Intermittency in a catalytic random medium

J. Gärtner [*]
F. den Hollander [†]

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Abstract

In this paper we study intermittency for the parabolic Anderson equation $\partial u/\partial t = \kappa\Delta u + \xi u$, where $u\colon \mathbb{Z}^d \times [0,\infty) \to \mathbb{R}$, κ is the diffusion constant, Δ is the discrete Laplacian, and $\xi\colon \mathbb{Z}^d \times [0,\infty) \to \mathbb{R}$ is a space-time random medium. We focus on the case where ξ is γ times the random medium that is obtained by running independent simple random walks with diffusion constant ρ starting from a Poisson random field with intensity ν. The solution of the equation describes the evolution of a "reactant" u under the influence of a "catalyst" ξ.

We consider the annealed Lyapunov exponents, i.e., the exponential growth rates of the successive moments of u, and show that they display an interesting dependence on the dimension d and on the parameters κ and γ, ρ, ν, with qualitatively different intermittency behavior in $d = 1, 2$, in $d = 3$ and in $d \geq 4$. Special attention is given to the asymptotics of these Lyapunov exponents for $\kappa \downarrow 0$ and $\kappa \to \infty$.

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[*]Institut für Mathematik, Technische Universität Berlin, Strasse des 17. Juni 136, D-10623 Berlin, Germany *jg@math.tu-berlin.de*
[†]EURANDOM, P.O. Box 513, 5600 MB Eindhoven, The Netherlands, *denhollander@eurandom.tue.nl*

1 Introduction and main results

1.1 Motivation

The parabolic Anderson equation is the partial differential equation

$$\frac{\partial}{\partial t}u(x,t) = \kappa\Delta u(x,t) + \xi(x,t)u(x,t), \qquad x \in \mathbb{Z}^d,\, t \geq 0. \tag{1.1.1}$$

Here, the u-field is $\mathbb{R}$-valued, $\kappa \in [0,\infty)$ is the diffusion constant, Δ is the discrete Laplacian, acting on u as

$$\Delta u(x,t) = \sum_{\substack{y\in\mathbb{Z}^d \\ \|y-x\|=1}} [u(y,t) - u(x,t)], \tag{1.1.2}$$

while

$$\xi = \{\xi(x,\cdot)\colon\ x \in \mathbb{Z}^d\} \tag{1.1.3}$$

is an $\mathbb{R}$-valued random field that evolves with time and that drives the equation.

Equation (1.1.1) is the parabolic analogue of the Schrödinger equation in a random potential. It is a discrete heat equation with the ξ-field playing the role of a source or sink. One interpretation, coming from population dynamics, is that $u(x,t)$ is the average number of particles at site x at time t when particles perform independent simple random walks at rate κ, split into two at rate $\xi(x,t)$ when $\xi(x,t) > 0$ (source term) and die at rate $-\xi(x,t)$ when $\xi(x,t) < 0$ (sink term). For more background on applications, the reader is referred to the monograph by Carmona and Molchanov [4] (Chapter I).

What makes (1.1.1) particularly interesting is that the two terms in the right-hand side *compete with each other*: the diffusion induced by Δ tends to make u flat, while the branching induced by ξ tends to make u irregular. Consequently, in the population dynamics context, there is a competition between particles spreading out by diffusion and particles clumping around the areas where the sources are large.

A systematic study of the parabolic Anderson model for *time-independent* random fields ξ has been carried out by Gärtner and Molchanov [16], [17], [18], Gärtner and den Hollander [11], Gärtner and König [12], Gärtner, König and Molchanov [14], [15], Biskup and König [1], [2] (for a survey, see Gärtner and König [13]). The focus of these papers is on the study of the dominant spatial peaks in the u-field in the limit of large t, in particular, the height, the shape and the location of these peaks. Both the discrete model on $\mathbb{Z}^d$ (with i.i.d. ξ-fields) and the continuous model on $\mathbb{R}^d$ (with Gaussian and Poisson-like ξ-fields) have been investigated, in the *quenched* setting (i.e., conditioned on ξ) as well as in the *annealed* setting (i.e., averaged over ξ).

Most of the theory currently available for *time-dependent* random fields ξ is restricted to the situation where the components of the ξ-field are *uncorrelated in space and time*. Carmona and Molchanov [4] (Chapter III) have obtained an essentially complete qualitative description of the annealed Lyapunov exponents, i.e., the exponential growth rates of the successive moments of $u(0,t)$ averaged w.r.t. ξ, for the case where the components of ξ are independent Brownian noises. The quenched Lyapunov exponent, i.e., the exponential growth rate of $u(0,t)$ conditioned on ξ, is harder to come by. Carmona, Molchanov and Viens [5], Carmona, Koralov and Molchanov [3], Cranston, Mountford and Shiga [6] have computed the asymptotics for $\kappa \downarrow 0$ of the quenched Lyapunov exponent for independent Brownian noises, which turns out to be singular. Cranston, Mountford and Shiga [7] have extended this result to independent Lévy noises. Further refinements for independent Brownian noises are obtained in Greven and den Hollander [19], including sharp bounds on the critical values of κ where the annealed Lyapunov exponents change from positive to zero, respectively, the quenched Lyapunov exponent changes from negative to zero, as well as a description of the equilibrium behavior when the quenched Lyapunov exponent is zero. These results are obtained from variational expressions for the Lyapunov exponents and are valid for general random walk transition kernels replacing Δ.

In the present paper we will be considering the situation where ξ is given by

$$\xi(x,t) = \gamma \sum_k \delta_{Y_k(t)}(x) \tag{1.1.4}$$

with $\gamma \in (0,\infty)$ a coupling constant and

$$\{Y_k(\cdot)\colon\ k \in \mathbb{N}\} \tag{1.1.5}$$

a collection of independent continuous-time simple random walks with diffusion constant $\rho \in (0,\infty)$ starting from a Poisson random field with intensity $\nu \in (0,\infty)$ (the index k is an arbitrary numbering). As initial condition for (1.1.1) we take for simplicity

$$u(\cdot,0) \equiv 1. \tag{1.1.6}$$

We are interested in computing the annealed Lyapunov exponents of u and studying their dependence on the parameters κ and γ,ρ,ν.

The population dynamics interpretation of (1.1.1) and (1.1.4–1.1.6) is as follows. Consider a spatially homogeneous system of two types of particles, A (catalyst) and B (reactant), performing independent continuous-time simple random walks such that:

(i) B-particles split into two at a rate that is γ times the number of A-particles present at the same location;

(ii) ρ and κ are the diffusion constants of the A- and B-particles, respectively;

(iii) ν and 1 are the initial intensities of the A- and B-particles, respectively.

Then

$$u(x,t) = \{\ \text{the average number of } B\text{-particles at site } x \text{ at time } t \text{ conditioned on the evolution of the } A\text{-particles}\ \}. \tag{1.1.7}$$

Kesten and Sidoravicius [21] recently investigated this model with additionally:

(iv) B-particles die at rate $\delta \in (0,\infty)$.

The latter amounts to the transformation

$$u(x,t) \to u(x,t)e^{-\delta t}. \tag{1.1.8}$$

We describe their results in Section 1.4.

1.2 Catalytic and intermittent behavior

Let $\langle\cdot\rangle$ denote expectation w.r.t. the ξ-field. For $p \in \mathbb{N}$ and $t > 0$, define

$$\Lambda_p(t) = \frac{1}{t} \log \left(e^{-\nu\gamma t} \langle u(0,t)^p \rangle^{1/p} \right). \tag{1.2.1}$$

This quantity monitors the effect of the randomness in the ξ-field on the growth of the p-th moment. Indeed, if in (1.1.1) we would replace $\xi(x,t)$ by its average value $\langle \xi(x,t) \rangle = \nu\gamma$ (according to (1.1.4)), then the solution would be $u(\cdot,t) \equiv e^{\nu\gamma t}$, resulting in $\Lambda_p(\cdot) \equiv 0$.

The key quantities of interest in the present paper are the following *Lyapunov exponents*:

$$\begin{aligned} \widehat{\lambda}_p &= \lim_{t\to\infty} \frac{1}{t} \log \Lambda_p(t), \\ \lambda_p &= \lim_{t\to\infty} \Lambda_p(t). \end{aligned} \tag{1.2.2}$$

(Note that λ_p is related to the moment Lyapunov exponent $\widetilde{\lambda}_p = \lim_{t\to\infty} \frac{1}{t} \log \langle u(0,t)^p \rangle$ via the relation $\lambda_p = \widetilde{\lambda}_p / p - \nu\gamma$.) The existence of the limits is not a priori evident and needs to be established. This will be done in Section 3 for $\widehat{\lambda}_p$ and in Section 4.1 for λ_p. From the Feynman-Kac representation for the moments of the solution of (1.1.1) and (1.1.4–1.1.6), given in Proposition 2.1.1 in Section 2.1, it will follow that $t \mapsto t\Lambda_p(t)$ is strictly positive and strictly increasing on $(0,\infty)$. Hence $\widehat{\lambda}_p, \lambda_p \geq 0$. We further have $\Lambda_p(t) \geq \Lambda_{p-1}(t)$ by Hölder's inequality applied to the definition of $\Lambda_p(t)$. Hence $\lambda_p \geq \lambda_{p-1}$. We will see in Section 4.3 that $\lambda_p > 0$.

Depending on the values of these Lyapunov exponents, we distinguish the following types of behavior.

Definition 1.2.1 *For $p \in \mathbb{N}$, we say that the system is:*
(a) strongly p-catalytic if $\widehat{\lambda}_p > 0$.
(b) weakly p-catalytic if $\widehat{\lambda}_p = 0$.

Strongly catalytic means that the moments of the u-field grow much faster in the random medium ξ than in the average medium $\langle \xi \rangle$, at a double exponential rate. Weakly catalytic corresponds to a slower rate. Strongly catalytic behavior comes from an extreme form of clumping in the ξ-field.

Definition 1.2.2 *For $p \in \mathbb{N} \setminus \{1\}$, we say that the system is:*
(a) strongly p-intermittent if either $\lambda_p = \infty$ or $\lambda_p > \lambda_{p-1}$.
(b) weakly p-intermittent if $\lambda_p < \infty$ and $\lambda_p = \lambda_{p-1}$.

Strongly p-intermittent means that $1/p$-th power of the p-th moment of the u-field grows faster than the $1/(p-1)$-st power of the $(p-1)$-st moment, at an exponential rate. Weakly p-intermittent corresponds to a slower rate. Strongly intermittent behavior also comes from clumping in the ξ-field, but in a less extreme form than for strongly catalytic behavior. Note that strong p-intermittency implies strong q-intermittency for all $q > p$ (see Gärtner and Molchanov [16]). Also note that weakly intermittent in our definition includes the possibility of no separation of the moments, which is usually called non-intermittent.

In the population dynamics context, both catalytic and intermittent behavior come from the B-particles clumping around the areas where the A-particles are clumping. It signals the appearance of rare high peaks in the u-field close to rare high peaks in the ξ-field. These peaks dominate the moments of the u-field (for more details, see Gärtner and Molchanov [16], Molchanov [24] (Lecture 8), den Hollander [20] (Chapter 8), Gärtner and König [13]).

1.3 Main theorems

Let

$$\widehat{\varphi}(k) = \sum_{\substack{x\in\mathbb{Z}^d \\ \|x\|=1}} [1-\cos(k\cdot x)], \qquad k \in [-\pi,\pi)^d. \tag{1.3.1}$$

For $\mu \geq 0$, define

$$R(\mu) = \frac{1}{(2\pi)^d} \int_{[-\pi,\pi)^d} \frac{dk}{\mu + \widehat{\varphi}(k)} \tag{1.3.2}$$

and put

$$r_d = \frac{1}{R(0)} \begin{cases} = 0 & \text{if } d = 1,2, \\ > 0 & \text{if } d \geq 3. \end{cases} \tag{1.3.3}$$

Note that $R(\mu)$ is the Fourier representation of the kernel of the resolvent $(\mu - \Delta)^{-1}$ at 0; $R(0)$ equals the Green function at the origin of simple random walk on $\mathbb{Z}^d$ jumping at rate $2d$, i.e., the Markov process generated by Δ.

The following fact, which is elementary and is well-known, is needed for Theorem 1.3.2 below (see Figure 1).

Lemma 1.3.1 *For $r \in (0,\infty)$, let*

$$\mu(r) = \sup \mathrm{Sp}(\Delta + r\delta_0) \tag{1.3.4}$$

denote the supremum of the spectrum of the operator $\Delta + r\delta_0$ in $\ell^2(\mathbb{Z}^d)$. Then:
(i) $\mathrm{Sp}(\Delta + r\delta_0) = [-4d, 0] \cup \{\mu(r)\}$ *with*

$$\mu(r) \begin{cases} = 0 & \textit{if } 0 < r \leq r_d, \\ > 0 & \textit{if } r > r_d. \end{cases} \tag{1.3.5}$$

(ii) For $r > r_d$, $\mu(r)$ is the unique solution of the equation $R(\mu) = 1/r$, and is an eigenvalue corresponding to a strictly positive eigenfunction.
(iii) On (r_d,∞), $r \mapsto \mu(r)/r$ is strictly increasing with $\lim_{r\to\infty} \mu(r)/r = 1$.
(iv) On $(0,\infty)$, $r \mapsto \mu(r)$ is convex.

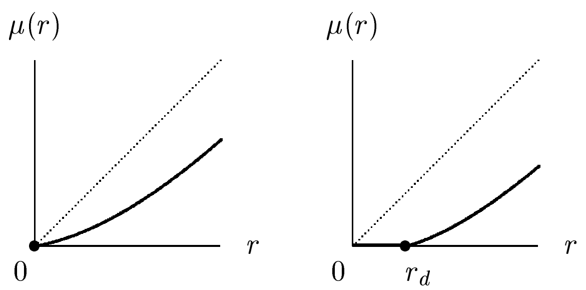

Fig. 1: $r \mapsto \mu(r)$ for $d = 1,2$, respectively, $d \geq 3$.

Our first result identifies the Lyapunov exponent $\widehat{\lambda}_p$.

Theorem 1.3.2 *Let $d \geq 1$ and $p \in \mathbb{N}$. For any choice of the parameters, the limit $\widehat{\lambda}_p$ exists, is finite, and equals*

$$\widehat{\lambda}_p = \rho\mu\left(p\gamma/\rho\right). \tag{1.3.6}$$

Thus, by (1.3.3) and (1.3.5), in $d = 1,2$ our system is always strongly p-catalytic, while in $d \geq 3$ it is strongly p-catalytic if and only if $p\gamma/\rho$ exceeds the critical threshold r_d. This dichotomy holds irrespectively of the values of κ and ν. Indeed, $\widehat{\lambda}_p$ in (1.3.6) does not depend on these parameters.

By (1.2.2), if $\widehat{\lambda}_p > 0$, then $\lambda_p = \infty$. Lemma 1.3.1 and Theorem 1.3.2 imply that when the system is strongly p-catalytic, it is strongly p-intermittent. Our second result looks at intermittency in the weakly catalytic regime, which corresponds to

$$d \geq 3, \qquad 0 < p\gamma/\rho \leq r_d. \tag{1.3.7}$$

Theorem 1.3.3 *Let $d \geq 3$ and $p \in \mathbb{N}$.*
(i) If $0 < p\gamma/\rho < r_d$, then the limit λ_p exists and is finite for any choice of κ and ν.
(ii) If $p\gamma/\rho = r_d$, then the limit λ_p exists and is infinite for any choice of κ and ν.

Our third result addresses the κ-dependence of the Lyapunov exponent $\lambda_p = \lambda_p(\kappa)$ and its asymptotics for small κ.

Theorem 1.3.4 *Let $d \geq 3$, $p \in \mathbb{N}$ and $0 < p\gamma/\rho < r_d$.*
(i) On $(0,\infty)$, $\kappa \mapsto \lambda_p(\kappa)$ is strictly decreasing and convex.
(ii)

$$\lim_{\kappa \downarrow 0} \lambda_p(\kappa) = \lambda_p(0) = \nu\gamma \, \frac{\frac{p\gamma}{\rho}}{r_d - \frac{p\gamma}{\rho}}. \tag{1.3.8}$$

Our fourth and fifth result concern the asymptotics of $\lambda_p(\kappa)$ for large κ.

Theorem 1.3.5 *Let $d \geq 4$, $p \in \mathbb{N}$ and $0 < p\gamma/\rho < r_d$. Then*

$$\lim_{\kappa \to \infty} \kappa\lambda_p(\kappa) = \frac{\nu\gamma^2}{r_d}. \tag{1.3.9}$$

Theorem 1.3.6 *Let $d = 3$, $p \in \mathbb{N}$ and $0 < p\gamma/\rho < r_3$. Then*

$$\lim_{\kappa \to \infty} \kappa\lambda_p(\kappa) = \frac{\nu\gamma^2}{r_3} + \left(\frac{\nu\gamma^2}{\rho}\, p\right)^{1/2} \mathcal{P} \tag{1.3.10}$$

with

$$\mathcal{P} = \sup_{\substack{f \in H^1(\mathbb{R}^3) \\ \|f\|_2 = 1}} \left[\left\| (-\Delta_{\mathbb{R}^3})^{-1/2} \, f^2 \right\|_2^2 - \|\nabla_{\mathbb{R}^3} f\|_2^2 \right] \in (0,\infty), \tag{1.3.11}$$

where $\nabla_{\mathbb{R}^3}$ and $\Delta_{\mathbb{R}^3}$ are the continuous (!) gradient and Laplacian, $\|\cdot\|_2$ is the L^2-norm, $H^1(\mathbb{R}^3) = \{f\colon \mathbb{R}^3 \to \mathbb{R}\colon\ f, \nabla_{\mathbb{R}^3} f \in L^2(\mathbb{R}^3)\}$, and

$$\left\| (-\Delta_{\mathbb{R}^3})^{-1/2} \, f^2 \right\|_2^2 = \int_{\mathbb{R}^3} dx\, f^2(x) \int_{\mathbb{R}^3} dy\, f^2(y) \, \frac{1}{4\pi|x-y|}. \tag{1.3.12}$$

The asymptotics is the same for all p when $d \geq 4$ (Theorem 1.3.5). The correction term with $\mathcal{P}$ is present only when $d = 3$ (Theorem 1.3.6).

Figure 2 shows the splitting of the graphs of the Lyapunov exponents $\lambda_p(\kappa)$ for small κ (Theorem 1.3.4(ii)) and the asymptotic behavior of the graphs for large κ (Theorems 1.3.5–1.3.6).

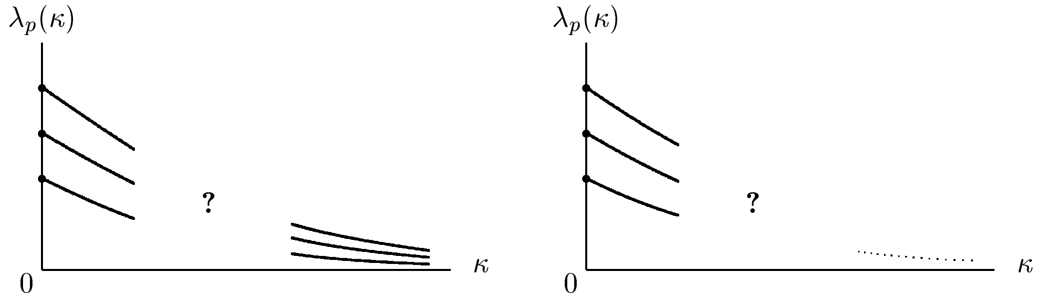

Fig. 2: Qualitative picture of $\kappa \mapsto \lambda_p(\kappa)$ for $p = 1, 2, 3$ and $d = 3$, respectively, $d \geq 4$. The dotted line represents the asymptotics given by (1.3.9).

The background of Theorems 1.3.5–1.3.6 is as follows. Let

$$\Lambda_1^*(t;\kappa) = \frac{1}{\kappa t} \log \mathbb{E}_0^X \left(\exp\left[\frac{\nu\gamma^2}{\kappa^2} \int_0^{\kappa t} ds \int_s^{\kappa t} du \; p\left(X(u) - X(s), \frac{\rho}{\kappa}(u-s)\right)\right]\right), \qquad (1.3.13)$$

where X is simple random walk on $\mathbb{Z}^d$ with generator Δ, starting at the origin, and $p(\cdot,\cdot)$ is its transition kernel. Define

$$\lambda_1^*(\kappa) = \lim_{t\to\infty} \Lambda_1^*(t;\kappa), \qquad \kappa > 0. \qquad (1.3.14)$$

We will see in Section 5 that

$$\lim_{\kappa\to\infty} \kappa\lambda_1(\kappa) = \lim_{\kappa\to\infty} \kappa^2\lambda_1^*(\kappa) = \begin{cases} \text{rhs (1.3.9)} & \text{if } d \geq 4, \\ \text{rhs (1.3.10)} & \text{if } d \geq 3. \end{cases} \qquad (1.3.15)$$

It will follow from a careful analysis of the double integral in the right-hand side of (1.3.13) that the regime $0 \leq u - s \ll \kappa^3$ gives rise to the term $\nu\gamma^2/r_d$ in (1.3.9) and (1.3.10), the regime $u - s \asymp \kappa^3$ gives rise to the variational term in (1.3.10) for $p = 1$, while the contribution of the regime $u - s \gg \kappa^3$ vanishes.

Interestingly, (1.3.11) is precisely the variational problem that arises in the so-called *polaron model*. Here, one takes Brownian motion W on $\mathbb{R}^3$ with generator $\Delta_{\mathbb{R}^3}$, starting at the origin, and for $\alpha > 0$ considers the quantity

$$\begin{aligned} \Theta(t;\alpha) &= \frac{1}{t}\log \mathbb{E}_0^W \left(\exp\left[\alpha \int_0^t ds \int_s^t du \; \frac{e^{-(u-s)}}{|W(u) - W(s)|}\right]\right) \\ &= \frac{1}{t}\log \mathbb{E}_0^W \left(\exp\left[\frac{1}{\alpha^2} \int_0^{\alpha^2 t} ds \int_s^{\alpha^2 t} du \; \frac{e^{-(u-s)/\alpha^2}}{|W(u) - W(s)|}\right]\right). \end{aligned} \qquad (1.3.16)$$

It was shown by Donsker and Varadhan [10] that

$$\theta(\alpha) = \lim_{t\to\infty} \Theta(t;\alpha), \quad \alpha > 0, \qquad (1.3.17)$$

with

$$\lim_{\alpha\to\infty}\frac{1}{\alpha^2}\,\theta(\alpha)=4\sqrt{\pi}\,\mathcal{P}. \tag{1.3.18}$$

It will turn out that the asymptotics in (1.3.15) and (1.3.18) are linked to each other as follows. If we consider the middle regime of the exponent in (1.3.13) and apply the Gaussian approximation to $p(\cdot,\cdot)$ and X through the scaling $t\to t/\kappa^2$, $x\to x/\kappa$ with $\kappa\to\infty$, then we obtain

$$\left(\frac{\kappa}{\rho}\right)^{-1}\frac{\nu\gamma^2}{\rho}\int_0^{t/\kappa}ds\int_s^{t/\kappa}du\;1\!\!1\{u-s\approx\kappa\}\,p_G\left(W(u)-W(s),\left(\frac{\kappa}{\rho}\right)^{-1}(u-s)\right) \tag{1.3.19}$$

with $p_G(x,t)=(4\pi t)^{-3/2}\exp[-\|x\|^2/4t]$. This expression is qualitatively similar to the exponent in the second line of (1.3.16) with α given by $\alpha^2=\kappa/\rho$. Although the two exponents are not the same, it turns out that they have the same large deviation behavior for $t\to\infty$ and $\kappa\to\infty$. Details can be found in Sections 5 and 7.

While Donsker and Varadhan use large deviations on the level of the process, we use large deviations on the level of the occupation time measure associated with the process.

It was shown by Lieb [23] that (1.3.11) has a unique maximizer modulo translations and that the centered maximizer is radially symmetric, radially non-increasing, strictly positive and smooth.

1.4 Discussion

Theorems 1.3.2–1.3.6 show that there is a delicate interplay between the various parameters in the model.

Catalytic behavior is controlled by γ/ρ, the ratio of the strength and the speed of the catalyst ξ, and is sensitive to this ratio only when $d\geq 3$. For large ratio the system is strongly catalytic, for small ratio the system is weakly catalytic. The high peaks in the reactant u develop at those sites where the catalyst ξ piles up. The analysis behind Theorem 1.3.2 shows that strongly catalytic behavior corresponds to the high peaks in the u-field being concentrated on *single sites*, whereas weakly catalytic plus strongly intermittent behavior corresponds to the high peaks being *spread out over islands containing several sites* (weakly intermittent behavior corresponds to no relevant high peaks). It follows from Lemma 1.3.1 and Theorem 1.3.2 that $\rho\mapsto\widehat{\lambda}_p(\rho)$ is strictly decreasing in the strongly catalytic regime. Thus, as the catalyst ξ moves faster it is less effective. Moreover, $\lim_{\rho\downarrow 0}\widehat{\lambda}_p(\rho)=p\gamma$. Here κ, the speed of the reactant u, plays no role, nor does ν, the intensity of the catalyst ξ.

Intermittent behavior is also sensitive to the parameters only when $d\geq 3$. Theorems 1.3.3–1.3.4 show that for small κ the reactant u has a range of high peaks, which grow at different exponential rates and determine the successive moments, and so the system is strongly intermittent. For large κ, on the other hand, the behavior depends on the dimension. The large diffusion of the reactant u prevents it to easily localize around the high peaks where the catalyst ξ piles up. As is clear from Theorems 1.3.5–1.3.6, in $d=3$ the system is strongly intermittent also for large κ, while in $d\geq 4$ it may or may not. To decide this issue we need a finer asymptotics than (1.3.9). We conjecture the following:

Conjecture 1.4.1 *In $d=3$, the system is strongly p-intermittent for all κ.*

Conjecture 1.4.2 *For $d\geq d_0\geq 4$, the system is weakly p-intermittent for $\kappa\geq\kappa_0(p)$.*

As promised at the end of Section 1.1, we discuss the results obtained by Kesten and Sidoravicius [21]. Their work was triggered by claims made in the physics literature: Shnerb, Louzoun, Bettelheim and Solomon [25], Shnerb, Bettelheim, Louzoun, Agam and Solomon [26].

