A new kind of switching process: the periodic switching process

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Abstract: In this paper, we propose a new kind of switching process: the periodic switching process. This process allows the distribution of the sojourn times not to be necessarily geometric, contrary to the Markov-switching case. It is therefore more flexible since it allows better control of the state changes. We specify the main characteristics of the periodic switching processes and we propose a method to simulate them. We also give some examples. Finally, with the help of an example, we exhibit empirically the long memory behavior of the periodic switching process for some well chosen sojourn times distribution. We then discuss some relations between the long memory behavior and the switching regime.

Keywords: Long memory; Markov-switching model; Regime switching; Simulation procedure

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1 Introduction

The switching process is nowadays frequently used in finance and economics. This kind of process takes into account the changes of state of a time series. In finance for instance, it is well known that the volatility of a time series could change, because of a depression, for example. One of the most popular models is the Markov-switching process introduced and developed by Hamilton (1989, 1990). A large literature exists concerning this model. One of its properties is that the change of state has an unique probability. This is due to the Markov definition of the model. Unfortunately, a consequence of this is that it is difficult to control the changes of state. Here we propose a new model built differently from the Markov-switching process. It will allow better control of the state changes.

For the Markov-switching model, it is possible to compute the distribution of the sojourn time in a state of the process. The distribution is then geometric. We recall in Section 2 the parameters of this distribution. For our model we permit the sojourn time to have a distribution that is not necessarily geometric. While in the case of the Markov-switching model, the distribution of the sojourn time is a consequence of the transition matrix, we prefer here to consider a particular sojourn time distribution and we study the transition matrix. We call this new kind of switching model, the *periodic switching process*.

Of course, we lose the main characteristics of the Markov chain. For example, we will see in Section 3, that the transition matrix is not unique (see Proposition 3.1). Moreover, the unobservable process (which determines the states) will depend on more than the last state. However, some very interesting characteristics will appear for this process, for example, better control of the state change. The difference between the two processes also occurs in their construction. In the case of the Markov-switching model, the process is built directly while in the case of the periodic switching process, we first build the unobservable process and from it we build the periodic switching process.

The plan of this paper is as follows. First we recall in Section 2, some definitions and properties of the Markov-switching process. In Section 3, we deal with the periodic switching process. In Subsection 3.1, we define it and we specify the transition matrix (Proposition 3.1) and the probability to be in a state (Proposition 3.2). We will remark also that this new switching process generalizes a class of Markov-switching process. In Subsection 3.2, we present a particular case for which the distribution is computable (Proposition 7). We also specify some of its statistical characteristics (expectation and variance). In Section 4, we turn to some simulations. In Subsection 4.1, we propose a procedure to simulate a periodic switching process. In Subsection 4.2, we illustrate this procedure with an example. After remarking that this kind of process exhibits long memory behavior, for some well chosen sojourn times, we empirically analyze this behavior in Subsection 4.3. In Section 5, we conclude and Section 6 is devoted to the proofs.

2 The Markov-switching model

The Markov-switching model has been introduced and studied by Hamilton (1989). Let $(X_t)_t$ be a Markov-switching model with r states, $r \geq 2$. It is therefore described by the following equations:

$$X_t = \begin{cases} m_1 + a_{0,1} + a_{1,1}X_{t-1} + \cdots + a_{p,1}X_{t-p} + \varepsilon_t & \text{for} \quad S_t = 1 \\ \vdots \qquad\qquad\qquad \vdots & \qquad\quad \vdots \\ m_r + a_{0,2} + a_{1,r}X_{t-1} + \cdots + a_{p,r}X_{t-p} + \varepsilon_t & \text{for} \quad S_t = r \end{cases}$$

where $(\varepsilon_t)_t$ is a centered Gaussian white noise (GWN) with finite variance σ_ε^2, $m_\alpha \in \mathbb{R}$ and $m_\alpha \neq m_\beta$ for $\alpha, \beta \in \{1, \ldots, r\}$ and $\alpha \neq \beta$, and $(a_{i,\alpha})_{1 \leq i \leq p}$ is a real sequence for every $\alpha \in \{1, \ldots, r\}$. The random variable S_t is a r-state Markov chain such that:

$$\mathbb{P}[S_t = \alpha | S_{t-1} = \beta] =: p_{\alpha\beta}, \quad \alpha, \beta \in \{1, \ldots, r\}. \tag{1}$$

