Reinsurance of large claims

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Abstract: The large claims reinsurance treaties ECOMOR and LCR are well known not to be very popular. They have been largely neglected by most reinsurers because of their technical complexity. In this paper, we derive some new mathematical results connected to distributional problems of these reinsurance forms. Perhaps these results can reopen the discussion on the usefulness of including the largest claims in the decision making procedure. Apart from asymptotic estimates for the tail of the distribution of the ECOMOR-quantity, we find its weak laws. We also deal with the weak laws of the LCR-quantity. Finally, we illustrate the outcomes with a number of simulations.

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1 Introduction

Let $\{X_1, X_2, \dots\}$ be a sequence of successive claim sizes that consists of independent and identically distributed (i.i.d.) random variables generated by the distribution F of a generic random variable X. Assume that consecutive claims occur according to a counting process $\{N(t);\ t \geq 0\}$, i.e. the random variable $N(t)$ counts the number of claims up to time t. Assume further that the claim number process $\{N(t);\ t \geq 0\}$ is independent of the claim size process $\{X_i;\ i \geq 1\}$.

Throughout the sequel, we denote by $(X_1^*, X_2^*, \dots, X_{N(t)}^*)$ the order statistics, arranged in increasing order, of the random vector $(X_1, X_2, \dots, X_{N(t)})$ of successive claim sizes up to time t. Further, these claims determine the aggregate claim amount in the random sum $S_{N(t)} := \sum_{i=1}^{N(t)} X_i$.

The main goal of a reinsurance treaty is the coverage against large claims. It is therefore somewhat remarkable that classical reinsurance contracts (proportional, surplus, excess-of-loss, stop-loss) are not expressed in terms of the largest claims. One possible reason why large claims reinsurance treaties are playing a minor (even non-existent) role is their mathematical intractability. In this paper we will try to indicate that extreme value theory is capable to overcome part of this problem. To avoid overloading the reader with technical details, we will restrict attention to two forms of large claims reinsurance, i.e. ECOMOR and LCR. For other reinsurance treaties based on the largest claims similar results can be obtained. We refer to Teugels [26] for an overview of most of the currently employed reinsurance forms with some of their properties.

We mainly will be interested in the reinsurance treaty ECOMOR (excédent du coût moyen relatif) introduced by the French actuary Thépaut [27]. The treaty is defined in terms of the upper order statistics of the random sample coming from the specific portfolio. More specifically, the reinsured amount in the ECOMOR treaty is defined by:

$$R_r(t) := \sum_{i=1}^{r} X_{N(t)-i+1}^* - rX_{N(t)-r}^* = \sum_{i=1}^{N(t)} \left\{ X_i - X_{N(t)-r}^* \right\}^+, \quad r \geq 1,\ t \geq 0 \tag{1}$$

if $N(t) > r$ and $R_r(t) := 0$ otherwise. The quantity $R_r(t)$ is thus a function of the $r+1$ upper order statistics $X_{N(t)-r}^* \leq \cdots \leq X_{N(t)}^*$ in the randomly indexed sample $X_1, \dots, X_{N(t)}$ of claim sizes up to time epoch t. The expression on the right in (1) shows how ECOMOR can also be considered as an excess-of-loss treaty with a random retention determined by the $(r+1)$th largest claim. In other words, the reinsurer covers the part of the r largest claims that overshoots the random retention $X_{N(t)-r}^*$.

We also will be interested in the LCR (largest claims reinsurance) treaty. As for ECOMOR, the distributional problems connected with this reinsurance form are quite hard to tackle. As a result and as far as we know, it has been hardly ever used in practice. If we just think about excessive claims then LCR is a possibility since it only deals with the largest claims. Indeed, the reinsured amount in the LCR treaty equals:

$$L_r(t) := \sum_{i=1}^{r} X_{N(t)-i+1}^*, \quad r \geq 1,\ t \geq 0 \tag{2}$$

if $N(t) \geq r$ and $L_r(t) := 0$ otherwise.

In the ECOMOR and LCR treaties, the number of reinsured claims is equal to the deterministic value r. However, in ECOMOR, the claim sizes are diminished by the random retention $X_{N(t)-r+1}^*$. Also, all the remaining claims end up to the first line reinsurer in LCR, whereas in ECOMOR, the first line reinsurer has to carry the responsibility for the random retention also.

In the sequel we will give new mathematical results relating to asymptotic distributional problems for the quantities $R_r(t)$ and $L_r(t)$ respectively defined in (1) and (2). As mentioned above, not much seems to be known about ECOMOR. Helbig [16] is one of the first actuaries to investigate some of the strengths and weaknesses of ECOMOR in a Poisson-Pareto setting. Within the framework of premium calculations, we mention Ammeter [1] who develops an approximation for the net risk premium under the Poisson-Pareto assumption. Kremer [17, 18] gives general asymptotic premium formulas and crude upper bounds for the net premium under a Pareto claim size distribution. Kremer [20] allows the claim sizes to be not necessarily independent but he only derives an upper premium bound. Also, Ammeter [1] points out the influence of excluding one or more largest claims on the expected amount of the remaining aggregate claim amount. He makes the important observation that - even within a portfolio with Pareto-distributed claim sizes - there is a preponderance of small claims. Finally, we cite Beirlant and Teugels [3] who give a full description of the asymptotic theory for the quantities $R_r(t)$, assuming that the number r of order statistics increases when t tends to infinity and that the distribution of the claim sizes belongs to the domain of attraction of either the Fréchet or the Gumbel distribution. Concerning LCR, some results have been obtained for premium calculations. Benktander [5] points out that these calculations quickly run into mathematically intractable formulas. In his paper, he also deals with a relation between excess-of-loss and largest claims situations. For the calculations of the net premium, we refer to Berglund [6] and its references. A comparison of the pure premium for the excess-of-loss cover at retention M with that of the largest claims cover at retention r has been investigated by Kupper [21], for the case where the claim size distribution is strict Pareto. Also, we cite Berliner [7] who has considered a set of interesting problems connected to LCR. Under the Poisson-Pareto assumption, he derives the joint distribution of two large claims and computes the covariance between them. He also considers the covariance between the largest claims cover and the total claim amount. Finally, Kremer [18] gives crude upper bounds for the net premium under a Pareto claim size distribution. The asymptotic efficiency of the LCR treaty is discussed in Kremer [19].

Since we pay special attention to large claims, we need to allow the largest claims (or extreme order statistics) to be extremal. We therefore assume that the claim size distribution F belongs to a specific class of distributions with heavy tail character. A first obvious candidate is the extremal class $\mathcal{C}_\gamma(a)$. Alternatively we can take the class $\mathcal{S}$ of subexponential distributions or even the more general class $\mathcal{L}$ of long-tailed distributions. All of these classes are natural candidates for claim size distributions with a heavy tail. The appropriate definitions will be recalled in Section 2.

In Section 3, we state our results for ECOMOR. In Subsection 3.1, we are interested in the *asymptotic equivalence* between the tail of the claim size distribution F and the tail of the distribution of $R_r(t)$ for a fixed $t \geq 0$. Depending on the tail character of F, different kinds of results will show up. In particular, we will head for accurate asymptotic equivalences and asymptotic bounds. In the subexponential case, we will derive a result showing the interplay between the random sum $S_{N(t)}$, the maximum $X^*_{N(t)}$ of the random sample and the quantity $R_r(t)$.

To avoid overloading this subsection with results we have restricted the proof of Theorem 2 to the case $r = 1$. The reason is that this special case amply illustrates how the imposed conditions come into play and this without burdening the proofs with additional technical details. The necessary adaptations of the proof to the case of general r have been combined in Section 6.

A further question is to see what happens with the distribution of the quantity $R_r(t)$ when t tends to infinity. We thus touch on the question of *convergence in distribution* for the random variable $R_r(t)$, i.e. an asymptotic evaluation of $\mathbb{P}[R_r(t) > a(t)s]$ when $t \to \infty$ for an appropriate normalizing quantity $a(t)$. Such expressions are particularly important when one wants to replace the complicated distribution of $R_r(t)$ by a much simpler expression. Subsection 3.2 will be devoted to answering this question for $R_1(t)$. In Subsection 3.3, we will give the limit of the Laplace transform of the normalized random variable $R_r(t)/a(t)$, since no general form is available for $\lim_{t\to\infty} \mathbb{P}[R_r(t) > a(t)s]$. To derive such results we need to assume that the claim size distribution F belongs to the extremal class $\mathcal{C}_\gamma(a)$, $\gamma \in \mathbb{R}$, and that the claim number process $\{N(t);\ t \geq 0\}$ is a mixed Poisson process.

Section 4 is devoted to results connected to LCR. Under the same assumptions as for the ECOMOR-quantity, we deal with the convergence in distribution for the random variable $L_r(t)$ when t tends to infinity. In Subsection 4.1, we will derive the limiting distribution for $L_1(t)$ properly normalized by some functions to be specified. In Subsection 4.2, we will use characteristic functions to give the general form for the weak limit for the appropriately normalized quantity $L_r(t)$.

In Section 5, we turn to a numerical verification of the accuracy of the approximations derived in Subsection 3.2 and Subsection 4.1. We will deal with two different cases depending on whether the claim size distribution is heavy or moderately heavy tailed. Some concluding remarks are given in Section 7.

2 Auxiliaries

We first introduce some useful notation. After recalling the definitions of some important classes of counting processes, we specify the types of claim size distributions that will be used. Finally, we restate a theorem concerning first order results on the extremal class $\mathcal{C}_\gamma(a)$ that will play a substantial role in the convergence questions.

2.1 The Claim Counting Process

For a fixed time $t \geq 0$, we denote by $Q_t(z)$ the probability generating function of the random variable $N(t)$ that counts the number of claims up to this time. It is defined for $|z| \leq 1$ by:

$$Q_t(z) := \mathbb{E}\{z^{N(t)}\} = \sum_{n=0}^{\infty} p_n(t) z^n$$

with $p_n(t) := \mathbb{P}[N(t) = n]$ for all $n \in \mathbb{N}$. By $Q_t^{(m)}(z)$ we denote its partial derivative of order m with respect to z, which is defined for $|z| < 1$. In terms of expectation, this is equal to:

$$Q_t^{(m)}(z) = m!\mathbb{E}\left\{ \binom{N(t)}{m} z^{N(t)-m} \right\}. \tag{3}$$

Recall from classical actuarial tradition that a counting process $\{N(t);\ t \geq 0\}$ is called a *mixed Poisson process* if for each fixed $t \geq 0$, the random variable $N(t)$ has a mixed Poisson distribution, with mixing distribution H of a nonnegative random variable Λ, given by:

$$\mathbb{P}[N(t) = n] = \mathbb{E}\left\{ \frac{(\Lambda t)^n}{n!} e^{-\Lambda t} \right\} = \int_0^\infty \frac{(\lambda t)^n}{n!} e^{-\lambda t} dH(\lambda), \quad n \in \mathbb{N}. \tag{4}$$

For practice, the mixing is explained by the fact that claims come from a heterogeneous group of policyholders. For a general overview on mixed Poisson processes, we refer to the monograph by Grandell [14].