I. $d = 1, 2$: *For any choice of the parameters, the average number of B-particles per site tends to infinity faster than exponential.* This result is covered by our Theorem 1.3.2, because the inclusion of the death rate δ shifts λ_1 by $-\delta$ (recall (1.1.8)), but does not affect $\widehat{\lambda}_1$, while $\widehat{\lambda}_1 > 0$ in $d = 1, 2$ for any choice of the parameters.

II. $d \geq 3$: *For γ small enough and δ large enough, the average number of B-particles per site tends to zero exponentially fast.* This result is covered by our Theorems 1.3.3–1.3.4, because γ small corresponds to the weakly catalytic regime for which $0 < \lambda_1 < \infty$, so that exponentially fast extinction occurs when $\delta > \lambda_1$.

III. $d \geq 1$: *For γ large enough, conditioned on the evolution of the A-particles, there is a phase transition, namely, for small δ the B-particles locally survive, while for large δ they become locally extinct.* This result is not linked to our theorems because we have no information on the quenched Lyapunov exponent.

The main focus of Kesten and Sidoravicius [21] is on survival versus extinction, while our focus is on moment asymptotics. Their approach does not lead to the identification of Lyapunov exponents, but it is more robust under an adaptation of the model than our approach, which is based on the Feynman-Kac representation in Section 2.1.

For related work on catalytic branching models, focussing in particular on continuum models with a singular catalyst in a measure-valued context, we refer to the overview papers by Dawson and Fleischmann [8] and Klenke [22]. Related references can be found there as well.

1.5 Future challenges

One challenge is to understand the geometry and the location of the high peaks in the u-field that determine the Lyapunov exponents in the weakly catalytic regime. These peaks (which are spread out over islands containing several sites) move and grow with time. The question is how.

Another challenge is to compute the quenched Lyapunov exponent, i.e.,

$$\lambda = \lim_{t \to \infty} \frac{1}{t} \log u(0,t) \quad \xi - a.s., \tag{1.5.1}$$

and to study its dependence on the parameters.

Finally, the choice in (1.1.4) constitutes one of the simplest types of catalyst dynamics. What happens for other choices of the ξ-field, e.g. when $\xi(x,t)$ is γ times the occupation number at site x at time t of a system of particles performing a simple exclusion process in equilibrium (i.e., particles moving like simple random walks but not being allowed to sit on top of each other)? This extension, which constitutes one of the simplest examples of a catalyst with interaction, is already quite hard. Since particles cannot pile up in this model, there will be no strongly catalytic regime (i.e., $\widehat{\lambda}_p = 0$). However, we expect the weakly catalytic regime to again exhibit a delicate interplay of parameters controlling the intermittent behavior.

1.6 Outline

The outline of the rest of this paper is as follows. In Section 2 we formulate some preparatory results, including a Feynman-Kac representation for the moments of the solution of (1.1.1) under (1.1.4–1.1.6), a certain concentration estimate, and the proof of Lemma 1.3.1. In Section 3 we prove Theorem 1.3.2 for $\widehat{\lambda}_p$. Section 4 contains the proof of Theorem 1.3.3–1.3.4 for $\lambda_p = \lambda_p(\kappa)$ in three parts: existence, convexity, and behavior for small κ. Sections 5–8, which take up over half of the paper, contain the proof of Theorems 1.3.5–1.3.6: behavior for large κ.

2 Preparations

Section 2.1 contains a Feynman-Kac representation for the moments of $u(0,t)$, which serves as the starting point of our analysis. Section 2.2 derives a certain concentration estimate that is needed for the proof of Theorem 1.3.2, while Section 2.3 contains the proof of Lemma 1.3.1.

2.1 Feynman-Kac representation

The formal starting point of our analysis of (1.1.1) is the following Feynman-Kac representation for the p-th moment of the u-field.

Proposition 2.1.1 *For any $p \in \mathbb{N}$,*

$$\langle u(0,t)^p \rangle = e^{p\nu\gamma t}\, \mathbb{E}_{0,\dots,0}^{X_1,\dots,X_p} \left(\exp\left[\nu\gamma \int_0^t \sum_{q=1}^p w(X_q(s),s)\, ds \right] \right), \tag{2.1.1}$$

where $X_1,\dots,X_p$ are independent simple random walks on $\mathbb{Z}^d$ with step rate $2d\kappa$ starting from the origin, the expectation is taken with respect to these random walks, and $w\colon\ \mathbb{Z}^d \times [0,\infty) \to \mathbb{R}$ is the solution of the Cauchy problem

$$\frac{\partial}{\partial t} w(x,t) = \rho\Delta w(x,t) + \gamma \left[\sum_{q=1}^p \delta_{X_q(t)}(x) \right] \{w(x,t)+1\}, \qquad w(\cdot,0) \equiv 0. \tag{2.1.2}$$

Proof. We give the proof for $p=1$. Let X,Y be independent copies of X_1, Y_0. By applying the Feynman-Kac formula to (1.1.1) and (1.1.6), and inserting (1.1.4), we have

$$\begin{aligned} u(0,t) &= \mathbb{E}_0^X \left(\exp\left[\int_0^t \xi(X(s),t-s)\, ds \right] \right) \\ &= \mathbb{E}_0^X \left(\prod_k \exp\left[\gamma \int_0^t \delta_{Y_k(t-s)}(X(s))\, ds \right] \right). \end{aligned} \tag{2.1.3}$$

Next, we take the expectation over the ξ-field. This is done by first taking the expectation over the trajectories Y_k given the starting points $Y_k(0)$ and then taking the expectation over

$Y_k(0)$ according to a Poisson random field with intensity ν:

$$\begin{aligned}
\langle u(0,t)\rangle &= \left\langle \mathbb{E}_0^X \prod_k \mathbb{E}_{Y_k(0)}^{Y_k}\left(\exp\left[\gamma\int_0^t \delta_{Y_k(t-s)}(X(s))\,ds\right]\right)\right\rangle \\
&= \mathbb{E}_0^X \left\langle \prod_k v(Y_k(0),t)\right\rangle \\
&= \mathbb{E}_0^X \left(\prod_{y\in\mathbb{Z}^d}\sum_{n\in\mathbb{N}_0}\frac{[\nu v(y,t)]^n}{n!}e^{-\nu}\right) \\
&= \mathbb{E}_0^X \left(\prod_{y\in\mathbb{Z}^d}\exp\left[\nu\{v(y,t)-1\}\right]\right)
\end{aligned} \tag{2.1.4}$$

with

$$v(y,t) = \mathbb{E}_y^Y\left(\exp\left[\gamma\int_0^t \delta_{Y(t-s)}(X(s))\,ds\right]\right). \tag{2.1.5}$$

The latter is a functional of X and is the solution of the Cauchy problem

$$\frac{\partial}{\partial t}v(x,t) = \rho\Delta v(x,t) + \gamma\delta_{X(t)}(x)v(x,t), \qquad v(\cdot,0)\equiv 1. \tag{2.1.6}$$

The last expectation in the r.h.s. of (2.1.4) equals $\mathbb{E}_0^X(\exp[\nu\Sigma(t)])$ with $\Sigma(t) = \sum_{y\in\mathbb{Z}^d}\{v(y,t)-1\}$. But from (2.1.6) we see that

$$\frac{d}{dt}\Sigma(t) = 0 + \gamma v(X(t),t), \qquad \Sigma(0) = 0. \tag{2.1.7}$$

Hence $\Sigma(t) = \gamma\int_0^t v(X(s),s)ds$. Now put

$$w(x,t) = v(x,t) - 1 \tag{2.1.8}$$

to complete the proof. The extension to arbitrary p is straightforward by taking p independent copies of the random walk X rather than one and repeating the argument. ∎

It follows from (1.2.1) and Proposition 2.1.1 that

$$\Lambda_p(t) = \frac{1}{pt}\log\mathbb{E}_{0,\dots,0}^{X_1,\dots,X_p}\left(\exp\left[\nu\gamma\int_0^t\sum_{q=1}^p w(X_q(s),s)\,ds\right]\right). \tag{2.1.9}$$

This is the representation we will work with later. Note that

$$w = w_{X_1,\dots,X_p}, \tag{2.1.10}$$

i.e., $w(\cdot,t)$ is to be solved as a function of the trajectories $X_1,\dots,X_p$ up to time t (and of the parameters p,γ,ρ), and $\Lambda_p(t)$ is to be calculated after insertion of the solution into the Feynman-Kac representation (2.1.9). Thus, the study of $\Lambda_p(t)$ amounts to doing a *large deviation analysis for a time-inhomogeneous functional of p random walks having long-time correlations.*

Note that

$$w(x,t) > 0 \qquad \forall x\in\mathbb{Z}^d,\ t>0, \tag{2.1.11}$$

as can be seen from (2.1.2). Hence $t\mapsto t\Lambda_p(t)$ is strictly positive and strictly increasing on $(0,\infty)$, as was claimed in Section 1.2.

2.2 Concentration estimate

The following estimate will be needed later on. It shows that the solution of (2.1.2) is maximal when $X_1, \ldots, X_p$ stay at the origin.

Proposition 2.2.1 *For any $p \in \mathbb{N}$ and $X_1, \ldots, X_p$,*

$$w(x,t) \leq \bar{w}(0,t) \qquad \forall\, x \in \mathbb{Z}^d,\ t \geq 0, \tag{2.2.1}$$

where $\bar{w}\colon\ \mathbb{Z}^d \times [0,\infty) \to \mathbb{R}$ is the solution of the Cauchy problem

$$\frac{\partial}{\partial t}\bar{w}(x,t) = \rho\Delta\bar{w}(x,t) + p\gamma\delta(x)\{\bar{w}(x,t)+1\}, \qquad \bar{w}(\cdot,0) \equiv 0. \tag{2.2.2}$$

Proof. Recall (1.3.1). Abbreviate $\oint dk = (2\pi)^{-d}\int_{[-\pi,\pi)^d} dk$. Let

$$p_\rho(x,t) = \oint dk\, e^{-\rho t\widehat{\varphi}(k)}\, e^{-ik\cdot x}, \qquad x \in \mathbb{Z}^d,\ t \geq 0, \tag{2.2.3}$$

denote the Fourier representation of the transition kernel associated with $\rho\Delta$. From this representation we see that

$$\max_{x\in\mathbb{Z}^d} p_\rho(x,t) = p_\rho(0,t) \qquad \forall\, t \geq 0. \tag{2.2.4}$$

The solution of (2.1.2) has the (implicit) representation

$$w(x,t) = \gamma\sum_{q=1}^{p}\int_0^t ds\, p_\rho(x - X_q(s), t-s)\{w(X_q(s),s)+1\}. \tag{2.2.5}$$

Abbreviate

$$\widehat{\eta}(t) = \frac{1}{p}\sum_{q=1}^{p} w(X_q(t),t). \tag{2.2.6}$$

We first prove that

$$\widehat{\eta}(t) \leq \bar{w}(0,t) \qquad \forall\, t \geq 0. \tag{2.2.7}$$

To that end, take $x = X_r(t)$, $r = 1, \ldots, p$, in (2.2.5), sum over r, and use (2.2.4), to obtain

$$\widehat{\eta}(t) \leq p\gamma\int_0^t ds\, p_\rho(0,t-s)\,\{\widehat{\eta}(s)+1\}. \tag{2.2.8}$$

Define

$$h(t) = p\gamma p_\rho(0,t) \geq 0. \tag{2.2.9}$$

Then (2.2.8) can be rewritten as

$$\widehat{\eta} \leq h * \{\widehat{\eta}+1\}. \tag{2.2.10}$$

Next, put

$$\bar{\eta}(t) = \bar{w}(0,t). \tag{2.2.11}$$

Then the same formulas with $X_1(\cdot), \ldots, X_p(\cdot) \equiv 0$ yield the relation

$$\bar{\eta} = h * \{\bar{\eta}+1\}. \tag{2.2.12}$$

Thus, it remains to show that (2.2.10) and (2.2.12) imply (2.2.7), i.e.,

$$\widehat{\eta} \leq \bar{\eta}. \tag{2.2.13}$$

This goes as follows.

Put $\delta = \bar{\eta} - \widehat{\eta}$. Then (2.2.10) and (2.2.12) give

$$\delta \geq h * \delta. \tag{2.2.14}$$

Iteration gives $\delta \geq h^{*n} * \delta$, and so to prove (2.2.13) it suffices to show that h^{*n} tends to zero as $n \to \infty$ uniformly on compact time intervals. To that end, put $h_T = \max_{t\in[0,T]} h(t)$. Then

$$0 \leq h^{*n}(t) \leq h_T \int_0^t h^{*(n-1)}(s)ds, \qquad t \in [0,T], \tag{2.2.15}$$

which when iterated gives

$$0 \leq h^{*n}(t) \leq h_T^n \frac{t^{n-1}}{(n-1)!}, \qquad t \in [0,T]. \tag{2.2.16}$$

Let $n \to \infty$ to get the claim.

Finally, put

$$\eta(t) = \max_{x\in\mathbb{Z}^d} w(x,t), \qquad t \geq 0. \tag{2.2.17}$$

Then (2.2.4–2.2.6) and (2.2.13) give

$$\eta \leq h * \{\widehat{\eta} + 1\} \leq h * \{\bar{\eta} + 1\}. \tag{2.2.18}$$

Now use (2.2.12) to get

$$\eta \leq \bar{\eta}, \tag{2.2.19}$$

which via (2.2.17) implies (2.2.1), as desired. ∎

Proposition 2.2.2 *For any $p \in \mathbb{N}$, $t \mapsto \bar{w}(0,t)$ is non-decreasing and $\bar{w}(0) = \lim_{t\to\infty} \bar{w}(0,t)$ satisfies*

$$\bar{w}(0) = \begin{cases} \frac{\frac{p\gamma}{\rho}}{r_d - \frac{p\gamma}{\rho}} & \text{if } 0 < \frac{p\gamma}{\rho} < r_d, \\ \infty & \text{otherwise.} \end{cases} \tag{2.2.20}$$

Proof. Returning to (2.2.11–2.2.12) and recalling (2.2.9), we have

$$\bar{w}(0,t) = p\gamma \int_0^t ds\, p_\rho(0,s)\, \{\bar{w}(0,t-s) + 1\}. \tag{2.2.21}$$

From this we see that $t \mapsto \bar{w}(0,t)$ is non-decreasing. Using this fact in (2.2.21), we have

$$\bar{w}(0,t) \leq p\gamma \left(\int_0^\infty ds\, p_\rho(0,s)\right) \{\bar{w}(0,t) + 1\} = \frac{p\gamma}{\rho}\frac{1}{r_d}\{\bar{w}(0,t) + 1\} \tag{2.2.22}$$

(recall (1.3.3)) and hence

$$\bar{w}(0,t) \leq \text{rhs } (2.2.20). \tag{2.2.23}$$

Taking the limit $t \to \infty$ in (2.2.21) and using monotone convergence, we get

$$\bar{w}(0) = p\gamma \left(\int_0^\infty du\, p_\rho(0,u)\right) \{\bar{w}(0) + 1\} = \frac{p\gamma}{\rho}\frac{1}{r_d}\{\bar{w}(0) + 1\}, \tag{2.2.24}$$

which implies the claim. ∎

2.3 Proof of Lemma 1.3.1.

The proof is elementary.

(i–ii) For $r \in (0,\infty)$, let $\mathcal{H} = \Delta + r\delta_0$. This is a self-adjoint operator on $\ell^2(\mathbb{Z}^d)$. The Fourier transform of $\mathcal{H}$ is the operator on $L^2([-\pi,\pi)^d)$ given by

$$(\widehat{\mathcal{H}}\widehat{v})(k) = -\widehat{\varphi}(k)\widehat{v}(k) + r \oint \widehat{v}(l)dl, \tag{2.3.1}$$

where we recall (1.3.1). Since $\mathrm{Sp}(\mathcal{H}) = \mathrm{Sp}(\widehat{\mathcal{H}})$, (1.3.4) reads

$$\mu(r) = \sup \mathrm{Sp}(\widehat{\mathcal{H}}). \tag{2.3.2}$$

The spectrum of $\widehat{\mathcal{H}}$ consists of those $\lambda \in \mathbb{R}$ for which $\lambda - \widehat{\mathcal{H}}$ is not invertible. Consider therefore the equation

$$(\lambda - \widehat{\mathcal{H}})f = g. \tag{2.3.3}$$

Substituting (2.3.1) into (2.3.3), we get

$$(\lambda + \widehat{\varphi})f - r\oint f = g. \tag{2.3.4}$$

Now, the range of $\widehat{\varphi}$ is the interval $[0,4d]$. Thus, if $\lambda \in [-4d,0]$, then there exists $g \in L^2([-\pi,\pi)^d)$ for which (2.3.4), and hence (2.3.3), has no solution, i.e.,

$$\mathrm{Sp}(\widehat{\mathcal{H}}) \supset [-4d,0]. \tag{2.3.5}$$

Next, assume that $\lambda > 0$. Divide (2.3.3) by $\lambda + \widehat{\varphi}$ and integrate to get

$$[1 - rR(\lambda)]\oint f = \oint \frac{g}{\lambda + \widehat{\varphi}} \tag{2.3.6}$$

with R as defined in (1.3.2). If $rR(\lambda) = 1$, then there is again no solution, i.e.,

$$rR(\lambda) = 1 \quad \Longrightarrow \quad \lambda \in \mathrm{Sp}(\widehat{\mathcal{H}}). \tag{2.3.7}$$

If, on the other hand, $rR(\lambda) \neq 1$, then (2.3.6) yields a unique solution

$$f = \frac{1}{\lambda + \widehat{\varphi}}\left(g + \frac{r}{1 - rR(\lambda)}\oint \frac{g}{\lambda + \widehat{\varphi}}\right), \tag{2.3.8}$$

which is in $L^2([-\pi,\pi)^d)$, i.e.,

$$rR(\lambda) \neq 1 \quad \Longrightarrow \quad \lambda \notin \mathrm{Sp}(\widehat{\mathcal{H}}). \tag{2.3.9}$$

The same argument shows that

$$(-\infty,-4d) \cap \mathrm{Sp}(\widehat{\mathcal{H}}) = \emptyset. \tag{2.3.10}$$

Combining (2.3.5), (2.3.7), (2.3.9–2.3.10) and noting that $rR(\lambda) = 1$ has a unique solution $\lambda = \lambda(r) > 0$ if and only if $r > r_d$, we obtain assertions (i) and (ii). Note that if $r > r_d$, then

$$e = r(\mu(r) - \Delta)^{-1}\delta_0 \tag{2.3.11}$$

is a positive eigenfunction of $\mathcal{H}$ to the eigenvalue $\mu(r)$, normalized by $e(0) = 1$ (rather than by $\|e\|_2 = 1$ with $\|\cdot\|_2$ the ℓ^2-norm).

(iii) From (1.3.2) we have

$$\mu R(\mu) = \oint \frac{\mu}{\mu + \widehat{\varphi}}. \tag{2.3.12}$$

Differentiate this relation w.r.t. μ, to obtain

$$[\mu R(\mu)]' = \oint \frac{\widehat{\varphi}}{(\mu + \widehat{\varphi})^2} > 0. \tag{2.3.13}$$

Next, differentiate the relation $rR(\mu(r)) = 1$ w.r.t. r and use that $R' < 0$, to obtain

$$\mu'(r) = -\frac{R(\mu(r))}{rR'(\mu(r))} > 0. \tag{2.3.14}$$

From (2.3.13–2.3.14) we get

$$[\mu(r)/r]' = [\mu(r)R(\mu(r))]' = [\mu R(\mu)]'(r)\mu'(r) > 0, \tag{2.3.15}$$

which proves the first part of assertion (iii). The second part of assertion (iii) follows from the estimate

$$0 < \frac{1}{\mu} - R(\mu) = \oint \frac{\widehat{\varphi}}{\mu(\mu + \widehat{\varphi})} < \frac{1}{\mu^2} \oint \widehat{\varphi} = \frac{2d}{\mu^2} \tag{2.3.16}$$

after letting $\mu \to \infty$, corresponding to $r \to \infty$.

(iv) Differentiating (2.3.14) w.r.t. r, we obtain

$$\mu''(r) = \frac{[R(\mu(r))]^2 R''(\mu(r))}{r^2 [R'(\mu(r))]^3}. \tag{2.3.17}$$

Since $R' < 0$ and $R'' > 0$, this completes the proof.

An alternative way of seeing (iii) and (iv) is via the Rayleigh-Ritz formula:

$$\mu(r) = \sup_{\substack{f \in \ell^2(\mathbb{Z}^d) \\ \|f\|_2 = 1}} \left\{ rf(0) - \frac{1}{2} \sum_{\substack{x,y \in \mathbb{Z}^d \\ \|x-y\|=1}} [f(x) - f(y)]^2 \right\}. \tag{2.3.18}$$

Indeed, this formula shows that $r \mapsto \mu(r)$ is a supremum of linear functions, and therefore is convex. Moreover, it shows that $r \mapsto \mu(r)/r$ is non-decreasing, and since the supremum is attained when $r > r_d$, it in fact gives that $r \mapsto \mu(r)/r$ is strictly increasing on (r_d, ∞) (and tends to 1 as $r \to \infty$).

3 Proof of Theorem 1.3.2

The proof uses spectral analysis.

3.1 Upper and lower bounds

Let $\mathcal{H} = \rho\Delta + p\gamma\delta_0$. This is a self-adjoint operator on $\ell^2(\mathbb{Z}^d)$. Equation (2.2.2) reads

$$\frac{\partial}{\partial t}\bar{w} = \mathcal{H}\bar{w} + p\gamma\delta_0, \qquad \bar{w}(\cdot,0) \equiv 0. \tag{3.1.1}$$

By (1.3.4),

$$\sup \mathrm{Sp}(\mathcal{H}) = \rho\mu(p\gamma/\rho). \tag{3.1.2}$$

Suppose first that $\rho\mu(p\gamma/\rho) > 0$. Then, by Lemma 1.3.1, this is an eigenvalue of $\mathcal{H}$ corresponding to a strictly positive eigenfunction $e \in \ell^2(\mathbb{Z}^d)$ (normalized as $\|e\|_2 = 1$). From (2.1.9) and Proposition 2.2.1 we have

$$-2d\kappa + \nu\gamma\frac{1}{t}\int_0^t \bar{w}(0,s)ds \le \Lambda_p(t;\kappa) \le \nu\gamma\frac{1}{t}\int_0^t \bar{w}(0,s)ds, \tag{3.1.3}$$

where we use that

$$\mathbb{P}_{0,\dots,0}^{X_1,\dots,X_p}\Big(X_q(s) = 0\ \forall s \in [0,t]\ \forall q = 1,\dots,p\Big) = e^{-2d\kappa pt}. \tag{3.1.4}$$

From (3.1.1) we have

$$\bar{w}(\cdot,t) = p\gamma\int_0^t ds\, \Big(e^{(t-s)\mathcal{H}}\delta_0\Big)(\cdot). \tag{3.1.5}$$

Moreover, from the spectral representation of $e^{(t-s)\mathcal{H}}$ and (3.1.2) we have

$$e^{(t-s)\rho\mu(p\gamma/\rho)}\langle e,\delta_0\rangle \le \Big\langle e^{(t-s)\mathcal{H}}\delta_0,\delta_0\Big\rangle \le e^{(t-s)\rho\mu(p\gamma/\rho)}\|\delta_0\|_2^2. \tag{3.1.6}$$

Combining (3.1.3) and (3.1.5–3.1.6), we arrive at

$$\widehat{\lambda}_p = \lim_{t\to\infty}\frac{1}{t}\log\Lambda_p(t;\kappa) = \rho\mu(p\gamma/\rho). \tag{3.1.7}$$

Suppose next that $\rho\mu(p\gamma/\rho) = 0$. Then the upper bound in (3.1.6) remains valid (despite the fact that no eigenfunction $e \in \ell^2(\mathbb{Z}^d)$ with eigenvalue 0 may exist), and so the limit equals zero.

4 Proof of Theorem 1.3.3–1.3.4

In Section 4.1 we prove Theorem 1.3.3, in Sections 4.2–4.3 we prove Theorem 1.3.4.

4.1 Existence of λ_p

We already know that λ_p exists and is infinite in the strongly catalytic regime, i.e., when $d = 1,2$ or $d \ge 3$, $p\gamma/\rho > r_d$ (see the remarks below Theorem 1.3.2). At the end of Section 4.3 we will see that the same is true at the boundary of the weakly catalytic regime, i.e., when $d \ge 3$, $p\gamma/\rho = r_d$, as is claimed in Theorem 1.3.3(ii). The following lemma proves Theorem 1.3.3(i).

