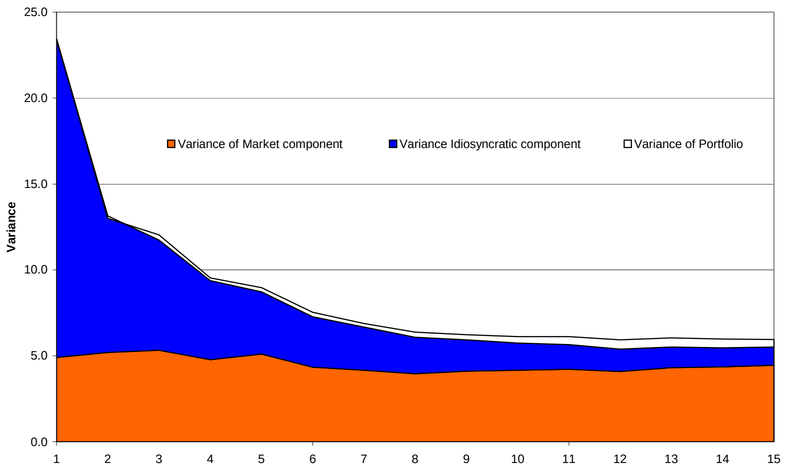

Figure 3: Variance Decomposition

takes approximately double the number of securities to eliminate the variance part contributed by the idiosyncratic risk part (which is consistent with the value of the tail index), cf. Elton and Grubel (1995). Hence, the diversification speed is higher for the downside risk measure. This corroborates the theoretical predictions above. Interestingly as noted at the end of the previous paragraph, another remarkable difference between the last figure and the first two figures is the size of the residual risk driven by the factors other than the market factor. While this component is relatively minor for the variance risk measure, it is even larger than the market risk component for the downside risk measure. This points to the presence of another factor F uncorrelated with R as in (7). This other factor induces correlation between the residuals, see Figure 3. This correlation is small but not negligible. But it the importance of this other factor is really with respect to the downside risk. In future research we hope to relate this factor to economic variables.

6 Out-of-sample, Out-of-portfolio

The semi-parametric approach we followed to construct the downside risk measure can also be used to go beyond the sample. We consider two possible applications of this technique which might be of use to risk managers. The first application asks the question how much extra diversification benefits could be derived from adding more securities, without having observations on these securities. By making an assumption regarding the value of the average beta and the average scale of the residual risk factors in the enlarged portfolio, one can use (6) to extrapolate to larger than sample size portfolios. A second application is to increase the loss levels at which one wants to evaluate the downside risk level beyond the worst case in sample. Moreover, even at the border of the sample our approach has real benefits. By its very nature the empirical distribution is bounded by the worst case and hence has its limitations, since the worst case is a bad estimator of the quantile at the $1/n$ probability level (and vice versa). Thus increasing the loss level x in (6) beyond the worst case gives an idea about the risk of observing even higher losses.

In Table 5 the block denoted as Case I just summarizes some information from the previous Table 4. The Case III block addresses the first application by increasing the number of securities k beyond the sample value of 15. We assumed the following average beta values: $\overline{\beta} = 0.7$, 0.83 and 0.9. The Case II block increases the loss return level. In Table 4 we used 15.97 as the highest loss level. Above this level many securities have no observations. There is one equity with much higher loss returns and we used this one to provide the 'out of sample' loss levels of 22.03, 25.21, 33.69 and 40.45 respectively. To interpret Case III, note that the inclusion of more stocks that have a close correlation with the market component increases the loss probability for a given VaR level. For example consider a portfolio of $k = 30$ stocks, at the -15.97 quantile when $\overline{\beta} = 0.7$ the probability is 0.0169 but when $\overline{\beta} = 0.9$ the probability increases to 0.0381.