We define the auxiliary quantities $q_m(w) := \mathbb{E}\left\{\Lambda^m e^{-w\Lambda}\right\}$, $m \in \mathbb{N}$, $w \geq 0$. Notice that for all $0 \leq w < t$ and $m \in \mathbb{N}$, we have the identity:

$$\frac{1}{t^m} Q_t^{(m)}(1 - w/t) = q_m(w) \tag{5}$$

where the right hand side does not depend on t. In many of the limiting results below we actually do not need equality in (5). It often suffices to require that:

$$\lim_{t\to\infty} \frac{1}{t^m} Q_t^{(m)}(1 - w/t) = q_m(w)$$

which itself follows from:

$$\frac{N(t)}{t} \xrightarrow{\mathcal{D}} \Lambda \quad \text{as} \quad t \to \infty \tag{6}$$

where $\Lambda \not\equiv 0$ is a nonnegative random variable and $\xrightarrow{\mathcal{D}}$ means convergence in distribution. If (6) holds we will say that the counting process $\{N(t);\ t \geq 0\}$ *averages in time.*

If the distribution H in (4) is degenerate at a single point λ, then we retrieve the (homogeneous) Poisson process with intensity λt. The latter plays a crucial role in applications since it is the most popular among all claim number processes in the actuarial literature. Also, the mixed Poisson process, introduced to actuaries by Dubourdieu [11], has always been very popular among (re)insurance modelers. It has found many applications in (re)insurance mathematics because of its flexibility, its success in actuarial data fitting and its property of being more dispersed than the Poisson process.

2.2 The Claim Size Process

Let us turn to the claim size distribution F. We first introduce the classes of claim size distributions that will be used in what follows.

Definition 1 *A distribution F on $\mathbb{R}$ and satisfying $F(x) < 1$ for all $x \in \mathbb{R}$ belongs to the class $\mathcal{L}$ of long-tailed distributions if for all $y \in \mathbb{R}$:*

$$\lim_{x\to\infty} \frac{1 - F(x+y)}{1 - F(x)} = 1.$$

Definition 2 *A distribution F on $\mathbb{R}_+$ and satisfying $F(x) < 1$ for all $x \geq 0$ belongs to the class $\mathcal{S}$ of subexponential distributions if:*

$$\lim_{x\to\infty} \frac{1 - F^{*2}(x)}{1 - F(x)} = 2$$

where F^{*2} denotes the 2-fold convolution of F with itself.

A distribution that belongs to $\mathcal{S}$ or $\mathcal{L}$ has upper extreme values that are very dominant among sample values. Subexponential distributions are often suggested as appropriate models for heavy-tailed distributions.

The subexponential class $\mathcal{S}$ has been introduced in Chistyakov [9]. It incorporates a wealth of important parametrized families of distributions. The potential role of subexponential distributions within risk or queueing theory was recognized by Pakes [22] and Teugels [25]. For textbook treatments of subexponential distributions, see Embrechts e.a. [12] or Rolski e.a. [23]. For a survey on subexponential distributions, see Goldie and Klüppelberg [13].

It is well known that $\mathcal{S}$ is a proper subset of $\mathcal{L}$ on the positive half-line. The family $\mathcal{L}$ of long-tailed distributions seems to be the largest class of heavy-tailed distributions for which one can derive asymptotic results. A famous subclass of $\mathcal{S}$ is the class of distributions on the positive half-line that have a regularly varying tail, or equivalently that are of Pareto-type, with negative index. Recall that a measurable and ultimately positive function g on $\mathbb{R}$ is regularly varying with index $\alpha \in \mathbb{R}$ (written $g \in \mathrm{RV}_\alpha$) if for all $t > 0$, $\lim_{x\to\infty} g(tx)/g(x) = t^\alpha$. For the latter concept, we refer to Bingham e.a. [8].

In the following, we denote by U the *tail quantile function* of the claim size distribution F which is defined by $U(y) := \inf\{x : F(x) \geq 1 - 1/y\}$. Also, we denote by $U^{\leftarrow}$ the left-continuous inverse function of U defined by $U^{\leftarrow}(x) := \inf\{y : U(y) \geq x\}$.

Definition 3 *A distribution F on $\mathbb{R}$ with tail quantile function U belongs to the extremal class $\mathcal{C}_\gamma(a)$ if there exists a constant $\gamma \in \mathbb{R}$ and an ultimately positive auxiliary function $a(.)$ such that:*

$$\lim_{x\to\infty} \frac{U(ux) - U(x)}{a(x)} = \int_1^u v^{\gamma-1} dv =: h_\gamma(u)$$

for all $u > 0$.

The choice of the extremal class $\mathcal{C}_\gamma(a)$ is not surprising if one knows that $F \in \mathcal{C}_\gamma(a)$ is a necessary and sufficient condition for the convergence in distribution of the properly normalized maximum of a random sample from the distribution F. Therefore, this class of distributions naturally appears when one deals with upper order statistics. Also, the following equivalences hold for any $\gamma > 0$:

$$F \in \mathcal{C}_\gamma(a) \Leftrightarrow 1 - F \in \mathrm{RV}_{-1/\gamma} \Leftrightarrow U \in \mathrm{RV}_\gamma$$

where U is the tail quantile function of the distribution F and $a(x) \sim \gamma U(x)$ as $x \to \infty$.

The proofs of the results concerning convergence in distribution rely heavily on the following theorem that deals with first order aspect of the extremal class $\mathcal{C}_\gamma(a)$.

Theorem 1 *Let $\gamma \in \mathbb{R}$ be a real-valued constant and $a(.)$ an ultimately positive auxiliary function. Then the following assertions are equivalent:*

(i) $F \in \mathcal{C}_\gamma(a)$;

(ii) For all u for which $1 + \gamma u > 0$:

$$\lim_{x\to\infty} \frac{1 - F(x + uh(x))}{1 - F(x)} = \frac{1}{(h_\gamma)^{-1}(u)} =: \eta_\gamma(u)$$

where $h \circ U = a$;

(iii) For all u for which $1 + \gamma u > 0$:

$$\lim_{x\to\infty} x\{1 - F(U(x) + ua(x))\} = \eta_\gamma(u).$$

In the above $\eta_\gamma(u) = (1 + \gamma u)^{-1/\gamma}$ if $\gamma \neq 0$, while $\eta_0(u) = e^{-u}$ and $h_0(u) = \log u$.

This result is due to de Haan [15]. It provides alternative conditions for a distribution F to belong to the extremal class $\mathcal{C}_\gamma(a)$ and offers us the possibility to switch from extremal properties of F to that of its tail quantile function U and conversely. See also Beirlant e.a. [4].

3 Results for ECOMOR

In the first part, we deal with asymptotic relations between the tail of the distribution of the generic claim size X and that of the random variable $R_r(t)$ for a fixed $t \geq 0$. We derive detailed asymptotic equivalences and asymptotic bounds for the different cases when the claim size distribution F belongs to one of the various heavy-tailed classes mentioned in Section 2. We do not make distributional assumptions on the number of claims $N(t)$. In the second part, we deal with convergence in distribution for $R_1(t)$ when $t \to \infty$. We provide a result assuming that F belongs to the extremal class $\mathcal{C}_\gamma(a)$ with $\gamma \in \mathbb{R}$. In the third part, we give the limit of the Laplace transform of the normalized random variable $R_r(t)/a(t)$. In both of these cases we need to make asymptotic assumptions on the claim number process.

3.1 Asymptotic Equivalence and Bounds

We start by deriving a general result that gives a necessary and sufficient condition for the asymptotic relation between the tail of the distribution of the largest claim $X^*_{N(t)}$ of the random sample and that of the generic claim size X.

Lemma 1 *Let F satisfy $F(x) < 1$ for all $x \in \mathbb{R}$ and $t \geq 0$ be fixed. Then:*

$$\left(\mathbb{P}[X^*_{N(t)} > s] \sim \mathbb{E}N(t)\mathbb{P}[X > s] \quad as \quad s \to \infty\right) \Leftrightarrow \mathbb{E}N(t) < \infty.$$

Proof :
Let $t \geq 0$ be fixed. By using $Q_t(z)$, we may rewrite $\mathbb{P}[X^*_{N(t)} > s]$ for each $s \in \mathbb{R}$ as:

$$\mathbb{P}[X^*_{N(t)} > s] = \sum_{n=0}^{\infty}(1 - F^n(s))p_n(t) = 1 - Q_t(F(s)).$$

Then:

$$\lim_{s\to\infty} \frac{\mathbb{P}[X^*_{N(t)} > s]}{\mathbb{P}[X > s]} = \lim_{s\to\infty} \frac{1 - Q_t(F(s))}{1 - F(s)} = \lim_{u\to 1^-} \frac{1 - Q_t(u)}{1 - u} = Q_t^{(1)}(1^-) = \mathbb{E}N(t) \leq \infty.$$

Thus, the left hand term of the claim obviously holds if and only if the expectation of $N(t)$ is finite, and this ends the proof. ■

Now, we give an asymptotic upper bound for the ratio of the tail of the distribution of $R_r(t)$ and that of the generic random variable X.

Lemma 2 *Assume that $F \in \mathcal{S}$ and let $t \geq 0$ and $r \geq 1$ be fixed. Assume further that $Q_t(z)$ is analytic at the point $z = 1$. Then:*

$$\limsup_{s\to\infty} \frac{\mathbb{P}[R_r(t) > s]}{\mathbb{P}[X > s]} \leq \mathbb{E}N(t).$$

Proof :
Let $t \geq 0$ be fixed. Recalling the assumptions, we know that:

$$\mathbb{P}[S_{N(t)} > s] \sim \mathbb{E}N(t)\mathbb{P}[X > s] \quad \text{as} \quad s \to \infty$$

(see for instance Rolski e.a. [23] page 102).

Let $r \geq 1$ be fixed. Since $R_r(t) = \sum_{i=1}^{r} X^*_{N(t)-i+1} - rX^*_{N(t)-r} \leq X^*_{N(t)} + X^*_{N(t)-1} + \cdots + X^*_1 = X_1 + \cdots + X_{N(t)} = S_{N(t)}$, the random variable $R_r(t)$ is bounded above by the random sum $S_{N(t)}$. Thus, the inequality $\mathbb{P}[R_r(t) > s] \leq \mathbb{P}[S_{N(t)} > s]$ holds for each $s \geq 0$ and the proof follows easily from the asymptotic equivalence given above. ■

A neat connection between factorial moments of the claim counting process and integrals of its generating function is given in the next auxiliary result.

Proposition 1 *Let $t \geq 0$, $\alpha \geq 0$ and $m \in \mathbb{N}\backslash\{0\}$ be fixed. Then:*

$$\int_0^1 Q_t^{(m)}(z) z^{\alpha} dz = \mathbb{E}\left\{\frac{N(t)!}{(N(t) - m)!(N(t) - m + \alpha + 1)}\right\}.$$

Moreover, both sides are finite if $\mathbb{E}\left\{N(t)^{m-1}\right\} < \infty$.

Proof :
Let $t \geq 0$, $\alpha \geq 0$ and $m \in \mathbb{N}\backslash\{0\}$ be fixed. From (3), we get by an application of Fubini's theorem:

$$\begin{aligned}
\int_0^1 Q_t^{(m)}(z) z^{\alpha} dz &= m! \int_0^1 \mathbb{E}\left\{\binom{N(t)}{m} z^{N(t)-m}\right\} z^{\alpha} dz \\
&= m!\mathbb{E}\left\{\binom{N(t)}{m} \int_0^1 z^{N(t)-m+\alpha} dz\right\} \\
&= m!\mathbb{E}\left\{\binom{N(t)}{m} \frac{1}{N(t) - m + \alpha + 1}\right\}
\end{aligned}$$

$$= \quad \mathbb{E}\left\{\frac{N(t)!}{(N(t)-m)!(N(t)-m+\alpha+1)}\right\}.$$

The random variable $N(t)!\{(N(t)-m)!(N(t)-m+\alpha+1)\}^{-1}$ is bounded above by $N(t)^{m-1}$. Hence, its expectation is finite if we assume $\mathbb{E}\left\{N(t)^{m-1}\right\} < \infty$, and the proof is finished. ■

Here is the first of our main results.