Lemma 4.1.1 *Let $d \ge 3$ and $p \in \mathbb{N}$. If $0 < p\gamma/\rho < r_d$, then the limit λ_p exists and is finite.*

Proof. Fix $d \geq 3$ and $p \in \mathbb{N}$, and return to (2.1.3). We have

$$u(0,t) = \sum_{x \in \mathbb{Z}^d} S_x(t) \tag{4.1.1}$$

with

$$S_x(t) = \mathbb{E}_0^X \left(\exp\left[\int_0^t \xi(X(s), t-s) ds \right] \delta_x(X(t)) \right). \tag{4.1.2}$$

Hence

$$\begin{aligned} \langle u(0,t)^p \rangle &= \left\langle \left[\sum_{x \in \mathbb{Z}^d} S_x(t) \right]^p \right\rangle \\ &\leq \left\langle \left[\sum_{x \in Q_{t \log t}} S_x(t) \right]^p \right\rangle + p \left\langle \sum_{x_1 \notin Q_{t \log t}} S_{x_1}(t) \left[\sum_{x \in \mathbb{Z}^d} S_x(t) \right]^{p-1} \right\rangle \\ &= \left\langle \left[\sum_{x \in Q_{t \log t}} S_x(t) \right]^p \right\rangle + p \sum_{x_1 \notin Q_{t \log t}} \sum_{x_2, \ldots, x_p \in \mathbb{Z}^d} \left\langle \prod_{q=1}^p S_{x_q}(t) \right\rangle, \end{aligned} \tag{4.1.3}$$

where $Q_{t \log t} = [-t \log t, t \log t]^d \cap \mathbb{Z}^d$. By Jensen's inequality, the first term in the r.h.s. of (4.1.3) is bounded above by

$$\begin{aligned} &|Q_{t \log t}|^{p-1} \left\langle \sum_{x \in Q_{t \log t}} [S_x(t)]^p \right\rangle \\ &= e^{p\nu\gamma t} \, |Q_{t \log t}|^{p-1} \sum_{x \in Q_{t \log t}} \mathbb{E}_{0,\ldots,0}^{X_1,\ldots,X_p} \left(\exp\left[\nu\gamma \sum_{q=1}^p \int_0^t w(X_q(s), s) \, ds \right] \prod_{q=1}^p \delta_x(X_q(t)) \right), \end{aligned} \tag{4.1.4}$$

where the last line follows the calculation in the proof of Proposition 2.1.1. The second term in the r.h.s. of (4.1.3) is bounded above by

$$p e^{p\nu\gamma\{\bar{w}(0)+1\}t} \, \mathbb{P}_0^{X_1} \left(X_1(t) \notin Q_{t \log t} \right), \tag{4.1.5}$$

where we use that $w(x,t) \leq \bar{w}(0,t) \leq \bar{w}(0)$ by Propositions 2.2.1–2.2.2, with $\bar{w}(0) < \infty$ strictly inside the weakly p-catalytic regime considered here. Now define

$$\underline{\Lambda}_p(t) = \max_{x \in \mathbb{Z}^d} \frac{1}{pt} \log \mathbb{E}_{0,\ldots,0}^{X_1,\ldots,X_p} \left(\exp\left[\nu\gamma \sum_{q=1}^p \int_0^t w(X_q(s), s) \, ds \right] \prod_{q=1}^p \delta_x(X_q(t)) \right). \tag{4.1.6}$$

Since the probability in (4.1.5) is superexponentially small (SES) in t, we see that a comparison of (2.1.9) and (4.1.6) yields the sandwich (combine 1.2.1) and (4.1.3–4.1.5))

$$\underline{\Lambda}_p(t) \leq \Lambda_p(t) \leq \frac{1}{pt} \log \left(|Q_{t \log t}|^p e^{pt \underline{\Lambda}_p(t)} + \text{SES} \right), \tag{4.1.7}$$

so that

$$\lim_{t \to \infty} \left[\Lambda_p(t) - \underline{\Lambda}_p(t) \right] = 0. \tag{4.1.8}$$

To prove existence of λ_p, it therefore suffices to prove existence of

$$\bar{\lambda}_p = \lim_{t\to\infty} \underline{\Lambda}_p(t), \tag{4.1.9}$$

after which we conclude that $\lambda_p = \bar{\lambda}_p$.

The proof of existence of (4.1.9) goes as follows. Write

$$w(x,s) = w_{X_1[0,t],\dots,X_p[0,t]}(x,s), \qquad s \in [0,t], \tag{4.1.10}$$

to exhibit the dependence of w on the p trajectories. We have, for any $s,t \geq 0$,

$$w_{X_1[0,s+t],\dots,X_p[0,s+t]}(x,u) \begin{cases} = w_{X_1[0,s],\dots,X_p[0,s]}(x,u) & \text{for } u \in [0,s] \\ \geq w_{X_1[s,s+t],\dots,X_p[s,s+t]}(x,u-s) & \text{for } u \in [s,s+t]. \end{cases} \tag{4.1.11}$$

Here, the inequality arises by resetting the initial condition to $\equiv 0$ at time s and using that the solution of (2.1.2) is monotone in the initial condition. It follows from (4.1.6) and (4.1.11) that

$$\begin{aligned} p(s+t)\underline{\Lambda}_p(s+t) &\geq \max_{x,y\in\mathbb{Z}^d} \log \mathbb{E}_{0,\dots,0}^{X_1,\dots,X_p}\Big(\exp\Big[\nu\gamma \sum_{q=1}^p \int_0^{s+t} w_{X_1[0,s+t],\dots,X_p[0,s+t]}(X_q(u),u)\,du\Big] \\ &\qquad\qquad \times \prod_{q=1}^p \delta_y(X_q(s)) \prod_{q=1}^p \delta_x(X_q(s+t))\Big) \\ &\geq \max_{x,y\in\mathbb{Z}^d} \Big\{ \log \mathbb{E}_{0,\dots,0}^{X_1,\dots,X_p}\Big(\exp\Big[\nu\gamma \sum_{q=1}^p \int_0^{s} w_{X_1[0,s],\dots,X_p[0,s]}(X_q(u),u)\,du\Big] \prod_{q=1}^p \delta_y(X_q(s))\Big) \\ &\qquad + \log \mathbb{E}_{0,\dots,0}^{X_1,\dots,X_p}\Big(\exp\Big[\nu\gamma \sum_{q=1}^p \int_0^{t} w_{X_1[0,t],\dots,X_p[0,t]}(X_q(u),u)\,du\Big] \prod_{q=1}^p \delta_{x-y}(X_q(t))\Big) \Big\} \\ &= ps\underline{\Lambda}_p(s) + pt\underline{\Lambda}_p(t), \end{aligned} \tag{4.1.12}$$

where we use that $w_{y+X_1[0,t],\dots,y+X_p[0,t]}$ does not depend on y. Thus, $t \mapsto t\underline{\Lambda}_p(t)$ is superadditive and so the limit in (4.1.9) indeed exists. It follows from Proposition 2.2.2 and (3.1.3) that $\lambda_p \leq p\nu\gamma\bar{w}(0)$, which proves that λ_p is finite strictly inside the weakly p-catalytic regime. ■

4.2 Convexity in κ

We will write down a formal expansion of the expectation in the r.h.s. of (2.1.9). From this expansion it will immediately follow that $\Lambda_p(t)$ is a convex function of κ, for any p, t and γ, ρ, ν. After that we can pass to the limit $t \to \infty$ to conclude that $\lambda_p = \lim_{t\to\infty} \Lambda_p(t)$ is a convex function of κ too.

Proposition 4.2.1 *For any* $p \in \mathbb{N}$,

$$\mathbb{E}_{0,\ldots,0}^{X_1,\ldots,X_p}\left(\exp\left[\nu\gamma\int_0^t \sum_{q=1}^p w(X_q(s),s)\,ds\right]\right)$$

$$= \sum_{n=0}^{\infty}(\nu\gamma)^n\left(\prod_{m=1}^n\int_0^{s_{m-1}} ds_m\right)\left(\prod_{m=1}^n\sum_{r_m=1}^p\sum_{l_m=1}^{\infty}\right)\gamma^{\sum_{m=1}^n l_m}\left(\prod_{\alpha=1}^n\prod_{\beta=1}^{l_\alpha}\int_0^{u_{\alpha,\beta-1}} du_{\alpha,\beta}\oint dk_{\alpha,\beta}\right)$$

$$\times\exp\left[-\rho\sum_{\alpha=1}^n\sum_{\beta=1}^{l_\alpha}(u_{\alpha,\beta-1}-u_{\alpha,\beta})\widehat{\varphi}(k_{\alpha,\beta})\right]\left(\prod_{\alpha=1}^n\prod_{\gamma=1}^{l_\alpha}\sum_{r_{\alpha,\gamma}=1}^p\right)$$

$$\times\exp\left[-\kappa\sum_{q=1}^p\int_0^t dv\;\widehat{\varphi}\left(\sum_{\alpha=1}^n\sum_{\beta=1}^{l_\alpha}k_{\alpha,\beta}\left\{\delta_{r_{\alpha,\beta},q}\mathbb{1}_{[0,u_{\alpha,\beta}]}(v)-\delta_{r_{\alpha,\beta-1},q}\mathbb{1}_{[0,u_{\alpha,\beta-1}]}(v)\right\}\right)\right], \tag{4.2.1}$$

with the convention $s_0 = t$, $r_{\alpha,0} = r_\alpha$ *and* $u_{\alpha,0} = s_\alpha$, $\alpha \in \mathbb{N}$.

Proof. By Taylor expansion, we have

$$\begin{aligned}&\mathbb{E}_{0,\ldots,0}^{X_1,\ldots,X_p}\left(\exp\left[\nu\gamma\int_0^t \sum_{q=1}^p w(X_q(s),s)\,ds\right]\right)\\ &= \sum_{n=0}^{\infty}(\nu\gamma)^n\left(\prod_{m=1}^n\int_0^{s_{m-1}} ds_m\right)\mathbb{E}_{0,\ldots,0}^{X_1,\ldots,X_p}\left(\prod_{m=1}^n\sum_{q=1}^p w(X_q(s_m),s_m)\right),\end{aligned} \tag{4.2.2}$$

with $s_0 = t$. To compute the n-point correlation under the integral, we return to (2.2.5). By substituting (2.2.3) into (2.2.5) and iterating the resulting equation, we obtain the expansion

$$\begin{aligned}w(X_r(t),t) = \sum_{l=1}^{\infty}\gamma^l&\left(\prod_{\beta=1}^l\int_0^{u_{\beta-1}} du_\beta\oint dk_\beta\right)\\ &\times\exp\left[-\rho\sum_{\beta=1}^l(u_{\beta-1}-u_\beta)\widehat{\varphi}(k_\beta)\right]\\ &\times\left(\prod_{\gamma=1}^l\sum_{r_\gamma=1}^p\right)\exp\left\{i\sum_{\beta=1}^l k_\beta\cdot[X_{r_\beta}(u_\beta)-X_{r_{\beta-1}}(u_{\beta-1})]\right\},\end{aligned} \tag{4.2.3}$$

with $u_0 = t$ and $r_0 = r$. This expansion is convergent because the summand is bounded above by $(\gamma t p)^l/l!$. With the help of (4.2.3), we get

$$
\begin{aligned}
&\mathbb{E}_{0,\ldots,0}^{X_1,\ldots,X_p}\left(\prod_{m=1}^{n}\sum_{q=1}^{p} w(X_q(s_m), s_m)\right)\\
&\quad = \left(\prod_{m=1}^{n}\sum_{r_m=1}^{p}\sum_{l_m=1}^{\infty}\right)\gamma^{\sum_{m=1}^{n} l_m}\left(\prod_{\alpha=1}^{n}\prod_{\beta=1}^{l_\alpha}\int_0^{u_{\alpha,\beta-1}} du_{\alpha,\beta}\oint dk_{\alpha,\beta}\right)\\
&\quad\times \exp\left[-\rho\sum_{\alpha=1}^{n}\sum_{\beta=1}^{l_\alpha}(u_{\alpha,\beta-1}-u_{\alpha,\beta})\widehat{\varphi}(k_{\alpha,\beta})\right]\\
&\quad\times \left(\prod_{\alpha=1}^{n}\prod_{\gamma=1}^{l_\alpha}\sum_{r_{\alpha,\gamma}=1}^{p}\right)\mathbb{E}_{0,\ldots,0}^{X_1,\ldots,X_p}\left(\exp\left\{i\sum_{\alpha=1}^{n}\sum_{\beta=1}^{l_\alpha}k_{\alpha,\beta}\cdot[X_{r_{\alpha,\beta}}(u_{\alpha,\beta})-X_{r_{\alpha,\beta-1}}(u_{\alpha,\beta-1})]\right\}\right),
\end{aligned}
\tag{4.2.4}
$$

with $r_{\alpha,0} = r_\alpha$ and $u_{\alpha,0} = s_\alpha$, $\alpha = 1,\ldots,n$. To complete the proof it therefore suffices to show that

$$
\begin{aligned}
&\mathbb{E}_{0,\ldots,0}^{X_1,\ldots,X_p}\left(\exp\left\{i\sum_{\alpha=1}^{n}\sum_{\beta=1}^{l_\alpha}k_{\alpha,\beta}\cdot[X_{r_{\alpha,\beta}}(u_{\alpha,\beta})-X_{r_{\alpha,\beta-1}}(u_{\alpha,\beta-1})]\right\}\right)\\
&\quad = \exp\left[-\kappa\sum_{q=1}^{p}\int_0^t dv\;\widehat{\varphi}\left(\sum_{\alpha=1}^{n}\sum_{\beta=1}^{l_\alpha}k_{\alpha,\beta}\left\{\delta_{r_{\alpha,\beta},q}1\!\!1_{[0,u_{\alpha,\beta}]}(v)-\delta_{r_{\alpha,\beta-1},q}1\!\!1_{[0,u_{\alpha,\beta-1}]}(v)\right\}\right)\right].
\end{aligned}
\tag{4.2.5}
$$

By writing

$$
\begin{aligned}
X_{r_{\alpha,\beta}}(u_{\alpha,\beta})-X_{r_{\alpha,\beta-1}}(u_{\alpha,\beta-1}) &= \sum_{q=1}^{p}\left\{\delta_{r_{\alpha,\beta},q}X_q(u_{\alpha,\beta})-\delta_{r_{\alpha,\beta-1},q}X_q(u_{\alpha,\beta-1})\right\}\\
&= \sum_{q=1}^{p}\int_0^t\left\{\delta_{r_{\alpha,\beta},q}1\!\!1_{[0,u_{\alpha,\beta}]}(v)-\delta_{r_{\alpha,\beta-1},q}1\!\!1_{[0,u_{\alpha,\beta-1}]}(v)\right\}\,dX_q(v)
\end{aligned}
\tag{4.2.6}
$$

and noting that the increments $dX_q(v)$, $q = 1,\ldots,p$, are independent, we see that (4.2.5) is a special case of the relation

$$
\mathbb{E}_0^{X_q}\left(\exp\left[i\int_0^t f(v)\cdot dX_q(v)\right]\right) = \exp\left[-\kappa\int_0^t \widehat{\varphi}(f(v))\,dv\right], \qquad q = 1,\ldots,p, \tag{4.2.7}
$$

which holds for any $f\colon \mathbb{R}^d \to \mathbb{R}$ that is piecewise continuous and has bounded jumps. To see why (4.2.7) is true, we note that

$$
\mathbb{E}_0^{X_q}\left(\exp\left[ik\cdot X_q(t)\right]\right) = \sum_{x\in\mathbb{Z}^d} e^{ik\cdot x}p_\kappa(x,t) \tag{4.2.8}
$$

with p_κ the transition kernel associated with $\kappa\Delta$. It follows from (2.2.3) that

$$\mathbb{E}_0^{X_q}\left(\exp\left[ik\cdot X_q(t)\right]\right)=\exp\left[-\kappa t\widehat{\varphi}(k)\right]. \tag{4.2.9}$$

From this relation, together with the fact that the increments of the process X_q over disjoint time intervals are independent, we get (4.2.7). ∎

The expression in Proposition 4.2.1 is complicated, but the relevant point is that the right-hand side is a linear combination with nonnegative coefficients of functions that are negative exponentials in κ. Such a quantity is log-convex in κ, which tells us that $\Lambda_p(t)$ is convex in κ (recall (2.1.9)). Consequently, $\lambda_p=\lim_{t\to\infty}\Lambda_p(t)$ is convex in κ too.

4.3 Small κ

If $\kappa=0$, then $X_1,\ldots,X_p$ stay at the origin and so we have from (2.1.9) and (2.2.1) that

$$\Lambda_p(t;0)=\nu\gamma\frac{1}{t}\int_0^t\bar{w}(0,s)ds. \tag{4.3.1}$$

Since $t\mapsto\bar{w}(0,t)$ is non-decreasing by Proposition 2.2.2, we have

$$\lambda_p(0)=\nu\gamma\bar{w}(0) \tag{4.3.2}$$

with $\bar{w}(0)=\lim_{t\to\infty}\bar{w}(0,t)$ given by (2.2.20). This proves the second equality in (1.3.8) in Theorem 1.3.4(ii). It follows from (3.1.3) and (4.3.1) that

$$\lambda_p(0)-2d\kappa\leq\lambda_p(\kappa)\leq\lambda_p(0). \tag{4.3.3}$$

Hence $\kappa\mapsto\lambda_p(\kappa)$ is continuous at 0 and is bounded on $[0,\infty)$. This proves the first equality in (1.3.8) in Theorem 1.3.4(ii). Since $\kappa\mapsto\lambda_p(\kappa)$ is convex, as was shown in Section 4.2, it must be continuous and non-increasing on $[0,\infty)$. Since it tends to zero like $1/\kappa$ as $\kappa\to\infty$, as stated in Theorems 1.3.5–1.3.6, which will be proved in Section 5, it must be strictly positive and strictly decreasing on $[0,\infty)$. Thus we have proved Theorem 1.3.4(i).

By Proposition 2.2.2 and (4.3.2), $\lambda_p(0)=\infty$ when $d\geq 3$, $p\gamma/\rho=r_d$. It therefore follows from (4.3.3) that $\lambda_p(\kappa)=\infty$. Thus we have proved Theorem 1.3.3(ii). The proof of Theorem 1.3.3(i) was already achieved with Lemma 4.1.1.

5 Proof of Theorems 1.3.5–1.3.6

The proof is long and technical. In Section 5.1 we do an appropriate scaling in κ. In Section 5.2 we formulate seven key lemmas that are the main ingredients in the proof. In Section 5.3 we prove Theorems 1.3.5–1.3.6 subject to these lemmas. The proof of the lemmas is deferred to Section 6–8.

5.1 Scaling

To exhibit the dependence on the parameters, we henceforth write

$$\Lambda_p(T)=\Lambda_p(T;\kappa,\gamma,\rho,\nu), \tag{5.1.1}$$

where $\Lambda_p(T)$ is defined in (1.2.1). Substituting (2.2.5) into (2.1.9), we find that

$$\Lambda_p(T;\kappa,\gamma,\rho,\nu) = \frac{1}{pT}\log \mathbb{E}_{0,\dots,0}^{X_1^\kappa,\dots,X_p^\kappa}\Bigg(\exp\Bigg[\nu\gamma^2\sum_{k,l=1}^{p}\int_0^T ds\int_s^T dt \\ p_\rho\left(X_l^\kappa(t)-X_k^\kappa(s),t-s\right)\left(1+w(X_k^\kappa(s),s)\right)\Bigg]\Bigg). \tag{5.1.2}$$

In this formula, $X_1^\kappa,\dots,X_p^\kappa$ are independent simple random walks on $\mathbb{Z}^d$ with diffusion constant κ (i.e., step rate $2d\kappa$), the expectation is over these random walks starting at 0, p_ρ is the transition kernel associated with $\rho\Delta$, and w denotes the solution of the Cauchy problem

$$\frac{\partial w}{\partial t} = \rho\Delta w + \gamma\left(\sum_{k=1}^{p}\delta_{X_k^\kappa(t)}\right)(1+w), \qquad w(\cdot,0)\equiv 0. \tag{5.1.3}$$

In Sections 2–4 the upper index κ was suppressed. We introduce it here because now we want to remove the dependence of the random walks on κ. Indeed, in (5.1.2) we perform a time scaling $t\to t/\kappa$, to obtain that

$$\Lambda_p(T;\kappa,\gamma,\rho,\nu) = \kappa\Lambda_p(\kappa T;1,\gamma/\kappa,\rho/\kappa,\nu). \tag{5.1.4}$$

Hence

$$\Lambda_p(T;\kappa,\gamma,\rho,\nu) = \kappa\Lambda_p^*(\kappa T;\kappa,\gamma,\rho,\nu), \tag{5.1.5}$$

where

$$\Lambda_p^*(T;\kappa,\gamma,\rho,\nu) = \frac{1}{pT}\log \mathbb{E}_{0,\dots,0}^{X_1,\dots,X_p}\Bigg(\exp\Bigg[\frac{\nu\gamma^2}{\kappa^2}\sum_{k,l=1}^{p}\int_0^T ds\int_s^T dt \\ p_{\rho/\kappa}(X_l(t)-X_k(s),t-s)\left(1+w^*(X_k(s),s)\right)\Bigg]\Bigg), \tag{5.1.6}$$

$X_1,\dots,X_p$ are simple random walks on $\mathbb{Z}^d$ with diffusion constant 1, w^* solves

$$\frac{\partial w^*}{\partial t} = \frac{\rho}{\kappa}\Delta w^* + \frac{\gamma}{\kappa}\left(\sum_{k=1}^{p}\delta_{X_k(t)}\right)(1+w^*), \qquad w^*(\cdot,0)\equiv 0, \tag{5.1.7}$$

and satisfies $w^*\geq 0$.

The Lyapunov exponents in Theorem 1.3.3 are (recall (1.2.2))

$$\lambda_p = \lambda_p(\kappa,\gamma,\rho,\nu) = \lim_{T\to\infty}\Lambda_p(T;\kappa,\gamma,\rho,\nu). \tag{5.1.8}$$

Because of (5.1.5), these are related to the rescaled Lyapunov exponents

$$\lambda_p^*(\kappa,\gamma,\rho,\nu) = \lim_{T\to\infty}\Lambda_p^*(T;\kappa,\gamma,\rho,\nu) \tag{5.1.9}$$

via

$$\lambda_p(\kappa,\gamma,\rho,\nu) = \kappa\lambda_p^*(\kappa,\gamma,\rho,\nu). \tag{5.1.10}$$

Note also that (5.1.4) leads to the scaling

$$\lambda_p(\kappa,\gamma,\rho,\nu) = \kappa\lambda_p(1,\gamma/\kappa,\rho/\kappa,\nu). \tag{5.1.11}$$

We will frequently suppress the parameters γ,ρ,ν from the notation and write $\Lambda_p(T;\kappa)$, $\Lambda_p^*(T;\kappa)$ and $\lambda_p(\kappa)$, $\lambda_p^*(\kappa)$.

5.2 Main ingredients in the proof

The assertion of Theorems 1.3.5–1.3.6 may now be restated as follows.

Theorem 5.2.1 *Let $d \geq 3$, $p \in \mathbb{N}$ and*

$$0 < \frac{p\gamma}{\rho} < r_d. \tag{5.2.1}$$

(i) For $d \geq 4$,

$$\lim_{\kappa\to\infty} \kappa^2 \lambda_p^*(\kappa) = \frac{\nu\gamma^2}{r_d}. \tag{5.2.2}$$

(ii) For $d = 3$,

$$\lim_{\kappa\to\infty} \kappa^2 \lambda_p^*(\kappa) = \frac{\nu\gamma^2}{r_3} + \left(\frac{\nu\gamma^2}{\rho}p\right)^{1/2} \mathcal{P} \tag{5.2.3}$$

with $\mathcal{P}$ the constant defined in (1.3.11).

The proof of Theorem 5.2.1 is based on seven lemmas, which are stated below and which provide lower and upper bounds for various parts contributing to (5.1.6). The guiding idea behind these lemmas is that the expectation in (5.1.6) can be moved to the exponential in the limit as $\kappa \to \infty$ uniformly in T, except for the part that produces the constant $\mathcal{P}$, which needs a large deviation analysis. This idea, though simple, is technically rather involved.

In the statement of the lemmas below three auxiliary parameters appear:

$$0 < a < \infty, \qquad 0 < \epsilon < K < \infty. \tag{5.2.4}$$

These parameters are needed to separate various time regimes. Four lemmas involve one random walk (X), one lemma involves two random walks (X, Y), and two lemmas involve p random walks $(X_1, \ldots, X_p)$. We use upper indices $-$ and $+$ for lim inf and lim sup, respectively.

5.2.1 Lower bound

The first lemma concerns the "diagonal term" $(0 \leq t - s \leq a\kappa^3)$. Let

$$\Lambda_{\text{diag}}^-(T; a, \kappa) = -\frac{1}{T} \log \mathbb{E}_0^X \left(\exp\left[-\frac{\nu\gamma^2}{\kappa^2} \int_0^T ds \int_s^{s+a\kappa^3} dt\; p_{\rho/\kappa}(X(t) - X(s), t - s)\right]\right) \tag{5.2.5}$$

and

$$\lambda_{\text{diag}}^-(a, \kappa) = \liminf_{T\to\infty} \Lambda_{\text{diag}}^-(T; a, \kappa). \tag{5.2.6}$$

Lemma 5.2.2 (Lower bound for the diagonal term)
For $d \geq 3$,

$$\liminf_{\kappa\to\infty} \kappa^2 \lambda_{\text{diag}}^-(a, \kappa) \geq \frac{\nu\gamma^2}{r_d} \qquad \forall\, 0 < a < \infty. \tag{5.2.7}$$

The second lemma concerns the "variational term" ($\epsilon\kappa^3 \leq t-s \leq K\kappa^3$), which involves p random walks and which will turn out to be responsible for the second term in the right-hand side of (5.2.3). Let

$$\begin{aligned}&\Lambda_{\text{var}}(T;\epsilon,K,\kappa)\\&= \frac{1}{pT}\log \mathbb{E}_{0,\ldots,0}^{X_1,\ldots,X_p}\left(\exp\left[\frac{\nu\gamma^2}{\kappa^2}\sum_{k,l=1}^{p}\int_0^T ds \int_{s+\epsilon\kappa^3}^{s+K\kappa^3} dt\ p_{\rho/\kappa}(X_l(t)-X_k(s),t-s)\right]\right)\end{aligned} \tag{5.2.8}$$

and

$$\lambda_{\text{var}}^{-}(\epsilon,K,\kappa) = \liminf_{T\to\infty}\Lambda_{\text{var}}(T;\epsilon,K,\kappa). \tag{5.2.9}$$

Lemma 5.2.3 (Lower bound for the variational term)
For $d=3$,

$$\liminf_{\kappa\to\infty}\kappa^2\lambda_{\text{var}}^{-}(\epsilon,K,\kappa) \geq \mathcal{P}_p(\epsilon,K;\gamma,\rho,\nu) \qquad \forall\, 0<\epsilon<K<\infty, \tag{5.2.10}$$

where

$$\begin{aligned}&\mathcal{P}_p(\epsilon,K;\gamma,\rho,\nu)\\&= \sup_{\substack{f\in H^1(\mathbb{R}^3)\\ \|f\|_2=1}}\left[\frac{\nu\gamma^2}{\rho}p\int_{\mathbb{R}^3}dx\ f^2(x)\int_{\mathbb{R}^3}dy\ f^2(y)\int_{\epsilon\rho}^{K\rho}dt\ p_G(x-y,t) - \|\nabla_{\mathbb{R}^3}f\|_2^2\right]\end{aligned} \tag{5.2.11}$$

with $p_G(x,t)=(4\pi t)^{-3/2}\exp[-\|x\|^2/4t]$ the Gaussian transition kernel associated with $\Delta_{\mathbb{R}^3}$.