7 Conclusion

Risk managers use portfolios to diversify away the unpriced risk of individual securities. In this paper we study the benefits of portfolio diversification with respect to extreme downside risk, or the VaR (Value at Risk) risk measure. The risk of a security is decomposed into a part which is attributable to the market risk and an orthogonal risk factor. The orthogonal part consists of an idiosyncratic part and a part which is attributable to dependency between assets but which cannot be explained by the market risk, i.e there are other factors. We compare the diversification speed of the downside risk measure with the diversification speed regarding the variance and find that the idiosyncratic downside risk evaporates at a higher speed. For this we offer a theoretical explanation on basis of the heavy tail properties of the asset return distributions. We also find that the non-diversifiable non-market factors are more important for the downside risk than for the global risk measure. Furthermore, we provide predictions for the downside risk diversification benefits beyond the range of the empirical distribution function.

This research can be extended in several directions. For theoretical reasons as well for empirical reasons it is of interest to extend the analysis to portfolios which include securities with heterogenous tail indices. Given the large gaps in Figures 1 and 2 between the total downside risk and the market factor downside risk contribution, it is of interest to see whether one can identify the remaining risk factors F as in (7). Moreover, one would like to explain why these remaining risk factors are relatively unimportant for the global risk measure.

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Table 1. Selected Stocks and Summary Statistics of Excess returns

Series	Name	μ_1	μ_2	μ_3	μ_4
m	S&P 500 Index	.0747	2.52	-2.31	55.49
1	ALCOA	.0707	4.84	-0.26	13.39
2	AT & T	.0392	4.33	-0.35	16.41
3	BLACK & DECKER	-.0168	5.61	-0.32	10.57
4	CAMPBELL SOUP	.0897	4.37	0.28	9.06
5	DISNEY (WALT)	.0981	4.86	-1.30	29.82
6	ENTERGY	.0454	4.06	-0.97	23.66
7	GEN.DYNAMICS	.0764	4.53	0.26	10.24
8	HEINZ HJ	.0968	3.99	0.11	6.35
9	JOHNSON & JOHNSON	.1053	4.08	-0.32	9.45
10	MERCK	.1212	3.96	-0.03	6.31
11	PEPSICO	.1170	4.43	-0.04	7.82
12	RALSTON PURINA	.1077	4.08	0.70	15.41
13	SEARS ROEBUCK	.0542	4.91	-0.24	16.83
14	UNITED TECHNOLOGIES	.0851	4.19	-0.10	6.83
15	XEROX	-.0423	5.48	-1.78	33.74

Note: Observations cover 01/01/1980 - 03/06/2001, giving 5526 daily observations. The μ_1, μ_2, μ_3 and μ_4 denote the sample mean, standard error, skewness and kurtosis of annualized excess returns, respectively. The estimates are reported in terms of the excess returns above the risk free interest rate (US Treasury bill 3 months).

Table 2. Left Tail Parameter Estimates and Beta

Series	α	A	m	β
R_m	2.963	2.522	298	1
1	3.789	110.117	113	0.877
2	2.785	7.953	289	0.929
3	3.220	58.601	136	0.938
4	3.505	48.766	68	0.719
5	2.549	6.211	496	1.012
6	1.981	1.339	682	0.475
7	3.218	27.687	140	0.710
8	3.404	25.811	197	0.640
9	3.377	23.663	292	0.927
10	4.035	104.724	62	0.854
11	3.789	103.171	71	0.867
12	3.136	14.106	190	0.669
13	3.166	28.244	256	1.074
14	4.335	288.036	66	0.895
15	2.098	2.999	537	0.949

Note: The values in α, A, m, and β are respectively the tail index, the scale parameter, the estimated optimal number of order statistics and market model beta.

Table 3. Left Tail Parameter Estimates

	Excess returns		Residuals	
Series	A	m	A	m
T	23.0	16.3	19.6	1021
R_m	-	-	4.3	15
1	26.2	122	24.7	86
2	19.5	91	15.2	53
3	46.4	216	42.2	147
4	22.7	106	19.5	68
5	24.0	112	22.1	77
6	14.4	67	14.9	52
7	25.3	118	25.0	87
8	16.3	76	14.9	52
9	13.9	65	10.6	37
10	15.7	73	11.5	40
11	24.2	113	18.7	65
12	15.0	70	16.4	57
13	29.0	135	17.5	61
14	20.2	94	13.2	46
15	32.4	151	26.7	93

Note: The values in row T give estimates from the pooled series imposing scale homogeneity. The values in rows $R_m, 1, 2, ..., 15$ give estimates for the market returns and the individual stock series for the total excess returns and the residual parts. The values in columns A and m are the scale parameter and the estimated optimal number of order statistics imposing identical tail indices.