Theorem 2 *Assume that $F \in \mathcal{L}$ and let $t \geq 0$ and $r \geq 1$ be fixed. Assume further that $Q_t^{(r)}(1) < \infty$. Then:*

$$\liminf_{s\to\infty} \frac{\mathbb{P}[R_r(t) > s]}{\mathbb{P}[X > s]} \geq \mathbb{E}N(t).$$

Proof :
Here we restrict the proof to the case $r = 1$. The proof for the general case $r \geq 1$ is postponed to Section 6.

Let $t \geq 0$ and $y \in \mathbb{R}$ be fixed. For each $s \geq 0$, we have:

$$\begin{aligned}\mathbb{P}[R_1(t) > s] &\geq \mathbb{P}[X^*_{N(t)} > s+y, R_1(t) > s] \\ &= \mathbb{P}[X^*_{N(t)} > s+y] - \mathbb{P}[X^*_{N(t)} > s+y, R_1(t) \leq s].\end{aligned}$$

Consider the second term on the right-hand side. We get:

$$\begin{aligned}\mathbb{P}[X^*_{N(t)} > s+y, R_1(t) \leq s] &= \mathbb{E}\left\{N(t)(N(t)-1)\int_{s+y}^{\infty}\int_{x_1-s}^{x_1} F^{N(t)-2}(x_2)dF(x_2)dF(x_1)\right\} \\ &= \mathbb{E}\left\{N(t)\int_{s+y}^{\infty}\left[F^{N(t)-1}(x_1) - F^{N(t)-1}(x_1-s)\right]dF(x_1)\right\} \\ &\leq \mathbb{E}\left\{N(t)\int_{s+y}^{\infty}\left[1 - F^{N(t)-1}(x_1-s)\right]dF(x_1)\right\} \\ &\leq \left[1-F(s+y)\right]\mathbb{E}\left\{N(t)\left[1-F^{N(t)-1}(y)\right]\right\}.\end{aligned}$$

Hence, since $F \in \mathcal{L}$, we get the following inequality:

$$\begin{aligned}\limsup_{s\to\infty}\frac{\mathbb{P}[X^*_{N(t)} > s+y, R_1(t) \leq s]}{\mathbb{E}N(t)\mathbb{P}[X > s]} &\leq \limsup_{s\to\infty}\frac{1-F(s+y)}{1-F(s)}\frac{\mathbb{E}\left\{N(t)\left[1-F^{N(t)-1}(y)\right]\right\}}{\mathbb{E}N(t)} \\ &= \frac{\mathbb{E}\left\{N(t)\left[1-F^{N(t)-1}(y)\right]\right\}}{\mathbb{E}N(t)}.\end{aligned}$$

Therefore, we get:

$$\begin{aligned}\liminf_{s\to\infty}\frac{\mathbb{P}[R_1(t) > s]}{\mathbb{E}N(t)\mathbb{P}[X > s]} &\geq \liminf_{s\to\infty}\left\{\frac{\mathbb{P}[X^*_{N(t)} > s+y]}{\mathbb{E}N(t)\mathbb{P}[X > s]} - \frac{\mathbb{P}[X^*_{N(t)} > s+y, R_1(t) \leq s]}{\mathbb{E}N(t)\mathbb{P}[X > s]}\right\} \\ &\geq \liminf_{s\to\infty}\frac{\mathbb{P}[X^*_{N(t)} > s+y]}{\mathbb{E}N(t)\mathbb{P}[X > s]} + \liminf_{s\to\infty}\left\{-\frac{\mathbb{P}[X^*_{N(t)} > s+y, R_1(t) \leq s]}{\mathbb{E}N(t)\mathbb{P}[X > s]}\right\} \\ &= \liminf_{s\to\infty}\frac{\mathbb{P}[X^*_{N(t)} > s+y]}{\mathbb{E}N(t)\mathbb{P}[X > s]} - \limsup_{s\to\infty}\frac{\mathbb{P}[X^*_{N(t)} > s+y, R_1(t) \leq s]}{\mathbb{E}N(t)\mathbb{P}[X > s]} \\ &\geq \liminf_{s\to\infty}\frac{\mathbb{P}[X^*_{N(t)} > s+y]}{\mathbb{E}N(t)\mathbb{P}[X > s]} - \frac{\mathbb{E}\left\{N(t)\left[1-F^{N(t)-1}(y)\right]\right\}}{\mathbb{E}N(t)}.\end{aligned}$$

Considering the first term on the right-hand side, we get:

$$\liminf_{s\to\infty}\frac{\mathbb{P}[X^*_{N(t)} > s+y]}{\mathbb{E}N(t)\mathbb{P}[X > s]} = \liminf_{s\to\infty}\frac{\mathbb{P}[X^*_{N(t)} > s+y]}{\mathbb{E}N(t)\mathbb{P}[X > s+y]}\frac{\mathbb{P}[X > s+y]}{\mathbb{P}[X > s]}$$

$$= \lim_{s\to\infty} \frac{1-F(s+y)}{1-F(s)} = 1$$

by Lemma 1 and the assumption $F \in \mathcal{L}$. Therefore, we obtain:

$$\liminf_{s\to\infty} \frac{\mathbb{P}[R_1(t) > s]}{\mathbb{E}N(t)\mathbb{P}[X > s]} \quad \geq \quad 1 - \frac{\mathbb{E}\left\{N(t)\left[1-F^{N(t)-1}(y)\right]\right\}}{\mathbb{E}N(t)}.$$

Thus, if we take the limit as $y \to \infty$ on both sides, applying monotone convergence theorem, the claim of the theorem is proved.

Indeed, we prove that the second term on the right-hand side goes to 0 when $y \to \infty$. The random variable $N(t)\left[1-F^{N(t)-1}(y)\right]$ is bounded above by $N(t)$ for all $y \in \mathbb{R}$ and is monotone decreasing in y, converging to 0 as $y \to \infty$. Also, $\mathbb{E}N(t) = Q_t^{(1)}(1) < \infty$ by assumption. Hence, applying monotone convergence theorem, we deduce that the second term on the right-hand side goes to 0 when $y \to \infty$. ∎

Now, we derive an asymptotic equivalence between the tail of the distribution of the quantity $R_1(t)$ and that of the generic claim size X under the long-tailed assumption.

Theorem 3 *Assume that $F \in \mathcal{L}$ is lying on $\mathbb{R}_+$ and let $t \geq 0$ be fixed. Assume further that $\mathbb{E}N(t) < \infty$. Then:*

$$\mathbb{P}[R_1(t) > s] \sim \mathbb{E}N(t)\mathbb{P}[X > s] \quad \textit{as} \quad s \to \infty.$$

Proof :
Let $t \geq 0$ be fixed. For each $s \geq 0$, we easily compute:

$$\frac{\mathbb{P}[R_1(t) > s]}{\mathbb{P}[X > s]} = \int_0^\infty Q_t^{(2)}(F(y))\frac{1-F(s+y)}{1-F(s)}dF(y).$$

We have $Q_t^{(2)}(F(y))\frac{1-F(s+y)}{1-F(s)} \leq Q_t^{(2)}(F(y))$ for all $s \geq 0$, and $\int_0^\infty Q_t^{(2)}(F(y))dF(y) = \int_0^1 Q_t^{(2)}(z)dz = \mathbb{E}N(t) < \infty$ by Proposition 1. Also, $Q_t^{(2)}(F(y))\frac{1-F(s+y)}{1-F(s)} \to Q_t^{(2)}(F(y))$ as $s \to \infty$ pointwise for all $y \geq 0$ since $F \in \mathcal{L}$. Thus, applying Lebesgue's theorem on dominated convergence, we get:

$$\lim_{s\to\infty} \frac{\mathbb{P}[R_1(t) > s]}{\mathbb{P}[X > s]} = \int_0^\infty Q_t^{(2)}(F(y))dF(y) = \mathbb{E}N(t)$$

which ends the proof. ∎

As a consequence of the previous claims, we get the following results for the subexponential class $\mathcal{S}$. The first two results deal with asymptotic equivalences and the last result shows the interplay between the quantity $R_r(t)$, the maximum $X^*_{N(t)}$ of the random sample and the random sum $S_{N(t)}$.

Corollary 1 *Assume that $F \in \mathcal{S}$ and let $t \geq 0$ be fixed.*

(i) If $\mathbb{E}N(t) < \infty$, then:

$$\mathbb{P}[R_1(t) > s] \sim \mathbb{E}N(t)\mathbb{P}[X > s] \quad \textit{as} \quad s \to \infty.$$

(ii) If $Q_t(z)$ is analytic at the point $z = 1$, then for every fixed $r \geq 2$:

$$\mathbb{P}[R_r(t) > s] \sim \mathbb{E}N(t)\mathbb{P}[X > s] \quad \textit{as} \quad s \to \infty.$$

(iii) If $Q_t(z)$ is analytic at the point $z = 1$, then for every fixed $r \geq 1$:

$$\mathbb{P}[R_r(t) > s] \sim \mathbb{P}[X^*_{N(t)} > s] \sim \mathbb{P}[S_{N(t)} > s] \sim \mathbb{E}N(t)\mathbb{P}[X > s] \quad \textit{as} \quad s \to \infty.$$

Proof :
Let $t \geq 0$ be fixed. Recall that $\mathcal{S}$ is a proper subset of $\mathcal{L}$ on the positive half-line.

(*i*) Consequence of Theorem 3.

(*ii*) By using Theorem 2 and Lemma 2 for a fixed $r \geq 2$, we get:

$$\mathbb{E}N(t) \leq \liminf_{s\to\infty} \frac{\mathbb{P}[R_r(t) > s]}{\mathbb{P}[X > s]} \leq \limsup_{s\to\infty} \frac{\mathbb{P}[R_r(t) > s]}{\mathbb{P}[X > s]} \leq \mathbb{E}N(t)$$

and the claim follows easily.

(*iii*) By using (*i*) for the case $r = 1$, (*ii*) for the case $r \geq 2$, Lemma 1 and the assumptions (as in the proof of Lemma 2). ■

3.2 Convergence in Distribution for $R_1(t)$

In this subsection, we deal with the asymptotic behavior of $\mathbb{P}[R_1(t) > a(t)s]$ for an appropriate norming function $a(t)$ when $t \to \infty$. To get such results, we base our approach on Theorem 1 given in Section 2. It will be natural to assume that the claim size distribution F belongs to the extremal class $\mathcal{C}_\gamma(a)$ with $\gamma \in \mathbb{R}$. Indeed, this assumption is crucial in dealing with the large order statistics when $N(t)$ is deterministic. In addition, we need to make an appropriate assumption on the claim number process $\{N(t);\ t \geq 0\}$.