5.2.2 Upper bound

The third lemma is the counterpart of Lemma 5.2.2. Let

$$\Lambda_{\text{diag}}^{+}(T;a,\kappa) = \frac{1}{T}\log\mathbb{E}_0^X\left(\exp\left[\frac{\nu\gamma^2}{\kappa^2}\int_0^T ds\int_s^{s+a\kappa^3}dt\ p_{\rho/\kappa}(X(t)-X(s),t-s)\right]\right) \tag{5.2.12}$$

and

$$\lambda_{\text{diag}}^{+}(a,\kappa) = \limsup_{T\to\infty}\Lambda_{\text{diag}}^{+}(T;a,\kappa). \tag{5.2.13}$$

Lemma 5.2.4 (Upper bound for the diagonal term)
(i) For $d\geq 4$,

$$\limsup_{\kappa\to\infty}\kappa^2\lambda_{\text{diag}}^{+}(a,\kappa) \leq \frac{\nu\gamma^2}{r_d} \qquad \forall\, 0<a<\infty. \tag{5.2.14}$$

(ii) For $d=3$,

$$\limsup_{a\downarrow 0}\limsup_{\kappa\to\infty}\kappa^2\lambda_{\text{diag}}^{+}(a,\kappa) \leq \frac{\nu\gamma^2}{r_3}. \tag{5.2.15}$$

The fourth lemma is the counterpart of Lemma 5.2.3. Let

$$\lambda_{\text{var}}^{+}(\epsilon,K,\kappa) = \limsup_{T\to\infty}\Lambda_{\text{var}}(T;\epsilon,K,\kappa). \tag{5.2.16}$$

Lemma 5.2.5 (Upper bound for the variational term)
(i) For $d \geq 4$,

$$\lim_{\kappa\to\infty} \kappa^2 \lambda^+_{\mathrm{var}}(\epsilon, K, \kappa) = 0 \qquad \forall\, 0 < \epsilon < K < \infty. \tag{5.2.17}$$

(ii) For $d = 3$,

$$\limsup_{\kappa\to\infty} \kappa^2 \lambda^+_{\mathrm{var}}(\epsilon, K, \kappa) \leq \mathcal{P}_p(\epsilon, K; \gamma, \rho, \nu) \qquad \forall\, 0 < \epsilon < K < \infty. \tag{5.2.18}$$

Three more lemmas deal with the upper bound, all of which turn out to involve terms that are negligible in the limit as $\kappa \to \infty$. The fifth lemma concerns the "off-diagonal" term ($t - s > a\kappa^3$). Let

$$\Lambda_{\mathrm{off}}(T; a, \kappa) = \frac{1}{T} \log \mathbb{E}_0^X \left(\exp\left[\frac{\nu\gamma^2}{\kappa^2} \int_0^T ds \int_{s+a\kappa^3}^\infty dt\; p_{\rho/\kappa}(X(t) - X(s), t - s) \right] \right) \tag{5.2.19}$$

and

$$\lambda^+_{\mathrm{off}}(a, \kappa) = \limsup_{T\to\infty} \Lambda_{\mathrm{off}}(T; a, \kappa). \tag{5.2.20}$$

Lemma 5.2.6 (Upper bound for the off-diagonal term)
(i) For $d \geq 4$,

$$\lim_{\kappa\to\infty} \kappa^2 \lambda^+_{\mathrm{off}}(a, \kappa) = 0 \qquad \forall\, 0 < a < \infty. \tag{5.2.21}$$

(ii) For $d = 3$,

$$\lim_{a\to\infty} \limsup_{\kappa\to\infty} \kappa^2 \lambda^+_{\mathrm{off}}(a, \kappa) = 0. \tag{5.2.22}$$

The sixth lemma concerns the "mixed" term and involves two random walks. Let

$$\Lambda_{\mathrm{mix}}(T; a, \kappa) = \frac{1}{T} \log \mathbb{E}_{0,0}^{X,Y} \left(\exp\left[\frac{\nu\gamma^2}{\kappa^2} \int_0^T ds \int_s^{s+a\kappa^3} dt\; p_{\rho/\kappa}(Y(t) - X(s), t - s) \right] \right) \tag{5.2.23}$$

with

$$\lambda^+_{\mathrm{mix}}(a, \kappa) = \limsup_{T\to\infty} \Lambda_{\mathrm{mix}}(T; a, \kappa). \tag{5.2.24}$$

Lemma 5.2.7 (Upper bound for the mixed term)
(i) For $d \geq 4$,

$$\lim_{\kappa\to\infty} \kappa^2 \lambda^+_{\mathrm{mix}}(\infty, \kappa) = 0. \tag{5.2.25}$$

(ii) For $d = 3$,

$$\lim_{\kappa\to\infty} \kappa^2 \lambda^+_{\mathrm{mix}}(a, \kappa) = 0 \qquad \forall\, 0 < a < a_0 \text{ with } a_0 \text{ small enough.} \tag{5.2.26}$$

The seventh lemma deals with a term that will be needed to handle the w^*-remainder in (5.1.6). Let

$$\begin{aligned} \Lambda_{\mathrm{rem}}(T; \kappa) = \frac{1}{T} \log \mathbb{E}_0^X \Bigg(\exp\Bigg[\frac{\nu\gamma^3}{\kappa^3} \int_0^T ds \\ \left(\int_s^\infty dt\; p_{\rho/\kappa}(X(t) - X(s), t - s) \right) \left(\int_0^s du\; p_{\rho/\kappa}(X(s) - X(u), s - u) \right) \Bigg] \Bigg) \end{aligned} \tag{5.2.27}$$

and

$$\lambda_{\rm rem}^{+}(\kappa) = \limsup_{T\to\infty} \Lambda_{\rm rem}(T;\kappa). \tag{5.2.28}$$

(Note the extra factor γ/κ in the exponent in the right-hand side of (5.2.27) compared to the previous definitions.)

Lemma 5.2.8 (Upper bound for the w^*-remainder)
For $d \geq 3$,

$$\lim_{\kappa\to\infty} \kappa^2 \lambda_{\rm rem}^{+}(\kappa) = 0. \tag{5.2.29}$$

The proof of Lemmas 5.2.2–5.2.8 is deferred to Section 6–8.

5.3 Proof of Theorem 5.2.1

Recall that the solution of (5.1.7) admits the (implicit) integral representation (compare with (2.2.5))

$$w^*(x,s) = \frac{\gamma}{\kappa}\sum_{l=1}^{p}\int_0^s du\; p_{\rho/\kappa}(x - X_l(u), s-u)\,(1 + w^*(X_l(u),u))\,. \tag{5.3.1}$$

Moreover, in the weakly catalytic regime given by (5.2.1), we have

$$w^*(x,s) \leq \bar{w}(0) = C^* = \frac{\frac{p\gamma}{\rho}}{r_d - \frac{p\gamma}{\rho}} < \infty \qquad \forall\, x \in \mathbb{Z}^d,\ s \geq 0 \tag{5.3.2}$$

(recall (2.2.1), (2.2.5) and (2.2.20)). Note that C^* does not depend on κ.

For $d \geq 3$ and $a \geq 0$, abbreviate

$$G_a(0) = \int_a^\infty dt\; p(0,t). \tag{5.3.3}$$

We have $G_0(0) = R(0) = 1/r_d$ (recall (1.3.3)), and there exists a constant $c_d > 0$ such that

$$G_a(0) \leq \frac{c_d}{r_d a^{(d-2)/2}}, \qquad a > 0. \tag{5.3.4}$$

5.3.1 Lower bound

Removing in (5.1.6) the terms with w^*, $t > s + K\kappa^3$, and $k \neq l$ for $t \leq s + \epsilon\kappa^3$, we get

$$\Lambda_p^*(T;\kappa,\gamma,\rho,\nu) \geq \frac{1}{pT}\log E_{0,\dots,0}^{X_1,\dots,X_p}\left(\exp[U + V - C]\right) \tag{5.3.5}$$

with

$$\begin{aligned} U &= \frac{\nu\gamma^2}{\kappa^2}\sum_{k=1}^{p}\int_0^T ds \int_s^{s+\epsilon\kappa^3} dt\, p_{\rho/\kappa}(X_k(t) - X_k(s), t-s),\\ V &= \frac{\nu\gamma^2}{\kappa^2}\sum_{k,l=1}^{p}\int_0^T ds \int_{s+\epsilon\kappa^3}^{s+K\kappa^3} dt\, p_{\rho/\kappa}(X_l(t) - X_k(s), t-s), \end{aligned} \tag{5.3.6}$$

where $C > 0$ is a constant that compensates for $t > T$ in (5.3.6). This constant may be chosen independently of T, as follows easily from rough estimates. By a reverse version of Hölder's inequality, we have

$$
\begin{aligned}
&E_{0,\dots,0}^{X_1,\dots,X_p}(\exp[U+V]) \geq \left(E_{0,\dots,0}^{X_1,\dots,X_p}(\exp[-\zeta U])\right)^{-1/\zeta}\left(E_{0,\dots,0}^{X_1,\dots,X_p}(\exp[\theta V])\right)^{1/\theta},\\
&\theta \in (0,1),\ \zeta = \frac{\theta}{1-\theta}.
\end{aligned}
\tag{5.3.7}
$$

Hence, recalling (5.2.5) and (5.2.8), we obtain

$$
\Lambda_p^*(T;\kappa,\gamma,\rho,\nu) \geq \frac{1}{\zeta}\,\Lambda_{\rm diag}^-(T;\epsilon,\kappa,\gamma,\rho,\zeta\nu) + \frac{1}{\theta}\,\Lambda_{\rm var}(T;\epsilon,K,\kappa,\gamma,\rho,\theta\nu). \tag{5.3.8}
$$

By letting $T\to\infty$, recalling (5.1.9), letting $\kappa\to\infty$, using Lemmas 5.2.2–5.2.3 for the corresponding terms in the right-hand side, and afterwards letting $\theta\uparrow 1$, we arrive at

$$
\liminf_{\kappa\to\infty}\kappa^2\lambda_p^*(\kappa) \geq \frac{\nu\gamma^2}{r_d} \qquad \text{if } d\geq 4 \tag{5.3.9}
$$

(drop the last term in (5.3.8)) and

$$
\liminf_{\kappa\to\infty}\kappa^2\lambda_p^*(\kappa) \geq \frac{\nu\gamma^2}{r_3} + \mathcal{P}_p(\epsilon,K;\gamma,\rho,\nu) \qquad \text{if } d=3 \tag{5.3.10}
$$

(keep the last term in (5.3.8)). In the latter, let $\epsilon\downarrow 0$ and $K\to\infty$, and use that

$$
\lim_{\epsilon\downarrow 0,\,K\to\infty}\mathcal{P}_p(\epsilon,K;\gamma,\rho,\nu) = \mathcal{P}_p(\gamma,\rho,\nu) \tag{5.3.11}
$$

with

$$
\begin{aligned}
&\mathcal{P}_p(\gamma,\rho,\nu)\\
&= \sup_{\substack{f\in H^1(\mathbb{R}^3)\\ \|f\|_2=1}}\left[\frac{\nu\gamma^2}{\rho}\,p\int_{\mathbb{R}^3}dx\ f^2(x)\int_{\mathbb{R}^3}dy\ f^2(y)\int_0^\infty dt\ p_G(x-y,t) - \|\nabla_{\mathbb{R}^3}f\|_2^2\right],
\end{aligned}
\tag{5.3.12}
$$

to obtain

$$
\liminf_{\kappa\to\infty}\kappa^2\lambda_p^*(\kappa) \geq \frac{\nu\gamma^2}{r_3} + \mathcal{P}_p(\gamma,\rho,\nu) \qquad \text{if } d=3. \tag{5.3.13}
$$

Finally, a straightforward scaling argument shows that

$$
\mathcal{P}_p(\gamma,\rho,\nu) = \left(\frac{\nu\gamma^2}{\rho}\,p\right)^{1/2}\mathcal{P} \tag{5.3.14}
$$

with $\mathcal{P}$ the constant defined in (1.3.11). This completes the proof of the lower bound in Theorem 5.2.1.

The fact that (5.3.11) holds is an immediate consequence of the fact that (1.3.11), and hence (5.3.12), has a maximizer $\bar f$, as shown by Lieb [23]. Indeed, we have

$$
\begin{aligned}
0 &\leq \mathcal{P}_p(\gamma,\rho,\nu) - \mathcal{P}_p(\epsilon,K;\gamma,\rho,\nu)\\
&\leq \frac{\nu\gamma^2}{\rho}\,p\int_{\mathbb{R}^3}dx\,\bar f^{\,2}(x)\int_{\mathbb{R}^3}dy\,\bar f^{\,2}(y)\int_{(0,\epsilon\rho)\cup(K\rho,\infty)}dt\ p_G(x-y,t),
\end{aligned}
\tag{5.3.15}
$$

and the right-hand side tends to zero as $\epsilon\downarrow 0$ and $K\to\infty$ because the full integral is finite.

5.3.2 Upper bound

We begin by splitting the exponent in the right-hand side of (5.1.6) into various parts. The splitting is done with the various lemmas in Section 5.2.2 in mind, and uses the parameters in (5.2.4) with $a = \epsilon$ or $a = K$.

Lemma 5.3.1 *For any* $p \in \mathbb{N}$,

$$\sum_{k,l=1}^{p} \int_0^T ds \int_s^T dt\ p_{\rho/\kappa}(X_l(t) - X_k(s), t-s)(1 + w^*(X_k(s), s))$$

$$\leq \left(1 + \frac{D_\epsilon}{\kappa^{d-2}}\right)(I + II + III) + \left(1 + \frac{D_\epsilon}{\kappa^{d-2}} + 2(1+C^*)\frac{\gamma p}{r_d \rho}\right) IV + (1+C^*)\frac{\gamma}{\kappa} V, \tag{5.3.16}$$

where C^* *is the constant in* (5.3.2),

$$D_\epsilon = \frac{(1+C^*)c_d \gamma p}{r_d \rho^{d/2} \epsilon^{(d-2)/2}} \tag{5.3.17}$$

with c_d *the constant in* (5.3.4), *and*

$$\begin{aligned}
I &= \sum_{k=1}^{p} \int_0^T ds \int_s^{s+\epsilon\kappa^3} dt\ p_{\rho/\kappa}(X_k(t) - X_k(s), t-s), \\
II &= \sum_{k,l=1}^{p} \int_0^T ds \int_{s+\epsilon\kappa^3}^{s+K\kappa^3} dt\ p_{\rho/\kappa}(X_l(t) - X_k(s), t-s), \\
III &= \sum_{k,l=1}^{p} \int_0^T ds \int_{s+K\kappa^3}^{\infty} dt\ p_{\rho/\kappa}(X_l(t) - X_k(s), t-s), \\
IV &= \sum_{\substack{k,l=1 \\ k\neq l}}^{p} \int_0^T ds \int_s^{s+\epsilon\kappa^3} dt\ p_{\rho/\kappa}(X_l(t) - X_k(s), t-s), \\
V &= \sum_{k=1}^{p} \int_0^T ds \left(\int_0^s dr\, p_{\rho/\kappa}(X_k(s) - X_k(r), s-r)\right)\left(\int_s^\infty dt\, p_{\rho/\kappa}(X_k(t) - X_k(s), t-s)\right).
\end{aligned} \tag{5.3.18}$$

Proof. For the term without w^* we bound

$$\sum_{k,l=1}^{p} \int_0^T ds \int_s^T dt\ p_{\rho/\kappa}(X_l(t) - X_k(s), t-s) \leq I + II + III + IV. \tag{5.3.19}$$

For the term with w^* we bound, with the help of (5.3.1–5.3.2),

$$\begin{aligned}
&\sum_{k,l=1}^{p} \int_0^T ds \int_s^T dt\ p_{\rho/\kappa}(X_l(t) - X_k(s), t-s) w^*(X_k(s), s) \\
&\leq (1+C^*)\frac{\gamma}{\kappa} \sum_{j,k,l=1}^{p} \int_0^T ds \\
&\qquad \left(\int_0^s dr\ p_{\rho/\kappa}(X_k(s) - X_j(r), s-r)\right)\left(\int_s^T dt\ p_{\rho/\kappa}(X_l(t) - X_k(s), t-s)\right).
\end{aligned} \tag{5.3.20}$$

By (5.3.4),

$$\int_{\epsilon\kappa^3}^{\infty} du\ p_{\rho/\kappa}(0,u) \leq \frac{C_\epsilon}{\kappa^{d-3}} \quad \text{with} \quad C_\epsilon = \frac{c_d}{r_d \rho^{d/2} \epsilon^{(d-2)/2}}. \tag{5.3.21}$$

Hence, by (2.2.4),

$$\begin{aligned} \int_0^{(s-\epsilon\kappa^3)\vee 0} dr\ p_{\rho/\kappa}(X_k(s)-X_j(r),s-r) &\leq \int_{-\infty}^{s-\epsilon\kappa^3} dr\ p_{\rho/\kappa}(0,s-r) \leq \frac{C_\epsilon}{\kappa^{d-3}}, \\ \int_{(s+\epsilon\kappa^3)\wedge T}^{T} dt\ p_{\rho/\kappa}(X_l(t)-X_k(s),t-s) &\leq \int_{s+\epsilon\kappa^3}^{\infty} dt\ p_{\rho/\kappa}(0,t-s) \leq \frac{C_\epsilon}{\kappa^{d-3}}. \end{aligned} \tag{5.3.22}$$

Splitting the integrals in the two factors in the right-hand side of (5.3.20) into two parts accordingly, and inserting (5.3.22), we find that

$$\begin{aligned} &\text{rhs } (5.3.20) \\ &\leq (1+C^*)\frac{\gamma}{\kappa} \sum_{j,k,l=1}^{p} \int_0^T ds \left(\int_{(s-\epsilon\kappa^3)\vee 0}^{s} dr\ p_{\rho/\kappa}(X_k(s)-X_j(r),s-r) \right) \\ &\qquad\qquad \times \left(\int_s^{(s+\epsilon\kappa^3)\wedge T} dt\ p_{\rho/\kappa}(X_l(t)-X_k(s),t-s) \right) \\ &\quad + \frac{D_\epsilon}{\kappa^{d-2}} \sum_{k,l=1}^{p} \int_0^T ds \int_s^T dt\ p_{\rho/\kappa}(X_l(t)-X_k(s),t-s), \end{aligned} \tag{5.3.23}$$

with $D_\epsilon = 2(1+C^*)C_\epsilon\gamma p$. Indeed, the second term in the right-hand side of (5.3.23) is obtained from the cross products by estimating, via (5.3.22),

$$\begin{aligned} &\sum_{j,k,l=1}^{p} \int_0^T ds \left(\int_0^s dr\ p_{\rho/\kappa}(X_k(s)-X_j(r),s-r) \right) \\ &\qquad\qquad \times \left(\int_{(s+\epsilon\kappa^3)\wedge T}^{T} dt\ p_{\rho/\kappa}(X_l(t)-X_k(s),t-s) \right) \\ &\qquad \leq \frac{C_\epsilon p}{\kappa^{d-3}} \sum_{j,k=1}^{p} \int_0^T ds \int_0^s dr\ p_{\rho/\kappa}(X_k(s)-X_j(r),s-r) \end{aligned} \tag{5.3.24}$$

and

$$\begin{aligned} &\sum_{j,k,l=1}^{p} \int_0^T ds \left(\int_0^{(s-\epsilon\kappa^3)\vee 0} dr\ p_{\rho/\kappa}(X_k(s)-X_j(r),s-r) \right) \\ &\qquad\qquad \times \left(\int_s^T dt\ p_{\rho/\kappa}(X_l(t)-X_k(s),t-s) \right) \\ &\qquad \leq \frac{C_\epsilon p}{\kappa^{d-3}} \sum_{k,l=1}^{p} \int_0^T ds \int_s^T dt\ p_{\rho/\kappa}(X_l(t)-X_k(s),t-s), \end{aligned} \tag{5.3.25}$$

and adding (5.3.24–5.3.25) after renaming indices $j \to k$ and $k \to l$ in the former.

The second term in the right-hand side of (5.3.23) can be estimated with the help of (5.3.19). For the first term, split the sum over the indices into $j=k=l$, $j \neq k$ and $k \neq l$. For $k \neq l$ ($j \neq k$) we estimate the first (second) inner integral by $\kappa/r_d\rho$. As a result, we obtain

$$\text{lhs } (5.3.20) \leq \frac{D_\epsilon}{\kappa^{d-2}} (I + II + III + IV) + 2(1+C^*)\frac{\gamma p}{r_d \rho} IV + (1+C^*)\frac{\gamma}{\kappa} V. \tag{5.3.26}$$

Combining (5.3.19) and (5.3.26), we arrive at the claim. ∎

Our next step is to apply Hölder's inequality to separate the various summands appearing in (5.3.18), so that we can apply to them the lemmas in Section 5.2.2. We will separate all summands except the ones in II, since the latter produces the variational problem in (5.2.11) and requires a cooperation of the p random walks.

The total number of summands in (5.3.18) that are separated thus equals $q = p+1+p^2+p(p-1)+p = 2p^2+p+1$. Hence, substituting (5.3.18) into (5.3.16), substituting the resulting formula into (5.1.6), and applying Hölder's inequality

$$E\left(e^{\sum_{r=1}^{q} S_r}\right) \leq \left[E\left(e^{\theta S_1}\right)\right]^{1/\theta} \prod_{r=2}^{q} \left[E\left(e^{\zeta S_r}\right)\right]^{1/\zeta}, \qquad \theta \in (1,\infty),\ \zeta = \frac{\theta}{\theta-1}(q-1), \tag{5.3.27}$$

to the expectation in the right-hand side of (5.1.6) (with $r=1$ reserved for II), we find that

$$\begin{aligned} p\Lambda_p^*(T;\kappa,\gamma,\rho,\nu) \leq\ & \frac{p}{\zeta}\ \Lambda_{\text{diag}}^{+}\left(T;\epsilon,\kappa,\gamma,\rho,\left(1+\frac{D_\epsilon}{\kappa^{d-2}}\right)\zeta\nu\right) \\ & + \frac{1}{\theta}\ p\Lambda_{\text{var}}\left(T;\epsilon,K,\kappa,\gamma,\rho,\left(1+\frac{D_\epsilon}{\kappa^{d-2}}\right)\theta\nu\right) \\ & + \frac{p^2}{\zeta}\ \Lambda_{\text{off}}\left(T;K,\kappa,\gamma,\rho,\left(1+\frac{D_\epsilon}{\kappa^{d-2}}\right)\zeta\nu\right) \\ & + \frac{p(p-1)}{\zeta}\ \Lambda_{\text{mix}}\left(T;\epsilon,\kappa,\gamma,\rho,\left(1+\frac{D_\epsilon}{\kappa^{d-2}}+2(1+C^*)\frac{\gamma p}{r_d\rho}\right)\zeta\nu\right) \\ & + \frac{p}{\zeta}\ \Lambda_{\text{rem}}\left(T;\kappa,\gamma,\rho,(1+C^*)\zeta\nu\right). \end{aligned} \tag{5.3.28}$$

By letting $T\to\infty$, recalling (5.1.9), letting $\kappa\to\infty$, using Lemmas 5.2.4–5.2.8 for the corresponding terms in the right-hand side of (5.3.28), and afterwards letting $\theta\downarrow 1$, we arrive at

$$\limsup_{\kappa\to\infty} \kappa^2\lambda_p^*(\kappa) \leq \frac{\nu\gamma^2}{r_d} \qquad \text{if } d\geq 4, \tag{5.3.29}$$

and, after estimating $\mathcal{P}_p(\epsilon,K;\gamma,\rho,\nu) \leq \mathcal{P}_p(\gamma,\rho,\nu)$, using (5.2.15) with $a=\epsilon$, and letting $\epsilon\downarrow 0$,

$$\limsup_{\kappa\to\infty} \kappa^2\lambda_p^*(\kappa) \leq \frac{\nu\gamma^2}{r_3} + \mathcal{P}_p(\gamma,\rho,\nu) \qquad \text{if } d=3. \tag{5.3.30}$$

For the second term in the right-hand side of (5.3.30) we may use (5.3.14). This completes the proof of the upper bound in Theorem 5.2.1.

6 Proof of Lemmas 5.2.2 and 5.2.4

As we saw in Section 5.3, the "diagonal" contributions to the lower and the upper bound in the proof of Theorem 5.2.1 come from Lemmas 5.2.2 and 5.2.4, respectively. In this section we prove these two lemmas. Let $p(x,t)$ denote the transition kernel associated with Δ. Then $p_{\rho/\kappa}(x,t) = p\left(x,\frac{\rho}{\kappa}t\right)$.