Table 4. Lower Tail Probabilities in Percentages

s	-7.10				-11.69			
k	1	5	10	15	1	5	10	15
EMP	4.995	1.195	0.633	0.579	0.995	0.253	0.145	0.145
NOR	7.325	0.934	0.225	0.198	0.817	0.005	0.000	0.000
FAT	6.551	1.181	0.741	0.706	0.988	0.265	0.185	0.171
CDp	-	0.633	0.392	0.423	-	0.125	0.078	0.084

s	-13.33				-15.97			
k	1	5	10	15	1	5	10	15
EMP	0.489	0.163	0.109	0.127	0.235	0.109	0.090	0.090
NOR	0.309	0.000	0.000	0.000	0.051	0.000	0.000	0.000
FAT	0.603	0.179	0.129	0.118	0.304	0.104	0.078	0.071
CDp	-	0.082	0.051	0.055	-	0.046	0.028	0.030

Note: The entries in rows EMP are the probabilities from the empirical distribution. The rows NOR and FAT report the probabilities calculated directly from the parameters of the averaged series itself, where in the former case one uses the presumption of normality and in the latter case regular variation is imposed. The numbers in rows CDp are the probabilities estimated using the pooled series. The k denotes the number of individual stocks included in the averaged series, and s is the loss quantile. Note probabilities are written in percentage format.

Table 5. Lower Tail Probabilities: Beyond the Sample and the Market

	s	-7.10	-11.69	-13.33	-15.97	-22.03	-25.21	-33.69	-40.45
	%	5.0	1.0	0.5	0.25	0.090	0.054	0.018	0.009
k		CASE I				CASE II			
	EMP	1.1946	.2534	.1629	.1086	.0362	.0362	.0181	.0181
5	FAT	1.1900	.2660	.1798	.1045	.0397	.0265	.0111	.0064
	CDp	.6093	.1205	.0789	.0439	.0154	.0100	.0039	.0021
	EMP	.6335	.1448	.1086	.0905	.0181	.0181	.0181	.0181
10	FAT	.6800	.1490	.1001	.0578	.0217	.0144	.0060	.0034
	CDp	.3914	.0774	.0507	.0282	.0099	.0064	.0025	.0014
	EMP	.5792	.1448	.1267	.0905	.0181	.0181	.0181	.0181
15	FAT	.7087	.1722	.1189	.0712	.0286	.0195	.0086	.0051
	CDp	.4227	.0836	.0547	.0304	.0107	.0069	.0027	.0015
		CASE III							
	CDp1	.2375	.0470	.0307	.0171				
20	CDp2	.4190	.0829	.0543	.0302				
	CDp3	.5318	.1052	.0689	.0383				
	CDp1	.2359	.0467	.0305	.0170				
25	CDp2	.4175	.0826	.0541	.0301				
	CDp3	.5302	.1049	.0687	.0382				
	CDp1	.2350	.0465	.0304	.0169				
30	CDp2	.4166	.0824	.0540	.0300				
	CDp3	.5294	.1047	.0686	.0381				

Note: The entries in rows EMP are the probabilities from the empirical distribution. The numbers in rows FAT are the probabilities calculated directly from the parameters of averaged series itself. The numbers in row CDp are the probabilities from the fat tail market model (6). The numbers in rows CDp1,2 and 3 are calculated by imposing $\overline{\beta} = 0.7, 0.8358$ and 0.9, respectively. The k denotes the number of individual stocks included in the averaged series, and s gives the loss quantile. Note probabilities are written in percentage format.