Proposition 2 *Assume that $F \in \mathcal{C}_\gamma(a)$ with $\gamma \in \mathbb{R}$ and that $\{N(t);\ t \geq 0\}$ is a mixed Poisson process. Then:*

$$\lim_{t\to\infty} \mathbb{P}[R_1(t) > a(t)s] = \int_0^\infty wq_2(w)\eta_\gamma(sw^\gamma)dw =: I_\gamma(s), \quad s \geq 0$$

provided $1 + \gamma sw^\gamma > 0$. Specifically:

(*i*) $\gamma = 0$: $I_0(s) = e^{-s}$;

(*ii*) $\gamma > 0$: $I_\gamma(s) = \int_0^\infty wq_2(w)(1 + \gamma sw^\gamma)^{-1/\gamma}dw$;

(*iii*) $\gamma < 0$: $I_\gamma(s) = \int_{(-\gamma s)^{-1/\gamma}}^\infty wq_2(w)(1 + \gamma sw^\gamma)^{-1/\gamma}dw$.

Proof :
Let $\gamma \in \mathbb{R}$ and $s \geq 0$ be fixed. For each $t \geq 0$, we easily compute:

$$\begin{aligned}
\mathbb{P}[R_1(t) > a(t)s] &= \int_{-\infty}^\infty Q_t^{(2)}(F(y))(1 - F(y + a(t)s))dF(y) \\
&= \int_0^\infty w\frac{Q_t^{(2)}(1 - w/t)}{t^2}\frac{t}{w}\left\{1 - F\left(U(t/w) + a(t/w)\frac{a(t)}{a(t/w)}s\right)\right\}1_{[0,t[}(w)dw \\
&= \int_0^\infty wq_2(w)\frac{t}{w}\left\{1 - F\left(U(t/w) + a(t/w)\frac{a(t)}{a(t/w)}s\right)\right\}1_{[0,t[}(w)dw.
\end{aligned}$$

Define $f_{t,w}(x) := \frac{t}{w}\{1 - F(U(t/w) + a(t/w)x)\}$ and $g_t(w) := \frac{a(t)}{a(t/w)}s$. We know that $f_{t,w}(x) \to \eta_\gamma(x) =: f(x)$ as $t \to \infty$ pointwise for all $1 + \gamma x > 0$, and we get $g_t(w) \to sw^\gamma =: g(w)$ as $t \to \infty$ pointwise for all $w \geq 0$.

We have $|wq_2(w)f_{t,w}(g_t(w))1_{[0,t[}(w)| \leq wq_2(w)$ for all $t \geq 0$, and $\int_0^\infty wq_2(w)dw = 1 < \infty$. We have to check $f_{t,w}(g_t(w)) \to f(g(w))$ as $t \to \infty$ pointwise for all w such that $1 + \gamma sw^\gamma > 0$.

Fix $w \geq 0$ such that $1 + \gamma sw^\gamma > 0$ and write the following triangular inequality:

$$|f_{t,w}(g_t(w)) - f(g(w))| \leq |f_{t,w}(g_t(w)) - f(g_t(w))| + |f(g_t(w)) - f(g(w))|.$$

First, we get $|f(g_t(w)) - f(g(w))| \to 0$ as $t \to \infty$, since f is continuous and $g_t(w) \to g(w)$ as $t \to \infty$. Now, for t large enough, there exists reals a, b with $-1/\max(0,\gamma) < a < g(w) < b < 1/\max(-\gamma, 0)$ such that $g_t(w) \in [a,b]$. Then, $|f_{t,w}(g_t(w)) - f(g_t(w))| \to 0$ as $t \to \infty$ iff $\lim_{t\to\infty} \sup_{x\in[a,b]} |f_{t,w}(x) - f(x)| = 0$ iff $\lim_{t\to\infty} \sup_{x\in[a,b]} |f_{t,1}(x) - f(x)| = 0$. The last equivalence is true since Theorem 1 (iii) holds locally uniformly for $x \in]-1/\max(0,\gamma), 1/\max(-\gamma,0)[$.

Thus, by Lebesgue's theorem on dominated convergence, we get:

$$\lim_{t\to\infty} \mathbb{P}[R_1(t) > a(t)s] = \int_0^\infty wq_2(w)\eta_\gamma(sw^\gamma)dw$$

provided $1 + \gamma sw^\gamma > 0$. Specifically:

(i) $\gamma = 0$: $\eta_0(s) = e^{-s}$ and $e^{-s}\int_0^\infty wq_2(w)dw = e^{-s}$;

(ii) $\gamma > 0$: $\eta_\gamma(sw^\gamma) = (1 + \gamma sw^\gamma)^{-1/\gamma}$ and $1 + \gamma sw^\gamma$ is positive for all $w \geq 0$;

(iii) $\gamma < 0$: $\eta_\gamma(sw^\gamma) = (1 + \gamma sw^\gamma)^{-1/\gamma}$ and $1 + \gamma sw^\gamma > 0 \Leftrightarrow w > (-\gamma s)^{-1/\gamma}$. ∎

The appearance of the exponential distribution in the case where $\gamma = 0$ is pleasing. For the other values of γ no simplification seems to be possible.

3.3 Weak Limit for $R_r(t)$

In the following result, we derive the general form for the weak limit for the random variable $R_r(t)$ normalized by the auxiliary function $a(t)$ from the domain of extremal attraction when $t \to \infty$. Note that the very definition of $R_r(t)$ makes a further centering unnecessary. Also, the normalized ECOMOR-quantity being always nonnegative, we use the concept of Laplace transform.

Theorem 4 *Let $r \geq 1$ be fixed. Assume that $F \in \mathcal{C}_\gamma(a)$ with $\gamma \in \mathbb{R}$ and that $\{N(t);\ t \geq 0\}$ is a mixed Poisson process. Then:*

$$\lim_{t\to\infty} \mathbb{E}\left\{\exp\left(-\theta\frac{R_r(t)}{a(t)}\right)\right\} = \frac{1}{r!}\int_0^\infty w^r q_{r+1}(w)\left(\int_0^1 e^{-\theta w^{-\gamma}h_\gamma(1/z)}dz\right)^r dw, \quad \theta \geq 0.$$

<u>*Proof*</u> :
Let $\gamma \in \mathbb{R}$, $r \geq 1$ and $\theta \geq 0$ be fixed.

$$\begin{aligned}
\lim_{t\to\infty} \mathbb{E}\left\{\exp\left(-\theta\frac{R_r(t)}{a(t)}\right)\right\} &= \lim_{t\to\infty}\frac{1}{r!}\int_{-\infty}^\infty Q_t^{(r+1)}(F(y))\left(\int_y^\infty e^{-\theta\{z-y\}/a(t)}dF(z)\right)^r dF(y)\\
&= \lim_{t\to\infty}\frac{1}{r!}\int_0^\infty \frac{Q_t^{(r+1)}(1-w/t)}{t^{r+1}}\left(\int_0^w e^{-\theta\{U(t/x)-U(t/w)\}/a(t)}dx\right)^r 1_{[0,t[}(w)dw\\
&= \lim_{t\to\infty}\frac{1}{r!}\int_0^\infty q_{r+1}(w)\left(\int_0^w e^{-\theta\{U(t/x)-U(t/w)\}/a(t)}dx\right)^r 1_{[0,t[}(w)dw.
\end{aligned}$$

Define $f_t(w) := q_{r+1}(w)\left(\int_0^w e^{-\theta\{U(t/x)-U(t/w)\}/a(t)}dx\right)^r 1_{[0,t[}(w)$ and $g_t(x) := e^{-\theta\{U(t/x)-U(t/w)\}/a(t)}$.

Suppose that there exists a function f such that $f_t(w) \to f(w)$ as $t \to \infty$ pointwise for all $w \geq 0$. We have $|f_t(w)| \leq w^r q_{r+1}(w)$ for all $t \geq 0$ (since $U(t/x) - U(t/w) \geq 0$ for all $0 \leq x \leq w < t$) and $\int_0^\infty w^r q_{r+1}(w)dw = r! < \infty$. Thus, by Lebesgue's theorem on dominated convergence, we get:

$$\lim_{t\to\infty}\frac{1}{r!}\int_0^\infty f_t(w)dw = \frac{1}{r!}\int_0^\infty f(w)dw.$$

We then have to check that $f_t(w) \to f(w)$ as $t \to \infty$ pointwise for all $w \geq 0$. Let w be fixed with $0 \leq x \leq w < t$. The functions g_t are uniformly bounded by 1 on $[0, w]$, and $\int_0^w dx = w < \infty$. Also, $g_t(x) \to e^{-\theta\{h_\gamma(1/x)-h_\gamma(1/w)\}} =: g(x)$ as $t \to \infty$ pointwise for all $x \in [0, w]$. Hence, by bounded convergence theorem, we get:

$$\lim_{t\to\infty} \int_0^w g_t(x)dx = \int_0^w g(x)dx$$

and we thus obtain:

$$\lim_{t\to\infty} f_t(w) = q_{r+1}(w) \left(\int_0^w g(x)dx\right)^r =: f(w).$$

Therefore, we get:

$$\begin{aligned}\lim_{t\to\infty} \mathbb{E}\left\{\exp\left(-\theta\frac{R_r(t)}{a(t)}\right)\right\} &= \frac{1}{r!}\int_0^\infty q_{r+1}(w)\left(\int_0^w e^{-\theta\{h_\gamma(1/x)-h_\gamma(1/w)\}}dx\right)^r dw \\ &= \frac{1}{r!}\int_0^\infty w^r q_{r+1}(w)\left(\int_0^1 e^{-\theta w^{-\gamma}h_\gamma(1/z)}dz\right)^r dw =: \varphi_r(\theta).\end{aligned}$$

Also, for each $\theta \geq 0$, we have:

$$\varphi_r(\theta) \leq \frac{1}{r!}\int_0^\infty w^r q_{r+1}(w)dw = 1 < \infty.$$

Finally, since the functions $k_\theta(z) := e^{-\theta w^{-\gamma}h_\gamma(1/z)}$ are uniformly bounded by 1 on $[0, 1]$ for each fixed $w \geq 0$, it is easy to prove that $\varphi_r(\theta) \to 1$ as $\theta \to 0$, using bounded convergence theorem and Lebesgue's theorem on dominated convergence. ■

We point out that to prove Theorem 4 we now use Definition 3 rather than Theorem 1 (*iii*) to prove Proposition 2. The limiting distribution given in the case $r = 1$ by Proposition 2 also follows from Theorem 4, but no inversion of the Laplace transform seems possible for general $r \geq 2$.

Special cases Two values of γ seem to give something special.

(*i*) If $\gamma = 0$, then we get a simple expression in that then:

$$\lim_{t\to\infty} \mathbb{E}\left\{\exp\left(-\theta\frac{R_r(t)}{a(t)}\right)\right\} = \frac{1}{(1+\theta)^r}, \quad \theta \geq 0$$

which can be directly interpreted as the Laplace transform of a gamma distribution $\Gamma(r, 1)$. This can also be written as:

$$\lim_{t\to\infty} \mathbb{P}[R_r(t) > a(t)s] = \frac{1}{(r-1)!}\int_s^\infty e^{-x}x^{r-1}dx, \quad s \geq 0.$$

The difficult distribution of $R_r(t)$ can then be approximated for $y \geq 0$ as:

$$\mathbb{P}[R_r(t) > y] \sim \frac{1}{(r-1)!}\int_{y/a(t)}^\infty e^{-x}x^{r-1}dx \quad \text{as} \quad t \to \infty.$$

The case $\gamma = 0$ applies to a very wide class of distributions like exponential, gamma, normal and lognormal. As an illustration, we specify in Table 1 the expression of the auxiliary function $a(t)$ associated with each of these distributions. This then permits us to approximate $\mathbb{P}[R_r(t) > y]$, for large t, using the relation given above.