6.1 Proof of Lemma 5.2.2

Proof. Let $a, A > 0$ be arbitrary. Estimate

$$\begin{aligned}&\int_0^T ds \int_s^{s+a\kappa^3} dt\; p_{\rho/\kappa}(X(t)-X(s),t-s)\\ &\geq \left(\sum_{\substack{k=1\\ \text{even}}}^{\lfloor T/a\rfloor} + \sum_{\substack{k=1\\ \text{odd}}}^{\lfloor T/a\rfloor}\right) \int_{(k-1)a}^{ka} ds \int_s^{s+A} dt\; p_{\rho/\kappa}(X(t)-X(s),t-s) \qquad \forall\, \kappa \geq \kappa_0(a,A).\end{aligned} \tag{6.1.1}$$

Note that the summands in each of the two sums are i.i.d. Hence, substituting (6.1.1) into (5.2.5) and applying the Cauchy-Schwarz inequality, we find that

$$\Lambda^-_{\text{diag}}(T;a,\kappa) \geq -\frac{\lfloor T/a\rfloor}{2T}\,\log \mathbb{E}_0^X\left(\exp\left[-2W(a,A,\kappa)\right]\right) \tag{6.1.2}$$

with

$$W(a,A,\kappa) = \frac{\nu\gamma^2}{\kappa^2}\int_0^a ds \int_s^{s+A} dt\; p_{\rho/\kappa}(X(t)-X(s),t-s). \tag{6.1.3}$$

Next, note that, by (2.2.4),

$$W(a,A,\kappa) \leq \frac{\nu\gamma^2}{\kappa^2}\int_0^a ds \int_s^{s+A} dt\; p_{\rho/\kappa}(0,t-s) \leq \frac{\nu\gamma^2}{\kappa^2}\,a\,\frac{\kappa}{\rho}\,\frac{1}{r_3}. \tag{6.1.4}$$

Since, for fixed a, the right-hand side tends to zero as $\kappa\to\infty$, it follows that

$$\mathbb{E}_0^X\left(\exp\left[-2W(a,A,\kappa)\right]\right) \leq \exp\left[-2\theta\,\mathbb{E}_0^X\left(W(a,A,\kappa)\right)\right] \qquad \forall\,\theta\in(0,1),\ \kappa\geq\kappa_1(\theta,a,A). \tag{6.1.5}$$

Moreover, since

$$\mathbb{E}_0^X\left(p_{\rho/\kappa}(X(t)-X(s),t-s)\right) = \mathbb{E}_0^X\left(p\left(X(t)-X(s),\frac{\rho}{\kappa}(t-s)\right)\right) = p\left(0,\left(1+\frac{\rho}{\kappa}\right)(t-s)\right), \tag{6.1.6}$$

it follows from (6.1.3) that

$$\mathbb{E}_0^X\left(W(a,A,\kappa)\right) = \frac{\nu\gamma^2}{\kappa^2}a\int_0^A du\; p\left(0,\left(1+\frac{\rho}{\kappa}\right)u\right). \tag{6.1.7}$$

Inserting (6.1.5) and (6.1.7) into (6.1.2), and letting $T\to\infty$, we find that

$$\lambda^-_{\text{diag}}(a,\kappa) \geq \theta\,\frac{\nu\gamma^2}{\kappa^2}\left(1+\frac{\rho}{\kappa}\right)^{-1}\int_0^{(1+\frac{\rho}{\kappa})A} du\; p(0,u). \tag{6.1.8}$$

Hence

$$\liminf_{\kappa\to\infty}\kappa^2\lambda^-_{\text{diag}}(a,\kappa) \geq \theta\,\nu\gamma^2\int_0^A du\; p(0,u). \tag{6.1.9}$$

Now let $A\to\infty$ and $\theta\uparrow 1$, to get the claim in (5.2.7). ∎

6.2 Proof of Lemma 5.2.4

The proof of Lemma 5.2.4 relies on Lemma 6.2.1 below. For $a > 0$, define

Define

$$\Lambda_a(\gamma,\rho,\nu) = \limsup_{\kappa\to\infty} \frac{1}{a\kappa} \log \mathbb{E}_0^X \left(\exp\left[\frac{\nu\gamma^2}{\kappa^2} \int_0^{a\kappa^3} ds \int_s^\infty dt\ p_{\rho/\kappa}(X(t)-X(s),t-s) \right] \right). \tag{6.2.1}$$

Lemma 6.2.1 *(a) If $d \geq 4$, then*

$$\Lambda_a(\gamma,\rho,\nu) \leq \frac{\nu\gamma^2}{r_d} \qquad \forall\, 0 < a < \infty. \tag{6.2.2}$$

(b) If $d = 3$, then

$$\Lambda_a(\gamma,\rho,\nu) \leq \frac{1 + Ca^{1/4}\frac{1}{r_d}}{1 - Ca^{1/4}(1+Ca^{1/4})\frac{1}{r_d}}\, \nu\gamma^2, \tag{6.2.3}$$

provided $a > 0$ is so small that

$$Ca^{1/4}(1+Ca^{1/4})\frac{1}{r_d} < 1, \tag{6.2.4}$$

where

$$C = C(\gamma,\rho,\nu) = \left(\frac{2c_3\nu\gamma^2}{\sqrt{\rho}} \right)^{1/2}. \tag{6.2.5}$$

Before giving the proof of Lemma 6.2.1, we first prove Lemma 5.2.4.

Proof. Split the integral in the right-hand side of (5.2.12) as follows:

$$\begin{aligned} &\int_0^T ds \int_s^{s+a\kappa^3} dt\ p_{\rho/\kappa}(X(t)-X(s),t-s) \\ &\leq \left(\sum_{\substack{k=1\\ \text{even}}}^{\lceil T/a\kappa^3\rceil} + \sum_{\substack{k=1\\ \text{odd}}}^{\lceil T/a\kappa^3\rceil} \right) \int_{(k-1)a\kappa^3}^{ka\kappa^3} ds \int_s^{s+a\kappa^3} dt\ p_{\rho/\kappa}(X(t)-X(s),t-s). \end{aligned} \tag{6.2.6}$$

Note that the summands in each of the two sums are i.i.d. Hence, substituting (6.2.6) into (5.2.12) and applying the Cauchy-Schwarz inequality, we find that

$$\begin{aligned} &\Lambda^+_{\text{diag}}(T;a,\kappa) \\ &\leq \frac{\lceil T/a\kappa^3\rceil}{2T} \log \mathbb{E}_0^X \left(\exp\left[\frac{2\nu\gamma^2}{\kappa^2} \int_0^{a\kappa^3} ds \int_s^{s+a\kappa^3} dt\ p_{\rho/\kappa}(X(t)-X(s),t-s) \right] \right). \end{aligned} \tag{6.2.7}$$

Letting $T \to \infty$, we arrive at

$$\lambda^+_{\text{diag}}(a,\kappa) \leq \frac{1}{2a\kappa^3} \log \mathbb{E}_0^X \left(\exp\left[\frac{2\nu\gamma^2}{\kappa^2} \int_0^{a\kappa^3} ds \int_s^{s+a\kappa^3} dt\ p_{\rho/\kappa}(X(t)-X(s),t-s) \right] \right). \tag{6.2.8}$$

Assertion (5.2.14) follows from (6.2.8) after extending the second integral to infinity and applying Lemma 6.2.1(a) with ν replaced by 2ν. Assertion (5.2.15) follows similarly by applying Lemma 6.2.1(b). ∎

6.3 Proof of Lemma 6.2.1

The proof of Lemma 6.2.1 is based on two further lemmas. Recall (5.3.3).

Lemma 6.3.1 *For any* $\alpha > 0$ *and* $M \in \mathbb{N}$,

$$\begin{aligned} &\mathbb{E}_0^X \left(\exp\left[\alpha \sum_{k=1}^M \int_0^\infty dt\; p_{\rho/\kappa}(Z_{k-1}(t) - Z_{k-1}(0), t) \right] \right) \\ &\leq \prod_{k=1}^M \max_{y_1,\dots,y_{k-1}} \mathbb{E}_0^X \left(\exp\left[\alpha \sum_{l=0}^{k-1} \int_0^\infty dt\; p_{\rho/\kappa}\left(X(t) + y_l, \frac{l}{M}T + t \right) \right] \right), \end{aligned} \tag{6.3.1}$$

where $Z_k(t) = X(\frac{k}{M}T + t)$, $k \in \mathbb{N}_0$, *and* $y_0 = 0$.

Lemma 6.3.2 *Let* $d \geq 3$. *For any* $\alpha > 0$, $M \in \mathbb{N}$, $k \in \mathbb{N}_0$ *and* $y_0, \dots, y_k \in \mathbb{Z}^d$,

$$\mathbb{E}_0^X \left(\exp\left[\alpha \sum_{l=0}^k \int_0^\infty dt\; p_{\rho/\kappa}\left(X(t) + y_l, \frac{l}{M}T + t \right) \right] \right) \leq \exp\left[\frac{\alpha \sum_{l=0}^k G_{\frac{\rho T}{\kappa M} l}(0)}{1 - \alpha \sum_{l=0}^k G_{\frac{\rho T}{\kappa M} l}(0)} \right], \tag{6.3.2}$$

provided that α *is so small that*

$$\alpha \sum_{l=0}^k G_{\frac{\rho T}{\kappa M} l}(0) < 1. \tag{6.3.3}$$

Before giving the proof of Lemmas 6.3.1–6.3.2, we first prove Lemma 6.2.1.
Proof. Let $M \in \mathbb{N}$ be arbitrary and abbreviate

$$Z_k(t) = X\left(\frac{k}{M} a\kappa^3 + t \right), \qquad k \in \mathbb{N}_0, \tag{6.3.4}$$

which is the same as below (6.3.1) with $T = a\kappa^3$. Then

$$\begin{aligned} &\mathbb{E}_0^X \left(\exp\left[\frac{\nu\gamma^2}{\kappa^2} \int_0^{a\kappa^3} ds \int_s^\infty dt\; p_{\rho/\kappa}(X(t) - X(s), t - s) \right] \right) \\ &= \mathbb{E}_0^X \left(\exp\left[\frac{\nu\gamma^2}{\kappa^2} \int_0^{\frac{a\kappa^3}{M}} ds \sum_{k=1}^M \int_s^\infty dt\; p_{\rho/\kappa}(Z_{k-1}(t) - Z_{k-1}(s), t - s) \right] \right). \end{aligned} \tag{6.3.5}$$

After applying Jensen's inequality, we get

$$\begin{aligned} \text{rhs (6.3.5)} &\leq \frac{M}{a\kappa^3} \int_0^{\frac{a\kappa^3}{M}} ds\; \mathbb{E}_0^X \left(\exp\left[\frac{\nu\gamma^2}{\kappa^2} \frac{a\kappa^3}{M} \sum_{k=1}^M \int_s^\infty dt\; p_{\rho/\kappa}(Z_{k-1}(t) - Z_{k-1}(s), t - s) \right] \right) \\ &= \mathbb{E}_0^X \left(\exp\left[\nu\gamma^2 \frac{a\kappa}{M} \sum_{k=1}^M \int_0^\infty dt\; p_{\rho/\kappa}(Z_{k-1}(t) - Z_{k-1}(0), t) \right] \right). \end{aligned} \tag{6.3.6}$$

To the espression in the right-hand side we may first apply Lemma 6.3.1 and then Lemma 6.3.2, both with $\alpha = \nu\gamma^2(a\kappa/M)$ and $T = a\kappa^3$. As a result, we obtain from (6.3.5) that

$$
\begin{aligned}
&\frac{1}{a\kappa}\log \mathbb{E}_0^X\left(\exp\left[\frac{\nu\gamma^2}{\kappa^2}\int_0^{a\kappa^3} ds \int_s^\infty dt\; p_{\rho/\kappa}(X(t)-X(s),t-s)\right]\right)\\
&\quad\leq \frac{1}{a\kappa}\sum_{k=1}^{M}\frac{\nu\gamma^2\frac{a\kappa}{M}\sum_{l=0}^{k-1}G_{\rho\frac{a\kappa^2}{M}l}(0)}{1-\nu\gamma^2\frac{a\kappa}{M}\sum_{l=0}^{k-1}G_{\rho\frac{a\kappa^2}{M}l}(0)} \leq \frac{\nu\gamma^2\sum_{l=0}^{M-1}G_{\rho\frac{a\kappa^2}{M}l}(0)}{1-\nu\gamma^2\frac{a\kappa}{M}\sum_{l=0}^{M-1}G_{\rho\frac{a\kappa^2}{M}l}(0)},
\end{aligned}
\tag{6.3.7}
$$

provided that

$$
\nu\gamma^2\frac{a\kappa}{M}\sum_{l=0}^{M-1}G_{\rho\frac{a\kappa^2}{M}l}(0) < 1. \tag{6.3.8}
$$

(a) Let $d \geq 4$. Then, by (5.3.4),

$$
\sum_{l=0}^{M-1}G_{\rho\frac{a\kappa^2}{M}l}(0) \leq G_0(0) + \left(\sum_{l=1}^{M-1}\frac{c_d}{\rho\frac{a\kappa^2}{M}l}\right)G_0(0) \leq \left(1+\widetilde{c}_d\,\frac{M\log M}{\rho a\kappa^2}\right)\frac{1}{r_d} \tag{6.3.9}
$$

for some $\widetilde{c}_d > 0$ and all $M \in \mathbb{N}$, provided that $\rho(a\kappa^2/M) \geq 1$. Now choose

$$
M = M(\kappa) = \lfloor \kappa^{3/2} \rfloor. \tag{6.3.10}
$$

Then, substituting (6.3.9) into (6.3.7) and letting $\kappa \to \infty$, we arrive at (6.2.2).

(b) Let $d = 3$. Then, by (5.3.4),

$$
\sum_{l=0}^{M-1}G_{\rho\frac{a\kappa^2}{M}l}(0) \leq G_0(0) + \left(\sum_{l=1}^{M-1}\frac{c_3}{\sqrt{\rho\frac{a\kappa^2}{M}l}}\right)G_0(0) \leq \left(1+\frac{2c_3}{\sqrt{\rho a}}\,\frac{M}{\kappa}\right)\frac{1}{r_d} \tag{6.3.11}
$$

for all $M \in \mathbb{N}$. Now choose

$$
M = M(\kappa) = \left\lfloor \left(\frac{\nu\gamma^2\sqrt{\rho}}{2c_3}\right)^{1/2} a^{3/4}\kappa \right\rfloor. \tag{6.3.12}
$$

Then

$$
\lim_{\kappa\to\infty}\frac{2c_3}{\sqrt{\rho a}}\,\frac{M(\kappa)}{\kappa} = Ca^{1/4}, \tag{6.3.13}
$$

and

$$
\lim_{\kappa\to\infty}\nu\gamma^2\frac{a\kappa}{M(\kappa)} = Ca^{1/4} \tag{6.3.14}
$$

where C is given by (6.2.5). Substituting (6.3.11) into (6.3.7) and assumption (6.3.8), letting $\kappa \to \infty$, and taking into account (6.3.14–6.3.13), we arrive at (6.2.3) under assumption (6.2.4). ∎

6.4 Proof of Lemmas 6.3.1–6.3.2

The proof of Lemma 6.3.1 goes as follows.

Proof. We show that the function defined by

$$
\begin{aligned}
E(r) = \prod_{k=1}^{r} \max_{y_1,\dots,y_{k-1}} \mathbb{E}_0^X \Bigg(\exp\Bigg[\alpha \sum_{l=0}^{k-1} \int_0^\infty dt\; p_{\rho/\kappa}\Big(X(t) + y_l, \frac{l}{M}T + t \Big) \\
\times \max_{z_1,\dots,z_r} \mathbb{E}_0^X \Bigg(\exp\Bigg[\alpha \sum_{k=1}^{M-r} \int_0^\infty dt\; p_{\rho/\kappa}(Z_{k-1}(t) - Z_{k-1}(0), t) \\
+ \alpha \sum_{l=1}^{r} \int_0^\infty dt\; p_{\rho/\kappa}\Big(X(t) + z_l, \frac{l}{M}T + t \Big) \Bigg] \Bigg)
\end{aligned}
\tag{6.4.1}
$$

for $r = 0, \dots, M-1$ is non-decreasing in r. Then $E(0) \le E(M-1)$, which is the desired inequality. (Note that for $r = M-1$ the first term in the right-hand side of (6.4.1) corresponds to $k = 1, \dots, M-1$ in the right-hand side of (6.3.1), the second term to $k = M$, $l = 0$, and the third term to $k = M$, $l = 1, \dots, M-1$.)

Fix r arbitrarily. We want to show that $E(r) \le E(r+1)$. To this end, fix also $z_1, \dots, z_r$ arbitrarily. Separately handling the summand for $k = 1$, splitting the integral over $(0,\infty)$ into integrals over $(0, T/M)$ and $(T/M, \infty)$, shifting time by T/M for the latter, and using the Markov property of X at time T/M, we obtain

$$
\begin{aligned}
&\mathbb{E}_0^X \Bigg(\exp\Bigg[\alpha \sum_{k=1}^{M-r} \int_0^\infty dt\; p_{\rho/\kappa}(Z_{k-1}(t) - Z_{k-1}(0), t) \\
&\qquad + \alpha \sum_{l=1}^{r} \int_0^\infty dt\; p_{\rho/\kappa}\Big(X(t) + z_l, \frac{l}{M}T + t \Big) \Bigg] \Bigg) \\
&= \mathbb{E}_0^X \Bigg(\exp\Bigg[\alpha \int_0^{T/M} dt\; p_{\rho/\kappa}(X(t), t) + \alpha \int_0^\infty dt\; p_{\rho/\kappa}\Big(Z_1(t), \frac{1}{M}T + t \Big) \\
&\qquad + \alpha \sum_{k=1}^{M-(r+1)} \int_0^\infty dt\; p_{\rho/\kappa}(Z_k(t) - Z_k(0), t) \\
&\qquad + \alpha \sum_{l=1}^{r} \int_0^{T/M} dt\; p_{\rho/\kappa}\Big(X(t) + z_l, \frac{l}{M}T + t \Big) \\
&\qquad + \alpha \sum_{l=2}^{r+1} \int_0^\infty dt\; p_{\rho/\kappa}\Big(Z_1(t) + z_{l-1}, \frac{l}{M}T + t \Big) \Bigg] \Bigg)
\end{aligned}
\tag{6.4.2}
$$

and

$$
\begin{aligned}
&\text{rhs } (6.4.2)\\
&\le \mathbb{E}_0^X\Bigg(\exp\Bigg[\alpha\int_0^{T/M} dt\ p_{\rho/\kappa}(X(t),t)+\alpha\sum_{l=1}^{r}\int_0^{T/M} dt\ p_{\rho/\kappa}\Big(X(t)+z_l,\frac{l}{M}T+t\Big)\\
&\quad\times\max_{z_0}\mathbb{E}_0^X\Bigg(\exp\Bigg[\alpha\int_0^{\infty} dt\ p_{\rho/\kappa}\Big(X(t)+z_0,\frac{1}{M}T+t\Big)\\
&\qquad+\alpha\sum_{k=1}^{M-(r+1)}\int_0^{\infty} dt\ p_{\rho/\kappa}(Z_{k-1}(t)-Z_{k-1}(0),t)\\
&\qquad+\alpha\sum_{l=2}^{r+1}\int_0^{\infty} dt\ p_{\rho/\kappa}\Big(X(t)+z_0+z_{l-1},\frac{l}{M}T+t\Big)\Bigg]\Bigg).
\end{aligned}
\tag{6.4.3}
$$

In the last line we maximize over $Z_1(0)=X(T/M)$ after using the Markov property of X at time T/M. Hence

$$
\begin{aligned}
&\max_{z_1,\dots,z_r}\mathbb{E}_0^X\Bigg(\exp\Bigg[\alpha\sum_{k=1}^{M-r}\int_0^{\infty} dt\ p_{\rho/\kappa}(Z_{k-1}(t)-Z_{k-1}(0),t)\\
&\qquad\qquad+\alpha\sum_{l=1}^{r}\int_0^{\infty} dt\ p_{\rho/\kappa}\Big(X(t)+z_l,\frac{l}{M}T+t\Big)\Bigg]\Bigg)\\
&\le \max_{y_1,\dots,y_r}\mathbb{E}_0^X\Bigg(\exp\Bigg[\alpha\sum_{l=0}^{r}\int_0^{\infty} dt\ p_{\rho/\kappa}\Big(X(t)+y_l,\frac{l}{M}T+t\Big)\Bigg]\Bigg) \qquad (6.4.4)\\
&\quad\times\max_{z_1,\dots,z_{r+1}}\mathbb{E}_0^X\Bigg(\exp\Bigg[\alpha\sum_{k=1}^{M-(r+1)}\int_0^{\infty} dt\ p_{\rho/\kappa}(Z_{k-1}(t)-Z_{k-1}(0),t)\\
&\qquad\qquad+\alpha\sum_{l=1}^{r+1}\int_0^{\infty} dt\ p_{\rho/\kappa}\Big(X(t)+z_l,\frac{l}{M}T+t\Big)\Bigg]\Bigg).
\end{aligned}
$$

Here, we extend the first two integrals in the right-hand side of (6.4.3) from T/M to infinity, use that $y_0=0$, replace z_0 by z_1 and z_0+z_{l-1} by z_l. Substituting (6.4.4) into (6.4.1), we get that $E(r)\le E(r+1)$, as desired. ∎

The proof of Lemma 6.3.2 goes as follows.

Proof. A Taylor expansion of the exponential function yields

$$
\begin{aligned}
&\mathbb{E}_0^X\Bigg(\exp\Bigg[\alpha\int_0^{\infty} dt\sum_{l=0}^{k} p_{\rho/\kappa}\Big(X(t)+y_l,\frac{l}{M}T+t\Big)\Bigg]\Bigg)\\
&\quad=\sum_{m=0}^{\infty}\alpha^m\mathbb{E}_0^X\Bigg(\prod_{j=1}^{m}\int_{t_{j-1}}^{\infty} dt_j\sum_{l=0}^{k} p_{\rho/\kappa}\Big(X(t_j)+y_l,\frac{l}{M}T+t_j\Big)\Bigg)
\end{aligned}
\tag{6.4.5}
$$

with $t_0 = 0$. A successive application of the Markov property at times $t_{m-1}, \dots, t_1$ yields

$$\begin{aligned}
&\mathbb{E}_0^X \left(\prod_{j=1}^{m} \int_{t_{j-1}}^{\infty} dt_j \sum_{l=0}^{k} p_{\rho/\kappa} \left(X(t_j) + y_l, \frac{l}{M}T + t_j \right) \right) \\
&= \mathbb{E}_0^X \left(\prod_{j=1}^{m-1} \int_{t_{j-1}}^{\infty} dt_j \sum_{l=0}^{k} p_{\rho/\kappa} \left(X(t_j) + y_l, \frac{l}{M}T + t_j \right) \right) \\
&\qquad \times \int_{t_{m-1}}^{\infty} dt_m \sum_{l=0}^{k} p \left(X(t_{m-1}) + y_l, \frac{\rho}{\kappa} \left(\frac{l}{M}T + t_m \right) + t_m - t_{m-1} \right) \qquad (6.4.6) \\
&\le \mathbb{E}_0^X \left(\prod_{j=1}^{m-1} \int_{t_{j-1}}^{\infty} dt_j \sum_{l=0}^{k} p_{\rho/\kappa} \left(X(t_j) + y_l, \frac{l}{M}T + t_j \right) \right) \left(\sum_{l=0}^{k} G_{\frac{\rho T}{\kappa M} l}(0) \right) \\
&\le \cdots \le \left(\sum_{l=0}^{k} G_{\frac{\rho T}{\kappa M} l}(0) \right)^m .
\end{aligned}$$

In the first inequality we use that $p(x,t) \le p(0,t)$ and that $t \mapsto p(0,t)$ is non-increasing. Substituting (6.4.6) into (6.4.5), summing the geometric series and using the inequality $1+x \le e^x$, $x \in \mathbb{R}$, we arrive at (6.3.2). ∎

7 Proof of Lemmas 5.2.3 and 5.2.5

As we saw in Section 5.3, the "variational" contributions to the lower and the upper bound in the proof of Theorem 5.2.1 come from Lemmas 5.2.3 and 5.2.5, respectively. In this section we prove these two lemmas.

The proof of Lemma 5.2.5(i), which applies to $d \ge 4$, is easy. Indeed, in the right-hand side of (5.2.8) separate the p^2 summands with the help of Hölder's inequality (as in (5.3.28)). The terms with $k = l$ are negligible for $\kappa \to \infty$ by Lemma 5.2.6(i) with $a = \epsilon$, while the same is true for the terms with $k \ne l$ by Lemma 5.2.7(i). Lemmas 5.2.6–5.2.7 are proved in Section 8.

Thus, we may henceforth restrict to $d = 3$.