(*ii*) If $\gamma = -1$, as with the uniform distribution on $[0, 1]$, a simple calculation yields:

$$\lim_{t\to\infty} \mathbb{E}\left\{\exp\left(-\theta\frac{R_r(t)}{a(t)}\right)\right\} = \mathbb{E}\left\{\prod_{j=0}^r \frac{\Lambda}{\Lambda + j\theta}\right\}, \quad \theta \geq 0.$$

Distribution	Function $a(t)$
$\text{Exp}(\lambda)$, $\lambda > 0$	λ^{-1}
$\Gamma(\alpha, \lambda)$, $\alpha, \lambda > 0$	λ^{-1}
$\text{N}(\mu, \sigma^2)$, $\mu \in \mathbb{R}$, $\sigma > 0$	$\sigma(2\ln t - \ln\ln t - \ln 4\pi)^{-1/2}$
$\text{LN}(\mu, \sigma^2)$, $\mu \in \mathbb{R}$, $\sigma > 0$	$\sigma(2\ln t - \ln\ln t - \ln 4\pi)^{-1/2}\exp\{\mu + \sigma b(t)\}$ with $b(t) = (2\ln t)^{1/2} - \frac{\ln\ln t + \ln 4\pi}{2(2\ln t)^{1/2}}$

Table 1: Auxiliary functions $a(t)$ for some distributions with $\gamma = 0$.

Proof :
Set $\gamma = -1$ and let $r \geq 1$ and $\theta \geq 0$ be fixed.

$$\begin{aligned}
\lim_{t\to\infty} \mathbb{E}\left\{\exp\left(-\theta\frac{R_r(t)}{a(t)}\right)\right\} &= \frac{1}{r!}\int_0^\infty w^r q_{r+1}(w)\left(\int_0^1 e^{-\theta(w-wz)}dz\right)^r dw \\
&= \frac{1}{r!}\int_0^\infty w^r \int_0^\infty \lambda^{r+1}e^{-\lambda w}dH(\lambda)\left(e^{-\theta w}\frac{e^{\theta w}-1}{\theta w}\right)^r dw \\
&= \frac{1}{r!\theta^r}\int_0^\infty \lambda^{r+1}\int_0^\infty e^{-\lambda w}\left(1-e^{-\theta w}\right)^r dw dH(\lambda).
\end{aligned}$$

Now, using the beta function B, we get:

$$\begin{aligned}
\int_0^\infty e^{-\lambda w}\left(1-e^{-\theta w}\right)^r dw &= \frac{1}{\theta}\int_0^1 x^r(1-x)^{\lambda/\theta-1}dx \\
&= \frac{1}{\theta}\,B\left(r+1, \lambda/\theta\right) = \frac{1}{\theta}\frac{r\theta}{\lambda + r\theta}\,B\left(r, \lambda/\theta\right) \\
&\vdots \\
&= \frac{1}{\theta}\frac{r\theta}{\lambda+r\theta}\frac{(r-1)\theta}{\lambda+(r-1)\theta}\cdots\frac{\theta}{\lambda+\theta}\frac{\theta}{\lambda} \\
&= \frac{r!\theta^r}{\prod_{j=0}^r(\lambda+j\theta)}.
\end{aligned}$$

Thus, we deduce:

$$\lim_{t\to\infty} \mathbb{E}\left\{\exp\left(-\theta\frac{R_r(t)}{a(t)}\right)\right\} = \int_0^\infty \frac{\lambda^{r+1}}{\prod_{j=0}^r \lambda + j\theta}dH(\lambda) = \mathbb{E}\left\{\prod_{j=0}^r \frac{\Lambda}{\Lambda + j\theta}\right\}. \blacksquare$$

In particular, for degenerate Λ the limit distribution is a product of independent exponentials.

From Theorem 4, we can derive the expressions of the first few moments. For instance, we get for the mean:

$$\lim_{t\to\infty}\frac{\mathbb{E}R_r(t)}{a(t)} = \frac{1}{(r-1)!(1-\gamma)}\int_0^\infty w^{r-\gamma}q_{r+1}(w)dw = \frac{\Gamma(r-\gamma+1)}{(r-1)!(1-\gamma)}\mathbb{E}\{\Lambda^\gamma\}$$

under the condition that $\gamma < 1$, where Γ denotes the gamma function. Similarly for the moment of second order, we get:

$$\begin{aligned}
\lim_{t\to\infty}\frac{\mathbb{E}\{R_r(t)^2\}}{a^2(t)} &= \frac{1+r(1-2\gamma)}{(r-1)!(1-\gamma)^2(1-2\gamma)}\int_0^\infty w^{r-2\gamma}q_{r+1}(w)dw \\
&= \frac{\{1+r(1-2\gamma)\}\Gamma(r-2\gamma+1)}{(r-1)!(1-\gamma)^2(1-2\gamma)}\mathbb{E}\{\Lambda^{2\gamma}\}
\end{aligned}$$

where we need to assume that $\gamma < 1/2$. Rewriting the moments in terms of the structure variable Λ permits us to illustrate the role played by the mixing random variable.

4 Results for LCR

We deal with convergence in distribution of the random variable $L_r(t)$ assuming that the claim size distribution F belongs to the extremal class $\mathcal{C}_\gamma(a)$ with $\gamma \in \mathbb{R}$, and that the claim number process is mixed Poisson. In the first part, we derive the limiting distribution of $L_1(t)$ that is normalized by some functions. In the second part, we give the general form for the weak limit for the appropriately normalized quantity $L_r(t)$.

4.1 Convergence in Distribution for $L_1(t)$

In the following proposition, we derive the limiting distribution of $L_1(t)$ properly normalized. Contrary to the ECOMOR case, we need here a centering function.

Proposition 3 *Assume that $F \in \mathcal{C}_\gamma(a)$ with $\gamma \in \mathbb{R}$ and that $\{N(t);\ t \geq 0\}$ is a mixed Poisson process. Then:*

$$\lim_{t\to\infty} \mathbb{P}[L_1(t) - c(t) > d(t)s] = \int_0^{\varphi(s)} q_1(w)dw =: J_\gamma(s)$$

where one of the three following cases emerges necessarily:

(i) $\gamma = 0$: $c(t) = U(t)$, $d(t) = a(t)$ *and* $\varphi(s) = e^{-s}$, $s \in \mathbb{R}$;

(ii) $\gamma > 0$: $c(t) = 0$, $d(t) = U(t)$ *and* $\varphi(s) = s^{-1/\gamma}$, $s > 0$;

(iii) $\gamma < 0$: $c(t) = x_+ := U(\infty)$, $d(t) = x_+ - U(t)$ *and* $\varphi(s) = |s|^{1/|\gamma|}$, $s \leq 0$.

Proof :
Let $\gamma \in \mathbb{R}$ and $x \in \mathbb{R}$ be fixed. For each $t \geq 0$, we easily compute:

$$\begin{aligned}\mathbb{P}[L_1(t) - U(t) \leq a(t)x] &= p_0(t) + \int_0^\infty \frac{Q_t^{(1)}(1-w/t)}{t} 1_{[\psi_t(x),t[}(w)dw \\ &= p_0(t) + \int_0^\infty q_1(w) 1_{[\psi_t(x),t[}(w)dw\end{aligned}$$

where we define $\psi_t(x) := t\{1 - F(U(t) + a(t)x)\}$.

We have $q_1(w)1_{[\psi_t(x),t[}(w) \leq q_1(w)$ for all $t \geq 0$, and $\int_0^\infty q_1(w)dw < \infty$ by Proposition 1. Moreover, $q_1(w)1_{[\psi_t(x),t[}(w) \to q_1(w)1_{[\eta_\gamma(x),\infty[}(w)$ as $t \to \infty$ pointwise all $w \geq 0$ by Theorem 1. Therefore, since $p_0(t) \to 0$ as $t \to \infty$ and by applying Lebesgue's theorem on dominated convergence, we get:

$$\lim_{t\to\infty} \mathbb{P}[L_1(t) - U(t) \leq a(t)x] = \int_{\eta_\gamma(x)}^\infty q_1(w)dw$$

for all x such that $1 + \gamma x > 0$. Since $\int_0^\infty q_1(w)dw = 1$, we get:

$$\lim_{t\to\infty} \mathbb{P}[L_1(t) - U(t) > a(t)x] = \int_0^{\eta_\gamma(x)} q_1(w)dw.$$

To arrive at the required statement, we replace $a(t)/U(t)$ by its limit γ when $\gamma > 0$ or $a(t)/(x_+ - U(t))$ by its limit $-\gamma$ when $\gamma < 0$. An affine transformation depending on the value of γ then suffices. ∎

Remark that for deterministic $N(t)$ the above proposition yields the classical limit laws for the maximum of a sample. In this form, the result is a special case of a more general weak convergence result in Silvestrov and Teugels [24] where even the independence condition between claim counting and claim size processes is weakened considerably.

4.2 Weak Limit for $L_r(t)$

In the following result, we derive the general form for the limit in distribution for the appropriately normalized random variable $L_r(t)$ when $t \to \infty$. Characteristic functions are used here instead of Laplace transforms since the normalized LCR-quantity may assume negative values.

Theorem 5 *Let $r \geq 1$ be fixed. Assume that $F \in \mathcal{C}_\gamma(a)$ with $\gamma \in \mathbb{R}$ and that $\{N(t);\ t \geq 0\}$ is a mixed Poisson process. Then:*

$$\lim_{t\to\infty} \mathbb{E}\left\{\exp\left(i\theta \frac{L_r(t) - rU(t)}{a(t)}\right)\right\} = \frac{1}{r!}\int_0^\infty q_{r+1}(w)\left(\int_0^w e^{i\theta h_\gamma(1/z)}dz\right)^r dw, \quad \theta \in \mathbb{R}.$$

<u>*Proof* :</u>
Let $\gamma \in \mathbb{R}$, $r \geq 1$ and $\theta \in \mathbb{R}$ be fixed.

$$\begin{aligned}
\lim_{t\to\infty} \mathbb{E}\left\{\exp\left(i\theta \frac{L_r(t) - rU(t)}{a(t)}\right)\right\} &= \lim_{t\to\infty} \frac{1}{r!}\int_{-\infty}^\infty Q_t^{(r+1)}(F(y))\left(\int_y^\infty e^{i\theta\{x-U(t)\}/a(t)}dF(x)\right)^r dF(y) \\
&= \lim_{t\to\infty} \frac{1}{r!}\int_0^\infty \frac{Q_t^{(r+1)}(1-w/t)}{t^{r+1}}\left(\int_0^w e^{i\theta\{U(t/z)-U(t)\}/a(t)}dz\right)^r 1_{[0,t[}(w)dw \\
&= \lim_{t\to\infty} \frac{1}{r!}\int_0^\infty q_{r+1}(w)\left(\int_0^w e^{i\theta\{U(t/z)-U(t)\}/a(t)}dz\right)^r 1_{[0,t[}(w)dw.
\end{aligned}$$

Define $f_t(w) := q_{r+1}(w)\left(\int_0^w e^{i\theta\{U(t/z)-U(t)\}/a(t)}dz\right)^r 1_{[0,t[}(w)$ and $g_t(z) := e^{i\theta\{U(t/z)-U(t)\}/a(t)}$.

Suppose that there exists a function f such that $f_t(w) \to f(w)$ as $t \to \infty$ pointwise for all $w \geq 0$. We have $|f_t(w)| \leq w^r q_{r+1}(w)$ for all $t \geq 0$ and also $\int_0^\infty w^r q_{r+1}(w)dw = r! < \infty$. Thus, by Lebesgue's theorem on dominated convergence, we get:

$$\lim_{t\to\infty} \frac{1}{r!}\int_0^\infty f_t(w)dw = \frac{1}{r!}\int_0^\infty f(w)dw.$$

We then have to check that $f_t(w) \to f(w)$ as $t \to \infty$ pointwise for all $w \geq 0$. Let w be fixed with $0 \leq z \leq w < t$. We have $g_t(z) \to e^{i\theta h_\gamma(1/z)} =: g(z)$ as $t \to \infty$ pointwise for all $z \in [0, w]$. Also, the functions g_t are uniformly bounded by 1 on $[0, w]$ and $\int_0^w dz = w < \infty$. Applying bounded convergence theorem, we get:

$$\lim_{t\to\infty}\int_0^w g_t(z)dz = \int_0^w g(z)dz$$

and we thus obtain:

$$\lim_{t\to\infty} f_t(w) = q_{r+1}(w)\left(\int_0^w g(z)dz\right)^r =: f(w).$$

Therefore, we get:

$$\lim_{t\to\infty} \mathbb{E}\left\{\exp\left(i\theta \frac{L_r(t) - rU(t)}{a(t)}\right)\right\} = \frac{1}{r!}\int_0^\infty q_{r+1}(w)\left(\int_0^w e^{i\theta h_\gamma(1/z)}dz\right)^r dw =: \varphi_r(\theta).$$

Also, for each $\theta \in \mathbb{R}$, we have:

$$|\varphi_r(\theta)| \leq \frac{1}{r!}\int_0^\infty w^r q_{r+1}(w)dw = 1 < \infty.$$

Finally, since the functions $k_\theta(z) := e^{i\theta h_\gamma(1/z)}$ are uniformly bounded by 1 on $[0, w]$ for each fixed $w \geq 0$, it is easy to prove that φ_r is continuous at 0, using bounded convergence theorem and Lebesgue's theorem on dominated convergence. ■

As a special case for $\gamma = 0$, we get:

$$\int_0^w e^{i\theta h_0(1/z)} dz = \int_0^w z^{-i\theta} dz = \frac{w^{1-i\theta}}{1-i\theta}.$$

But then, using the structure variable, we find that:

$$\lim_{t\to\infty} \mathbb{E}\left\{\exp\left(i\theta \frac{L_r(t) - rU(t)}{a(t)}\right)\right\} = \frac{\Gamma(r(1-i\theta)+1)}{\Gamma(r+1)} \frac{\mathbb{E}\{\Lambda^{ri\theta}\}}{(1-i\theta)^r}, \quad \theta \in \mathbb{R}. \tag{7}$$

The reader may wonder why $L_r(t)$ is treated via characteristic functions. For $\gamma > 0$, we again could use Laplace transforms as for the ECOMOR-quantity. We find that we can replace θ by $i\theta$ in Theorem 5 to obtain:

$$\lim_{t\to\infty} \mathbb{E}\left\{\exp\left(-\theta \frac{L_r(t)}{U(t)}\right)\right\} = \frac{1}{r!} \int_0^\infty q_{r+1}(w) \left(\int_0^w e^{-\theta z^{-\gamma}} dz\right)^r dw, \quad \theta \geq 0 \tag{8}$$

where $a(t)/U(t)$ has been replaced by its limit γ. For $r = 1$, also the right hand side can easily be written as a Laplace transform. The resulting expression for the limit in distribution coincides with that of $L_1(t)$ from Proposition 3.

From (8), we can immediately deduce the first few moments for $\gamma > 0$. For instance, we restrict attention to the mean and to the moment of second order. Easy deductions yield:

$$\lim_{t\to\infty} \frac{\mathbb{E}L_r(t)}{U(t)} = \frac{1}{(r-1)!(1-\gamma)} \int_0^\infty w^{r-\gamma} q_{r+1}(w) dw = \frac{\Gamma(r-\gamma+1)}{(r-1)!(1-\gamma)} \mathbb{E}\{\Lambda^\gamma\}$$

where we need to assume that $0 < \gamma < 1$, and:

$$\begin{aligned}\lim_{t\to\infty} \frac{\mathbb{E}\{L_r(t)^2\}}{U^2(t)} &= \frac{\gamma^2 + r(1-2\gamma)}{(r-1)!(1-\gamma)^2(1-2\gamma)} \int_0^\infty w^{r-2\gamma} q_{r+1}(w) dw \\ &= \frac{\{\gamma^2 + r(1-2\gamma)\}\Gamma(r-2\gamma+1)}{(r-1)!(1-\gamma)^2(1-2\gamma)} \mathbb{E}\{\Lambda^{2\gamma}\}\end{aligned}$$

under the condition $0 < \gamma < 1/2$.

However, already for $\gamma = 0$ we run into problems. Replacing θ by $i\theta$ in (7), with $\theta \geq 0$, shows that we need to make an additional restriction on the mixing variable in that $\mathbb{E}\{\Lambda^{-r\theta}\} < \infty$.

The situation gets even worse when $\gamma < 0$. In the ECOMOR case, $X^*_{N(t)-r}$ plays the role of a random centering keeping $R_r(t)$ nonnegative. In the case of $L_r(t)$, the centering quantity $rU(t)$ is deterministic and $rU(t)/a(t) \to \infty$ as $t \to \infty$. Hence, the Laplace transform will not exist for $\theta > 0$. However, as before, we can prove that:

$$\lim_{t\to\infty} \mathbb{E}\left\{\exp\left(-\theta \frac{L_r(t) - rU(t)}{a(t)}\right)\right\} = \frac{1}{r!} \int_0^\infty q_{r+1}(w) \left(\int_0^w e^{-\theta h_\gamma(1/z)} dz\right)^r dw, \quad \theta \leq 0.$$

5 Simulations

The aim of this section is to illustrate Proposition 2 and Proposition 3 by performing a simulation study. We deal with two concrete examples for the distribution F of the claim size process $\{X_i;\ i \geq 1\}$. The first one is the Student distribution $t(\nu)$ with $\nu \in \{0.5, 1, 2, 5\}$ which is heavy-tailed ($\gamma > 0$). The second one is the standard normal distribution $\mathrm{N}(0, 1)$ which is moderately-tailed ($\gamma = 0$). The claim number process $\{N(t);\ t \geq 0\}$ is chosen to be a Poisson process such that, for each fixed $t \geq 0$, the random variable $N(t)$ has a Poisson distribution $\mathrm{Poi}(\lambda t)$ with parameter $\lambda = 1$.

In a first part, we fix the time $t \geq 0$. We give figures showing the quantities $\mathbb{P}[R_1(t) > a(t)s]$ and $I_\gamma(s)$, and $\mathbb{P}[L_1(t) - c(t) > d(t)s]$ and $J_\gamma(s)$ as functions of s. Figure 1 and Figure 2 deal with $R_1(t)$ for

$t \in \{5, 10, 50, 100\}$, respectively with $F \sim t(2)$ and $F \sim \mathrm{N}(0,1)$. Figure 3 and Figure 4 deal with $L_1(t)$ for $t \in \{5, 10, 50, 100\}$, respectively with $F \sim t(2)$ and $F \sim \mathrm{N}(0,1)$.

In a second part, we go into further details. The values of s are fixed such that $I_\gamma(s) = 0.5$ and $J_\gamma(s) = 0.5$. We propose tables showing the evolution of $\mathbb{P}[R_1(t) > a(t)s]$ in comparison with $I_\gamma(s)$, and $\mathbb{P}[L_1(t) - c(t) > d(t)s]$ in comparison with $J_\gamma(s)$, for $F \sim \{\mathrm{N}(0,1), t(5), t(2), t(1), t(0.5)\}$, with respect to increasing values of the time $t \geq 0$, with $t \in \{5, 10, 50, 100, 500, 1000, 5000, 10000, 50000\}$. Table 2 deals with $R_1(t)$ and Table 3 with $L_1(t)$.

For each of the claim size distributions, we recall the expression of the parameters necessary for the simulations. For the $t(\nu)$ distribution: $\gamma = 1/\nu$, $a(t) = \gamma(\gamma d_\nu t)^\gamma$ and $U(t) = (\gamma d_\nu t)^\gamma$ with $d_\nu = \frac{\nu^{\nu/2}}{\sqrt{\pi}} \frac{\Gamma((\nu+1)/2)}{\Gamma(\nu/2)}$. For the $\mathrm{N}(0,1)$ distribution: $\gamma = 0$, $a(t) = (2\log t - \log\log t - \log 4\pi)^{-1/2}$ and $U(t)$ is approximated numerically since for small values of t no simple expression is available. Also, with the Poisson assumption $\mathrm{Poi}(t)$ on $N(t)$, we get $q_1(w) = q_2(w) = e^{-w}$.

To get estimated values of $\mathbb{P}[R_1(t) > a(t)s]$ and $\mathbb{P}[L_1(t) - c(t) > d(t)s]$ for each distribution F and $t \geq 0$, we have simulated $n = 100000$ replications of the random variables $R_1(t)$ and $L_1(t)$ by using n random samples from F of random size $N(t)$ from $\mathrm{Poi}(t)$.

	$\mathrm{N}(0,1)$ $[s = 0.69315]$	$t(5)$ $[s = 0.68099]$	$t(2)$ $[s = 0.66187]$	$t(1)$ $[s = 0.64104]$	$t(0.5)$ $[s = 0.65242]$
$t = 5$	0.10813	0.74730	0.68235	0.61530	0.56184
$t = 10$	0.34797	0.66341	0.57547	0.52127	0.50440
$t = 50$	0.43015	0.56825	0.51577	0.50215	0.50012
$t = 100$	0.44980	0.54680	0.50305	0.50010	0.50000
$t = 500$	0.46688	0.51950	0.50044	0.50000	0.50000
$t = 1000$	0.47237	0.51408	0.50020	0.50000	0.50000
$t = 5000$	0.47501	0.50630	0.50003	0.50000	0.50000
$t = 10000$	0.47930	0.50421	0.50001	0.50000	0.50000
$t = 50000$	0.48305	0.50176	0.50000	0.50000	0.50000

Table 2: Estimated values of $\mathbb{P}[R_1(t) > a(t)s]$ for s such that $I_\gamma(s) = 0.5$ (s-values in brackets) with $t \in \{5, 10, 50, 100, 500, 1000, 5000, 10000, 50000\}$ and $F \sim \{\mathrm{N}(0,1), t(5), t(2), t(1), t(0.5)\}$.