7.1 Space-time scaling

We begin with a space-time scaling of the random walks. Let $\mathbb{Z}_\kappa^3 = \kappa^{-1}\mathbb{Z}^3$, and define

$$\begin{aligned}
X_k^{(\kappa)}(t) &= \kappa^{-1} X_k(\kappa^2 t), \quad t \ge 0,\ k = 1, \dots, p, \\
p^{(\kappa)}(x,t) &= \kappa^3 p(\kappa x, \kappa^2 t), \quad x \in \mathbb{Z}_\kappa^3,\ t \ge 0.
\end{aligned} \qquad (7.1.1)$$

Each $X_k^{(\kappa)}$ lives on $\mathbb{Z}_\kappa^3$, has generator

$$(\Delta^{(\kappa)} f)(x) = \kappa^2 \sum_{\substack{y \in \mathbb{Z}_\kappa^3 \\ \|y-x\| = \kappa^{-1}}} [f(y) - f(x)], \qquad x \in \mathbb{Z}_\kappa^3, \qquad (7.1.2)$$

and has transition kernel whose density is $p^{(\kappa)}$ w.r.t. the discrete Lebesgue measure on $\mathbb{Z}_\kappa^3$ where each site carries weight κ^{-3}. As $\kappa \to \infty$, each $X_k^{(\kappa)}$ converges weakly to Brownian motion, which has as generator the continuous Laplacian $\Delta_{\mathbb{R}^3}$, and $p^{(\kappa)}$ converges weakly to p_G, the density of the transition kernel associated with Brownian motion w.r.t. the continuous Lebesgue measure on $\mathbb{R}^3$. The last convergence is uniform on compacts, i.e., for every compact $C \subset \mathbb{R}^3 \times (0,\infty)$ and every $\theta \in (0,1)$ there exists $\kappa_0 = \kappa_0(C,\theta)$ such that

$$\theta\, p_G(x,t) \leq p^{(\kappa)}(x,t) \leq \frac{1}{\theta}\, p_G(x,t) \qquad \forall\, (x,t) \in C,\ \kappa \geq \kappa_0. \tag{7.1.3}$$

Further note that

$$\min_{x\in\mathbb{R}^3} \frac{p_G(x,u_2)}{p_G(x,u_1)} = \frac{p_G(0,u_2)}{p_G(0,u_1)} = \left(\frac{u_1}{u_2}\right)^{3/2} \qquad \forall\, u_2 \geq u_1. \tag{7.1.4}$$

7.2 Proof of Lemma 5.2.3

Fix $0 < \epsilon < K < \infty$, $\delta > 0$ small, and $\theta \in (0,1)$. Abbreviate

$$\begin{array}{lllll} L &=& L(\delta,\epsilon) &=& \lceil \epsilon/\delta \rceil, \\ M &=& M(\delta,K) &=& \lfloor K/\delta \rfloor, \\ N &=& N(T;\delta,\kappa) &=& \lfloor T/\delta\kappa^3 \rfloor. \end{array} \tag{7.2.1}$$

Fix a large open cube $Q \subset \mathbb{R}^3$, centered at the origin. Later we will take limits in the following order:

$$T \to \infty,\ \kappa \to \infty,\ \delta \downarrow 0,\ \theta \uparrow 1,\ Q \uparrow \mathbb{R}^3. \tag{7.2.2}$$

Let C_Q be the event

$$C_Q = C_Q(N,M,\delta,\kappa) = \left\{ X_k^{(\kappa)}(t) \in Q\ \forall\, 0 \leq t \leq (N+M)\delta\kappa,\ k = 1,\dots,p \right\}. \tag{7.2.3}$$

Then from (5.2.8), (7.1.1) and the lower bound in (7.1.3) we get

$$\Lambda_{\rm var}(T;\epsilon,K,\kappa) \geq \frac{1}{pT} \log \mathbb{E}_{0,\dots,0}^{X_1^{(\kappa)},\dots,X_p^{(\kappa)}} \left(\exp[U]\, 1\!\!1_{C_Q} \right) \tag{7.2.4}$$

with

$$\begin{aligned} U &= \frac{\nu\gamma^2}{\kappa} \sum_{k,l=1}^{p} \int_0^{T/\kappa^2} ds \int_{s+\epsilon\kappa}^{s+K\kappa} dt\ p^{(\kappa)}\left(X_l^{(\kappa)}(t) - X_k^{(\kappa)}(s), \frac{\rho}{\kappa}(t-s) \right) \\ &\geq \frac{\nu\gamma^2}{\kappa} \sum_{k,l=1}^{p} \int_0^{T/\kappa^2} ds \int_{s+\epsilon\kappa}^{s+K\kappa} dt\ \theta\, p_G\left(X_l^{(\kappa)}(t) - X_k^{(\kappa)}(s), \frac{\rho}{\kappa}(t-s) \right) \end{aligned} \tag{7.2.5}$$

for $\kappa \geq \kappa_0(C,\theta)$ with $C = 2\bar{Q} \times [\epsilon\rho, K\rho]$ (with $\bar{Q}$ the closure of Q). Moreover,

$$\begin{aligned} &\text{rhs (7.2.5)} \\ &\geq \frac{\nu\gamma^2}{\kappa} \sum_{k,l=1}^{p} \sum_{n=1}^{N} \int_{(n-1)\delta\kappa}^{n\delta\kappa} ds \int_{n\delta\kappa+\epsilon\kappa}^{(n-1)\delta\kappa+K\kappa} dt\ \theta\, p_G\left(X_l^{(\kappa)}(t) - X_k^{(\kappa)}(s), \frac{\rho}{\kappa}(t-s) \right) \\ &\geq \frac{\nu\gamma^2}{\kappa} \sum_{k,l=1}^{p} \sum_{n=1}^{N} \sum_{m=L+1}^{M-1} \int_{(n-1)\delta\kappa}^{n\delta\kappa} ds \int_{(n+m-1)\delta\kappa}^{(n+m)\delta\kappa} dt\ \theta\, p_G\left(X_l^{(\kappa)}(t) - X_k^{(\kappa)}(s), \frac{\rho}{\kappa}(t-s) \right). \end{aligned} \tag{7.2.6}$$

Next, note that $(m-1)\delta\rho \leq \frac{\rho}{\kappa}(t-s) \leq (m+1)\delta\rho$ for all s,t in the domain of integration corresponding to n,m, and use (7.1.4), to obtain

$$\Lambda_{\text{var}}(T;\epsilon,K,\kappa) \geq \frac{1}{pT}\log \mathbb{E}_{0,\dots,0}^{X_1^{(\kappa)},\dots,X_p^{(\kappa)}}\left(\exp[V]\,1\!\!1_{C_Q}\right) \tag{7.2.7}$$

with

$$\begin{aligned} V = \frac{\nu\gamma^2}{\kappa}\sum_{k,l=1}^{p}\sum_{n=1}^{N}\sum_{m=L+1}^{M-1}\int_{(n-1)\delta\kappa}^{n\delta\kappa} ds \int_{(n+m-1)\delta\kappa}^{(n+m)\delta\kappa} dt \\ \times\,\theta\left(\frac{L}{L+2}\right)^{3/2} p_G\left(X_l^{(\kappa)}(t)-X_k^{(\kappa)}(s),(m-1)\delta\rho\right). \end{aligned} \tag{7.2.8}$$

In this last expression, the time coordinate of the kernel is fixed for each m. Therefore, if we introduce the normalized occupation time measures

$$\Xi_{k,r}^{(\kappa)}(A) = \frac{1}{\delta\kappa}\int_{(r-1)\delta\kappa}^{r\delta\kappa} ds\,1\!\!1_A\left(X_k^{(\kappa)}(s)\right), \quad k=1,\dots,p,\ r=1,\dots,N+M,\ A\subset\mathbb{R}^3 \text{ Borel,} \tag{7.2.9}$$

then we may write

$$\begin{aligned} V = \theta\left(\frac{L}{L+2}\right)^{3/2}\frac{\nu\gamma^2}{\rho}\,\delta\kappa\sum_{k,l=1}^{p}\sum_{n=1}^{N}\sum_{m=L+1}^{M-1} \\ \times\int_Q \Xi_{k,n}^{(\kappa)}(dx)\int_Q \Xi_{l,n+m}^{(\kappa)}(dy)\ \delta\rho\ p_G(y-x,(m-1)\delta\rho). \end{aligned} \tag{7.2.10}$$

This representation brings us into a position where we can do a large deviation analysis, as follows.

For $\mu\in\mathcal{M}_1(Q)$, the set of probability measures on Q, let $\mathcal{U}_Q(\mu)\subset\mathcal{M}_1(Q)$ denote any weak open neighborhood of μ such that

$$\begin{aligned} \nu_1,\nu_2\in\mathcal{U}_Q(\mu) \Longrightarrow \int_Q \nu_1(dx)\int_Q \nu_2(dy)\ p_G(y-x,(m-1)\delta\rho) \\ \geq \theta\int_Q \mu(dx)\int_Q \mu(dy)\ p_G(y-x,(m-1)\delta\rho) \\ \forall\, m=L,\dots,M, \end{aligned} \tag{7.2.11}$$

and let $C_{Q,\mu}$ be the event

$$C_{Q,\mu} = \left\{\Xi_{k,r}^{(\kappa)}\in\mathcal{U}_Q(\mu)\ \forall\, k=1,\dots,p,\ r=1,\dots,N+M\right\}. \tag{7.2.12}$$

Then, for any $\mu\in\mathcal{M}_1(Q)$, we may bound, via (7.2.7) and (7.2.10),

$$\begin{aligned} \Lambda_{\text{var}}(T;\epsilon,K,\kappa) &\geq \frac{1}{pT}\log \mathbb{E}_{0,\dots,0}^{X_1^{(\kappa)},\dots,X_p^{(\kappa)}}\left(\exp[V]\,1\!\!1_{C_Q}\,1\!\!1_{C_{Q,\mu}}\right) \\ &\geq \frac{1}{pT}\theta^2\left(\frac{L}{L+2}\right)^{3/2}\frac{\nu\gamma^2}{\rho}p^2 N \\ &\quad\times\delta\kappa\int_Q \mu(dx)\int_Q \mu(dy)\sum_{m=L+1}^{M-1}\delta\rho\ p_G(y-x,(m-1)\delta\rho) \\ &\quad+\frac{1}{pT}\log \mathbb{P}_{0,\dots,0}^{X_1^{(\kappa)},\dots,X_p^{(\kappa)}}(C_Q\cap C_{Q,\mu}). \end{aligned} \tag{7.2.13}$$

By again appealing to (7.1.4), the sum in the first term in the right-hand side of (7.2.13) can be estimated as follows:

$$\sum_{m=L+1}^{M-1} \delta\rho\, p_G(y-x,(m-1)\delta\rho) \geq \left(\frac{L}{L+2}\right)^{3/2} \int_{(L-1)\delta\rho}^{(M-2)\delta\rho} du\, p_G(y-x,u)\,. \tag{7.2.14}$$

As to the second term in the right-hand side of (7.2.13), by using the independence of the p random walks and using the Markov property at times $r\delta\kappa$ for $r=1,\dots,N+M$, we may estimate (with $X^{(\kappa)} = X_1^{(\kappa)}$, $\Xi_r^{(\kappa)} = \Xi_{1,r}^{(\kappa)}$)

$$\begin{aligned}
&\mathbb{P}_{0,\dots,0}^{X_1^{(\kappa)},\dots,X_p^{(\kappa)}}(C_Q\cap C_{Q,\mu})\\
&= \Big[\mathbb{P}_0^{X^{(\kappa)}}\Big(X^{(\kappa)}(t)\in Q\ \forall\, 0\leq t\leq (N+M)\delta\kappa,\ \Xi_r^{(\kappa)}\in\mathcal{U}_Q(\mu)\,\forall\, r=1,\dots,N+M\Big)\Big]^p\\
&\geq \Big[\mathbb{P}_0^{X^{(\kappa)}}\Big(X^{(\kappa)}(t)\in Q\ \forall\, 0\leq t\leq (N+M)\delta\kappa,\\
&\qquad\qquad \Xi_r^{(\kappa)}\in\mathcal{U}_Q(\mu) \text{ and } X^{(\kappa)}(r\delta\kappa)\in\tfrac12 Q\ \forall\, r=1,\dots,N+M\Big)\Big]^p\\
&\geq \Big[\min_{x\in\mathbb{Z}_\kappa^3\cap\frac12 Q}\mathbb{P}_x^{X^{(\kappa)}}\Big(X^{(\kappa)}(t)\in Q\ \forall\, 0\leq t\leq \delta\kappa,\\
&\qquad\qquad \Xi^{(\kappa)}\in\mathcal{U}_Q(\mu) \text{ and } X^{(\kappa)}(\delta\kappa)\in\tfrac12 Q\Big)\Big]^{p(N+M)}.
\end{aligned} \tag{7.2.15}$$

The dependence on N is now pulled out of both terms in the right-hand side of (7.2.13), and so we can take the limit $T\to\infty$, to obtain from (5.2.9), (7.2.1) and (7.2.13–7.2.15) that

$$\begin{aligned}
&\kappa^2\lambda_{\text{var}}^-(\epsilon,K,\kappa) \geq \theta^2\left(\frac{L}{L+2}\right)^3 \frac{\nu\gamma^2}{\rho}\, p\int_Q\mu(dx)\int_Q\mu(dy)\int_{(L-1)\delta\rho}^{(M-2)\delta\rho} du\, p_G(y-x,u)\\
&+\frac{1}{\delta\kappa}\log\min_{x\in\mathbb{Z}_\kappa^3\cap\frac12 Q}\mathbb{P}_x^{X^{(\kappa)}}\Big(X^{(\kappa)}(t)\in Q\ \forall\, 0\leq t\leq\delta\kappa,\ \Xi^{(\kappa)}\in\mathcal{U}_Q(\mu) \text{ and } X^{(\kappa)}(\delta\kappa)\in\tfrac12 Q\Big)
\end{aligned} \tag{7.2.16}$$

for $\kappa\geq\kappa_0(C,\theta)$. The final step in the argument is the following large deviation bound:

Lemma 7.2.1 *For each $\mu\in\mathcal{M}_1(Q)$,*

$$\begin{aligned}
&\liminf_{\kappa\to\infty}\frac{1}{\delta\kappa}\log\min_{x\in\mathbb{Z}_\kappa^3\cap\frac12 Q}\mathbb{P}_x^{X^{(\kappa)}}\Big(X^{(\kappa)}(t)\in Q\ \forall\, 0\leq t\leq\delta\kappa,\ \Xi^{(\kappa)}\in\mathcal{U}_Q(\mu) \text{ and } X^{(\kappa)}(\delta\kappa)\in\tfrac12 Q\Big)\\
&\quad\geq -S_Q(\mu)
\end{aligned} \tag{7.2.17}$$

with $S_Q\colon \mathcal{M}_1(Q)\to[0,\infty]$ given by

$$S_Q(\mu) = \begin{cases} \|\nabla_{\mathbb{R}^3} f\|_2^2 & \text{if } \mu\ll dx \text{ and } \sqrt{\frac{d\mu}{dx}} = f(x) \text{ with } f\in H_0^1(Q),\\ \infty & \text{otherwise,}\end{cases} \tag{7.2.18}$$

where $H_0^1(Q)$ is the completion of $C_c^\infty(Q)$ (the space of C^∞-functions $f\colon Q\to\mathbb{R}$ with compact support) w.r.t. the H^1-norm $\|f\|_{H^1} = \|f\|_2 + \|\nabla f\|_2$.

The proof of Lemma 7.2.1 is deferred to Section 7.4. Letting $\kappa\to\infty$ in (7.2.16) using (7.2.17), letting $\delta\downarrow 0$, recalling (7.2.1), letting $\theta\uparrow 1$, and afterwards taking the supremum over $\mu \in \mathcal{M}_1(Q)$, we arrive at

$$\liminf_{\kappa\to\infty} \kappa^2 \lambda^-_{\text{var}}(\epsilon, K, \kappa) \geq \sup_{\substack{f\in H^1_0(Q)\\ \|f\|_2=1}} \left[\frac{\nu\gamma^2}{\rho}\, p \int_Q dx\, f^2(x) \int_Q dy\, f^2(y) \int_{\epsilon\rho}^{K\rho} du\; p_G(y-x,u) - \|\nabla_{\mathbb{R}^3} f\|_2^2\right]. \tag{7.2.19}$$

Finally, let $Q \uparrow \mathbb{R}^3$ and use a standard approximation argument to show that the variational expression in the right-hand side of (7.2.19) converges to

$$\sup_{\substack{f\in H^1(\mathbb{R}^3)\\ \|f\|_2=1}} \left[\frac{\nu\gamma^2}{\rho}\, p \int_{\mathbb{R}^3} dx\, f^2(x) \int_{\mathbb{R}^3} dy\, f^2(y) \int_{\epsilon\rho}^{K\rho} du\; p_G(y-x,u) - \|\nabla_{\mathbb{R}^3} f\|_2^2\right]. \tag{7.2.20}$$

The latter is precisely $\mathcal{P}_p(\epsilon, K; \gamma, \rho, \nu)$ defined in (5.2.11), so we have completed the proof of Lemma 5.2.3.

7.3 Proof of Lemma 5.2.5

At the beginning of Section 7 we already dealt with Lemma 5.2.5(i). Thus, we need only prove Lemma 5.2.5(ii).

Part of the argument runs parallel to Section 7.2. Fix $\epsilon, K, \delta, \theta$ as before. Retain (7.2.1), but with $\lceil\cdot\rceil$ and $\lfloor\cdot\rfloor$ interchanged. Let $Q \subset \mathbb{R}^3$ be a large closed cube, centered at the origin. Later we will again take limits in the order (7.2.2).

Let $l(Q)$ and $l(Q^{(\kappa)})$ denote the side length of Q, respectively, $Q^{(\kappa)} = Q \cap \mathbb{Z}^3_\kappa$. Let

$$\begin{aligned} &X_k^{(\kappa,Q)}(t), \quad t \geq 0,\, k = 1, \ldots, p,\\ &p^{(\kappa,Q)}(x,t), \quad x \in Q,\, t \geq 0, \end{aligned} \tag{7.3.1}$$

denote the Q-periodization of (7.1.1), i.e.,

$$\begin{aligned} X_k^{(\kappa,Q)}(t) &= X_k^{(\kappa)}(t) \mod(Q^{(\kappa)}),\\ p^{(\kappa,Q)}(x,t) &= \sum_{k\in\mathbb{Z}^3} p^{(\kappa)}\left(x + \frac{k}{\kappa}\, l(Q^{(\kappa)}), t\right). \end{aligned} \tag{7.3.2}$$

Similarly, let

$$p_G^{(Q)}(x,t), \quad x \in Q,\, t \geq 0, \tag{7.3.3}$$

denote the Q-periodization of the Gaussian kernel, i.e.,

$$p_G^{(Q)}(x,t) = \sum_{k\in\mathbb{Z}^3} p_G(x + k\, l(Q), t). \tag{7.3.4}$$

From (5.2.8), (7.3.2) and the upper bound in (7.1.3) (which carries over to the Q-periodized kernels) we get

$$\Lambda_{\text{var}}(T; \epsilon, K, \kappa) \leq \frac{1}{pT} \log \mathbb{E}_{0,\ldots,0}^{X_1^{(\kappa,Q)},\ldots,X_p^{(\kappa,Q)}}(\exp[U]), \tag{7.3.5}$$

with

$$U = \frac{\nu\gamma^2}{\kappa} \sum_{k,l=1}^{p} \int_0^{T/\kappa^2} ds \int_{s+\epsilon\kappa}^{s+K\kappa} dt\; p^{(\kappa,Q)} \left(X_l^{(\kappa,Q)}(t) - X_k^{(\kappa,Q)}(s), \frac{\rho}{\kappa}(t-s) \right)$$
$$\leq \frac{\nu\gamma^2}{\kappa} \sum_{k,l=1}^{p} \int_0^{T/\kappa^2} ds \int_{s+\epsilon\kappa}^{s+K\kappa} dt\; \frac{1}{\theta}\, p_G^{(Q)} \left(X_l^{(\kappa,Q)}(t) - X_k^{(\kappa,Q)}(s), \frac{\rho}{\kappa}(t-s) \right) \tag{7.3.6}$$

for $\kappa \geq \kappa_0 = \kappa_0(C,\theta)$ with $C = 2Q \times [\epsilon\rho, K\rho]$. Moreover,

rhs (7.3.6)

$$\leq \frac{\nu\gamma^2}{\kappa} \sum_{k,l=1}^{p} \sum_{n=1}^{N} \int_{(n-1)\delta\kappa}^{n\delta\kappa} ds \int_{(n-1)\delta\kappa+\epsilon\kappa}^{n\delta\kappa+K\kappa} dt\; \frac{1}{\theta}\, p_G^{(Q)} \left(X_l^{(\kappa,Q)}(t) - X_k^{(\kappa,Q)}(s), \frac{\rho}{\kappa}(t-s) \right)$$
$$\leq \frac{\nu\gamma^2}{\kappa} \sum_{k,l=1}^{p} \sum_{n=1}^{N} \sum_{m=L}^{M} \int_{(n-1)\delta\kappa}^{n\delta\kappa} ds \int_{(n+m-1)\delta\kappa}^{(n+m)\delta\kappa} dt\; \frac{1}{\theta}\, p_G^{(Q)} \left(X_l^{(\kappa,Q)}(t) - X_k^{(\kappa,Q)}(s), \frac{\rho}{\kappa}(t-s) \right). \tag{7.3.7}$$

This is the analogue of (7.2.5–7.2.6).

Next, use (7.1.4) to obtain

$$\Lambda_{\mathrm{var}}(T;\epsilon,K,\kappa) \leq \frac{1}{pT} \log \mathbb{E}_{0,\ldots,0}^{X_1^{(\kappa,Q)},\ldots,X_p^{(\kappa,Q)}} (\exp[V]), \tag{7.3.8}$$

with

$$V = \frac{\nu\gamma^2}{\kappa} \sum_{k,l=1}^{p} \sum_{n=1}^{N} \sum_{m=L}^{M} \int_{(n-1)\delta\kappa}^{n\delta\kappa} ds \int_{(n+m-1)\delta\kappa}^{(n+m)\delta\kappa} dt$$
$$\times \frac{1}{\theta} \left(\frac{L+1}{L-1} \right)^{3/2} p_G^{(Q)} \left(X_l^{(\kappa,Q)}(t) - X_k^{(\kappa,Q)}(s), (m+1)\delta\rho \right)$$
$$= \frac{1}{\theta} \left(\frac{L+1}{L-1} \right)^{3/2} \frac{\nu\gamma^2}{\rho}\, \delta\kappa \sum_{k,l=1}^{p} \sum_{n=1}^{N} \sum_{m=L}^{M}$$
$$\times \int_Q \Xi_{k,n}^{(\kappa,Q)}(dx) \int_Q \Xi_{l,n+m}^{(\kappa,Q)}(dy)\; \delta\rho\; p_G^{(Q)}(y-x,(m+1)\delta\rho), \tag{7.3.9}$$

which is the analogue of (7.2.8) and (7.2.10). Here,

$$\Xi_{k,r}^{(\kappa,Q)}(A) = \frac{1}{\delta\kappa} \int_{(r-1)\delta\kappa}^{r\delta\kappa} ds\; \mathbb{1}_A \left(X_k^{(\kappa,Q)}(s) \right), \quad k = 1,\ldots,p,\; r = 1,\ldots,N+M+1,\; A \subset Q \text{ Borel}. \tag{7.3.10}$$

is the analogue of (7.2.9).

For $\mu \in \mathcal{M}_1(Q)$, let $\mathcal{U}_Q(\mu) \subset \mathcal{M}_1(Q)$ be any weak neighborhood of μ such that for $\mu_1,\mu_2 \in \mathcal{M}_1(Q)$,

$$\nu_1 \in \mathcal{U}_Q(\mu_1),\ \nu_2 \in \mathcal{U}_Q(\mu_2) \Longrightarrow \int_Q \nu_1(dx) \int_Q \nu_2(dy)\; p_G^{(Q)}(y-x,u)$$
$$\leq \frac{1}{\theta} \int_Q \mu_1(dx) \int_Q \mu_2(dy)\; p_G^{(Q)}(y-x,u) \tag{7.3.11}$$
$$\forall\, u \in [\epsilon\rho, K\rho + 2\delta\rho],$$

which is the analogue of (7.2.11), and

$$\inf_{\mu' \in \mathcal{U}_Q(\mu)} \widehat{S}_Q(\mu') \geq \theta \, \widehat{S}_Q(\mu) \qquad \forall \, \mu \in \mathcal{M}_1(Q), \tag{7.3.12}$$

where $\widehat{S}_Q$ is the rate function defined in (7.3.21) below. The latter inequality can be achieved because $\mu \mapsto \widehat{S}_Q(\mu)$ is lower semi-continuous.

Since $\mathcal{M}_1(Q)$ is compact, there exist finitely many $\mu_1, \ldots, \mu_I \in \mathcal{M}_1(Q)$ (with I not depending on T, κ) such that

$$\mathcal{M}_1(Q) \subset \bigcup_{i=1}^{I} \mathcal{U}_Q(\mu_i). \tag{7.3.13}$$

Let

$$\mathcal{J} = \big\{ J \colon \{1, \ldots, p\} \times \{1, \ldots, N+M+1\} \to \{1, \ldots, I\} \big\}. \tag{7.3.14}$$

For $J \in \mathcal{J}$, let $C_{Q,J}$ be the event

$$C_{Q,J} = \Big\{ \Xi_{k,r}^{(\kappa,Q)} \in \mathcal{U}_Q\left(\mu_{J(k,r)}\right) \ \forall \, k = 1, \ldots, p, \, r = 1, \ldots, N+M+1 \Big\}. \tag{7.3.15}$$

Then, because of (7.3.13), we may bound

$$\Lambda_{\mathrm{var}}(T; \epsilon, K, \kappa) \leq \frac{1}{pT} \log \max_{J \in \mathcal{J}} \mathbb{E}_{0,\ldots,0}^{X_1^{(\kappa,Q)}, \ldots, X_p^{(\kappa,Q)}} \left(\exp[V] \, 1\!\!1_{C_{Q,J}} \right) + \frac{1}{pT} \log |\mathcal{J}|. \tag{7.3.16}$$

On $C_{Q,J}$ we have, via (7.3.9) and (7.3.11),

$$\begin{aligned} V \leq \frac{1}{\theta^2} \left(\frac{L+1}{L-1} \right)^{3/2} \frac{\nu \gamma^2}{\rho} \, \delta\kappa \sum_{k,l=1}^{p} \sum_{n=1}^{N} \sum_{m=L}^{M} \\ \times \int_Q \mu_{J(k,n)}(dx) \int_Q \mu_{J(l,n+m)}(dy) \ \delta\rho \ p_G^{(Q)}(y-x, (m+1)\delta\rho). \end{aligned} \tag{7.3.17}$$

Moreover, similarly as in (7.2.15),

$$\mathbb{P}_{0,\ldots,0}^{X_1^{(\kappa,Q)}, \ldots, X_p^{(\kappa,Q)}} (C_{Q,J}) \leq \prod_{k=1}^{p} \prod_{r=1}^{N+M+1} \max_{x \in \mathbb{Z}_\kappa^3 \cap Q} \mathbb{P}_x^{X^{(\kappa,Q)}} \left(\Xi^{(\kappa,Q)} \in \mathcal{U}_Q\left(\mu_{J(k,r)}\right) \right). \tag{7.3.18}$$

Combining (7.3.16–7.3.18), it follows that

$$\begin{aligned} \Lambda_{\mathrm{var}}(T; \epsilon, K, \kappa) \leq \frac{1}{pT} \max_{J \in \mathcal{J}} \Big[\frac{1}{\theta^2} \left(\frac{L+1}{L-1} \right)^{3/2} \frac{\nu \gamma^2}{\rho} \sum_{k,l=1}^{p} \sum_{n=1}^{N} \\ \times \delta\kappa \sum_{m=L}^{M} \int_Q \mu_{J(k,n)}(dx) \int_Q \mu_{J(l,n+m)}(dy) \ \delta\rho \ p_G^{(Q)}(y-x, (m+1)\delta\rho) \\ + \sum_{k=1}^{p} \sum_{r=1}^{N+M+1} \log \max_{x \in \mathbb{Z}_\kappa^3 \cap Q} \mathbb{P}_x^{X^{(\kappa,Q)}} \left(\Xi^{(\kappa,Q)} \in \mathcal{U}_Q\left(\mu_{J(k,r)}\right) \right) \Big] \\ + \frac{1}{pT} \log |\mathcal{J}| \end{aligned} \tag{7.3.19}$$

for $\kappa \geq \kappa_0(C, \theta)$.