	$\mathrm{N}(0,1)$ $[s = 0.36651]$	$t(5)$ $[s = 1.07606]$	$t(2)$ $[s = 1.20112]$	$t(1)$ $[s = 1.44270]$	$t(0.5)$ $[s = 2.08137]$
$t = 5$	0.23168	0.16200	0.42400	0.48640	0.51223
$t = 10$	0.41469	0.19200	0.45600	0.49780	0.50284
$t = 50$	0.46538	0.27700	0.47400	0.49960	0.50004
$t = 100$	0.47588	0.33100	0.49800	0.50000	0.50000
$t = 500$	0.48296	0.40100	0.50000	0.50000	0.50000
$t = 1000$	0.48293	0.42200	0.50000	0.50000	0.50000
$t = 5000$	0.48742	0.44700	0.50000	0.50000	0.50000
$t = 10000$	0.48901	0.47600	0.50000	0.50000	0.50000
$t = 50000$	0.49126	0.48200	0.50000	0.50000	0.50000

Table 3: Estimated values of $\mathbb{P}[L_1(t) - c(t) > d(t)s]$ for s such that $J_\gamma(s) = 0.5$ (s-values in brackets) with $t \in \{5, 10, 50, 100, 500, 1000, 5000, 10000, 50000\}$ and $F \sim \{\mathrm{N}(0,1), t(5), t(2), t(1), t(0.5)\}$.

As the tail goes heavier, we can see from Table 2 that the first order correction for $R_1(t)$ is good enough. For example, in case of $t(2)$ distribution, for $t = 50$, the approximation nearly coincides with the estimated probability. However, for standard normal distribution $\mathrm{N}(0,1)$, the first order approximation is still not good enough at $t = 1000$. This interpretation is confirmed by Figure 1 and Figure 2. So for practical purposes, in case of heavier tailed distributions, it is safe to use the first order approximation. But even for moderate values of t, the first order approximation is inappropriate for $\gamma = 0$. For this class of distributions, the second order approximation should be investigated to see the improvement. This aspect will be investigated in a forthcoming paper.

From Table 3, Figure 3 and Figure 4, we make the same conclusion for $L_1(t)$ as for $R_1(t)$. One also notices that for increasing ν the tail of the $t(\nu)$ distribution becomes less heavy and, at the same time, the first order correction gets worse as well. When $\nu = 5$, the approximation is worse than that of the standard normal distribution, even for large values of t.

6 Proof of Theorem 2

Let $t \geq 0$, $r \geq 1$ and $y \in \mathbb{R}$ be fixed. For each $s \geq 0$, we have:

$$\begin{aligned}\mathbb{P}[R_r(t) > s] &\geq \mathbb{P}[X^*_{N(t)} > s+y, R_r(t) > s] \\ &= \mathbb{P}[X^*_{N(t)} > s+y] - \mathbb{P}[X^*_{N(t)} > s+y, R_r(t) \leq s].\end{aligned}$$

Consider the second term on the right-hand side. We get:

$$\begin{aligned}\mathbb{P}[X^*_{N(t)} > s+y, R_r(t) \leq s] &= \mathbb{E}\Bigg\{N(t)(N(t)-1)\cdots(N(t)-r)\int_{s+y}^{\infty}\int_{x_1-s}^{x_1}\int_{\frac{x_1+x_2-s}{2}}^{x_2}\cdots \\ &\quad \int_{\frac{x_1+\cdots+x_{r-1}-s}{r-1}}^{x_{r-1}}\int_{\frac{x_1+\cdots+x_r-s}{r}}^{x_r} F(dx_{r+1})\cdots F(dx_1)F^{N(t)-(r+1)}(x_{r+1})\Bigg\} \\ &= \mathbb{E}\Bigg\{N(t)(N(t)-1)\cdots(N(t)-r+1)\int_{s+y}^{\infty}\int_{x_1-s}^{x_1}\int_{\frac{x_1+x_2-s}{2}}^{x_2}\cdots \\ &\quad \int_{\frac{x_1+\cdots+x_{r-1}-s}{r-1}}^{x_{r-1}} F(dx_r)\cdots F(dx_1)\left[F^{N(t)-r}(x_r) - F^{N(t)-r}\left((x_1+\cdots+x_r-s)/r\right)\right]\Bigg\} \\ &\leq \mathbb{E}\Bigg\{N(t)(N(t)-1)\cdots(N(t)-r+1)\int_{s+y}^{\infty}\int_{0}^{x_1}\int_{0}^{x_2}\cdots \\ &\quad \int_{0}^{x_{r-1}} F(dx_r)\cdots F(dx_1)\left[1 - F^{N(t)-r}\left((x_1+\cdots+x_r-s)/r\right)\right]\Bigg\} \\ &\leq \mathbb{E}\Bigg\{N(t)(N(t)-1)\cdots(N(t)-r+1)\int_{s+y}^{\infty}\int_{0}^{x_1}\int_{0}^{x_2}\cdots \\ &\quad \int_{0}^{x_{r-2}} F(dx_{r-1})\cdots F(dx_1)\left[1 - F^{N(t)-r}\left((x_1+\cdots+x_{r-1}-s)/r\right)\right]\Bigg\} \\ &\leq \cdots\cdots\cdots\cdots \\ &\vdots \\ &\leq \mathbb{E}\Bigg\{N(t)(N(t)-1)\cdots(N(t)-r+1)\int_{s+y}^{\infty} F(dx_1)\left[1 - F^{N(t)-r}((x_1-s)/r)\right]\Bigg\} \\ &\leq \Big[1-F(s+y)\Big]\mathbb{E}\Big\{N(t)(N(t)-1)\cdots(N(t)-r+1)\Big[1-F^{N(t)-r}(y/r)\Big]\Big\}.\end{aligned}$$

Hence, since $F \in \mathcal{L}$, we get the following inequality:

$$\begin{aligned}&\limsup_{s\to\infty}\frac{\mathbb{P}[X^*_{N(t)} > s+y, R_r(t) \leq s]}{\mathbb{E}N(t)\mathbb{P}[X > s]} \\ &\quad \leq \limsup_{s\to\infty}\frac{1-F(s+y)}{1-F(s)}\frac{\mathbb{E}\left\{N(t)(N(t)-1)\cdots(N(t)-r+1)\left[1-F^{N(t)-r}(y/r)\right]\right\}}{\mathbb{E}N(t)} \\ &\quad = \frac{\mathbb{E}\left\{N(t)(N(t)-1)\cdots(N(t)-r+1)\left[1-F^{N(t)-r}(y/r)\right]\right\}}{\mathbb{E}N(t)}.\end{aligned}$$

Therefore, we get:

$$\begin{aligned}
\liminf_{s\to\infty} \frac{\mathbb{P}[R_r(t)>s]}{\mathbb{E}N(t)\mathbb{P}[X>s]} &\geq \liminf_{s\to\infty}\left\{\frac{\mathbb{P}[X^*_{N(t)}>s+y]}{\mathbb{E}N(t)\mathbb{P}[X>s]} - \frac{\mathbb{P}[X^*_{N(t)}>s+y, R_r(t)\leq s]}{\mathbb{E}N(t)\mathbb{P}[X>s]}\right\} \\
&\geq \liminf_{s\to\infty}\frac{\mathbb{P}[X^*_{N(t)}>s+y]}{\mathbb{E}N(t)\mathbb{P}[X>s]} + \liminf_{s\to\infty}\left\{-\frac{\mathbb{P}[X^*_{N(t)}>s+y, R_r(t)\leq s]}{\mathbb{E}N(t)\mathbb{P}[X>s]}\right\} \\
&= \liminf_{s\to\infty}\frac{\mathbb{P}[X^*_{N(t)}>s+y]}{\mathbb{E}N(t)\mathbb{P}[X>s]} - \limsup_{s\to\infty}\frac{\mathbb{P}[X^*_{N(t)}>s+y, R_r(t)\leq s]}{\mathbb{E}N(t)\mathbb{P}[X>s]} \\
&\geq \liminf_{s\to\infty}\frac{\mathbb{P}[X^*_{N(t)}>s+y]}{\mathbb{E}N(t)\mathbb{P}[X>s]} - \frac{\mathbb{E}\left\{N(t)(N(t)-1)\cdots(N(t)-r+1)\left[1-F^{N(t)-r}(y/r)\right]\right\}}{\mathbb{E}N(t)}.
\end{aligned}$$

Considering the first term on the right-hand side, we get:

$$\begin{aligned}
\liminf_{s\to\infty}\frac{\mathbb{P}[X^*_{N(t)}>s+y]}{\mathbb{E}N(t)\mathbb{P}[X>s]} &= \liminf_{s\to\infty}\frac{\mathbb{P}[X^*_{N(t)}>s+y]}{\mathbb{E}N(t)\mathbb{P}[X>s+y]}\frac{\mathbb{P}[X>s+y]}{\mathbb{P}[X>s]} \\
&= \lim_{s\to\infty}\frac{1-F(s+y)}{1-F(s)} = 1
\end{aligned}$$

by Lemma 1 and the assumption $F\in\mathcal{L}$. Therefore, we obtain:

$$\liminf_{s\to\infty}\frac{\mathbb{P}[R_r(t)>s]}{\mathbb{E}N(t)\mathbb{P}[X>s]} \geq 1-\frac{\mathbb{E}\left\{N(t)(N(t)-1)\cdots(N(t)-r+1)\left[1-F^{N(t)-r}(y/r)\right]\right\}}{\mathbb{E}N(t)}.$$

Thus, if we take the limit as $y\to\infty$ on both sides, applying monotone convergence theorem, the claim of the theorem is proved.

Indeed, we prove that the second term on the right-hand side goes to 0 when $y\to\infty$. The random variable $N(t)(N(t)-1)\cdots(N(t)-r+1)\left[1-F^{N(t)-r}(y/r)\right]$ is bounded above by $N(t)(N(t)-1)\cdots(N(t)-r+1)$ for all $y\in\mathbb{R}$ and is monotone decreasing in y, converging to 0 as $y\to\infty$. Also, $\mathbb{E}\{N(t)(N(t)-1)\cdots(N(t)-r+1)\} = Q_t^{(r)}(1)<\infty$ by assumption. To divide by $\mathbb{E}N(t)$ is not a problem since $\mathbb{E}N(t)\leq Q_t^{(r)}(1)<\infty$. Hence, applying monotone convergence theorem, we deduce that the second term on the right-hand side goes to 0 when $y\to\infty$. ■

7 Conclusions and Remarks

In this paper, we have dealt with two large claims reinsurance treaties: ECOMOR and LCR. Reinsurance mathematics is one important field of mathematical risk theory and ECOMOR and LCR are typical examples of applications of extreme value theory to reinsurance. We have derived new mathematical results connected with asymptotic distributional problems for the quantities $R_r(t)$ and $L_r(t)$, that are defined as the reinsured amounts in ECOMOR and LCR, respectively.

7.1 There is some need to get further information on the accuracy of the approximations given in Subsection 3.2 and Subsection 4.1 dealing with the convergence in distribution for $R_1(t)$ and $L_1(t)$. It would be interesting to get remainder results for the case where the claim size distribution F belongs to the extremal class $\mathcal{C}_\gamma(a)$, $\gamma\in\mathbb{R}$, with remainder and where the claim number process $\{N(t);\ t\geq 0\}$ is a mixed Poisson process. As pointed out in Section 5, this problem will be investigated in a forthcoming paper.

7.2 We point out that - apart from the results in Subsection 3.1 - most of our results can be extended to the case where the counting process $\{N(t);\ t\geq 0\}$ averages in time, as defined in (6).

7.3 Another type of result would be to make a comparison of the random variable $R_r(t)$, or $L_r(t)$, with the random sum $S_{N(t)}$ when the claim size distribution F has a heavy tail. Such a comparison would show what percentage of the portfolio is reinsured under an ECOMOR treaty, or LCR. For a situation where the claim number is deterministic in ECOMOR, see Darling [10]. Also, it would be particularly interesting to deal with the general study on the ratio $R_r(t)/S_{N(t)}$, or $L_r(t)/S_{N(t)}$, since results will lead to better insight into the dominant terms in a portfolio. Moreover, quantities like $\mathbb{E}\left\{R_r(t)/S_{N(t)}\right\}$ and $\mathbb{E}\left\{L_r(t)/S_{N(t)}\right\}$ should tell us how ECOMOR and LCR compare with the more traditional proportional reinsurance treaty. As these questions need a totally different approach, we will deal with them in a separate publication.

7.4 As indicated in the article by Beirlant [2], asymptotic results for $t \to \infty$ are not always relevant in catastrophic reinsurance. For example, earthquake claims or claims resulting from windstorms and hurricanes are commonly settled quickly. However, traditionally the number of claims is then very high. It is worth noting that the condition $t \to \infty$ in results on convergence in distribution can be replaced by a condition of the form $\mathbb{E}N \to \infty$. The changes in the arguments are easily made.

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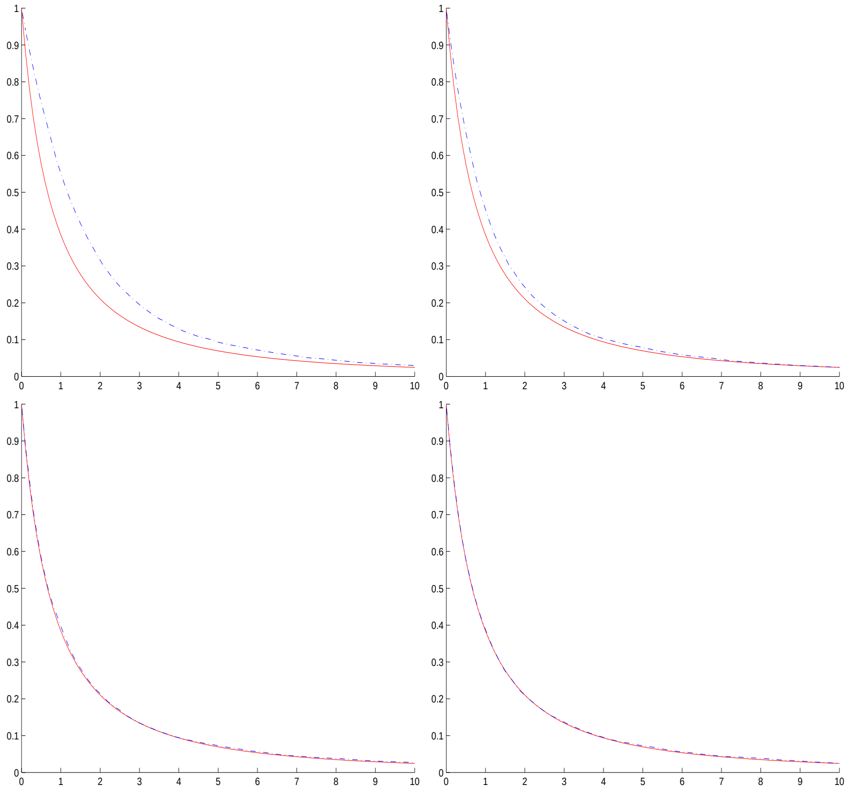

Figure 1: Quantities $\mathbb{P}[R_1(t) > a(t)s]$ in dashdot lines and $I_\gamma(s)$ in solid lines as functions of $s \geq 0$, for fixed $t \in \{5, 10, 50, 100\}$ [from the upper-left corner to the lower-right corner] and with $F \sim t(2)$.

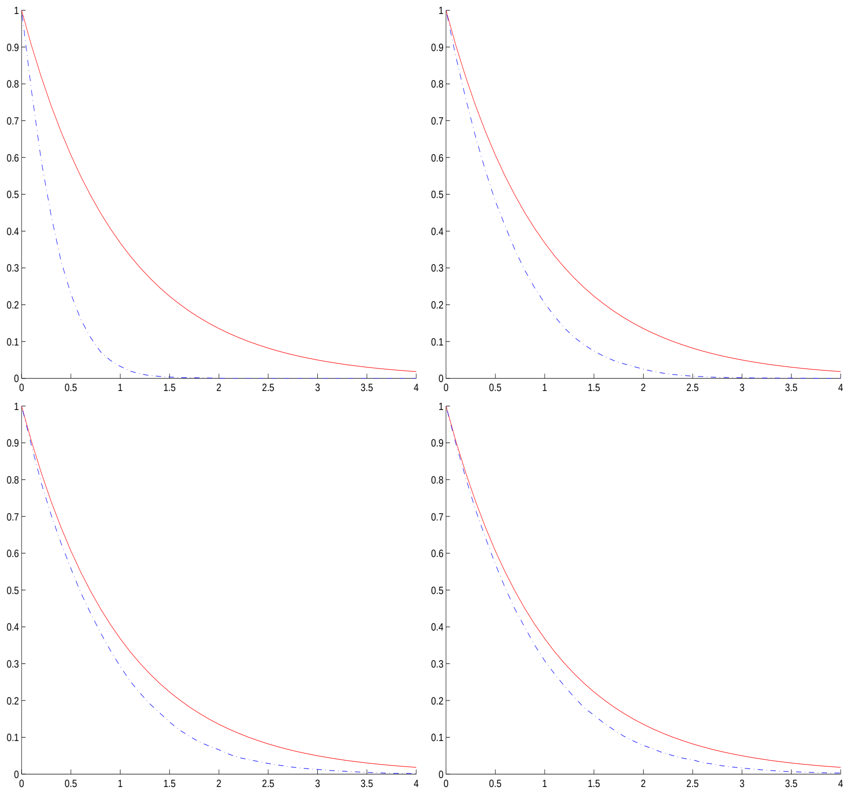

Figure 2: Quantities $\mathbb{P}[R_1(t) > a(t)s]$ in dashdot lines and $I_\gamma(s)$ in solid lines as functions of $s \geq 0$, for fixed $t \in \{5, 10, 50, 100\}$ [from the upper-left corner to the lower-right corner] and with $F \sim \mathrm{N}(0, 1)$.

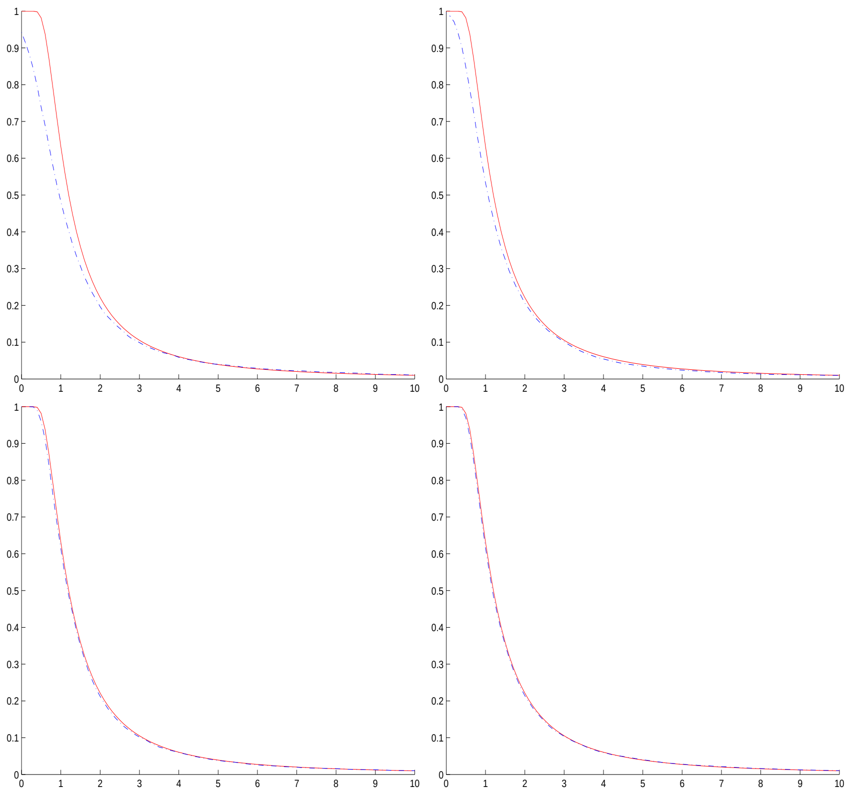

Figure 3: Quantities $\mathbb{P}[L_1(t) - c(t) > d(t)s]$ in dashdot lines and $J_\gamma(s)$ in solid lines as functions of $s > 0$, for fixed $t \in \{5, 10, 50, 100\}$ [from the upper-left corner to the lower-right corner] and with $F \sim t(2)$.

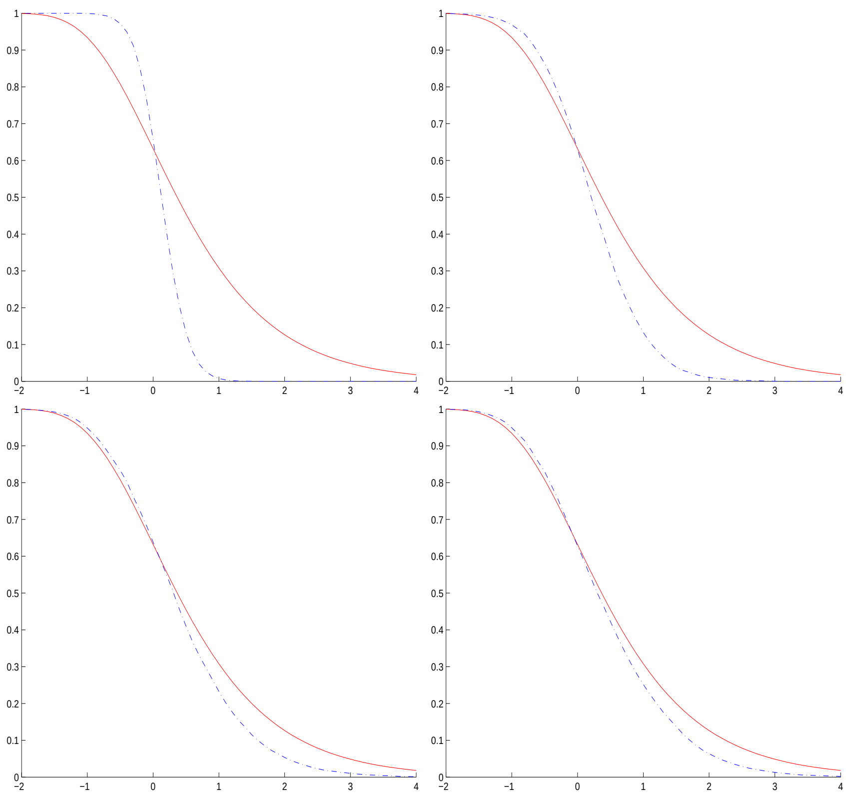

Figure 4: Quantities $\mathbb{P}[L_1(t) - c(t) > d(t)s]$ in dashdot lines and $J_\gamma(s)$ in solid lines as functions of $s \in \mathbb{R}$, for fixed $t \in \{5, 10, 50, 100\}$ [from the upper-left corner to the lower-right corner] and with $F \sim \mathrm{N}(0, 1)$.