Below we will need the following upper large deviation bound (with $\Xi^{(\kappa,Q)} = \Xi_{1,1}^{(\kappa,Q)}$), which is the reverse of Lemma 7.2.1.

Lemma 7.3.1 *For each $i \in \{1, \dots, I\}$,*

$$\limsup_{\kappa\to\infty} \frac{1}{\delta\kappa} \log \max_{x\in\mathbb{Z}_\kappa^3\cap Q} \mathbb{P}_x^{X^{(\kappa,Q)}}\Big(\Xi^{(\kappa,Q)} \in \mathcal{U}_Q(\mu_i)\Big) \le -\theta\, \widehat{S}_Q(\mu_i) \tag{7.3.20}$$

with $\widehat{S}_Q$ the Q-periodization of S_Q, i.e., $\widehat{S}_Q\colon \mathcal{M}_1(Q) \to [0,\infty]$ given by

$$\widehat{S}_Q(\mu) = \begin{cases} \|\nabla_{\mathbb{R}^3} f\|_2^2 & \text{if } \mu \ll dx \text{ and } \sqrt{\frac{d\mu}{dx}} = f(x) \text{ with } f \in H^1_{per}(Q),\\ \infty & \text{otherwise,}\end{cases} \tag{7.3.21}$$

where $H^1_{per}(Q)$ is the space of functions in $H^1(Q)$ with periodic boundary conditions.

The proof of Lemma 7.3.1 is deferred to Section 7.4.

Next, define

$$\mu^J_{k,s} = \mu_{J(k,r)} \quad \text{for } k = 1,\dots,p,\ r = 1,\dots,N+M+1,\ (r-1)\delta\kappa \le s < r\delta\kappa. \tag{7.3.22}$$

The measure-valued paths $s \mapsto \mu^J_{k,s}$ are piecewise constant and take values in $\{\mu_1,\dots,\mu_I\}$. Using once again (7.1.4), we may revert back time from sums to integrals, to obtain

$$\begin{aligned} &\sum_{n=1}^N \delta\kappa \sum_{m=L}^M \int_Q \mu_{J(k,n)}(dx) \int_Q \mu_{J(l,n+m)}(dy)\ \delta\rho\ p_G^{(Q)}(y-x,(m+1)\delta\rho)\\ &\le \frac{\rho}{\kappa}\left(\frac{L+3}{L+1}\right)^{3/2} \sum_{n=1}^N \int_{(n-1)\delta\kappa}^{n\delta\kappa} ds \sum_{m=L}^M \int_{(n+m-1)\delta\kappa}^{(n+m)\delta\kappa} dt\\ &\qquad \times \int_Q \mu^J_{k,s}(dx) \int_Q \mu^J_{l,t}(dy)\ p_G^{(Q)}\Big(y-x, \frac{\rho}{\kappa}(t-s)+2\delta\rho\Big)\\ &\le \frac{\rho}{\kappa}\left(\frac{L+1}{L-1}\right)^{3/2} \int_0^{N\delta\kappa} ds \int_{s+(L-1)\delta\kappa}^{s+(M+1)\delta\kappa} dt\\ &\qquad \times \int_Q \mu^J_{k,s}(dx) \int_Q \mu^J_{l,t}(dy)\ p_G^{(Q)}\Big(y-x, \frac{\rho}{\kappa}(t-s)+2\delta\rho\Big)\\ &\le \frac{\rho}{\kappa}\left(\frac{L+1}{L-1}\right)^{3/2} \int_0^{(N+M+1)\delta\kappa} ds \int_0^{(N+M+1)\delta\kappa} dt\ \ 1\!\!1_{\{(L-1)\delta\kappa \le t-s \le (M+1)\delta\kappa\}}\\ &\qquad \times \int_Q \mu^J_{k,s}(dx) \int_Q \mu^J_{l,t}(dy)\ p_G^{(Q)}\Big(y-x, \frac{\rho}{\kappa}(t-s)+2\delta\rho\Big) \end{aligned} \tag{7.3.23}$$

and, according to Lemma 7.3.1,

$$\begin{aligned} &\sum_{r=1}^{N+M+1} \log \max_{x\in\mathbb{Z}_\kappa^3\cap Q} \mathbb{P}_x^{X^{(\kappa,Q)}}\Big(\Xi^{(\kappa,Q)} \in \mathcal{U}_Q(\mu_{J(k,r)})\Big)\\ &\quad = \int_0^{(N+M+1)\delta\kappa} ds\ \frac{1}{\delta\kappa} \log \max_{x\in\mathbb{Z}_\kappa^3\cap Q} \mathbb{P}_x^{X^{(\kappa,Q)}}\Big(\Xi^{(\kappa,Q)} \in \mathcal{U}_Q(\mu^J_{k,s})\Big)\\ &\qquad \le -\theta^2 \int_0^{(N+M+1)\delta\kappa} ds\ \widehat{S}_Q(\mu^J_{k,s}) \end{aligned} \tag{7.3.24}$$

for $\kappa \geq \kappa_1(C,\theta) \geq \kappa_0(C,\theta)$. Inserting (7.3.23–7.3.24) into (7.3.19), we arrive at

$$
\begin{aligned}
&\Lambda_{\mathrm{var}}(T;\epsilon,K,\kappa) \\
&\leq \frac{1}{pT} \max_{J\in\mathcal{J}} \Bigg[\frac{1}{\theta^2} \left(\frac{L+1}{L-1}\right)^3 \frac{\nu\gamma^2}{\kappa} \sum_{k,l=1}^{p} \int_0^{(N+M+1)\delta\kappa} ds \int_0^{(N+M+1)\delta\kappa} dt \; 1\!\!1_{\{(L-1)\delta\kappa\leq t-s\leq(M+1)\delta\kappa\}} \\
&\qquad\qquad \times \int_Q \mu_{k,s}^J(dx) \int_Q \mu_{l,t}^J(dy) p_G^{(Q)}\left(y-x, \frac{\rho}{\kappa}(t-s)+2\delta\rho\right) \\
&\qquad\qquad\qquad - \theta^2 \sum_{k=1}^{p} \int_0^{(N+M+1)\delta\kappa} ds\; \widehat{S}_Q(\mu_{k,s}^J) \Bigg] \\
&\qquad\qquad\qquad + \frac{1}{pT} \log|\mathcal{J}|
\end{aligned}
\tag{7.3.25}
$$

for $\kappa \geq \kappa_1(C,\theta)$.

At this point we can do a time-diagonalization:

Lemma 7.3.2 *For every $A>0$ and $\mu_{k,s}\in\mathcal{M}_1(Q)$ with $k=1,\dots,p$, $0\leq s\leq (N+M+1)\delta\kappa$,*

$$
\begin{aligned}
&\frac{A}{\kappa}\int_0^{(N+M+1)\delta\kappa} ds \int_s^{(N+M+1)\delta\kappa} dt \; 1\!\!1_{\{(L-1)\delta\kappa\leq t-s\leq(M+1)\delta\kappa\}} \\
&\qquad \times \sum_{k,l=1}^{p} \int_Q \mu_{k,s}(dx) \int_Q \mu_{l,t}(dy)\; p_G^{(Q)}\left(y-x,\frac{\rho}{\kappa}(t-s)+2\delta\rho\right) \\
&\qquad\qquad - \sum_{k=1}^{p} \int_0^{(N+M+1)\delta\kappa} ds\; \widehat{S}_Q(\mu_{k,s}) \\
&\leq p(N+M+1)\delta\kappa \\
&\qquad \times \sup_{\nu\in\mathcal{M}_1(Q)} \left[\frac{A}{\kappa} p \int_Q \nu(dx) \int_Q \nu(dy) \int_{(L-1)\delta\kappa}^{(M+1)\delta\kappa} du\; p_G^{(Q)}\left(y-x,\frac{\rho}{\kappa}u+2\delta\rho\right) - \widehat{S}_Q(\nu)\right].
\end{aligned}
\tag{7.3.26}
$$

The proof is given below. Inserting (7.3.26) with $A=\theta^{-4}(\frac{L+1}{L-1})^3\nu\gamma^2$ into (7.3.25), inserting (7.2.18), letting $T\to\infty$ and recalling (5.2.16), we obtain

$$
\begin{aligned}
\kappa^2\lambda_{\mathrm{var}}^{+}(\epsilon,K,\kappa) &\leq \theta^2 \sup_{\substack{f\in H^1_{per}(Q)\\ \|f\|_2=1}} \Bigg[\frac{1}{\theta^4}\left(\frac{L+1}{L-1}\right)^3 \frac{\nu\gamma^2}{\rho} p \int_Q dx\, f^2(x) \int_Q dy\, f^2(y) \\
&\quad \times \int_{(L-1)\delta\rho}^{(M+1)\delta\rho} du\; p_G^{(Q)}(y-x,u+2\delta\rho) - \|\nabla_{\mathbb{R}^3} f\|_2^2 \Bigg] \\
&\qquad\qquad + \frac{1}{\delta\kappa}\log I,
\end{aligned}
\tag{7.3.27}
$$

where we note that $\log|\mathcal{J}| = p(N+M+1)\log I$ and recall the last line of (7.2.1). Now let

$\kappa \to \infty$, $\delta \downarrow 0$ (yielding $L \to \infty$, $(L-1)\delta \to \epsilon$ and $(M+1)\delta \to K$) and $\theta \uparrow 1$, to obtain

$$
\begin{aligned}
&\limsup_{\kappa\to\infty} \kappa^2 \lambda^+_{\mathrm{var}}(\epsilon, K, \kappa) \\
&\le \sup_{\substack{f\in H^1_{per}(Q)\\ \|f\|_2=1}} \left[\frac{\nu\gamma^2}{\rho}\, p \int_Q dx\, f^2(x) \int_Q dy\, f^2(y) \int_{\epsilon\rho}^{K\rho} du\; p_G^{(Q)}(y-x,u) - \|\nabla_{\mathbb{R}^3} f\|_2^2 \right] \qquad (7.3.28)\\
&= \mathcal{P}_p^{(Q)}(\epsilon, K; \gamma, \rho, \nu).
\end{aligned}
$$

Finally, let $Q \uparrow \mathbb{R}^3$ and use the following.

Lemma 7.3.3 *Let $\mathcal{P}_p(\epsilon, K; \gamma, \rho, \nu)$ be as defined in* (5.2.11). *Then*

$$
\limsup_{Q\uparrow\mathbb{R}^3} \mathcal{P}_p^{(Q)}(\epsilon, K; \gamma, \rho, \nu) \le \mathcal{P}_p(\epsilon, K; \gamma, \rho, \nu). \qquad (7.3.29)
$$

The proof of Lemma 7.3.3 is deferred to Section 7.4. Combining (7.3.28–7.3.29), we have completed the proof of Lemma 5.2.5.

We close this section by proving Lemma 7.3.2.

Proof. Abbreviate

$$
\nu_s = \frac{1}{p} \sum_{k=1}^{p} \mu_{k,s} \in \mathcal{M}_1(Q), \qquad 0 \le s \le (N+M+1)\delta\kappa. \qquad (7.3.30)
$$

Since $\mu \mapsto \widehat{S}_Q(\mu)$ is convex, we have

$$
\begin{aligned}
\text{lhs } (7.3.26) \le \frac{p^2}{2}\frac{A}{\kappa} &\int_0^{(N+M+1)\delta\kappa} ds \int_0^{(N+M+1)\delta\kappa} dt\; 1\!\!1_{\{(L-1)\delta\kappa \le |t-s| \le (M+1)\delta\kappa\}} \\
&\times \int_Q \nu_s(dx) \int_Q \nu_t(dy)\; p_G^{(Q)}\Big(y-x, \frac{\rho}{\kappa}|t-s| + 2\delta\rho\Big) \qquad (7.3.31)\\
&- p \int_0^{(N+M+1)\delta\kappa} ds\, \widehat{S}_Q(\nu_s),
\end{aligned}
$$

where we symmetrize the integrals w.r.t. s and t. Let $B > 0$ be the size of Q, i.e., $Q = [-B, B)^3$. Then $p_G^{(Q)}$ admits the Fourier representation

$$
p_G^{(Q)}(x,t) = \frac{1}{(2B)^3} \sum_{q\in\mathbb{Z}^3} e^{-(\pi/B)^2|q|^2 t} e^{-i(\pi/B)q\cdot x}, \qquad x \in Q,\ t > 0. \qquad (7.3.32)
$$

Let

$$
\widehat{\nu}_s(q) = \int_Q e^{i(\pi/B)q\cdot x} \nu_s(dx), \qquad q \in \mathbb{Z}^3. \qquad (7.3.33)
$$

Then we may rewrite

$$
\begin{aligned}
\text{rhs } (7.3.31) = \frac{p^2}{2}\frac{A}{\kappa} &\int_0^{(N+M+1)\delta\kappa} ds \int_0^{(N+M+1)\delta\kappa} dt\; 1\!\!1_{\{(L-1)\delta\kappa \le |t-s| \le (M+1)\delta\kappa\}} \\
&\times \frac{1}{(2B)^3} \sum_{q\in\mathbb{Z}^3} e^{-(\pi/B)^2|q|^2[\frac{\rho}{\kappa}|t-s| + 2\delta\rho]}\; \widehat{\nu}_s(q)\, \overline{\widehat{\nu}_t(q)} - p \int_0^{(N+M+1)\delta\kappa} ds\, \widehat{S}_Q(\nu_s).
\end{aligned}
\qquad (7.3.34)
$$

Since this expression is real-valued and

$$\mathrm{Re}\left(\widehat{\nu}_s(q)\,\overline{\widehat{\nu}_t(q)}\right) \le \frac{1}{2}|\widehat{\nu}_s(q)|^2 + \frac{1}{2}|\widehat{\nu}_t(q)|^2, \tag{7.3.35}$$

we find, after inserting (7.3.35) into (7.3.34) and undoing the symmetrization w.r.t. s and t afterwards, that

$$\begin{aligned}\text{rhs (7.3.34)} \le{}& p^2\,\frac{A}{\kappa}\int_0^{(N+M+1)\delta\kappa} ds \int_{s+(L-1)\delta\kappa}^{s+(M+1)\delta\kappa} dt\;\frac{1}{(2B)^3}\sum_{q\in\mathbb{Z}^3} e^{-(\pi/B)^2|q|^2[\frac{\rho}{\kappa}(t-s)+2\delta\rho]}\;|\widehat{\nu}_s(q)|^2\\ &- p\int_0^{(N+M+1)\delta\kappa} ds\,\widehat{S}_Q(\nu_s).\end{aligned} \tag{7.3.36}$$

Again using (7.3.32–7.3.33), we see that

$$\begin{aligned}\text{rhs (7.3.36)} ={}& p\int_0^{(N+M+1)\delta\kappa} ds\\ &\times\left(\frac{A}{\kappa}\,p\int_Q \nu_s(dx)\int_Q \nu_s(dy)\int_{(L-1)\delta\kappa}^{(M+1)\delta\kappa} du\; p_G^{(Q)}\left(y-x,\frac{\rho}{\kappa}u+2\delta\rho\right) - \widehat{S}_Q(\nu_s)\right).\end{aligned} \tag{7.3.37}$$

Clearly, this expression does not exceed the right-hand side of (7.3.26). ∎

7.4 Proof of Lemmas 7.2.1, 7.3.1 and 7.3.3

The proof of Lemma 7.2.1 runs as follows.

Proof. Let $X^{(\kappa)}$ be the scaled random walk on $\mathbb{Z}^3_\kappa$ (as in (7.1.1)), let $\tau^{(\kappa)}$ be the first time $X^{(\kappa)}$ exits Q, and let $\Xi^{(\kappa)}$ be the normalized occupation time measure of $X^{(\kappa)}$ (as in (7.2.9)). Define the conditional probability measures

$$\mathbb{Q}_x^{(\kappa)}(\cdot) = \mathbb{P}_x^{X^{(\kappa)}}\left(\Xi^{(\kappa)}\in\cdot\,\Big|\,\tau^{(\kappa)}>\delta\kappa,\ X^{(\kappa)}(\delta\kappa)\in\frac{1}{2}Q\right). \tag{7.4.1}$$

Let ζ_0 denote the principal eigenvalue of the Laplacian Δ_Q with Dirichlet boundary condition in $L^2(Q)$. We will prove the following:

(a)
$$\lim_{\kappa\to\infty}\frac{1}{\delta\kappa}\log\mathbb{P}_x^{X^{(\kappa)}}\left(\tau^{(\kappa)}>\delta\kappa,\ X^{(\kappa)}(\delta\kappa)\in\frac{1}{2}Q\right)=\zeta_0 \tag{7.4.2}$$
uniformly in $x\in\frac{1}{2}Q$.

(b) The family $(\mathbb{Q}_x^{(\kappa)})_{\kappa>0}$ satisfies the full large deviation principle on $\mathcal{M}_1(Q)$, uniformly in $x\in\frac{1}{2}Q$, with rate $\delta\kappa$ and with rate function $S_Q+\zeta_0$ (recall (7.2.18)).

As a consequence of (a) and (b), the family $(\widetilde{\mathbb{Q}}_x^{(\kappa)})_{\kappa>0}$ of sub-probability measures defined by

$$\widetilde{\mathbb{Q}}_x^{(\kappa)}(\cdot) = \mathbb{P}_x^{X^{(\kappa)}}\left(\Xi^{(\kappa)}\in\cdot\,,\ \tau^{(\kappa)}>\delta\kappa,\ X^{(\kappa)}(\delta\kappa)\in\frac{1}{2}Q\right) \tag{7.4.3}$$

satisfies the full large deviation principle on $\mathcal{M}_1(Q)$, uniformly in $x \in \frac{1}{2}Q$, with rate $\delta\kappa$ and with rate function S_Q. The latter in turn implies Lemma 7.2.1.

The proof of assertions (a) and (b) goes as follows. Given a potential $V \in C_c^\infty(Q)$, let $\zeta_0(V)$ denote the principal eigenvalue of $\Delta_Q + V$ with Dirichlet boundary condition in $L^2(Q)$. It is well known that $V \mapsto \zeta_0(V)$ is Gateaux differentiable and that S_Q has the following representation as a Legendre transform:

$$S_Q(\mu) = \sup_{V \in C_c^\infty(Q)} \left[\int_Q V \, d\mu - \zeta_0(V) \right], \qquad \mu \in C_c^\infty(Q)^*, \tag{7.4.4}$$

with $C_c^\infty(Q)^*$ the algebraic dual of $C_c^\infty(Q)$ equipped with the weak* topology ((7.4.4) is dual to the Rayleigh-Ritz formula for $\zeta_0(V)$). We may therefore apply a uniform (w.r.t. the starting point) version of Dawson and Gärtner [9], Theorem 3.4, to see that in order to prove (a) and (b) it is enough to show that

$$\lim_{\kappa\to\infty} \frac{1}{\delta\kappa} \log \mathbb{E}_x^{X^{(\kappa)}} \left(\exp\left[\int_0^{\delta\kappa} V(X^{(\kappa)}(s))\, ds \right] 1\!\!1 \left\{ \tau^{(\kappa)} > \delta\kappa,\, X^{(\kappa)}(\delta\kappa) \in \frac{1}{2}Q \right\} \right) = \zeta_0(V) \tag{7.4.5}$$

uniformly in $x \in \frac{1}{2}Q$ for all $V \in C_c^\infty(Q)$. (A similar argument as in [9], Section 3.5, shows that $S_Q(\mu) < \infty$, $\mu \in C_c^\infty(Q)^*$ implies $\mu \in \mathcal{M}_1(Q)$, which is needed for the application of [9], Theorem 3.4.) Note that assertion (a) coincides with (7.4.5) for $V = 0$.

Fix $V \in C_c^\infty(Q)$. Abbreviate

$$s_-^{(\kappa)}(t) = \log \inf_{x \in \frac{1}{2}Q} \mathbb{E}_x^{X^{(\kappa)}} \left(\exp\left[\int_0^t V(X^{(\kappa)}(s))\, ds \right] 1\!\!1 \left\{ \tau^{(\kappa)} > t,\, X^{(\kappa)}(t) \in \frac{1}{2}Q \right\} \right). \tag{7.4.6}$$

Fix $T > 0$. For $t = \delta\kappa$ split the integral in the right-hand side of (7.4.6) into the sum of $\lfloor \delta\kappa/T \rfloor$ integrals of length $T_\kappa = \delta\kappa / \lfloor \delta\kappa/T \rfloor$. Then, using the Markov property of $X^{(\kappa)}$ at the splitting points, we get

$$s_-^{(\kappa)}(\delta\kappa) \geq \lfloor \delta\kappa/T \rfloor \, s_-^{(\kappa)}(T_\kappa). \tag{7.4.7}$$

Hence

$$\begin{aligned} \liminf_{\kappa\to\infty} \frac{s_-^{(\kappa)}(\delta\kappa)}{\delta\kappa} &\geq \frac{1}{T} \liminf_{\kappa\to\infty} s_-^{(\kappa)}(T_\kappa) \\ &= \frac{1}{T} \log \inf_{x \in \frac{1}{2}Q} \mathbb{E}_x^W \left(\exp\left[\int_0^T V(W(s)) ds \right] 1\!\!1 \left\{ \tau > T,\, W(T) \in \frac{1}{2}Q \right\} \right), \end{aligned} \tag{7.4.8}$$

where W is Brownian motion on $\mathbb{R}^3$, with generator $\Delta_{\mathbb{R}^3}$, and τ denotes the first time W exits Q. To derive the last line of (7.4.8) we use a uniform version of Donsker's invariance principle. It is well known that the right-hand side of (7.4.8) tends to $\zeta_0(V)$ as $T \to \infty$. Therefore we arrive at the lower bound

$$\liminf_{\kappa\to\infty} \frac{s_-^{(\kappa)}(\delta\kappa)}{\delta\kappa} \geq \zeta_0(V). \tag{7.4.9}$$

To get the corresponding upper bound, abbreviate

$$s_+^{(\kappa)}(t) = \log \sup_{x \in Q} \mathbb{E}_x^{X^{(\kappa)}} \left(\exp\left[\int_0^t V(X^{(\kappa)}(s))\, ds \right] 1\!\!1 \left\{ \tau^{(\kappa)} > t \right\} \right). \tag{7.4.10}$$

Then, in analogy with the above considerations, we obtain through a superadditivity argument that

$$\limsup_{\kappa\to\infty} \frac{s_+^{(\kappa)}(\delta\kappa)}{\delta\kappa} \le \zeta_0(V). \tag{7.4.11}$$

Combine (7.4.9) and (7.4.11) to get (7.4.5). ∎

The proof of Lemma 7.3.1 runs as follows.

Proof. Let $\widehat{X}^{(\kappa)}$ denote the random walk on $Q^{(\kappa)} = Q \cap \mathbb{Z}^3_\kappa$ obtained by wrapping $X^{(\kappa)}$ around $Q^{(\kappa)}$. Let

$$\widehat{\Xi}^{(\kappa)}(A) = \frac{1}{\delta\kappa}\int_0^{\delta\kappa} ds\, 1\!\!1_A\left(\widehat{X}^{(\kappa)}(s)\right), \quad A \subset \mathbb{R}^3 \text{ Borel}, \tag{7.4.12}$$

and

$$\widehat{\mathbb{Q}}_x^{(\kappa)}(\cdot) = \mathbb{P}_x^{\widehat{X}^{(\kappa)}}\left(\Xi^{(\kappa)} \in \cdot\right). \tag{7.4.13}$$

Then the analogue of (b) reads:

(b') The family $(\widehat{\mathbb{Q}}_x^{(\kappa)})_{\kappa>0}$ satisfies the full large deviation principle on $\mathcal{M}_1(Q)$, uniformly in $x \in Q$, with rate $\delta\kappa$ and with rate function $\widehat{S}_Q$ (recall (7.3.21)).

The proof of assertion (b') goes along the same lines as the proof of assertions (a) and (b), and is in fact even simpler. It implies Lemma 7.3.1. ∎

The proof of Lemma 7.3.3 runs as follows.

Proof. Let $Q = Q_B = [-B, B)^3$. Write $Q_B(q) = Q_B + q$, $q \in \mathbb{R}^3$. Let

$$\widehat{p}^{(Q_B)}(x,t) = \sum_{k\in\mathbb{Z}^3} p_G(x + 2Bk, t) \tag{7.4.14}$$

denote the Q_B-periodization of the Gaussian transition kernel p_G. Recall that $H^1_{per}(Q_B)$ denotes the space of functions in $H^1(Q_B)$ with periodic boundary conditions.

Fix $B > 1$ and $f \in H^1_{per}(Q_B)$ with $\|f\|_2 = 1$. Put $A = B - \sqrt{B}$. Let $\widehat{f}$ be the Q_B-periodic extension of f to $\mathbb{R}^3$. Then

$$\frac{1}{|Q_B|}\int_{Q_B} dq \int_{Q_A(q)} dx\ \widehat{f}^{\,2}(x) = \frac{|Q_A|}{|Q_B|}, \tag{7.4.15}$$

and hence there exists $q \in Q_B$ (depending on B, f) such that

$$\int_{Q_A(q)} dx\ \widehat{f}^{\,2}(x) \ge \frac{|Q_A|}{|Q_B|}. \tag{7.4.16}$$

Let $h_B \colon \mathbb{R}^3 \to \mathbb{R}$ be a smooth function (depending on B, q) satisfying

$$0 \le h_B \le 1, \quad h_B = 1 \text{ on } Q_A(q), \quad h_B = 0 \text{ on } \mathbb{R}^3 \setminus Q_B(q). \tag{7.4.17}$$

We may assume that

$$D = \left\|\Delta(h_B(1-h_B)) + 2|\nabla h_B|^2\right\|_\infty < \infty \tag{7.4.18}$$

with D not dependent on B, q, f. Define

$$f_B = \frac{h_B \widehat{f}}{\|h_B \widehat{f}\|_2}. \tag{7.4.19}$$

Then $f_B \in H^1(\mathbb{R}^3)$ and $\|f_B\|_2 = 1$. Moreover, by (7.4.16–7.4.17) we have

$$\frac{|Q_A|}{|Q_B|} \le \|h_B \widehat{f}\|_2^2 \le 1. \tag{7.4.20}$$

Hence $\|h_B \widehat{f}\|_2 \to 1$ as $B \to \infty$.

Next, observe that

$$\|x - y + 2Bk\|_\infty \ge 2B(\|k\|_\infty - 1) + 2(B - A), \qquad x, y \in Q_A(q),\ k \in \mathbb{Z}^3 \setminus \{0\}. \tag{7.4.21}$$

Because

$$p_G(x,t) = (4\pi t)^{-3/2} \exp[-\|x\|^2/4t] \le (4\pi t)^{-3/2} \exp[-\|x\|_\infty^2/4t], \tag{7.4.22}$$

it follows from (7.4.21) that there exists δ_B (not depending on q, f), satisfying $\delta_B \to 0$ as $B \to \infty$, such that

$$\int_{\epsilon\rho}^{K\rho} \widehat{p}_G^{(Q_B)}(x-y,t)\, dt \le \int_{\epsilon\rho}^{K\rho} p_G(x-y,t)\, dt + \delta_B, \qquad x, y \in Q_A(q). \tag{7.4.23}$$

Moreover, from this it also follows that there exists a constant $C < \infty$ (not depending on $B \ge 1, q, f$) such that

$$\int_{\epsilon\rho}^{K\rho} \widehat{p}_G^{(Q_B)}(x-y,t)\, dt \le C, \qquad x, y \in Q_A(q). \tag{7.4.24}$$

With the above estimates in place, we next derive an upper bound for

$$\int_{Q_B} dx \int_{Q_B} dy \int_{\epsilon\rho}^{K\rho} dt\ \widehat{p}_G^{(Q_B)}(x-y,t)\, \widehat{f}^{\,2}(x) \widehat{f}^{\,2}(y). \tag{7.4.25}$$

Since $\widehat{f}$ is Q_B-periodic, we may replace the domain of integration $Q_B \times Q_B$ by $Q_B(q) \times Q_B(q)$. After that we may split the integral into two parts: $Q_A(q) \times Q_A(q)$ and $[Q_B(q) \times Q_B(q)] \setminus [Q_A(q) \times Q_A(q)]$. The latter coincides with the union of $[Q_B(q) \setminus Q_A(q)] \times Q_B(q)$ and $Q_B(q) \times [Q_B(q) \setminus Q_A(q)]$. Therefore, using (7.4.16) and (7.4.23–7.4.24), we obtain

$$\begin{aligned}
&\int_{Q_B} dx \int_{Q_B} dy \int_{\epsilon\rho}^{K\rho} dt\ \widehat{p}_G^{(Q_B)}(x-y,t)\, \widehat{f}^{\,2}(x) \widehat{f}^{\,2}(y) \\
&\le \int_{Q_A(q)} dx \int_{Q_A(q)} dy \int_{\epsilon\rho}^{K\rho} dt\ p_G(x-y,t)\, \widehat{f}^{\,2}(x) \widehat{f}^{\,2}(y) + \delta_B + 2C \int_{Q_B(q) \setminus Q_A(q)} dx\ \widehat{f}^{\,2}(x) \\
&\le \int_{Q_A(q)} dx \int_{Q_A(q)} dy \int_{\epsilon\rho}^{K\rho} dt\ p_G(x-y,t)\, f_B^2(x) f_B^2(y) + \delta_B + 2C\, \frac{|Q_B \setminus Q_A|}{|Q_B|} \\
&\le \frac{|Q_A|}{|Q_B|} \int_{Q_A(q)} dx \int_{Q_A(q)} dy \int_{\epsilon\rho}^{K\rho} dt\ p_G(x-y,t)\, f^2(x) f^2(y) + \delta_B + 3C\, \frac{|Q_B \setminus Q_A|}{|Q_B|},
\end{aligned} \tag{7.4.26}$$

where in the second inequality we use that $\widehat{f}^{\,2} = (h_B \widehat{f})^2 = \|h_B \widehat{f}\|_2^2 f_B^2 \le f_B^2$ on $Q_A(q)$.

Next, we derive a lower bound for $\|\nabla f\|_2^2$ in terms of f_B. First estimate

$$\begin{aligned}
\|\nabla f\|_2^2 &= \int_{Q_B(q)} dx\ |\nabla(h_B \widehat{f}) + \nabla((1-h_B)\widehat{f})|^2 \\
&\ge \int_{Q_B(q)} dx\ |\nabla(h_B \widehat{f})|^2 + 2 \int_{Q_B(q)} dx\ \nabla(h_B \widehat{f}) \cdot \nabla((1-h_B)\widehat{f}).
\end{aligned} \tag{7.4.27}$$

But

$$\nabla(h_B\widehat{f})\cdot\nabla((1-h_B)\widehat{f}) \geq (\widehat{f}\nabla(h_B(1-h_B)))\cdot\nabla\widehat{f} - |\nabla h_B|^2\widehat{f}^{\,2}, \tag{7.4.28}$$

and integration by parts shows that

$$\int_{Q_B(q)} dx\ (\widehat{f}\nabla(h_B(1-h_B)))\cdot\nabla\widehat{f} = -\frac{1}{2}\int_{Q_B(q)} dx\ \widehat{f}^{\,2}\,\Delta(h_B(1-h_B)). \tag{7.4.29}$$

Hence, recalling the definition of f_B and taking into account (7.4.16), (7.4.18) and (7.4.20), we obtain

$$\begin{aligned}\|\nabla f\|_2^2 &\geq \|h_B\widehat{f}\|_2^2\,\|\nabla f_B\|_2^2 - \int_{Q_B(q)\setminus Q_A(q)} dx\ \widehat{f}^{\,2}\ \left[\nabla(h_B(1-h_B)) + 2|\nabla h_B|^2\right]\\ &\geq \frac{|Q_A|}{|Q_B|}\,\|\nabla f_B\|_2^2 - \frac{|Q_B\setminus Q_A|}{|Q_B|}\,D.\end{aligned} \tag{7.4.30}$$

Combining (7.4.26) and (7.4.30), and abbreviating $\alpha = (\nu\gamma^2/\rho)p$, we arrive at

$$\begin{aligned}&\alpha\int_{Q_B} dx\int_{Q_B} dy\int_{\epsilon\rho}^{K\rho} dt\ \widehat{p}_G(x-y,t)\,f^2(x)f^2(y) - \|\nabla f\|_2^2\\ &\qquad\leq \frac{|Q_A|}{|Q_B|}\,\mathcal{P} + \alpha\delta_B + (3\alpha C + D)\,\frac{|Q_B\setminus Q_A|}{|Q_B|}.\end{aligned} \tag{7.4.31}$$

Since C, D and δ_B do not depend on f, we conclude that (recall (7.3.28))

$$\mathcal{P}_p^{(Q_B)}(\epsilon,K;\gamma,\rho,\nu) \leq \frac{|Q_A|}{|Q_B|}\,\mathcal{P}_p(\epsilon,K;\gamma,\rho,\nu) + \alpha\delta_B + (3\alpha C + D)\,\frac{|Q_B\setminus Q_A|}{|Q_B|}. \tag{7.4.32}$$

Now let $B\to\infty$ and use that $\delta_B\to 0$ and $|Q_A|/|Q_B|\to 1$, to get the claim in (7.3.29). ∎

8 Proof of Lemmas 5.2.6–5.2.8

In this section we prove Lemmas 5.2.6–5.2.8, which handle the terms that are asymptotically negligible as $\kappa\to\infty$.

8.1 Proof of Lemma 5.2.6

Proof. Using the rough bound

$$p_{\rho/\kappa}(X(t)-X(s),t-s) \leq p_{\rho/\kappa}(0,t-s) = p\left(0,\frac{\rho}{\kappa}(t-s)\right), \tag{8.1.1}$$

we conclude from (5.2.19–5.2.20) that

$$\kappa^2\lambda_{\text{off}}^{+}(a,\kappa) \leq \frac{\nu\gamma^2}{\rho}\kappa\int_{\rho a\kappa^2}^{\infty} dt\, p(0,t). \tag{8.1.2}$$

Because of (5.3.3–5.3.4), the expression in the right-hand side is bounded above by a constant times $a^{-(d-2)/2}\kappa^{-(d-3)}$. From this the claims in (5.2.21–5.2.22) follow. ∎

8.2 Proof of Lemma 5.2.7

For the proof of Lemma 5.2.7 we need two more lemmas. Let $\mathcal{G}$ denote the Green operator acting on functions $V\colon \mathbb{Z}^d \to [0,\infty)$ as

$$(\mathcal{G}V)(x) = \sum_{y\in\mathbb{Z}^d} G(y-x)V(y), \qquad x \in \mathbb{Z}^d, \tag{8.2.1}$$

with $G(z) = \int_0^\infty dt\, p(z,t)$. Let $\|\cdot\|_\infty$ denote the supremum norm.

Lemma 8.2.1 *For any $V\colon \mathbb{Z}^d \to [0,\infty)$ and $x \in \mathbb{Z}^d$,*

$$\mathbb{E}_x^X\left(\exp\left[\int_0^\infty dt\ V(X(t))\right]\right) \le (1-\|\mathcal{G}V\|_\infty)^{-1}, \tag{8.2.2}$$

provided that

$$\|\mathcal{G}V\|_\infty < 1. \tag{8.2.3}$$

Lemma 8.2.2 *For any $\alpha,\beta > 0$ and $a > 0$,*

$$\begin{aligned} &\mathbb{E}_{0,0}^{X,Y}\left(\exp\left[\alpha\int_0^T ds \int_s^{s+a\kappa^3} dt\ p_\beta(Y(t)-X(s),t-s)\right]\right)\\ &\qquad\le \mathbb{E}_0^X\left(\exp\left[\alpha\int_0^T ds\int_s^{s+a\kappa^3} dt\ p_\beta(X(s),t-s)\right]\right). \end{aligned} \tag{8.2.4}$$

Before giving the proof of Lemmas 8.2.1–8.2.2, we first prove Lemma 5.2.7.

Proof. Using Lemma 8.2.2, we get from (5.2.23) that

$$\begin{aligned} \Lambda_{\text{mix}}(T;a,\kappa) &\le \frac{1}{T}\log \mathbb{E}_0^X\left(\exp\left[\frac{\nu\gamma^2}{\kappa^2}\int_0^T ds\int_s^{s+a\kappa^3} dt\ p_{\rho/\kappa}(X(s),t-s)\right]\right)\\ &\le \frac{1}{T}\log\mathbb{E}_0^X\left(\exp\left[\int_0^\infty ds\ V_{a,\kappa}(X(s))\right]\right), \end{aligned} \tag{8.2.5}$$

where

$$V_{a,\kappa}(x) = \frac{\nu\gamma^2}{\rho\kappa}\int_0^{\rho a\kappa^2} dt\ p(x,t), \qquad x\in\mathbb{Z}^d. \tag{8.2.6}$$

It follows from (5.3.3–5.3.4) that, as $\kappa\to\infty$,

$$\|\mathcal{G}V_{a,\kappa}\|_\infty = \frac{\nu\gamma^2}{\rho\kappa}\int_0^{\rho a\kappa^2} dt\int_t^\infty ds\ p(0,s) \tag{8.2.7}$$

tends to zero for $d\ge 4$ and $0<a<\infty$ and tends to a constant times $a^{1/2}$ for $d=3$. Hence, by Lemma 8.2.1, for large κ the expectation in the right-hand side of (8.2.5) is finite for $0<a<a_0$ with $a_0=\infty$ for $d\ge 4$ and $a_0>0$ small enough for $d=3$. Thus, by letting $T\to\infty$ in (8.2.5), we conclude that

$$\lambda_{\text{mix}}^+(a,\kappa) = 0 \qquad \forall\, 0<a<a_0,\ \kappa\ge\kappa_0(a). \tag{8.2.8}$$

This yields (5.2.26). To prove (5.2.25), simply note that for all $0 < a < \infty$,

$$\Lambda_{\text{mix}}(T;\infty,\kappa) \leq \Lambda_{\text{mix}}(T;a,\kappa) + \frac{\nu\gamma^2}{\rho\kappa}\int_{\rho a\kappa^2}^{\infty} dt\; p(0,t) \tag{8.2.9}$$

and hence

$$\kappa^2\lambda^{+}_{\text{mix}}(\infty,\kappa) \leq \frac{\nu\gamma^2}{\rho}\kappa\int_{\rho a\kappa^2}^{\infty} dt\; p(0,t) \qquad \forall\, 0 < a < a_0,\, \kappa \geq \kappa_0(a). \tag{8.2.10}$$

Now proceed as with (8.1.2) to get the claim. ∎

8.3 Proof of Lemmas 8.2.1–8.2.2

The proof of Lemma 8.2.1 goes as follows.

Proof. A Taylor expansion of the exponential function yields

$$\begin{aligned}&\mathbb{E}_x^X\left(\exp\left[\int_0^\infty dt\; V(X(t))\right]\right)\\ &= \sum_{n=0}^{\infty}\int_0^\infty dt_1\int_{t_1}^\infty dt_2\cdots\int_{t_{n-1}}^\infty dt_n\;\mathbb{E}_x^X\left(V(X(t_1))V(X(t_2))\times\cdots\times V(X(t_n))\right).\end{aligned} \tag{8.3.1}$$

But

$$\begin{aligned}&\int_0^\infty dt_1\int_{t_1}^\infty dt_2\cdots\int_{t_{n-1}}^\infty dt_n\;\mathbb{E}_x^X\left(V(X(t_1))V(X(t_2))\times\cdots\times V(X(t_n))\right)\\ &= \sum_{y_1\in\mathbb{Z}^d}\int_0^\infty dt_1\; p(y_1-x,t_1)V(y_1)\sum_{y_2\in\mathbb{Z}^d}\int_{t_1}^\infty dt_2\; p(y_2-y_1,t_2-t_1)V(y_2)\\ &\qquad\times\cdots\times\sum_{y_n\in\mathbb{Z}^d}\int_{t_{n-1}}^\infty dt_n\; p(y_n-y_{n-1},t_n-t_{n-1})V(y_n)\\ &= \sum_{y_1\in\mathbb{Z}^d}G(y_1-x)V(y_1)\sum_{y_2\in\mathbb{Z}^d}G(y_2-y_1)V(y_2)\times\cdots\times\sum_{y_n\in\mathbb{Z}^d}G(y_n-y_{n-1})V(y_n)\\ &\leq \|\mathcal{G}V\|_\infty^n.\end{aligned} \tag{8.3.2}$$

Substituting this into (8.3.1) and summing the geometric series, we arrive at the claim in (8.2.2). ∎

The proof of Lemma 8.2.2 goes as follows.

Proof. Using the Fourier representation of the transition kernel (recall (2.2.3))

$$p_\beta(x,t) = \oint dk\, e^{-\beta t\widehat{\varphi}(k)}e^{-ik\cdot x} \tag{8.3.3}$$

and expanding the exponential function in a Taylor series, we find that

$$
\begin{aligned}
&\mathbb{E}_{0,0}^{X,Y}\left(\exp\left[\alpha\int_0^T ds\int_s^{s+a\kappa^3} dt\; p_\beta(Y(t)-X(s),t-s)\right]\right)\\
&=\sum_{n=0}^{\infty}\alpha^n\int_0^T ds_1\int_{s_1}^T ds_2\cdots\int_{s_{n-1}}^T ds_n\\
&\int_{s_1}^{s_1+a\kappa^3} dt_1\int_{s_2}^{s_2+a\kappa^3} dt_2\cdots\int_{s_n}^{s_n+a\kappa^3} dt_n\;\oint dk_1\oint dk_2\cdots\oint dk_n\;\exp\left[-\beta\sum_{j=1}^n(t_j-s_j)\widehat{\varphi}(k_j)\right]\\
&\times\mathbb{E}_0^Y\left(\exp\left[-i\sum_{j=1}^n k_j\cdot Y(t_j)\right]\right)\mathbb{E}_0^X\left(\exp\left[i\sum_{j=1}^n k_j\cdot X(s_j)\right]\right).
\end{aligned}
\tag{8.3.4}
$$

Here, to factorize the two expectations we have used that the random walks X and Y are independent. By symmetry of X and Y, these two expectations are real-valued. An explicit computation shows that the second expectation is strictly positive. (Use that the s_i are ordered and that X has independent increments, so that the expectation factors into a product.) The first expectation clearly is ≤ 1. Hence, the above expression can be bounded from above by the same expression with Y replaced by 0. This in turn yields (8.2.4). ∎

8.4 Proof of Lemma 5.2.8

We begin by noting two facts. First, define

$$\Lambda_{\text{full}}(T;\kappa)=\frac{1}{T}\log\mathbb{E}_0^X\left(\exp\left[\frac{\nu\gamma^2}{\kappa^2}\int_0^T ds\int_s^\infty dt\; p_{\rho/\kappa}(X(t)-X(s),t-s)\right]\right)\tag{8.4.1}$$

and

$$\lambda_{\text{full}}^+(\kappa)=\limsup_{T\to\infty}\Lambda_{\text{full}}(T;\kappa).\tag{8.4.2}$$

By splitting the second integral in the right-hand side of (8.4.1) into a diagonal, a variational and an off-diagonal part (in accordance with Lemmas 5.2.4–5.2.6), applying Hölder's inequality to separate the parts (similarly as in (5.3.28)), and applying Lemmas 5.2.4–5.2.6, we find that

$$\limsup_{\kappa\to\infty}\kappa^2\lambda_{\text{full}}^+(\kappa)\leq\frac{\nu\gamma^2}{r_d}\qquad\text{if } d\geq 4,\tag{8.4.3}$$

while

$$\limsup_{\kappa\to\infty}\kappa^2\lambda_{\text{full}}^+(\kappa)\leq\frac{\nu\gamma^2}{r_3}+\left(\frac{\nu\gamma^2}{\rho}\right)^{1/2}\mathcal{P}\qquad\text{if } d=3.\tag{8.4.4}$$

Second, note that Lemma 6.3.2 for $k=0$ yields the bound

$$\mathbb{E}_0^X\left(\exp\left[\alpha\int_0^\infty dt\; p_{\rho/\kappa}(X(t),t)\right]\right)\leq\exp\left[\frac{\alpha G_0(0)}{1-\alpha G_0(0)}\right]\leq\exp\left[\frac{2\alpha}{r_d}\right],\tag{8.4.5}$$

provided that

$$0\leq\alpha\leq\frac{r_d}{2}.\tag{8.4.6}$$

The proof of Lemma 5.2.8 goes as follows.

Proof. Using the rough bound (8.1.1), we have

$$\begin{aligned}
&\int_0^T ds \left(\int_s^\infty dt\ p_{\rho/\kappa}(X(t)-X(s),t-s)\right)\left(\int_0^s du\ p_{\rho/\kappa}(X(s)-X(u),s-u)\right)\\
&\leq \int_0^T ds \left(\int_s^{s+\kappa^{3/2}} dt\ p_{\rho/\kappa}(X(t)-X(s),t-s)\right)\left(\int_{s-\kappa^{3/2}}^s du\ p_{\rho/\kappa}(X(s)-X(u),s-u)\right)\\
&\quad + 2\left(\int_{\kappa^{3/2}}^\infty du\ p_{\rho/\kappa}(0,u)\right)\int_0^T ds\int_s^\infty dt\ p_{\rho/\kappa}(X(t)-X(s),t-s).
\end{aligned} \tag{8.4.7}$$

Substituting this into (5.2.27) and applying the Cauchy-Schwarz inequality, we find that

$$\Lambda_{\mathrm{rem}}(T;\kappa) \leq \Lambda^{(1)}_{\mathrm{rem}}(T;\kappa) + \Lambda^{(2)}_{\mathrm{rem}}(T;\kappa), \tag{8.4.8}$$

where

$$\begin{aligned}
\Lambda^{(1)}_{\mathrm{rem}}(T;\kappa) = \frac{1}{2T}\log\mathbb{E}_0^X\Bigg(\exp\Bigg[\frac{2\nu\gamma^3}{\kappa^3}\int_0^T ds\\
\left(\int_s^{s+\kappa^{3/2}} dt\ p_{\rho/\kappa}(X(t)-X(s),t-s)\right)\left(\int_{s-\kappa^{3/2}}^s du\ p_{\rho/\kappa}(X(s)-X(u),s-u)\right)\Bigg]\Bigg)
\end{aligned} \tag{8.4.9}$$

and

$$\begin{aligned}
&\Lambda^{(2)}_{\mathrm{rem}}(T;\kappa) = \frac{1}{2T}\log\mathbb{E}_0^X\Bigg(\exp\Bigg[\frac{\nu\gamma^2}{\kappa^2}\\
&\left(\frac{4\gamma}{\kappa}\int_{\kappa^{3/2}}^\infty du\ p_{\rho/\kappa}(0,u)\right)\int_0^T ds\int_s^\infty dt\ p_{\rho/\kappa}(X(t)-X(s),t-s)\Bigg]\Bigg).
\end{aligned} \tag{8.4.10}$$

To prove Lemma 5.2.8, it will be enough to show that

$$\lim_{\kappa\to\infty}\kappa^2\limsup_{T\to\infty}\Lambda^{(i)}_{\mathrm{rem}}(T;\kappa) = 0, \qquad i=1,2. \tag{8.4.11}$$

Since for $d \geq 3$,

$$\frac{4\gamma}{\kappa}\int_{\kappa^{3/2}}^\infty du\ p_{\rho/\kappa}(0,u) \to 0 \quad \text{as } \kappa\to\infty, \tag{8.4.12}$$

(8.4.11) for $i=2$ follows from (8.4.1–8.4.4) with ν replaced by ν times the integral in (8.4.12). To prove (8.4.11) for $i=1$, we split the integral in the right-hand side of (8.4.9) as follows:

$$\int_0^T ds = \left(\sum_{\substack{k=1\\ \text{even}}}^{\lceil T/2\kappa^{3/2}\rceil} + \sum_{\substack{k=1\\ \text{odd}}}^{\lceil T/2\kappa^{3/2}\rceil}\right)\int_{(k-1)2\kappa^{3/2}}^{k2\kappa^{3/2}} ds. \tag{8.4.13}$$

Note that the summands in each of the two sums are i.i.d. Hence, substituting (8.4.13) into

(8.4.9) and applying the Cauchy-Schwarz inequality, we find that

$$\begin{aligned}
&\Lambda^{(1)}_{\rm rem}(T;\kappa)\\
&\leq \frac{\lceil T/2\kappa^{3/2}\rceil}{4T}\log \mathbb{E}^X_0\left(\exp\left[\frac{4\nu\gamma^3}{\kappa^3}\int_0^{2\kappa^{3/2}} ds\right.\right.\\
&\left.\left.\left(\int_s^{s+\kappa^{3/2}} dt\ p_{\rho/\kappa}(X(t)-X(s),t-s)\right)\left(\int_{s-\kappa^{3/2}}^s du\ p_{\rho/\kappa}(X(s)-X(u),s-u)\right)\right]\right).
\end{aligned} \tag{8.4.14}$$

Letting $T\to\infty$ and applying Jensen's inequality, we arrive at

$$\begin{aligned}
&\limsup_{T\to\infty}\Lambda^{(1)}_{\rm rem}(T;\kappa)\\
&\leq \frac{1}{8\kappa^{3/2}}\log\mathbb{E}^{X,Y}_{0,0}\left(\exp\left[\frac{4\nu\gamma^3}{\kappa^3}2\kappa^{3/2}\left(\int_0^{\kappa^{3/2}} dt\ p_{\rho/\kappa}(X(t),t)\right)\left(\int_0^{\kappa^{3/2}} du\ p_{\rho/\kappa}(Y(u),u)\right)\right]\right),
\end{aligned} \tag{8.4.15}$$

where we use that the increments of X over the time intervals $[s,s+\kappa^{3/2}]$ and $[s-\kappa^{3/2},s]$ are independent in order to replace the expectation over the single random walk X by an expectation over the two independent random walks X,Y. Since for $d\geq 3$,

$$\frac{4\nu\gamma^3}{\kappa^3}2\kappa^{3/2}\int_0^\infty du\ p_{\rho/\kappa}(Y(u),u)\leq\frac{8\nu\gamma^3}{\kappa^{3/2}}\int_0^\infty du\ p_{\rho/\kappa}(0,u)=\frac{8\nu\gamma^3}{r_d\rho\kappa^{1/2}}\to 0\quad\text{as }\kappa\to\infty, \tag{8.4.16}$$

we may apply (8.4.5–8.4.6) with α equal to the left-hand side of (8.4.16) to see that, for large κ,

$$\text{rhs (8.4.15)}\leq\frac{1}{8\kappa^{3/2}}\log\mathbb{E}^Y_0\left(\exp\left[\frac{2}{r_d}\frac{8\nu\gamma^3}{\kappa^{3/2}}\int_0^\infty du\ p_{\rho/\kappa}(Y(u),u)\right]\right). \tag{8.4.17}$$

Now we may apply (8.4.5–8.4.6) once more, this time with $\alpha=16\nu\gamma^3/r_d\kappa^{3/2}$, to obtain that, for large κ,

$$\limsup_{T\to\infty}\Lambda^{(1)}_{\rm rem}(T;\kappa)\leq\frac{1}{8\kappa^{3/2}}\frac{2}{r_d}\left(\frac{16\nu\gamma^3}{r_d\kappa^{3/2}}\right)=\frac{4\nu\gamma^3}{r_d^2\kappa^3}. \tag{8.4.18}$$

This implies (8.4.11) for $i=1$. ∎

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