The probabilities $(p_{\alpha\beta})_{1 \leq \alpha, \beta \leq r}$ are called the *transition probabilities*. They satisfy the following property:

$$\sum_{\beta=1}^{r} p_{\alpha\beta} = 1 \quad \text{for} \quad \alpha \in \{1, \ldots, r\}.$$

We denote by U_t^α, $\alpha \in \{1,\dots,r\}$, the integer valued random variable whose events are defined as:

$$\{U_t^\alpha = 0\} := \{S_{t+1} = \beta\} | \{S_t = \alpha\} \cap \{S_{t-1} = \beta\}$$
$$\{U_t^\alpha = n\} := \{S_{t+n+1} = \beta\} \cap \left(\bigcap_{k=1}^{n} \{S_{t+k} = \alpha\}\right) \Big| \{S_t = \alpha\} \cap \{S_{t-1} = \gamma\}, \quad n \in \mathbb{N}^*$$

with $\beta, \gamma \in \{1,\dots,r\}$, $\beta \neq \alpha$ and $\gamma \neq \alpha$.

By definition, the process $(U_t^\alpha)_t$ represents the time that the process stays in the state α. In the following, we call $(U_t^\alpha)_t$ the *sojourn time process* in the state α. Using the Markov property of S_t, it is easy to show that $(U_t^\alpha)_t$ has a geometric distribution on $\mathbb{N}$ with parameter $(1 - p_{\alpha\alpha})$, i.e.:

$$\mathbb{P}[U_t^\alpha = n] = p_{\alpha\alpha}^n (1 - p_{\alpha\alpha}), \quad n \in \mathbb{N}$$

and then that:

$$\mathbb{E}[U_t^\alpha] = \frac{p_{\alpha\alpha}}{1 - p_{\alpha\alpha}} \quad \text{and} \quad \mathbb{V}[U_t^\alpha] = \frac{p_{\alpha\alpha}}{(1 - p_{\alpha\alpha})^2}.$$

Remark 2.1 *Assume that in the case of a 2-states Markov-switching process, we choose a probability of change of states closed to 0. This means that $p_{11} \simeq 1$ and $p_{22} \simeq 1$, with of course $p_{11} < 1$ and $p_{22} < 1$. In this context, the parameters of the geometric distributions of the sojourn times U_t^1 and U_t^2 are close to 0. Then, we have the following approximations for the expectation and the variance of these sojourn times:*

$$\mathbb{V}[U_t^1] \sim \mathbb{E}[U_t^1]^2 \quad \textit{and} \quad \mathbb{V}[U_t^2] \sim \mathbb{E}[U_t^2]^2.$$

Now, we assume that we have built a 2-state process with sojourn times V_t^1 and V_t^2, respectively distributed with a Poisson distribution with parameter θ_1 and θ_2, such that $\mathbb{E}[V_t^1]$ and $\mathbb{E}[V_t^2]$ are as close as possible to $\mathbb{E}[U_t^1]$ and $\mathbb{E}[U_t^2]$ respectively. Since the expectation and the variance are equal for a Poisson distribution, the variances of the sojourn times V_t^1 and V_t^2 are smaller than those of the sojourn times U_t^1 and U_t^2. As a consequence, the distribution of the sojourn times V_t^1 and V_t^2 will be more concentrated around their expectation. An important consequence of this is that the state changes will be more "regular" in the case of sojourn times with Poisson distributions than in the case of sojourn times having geometric distributions.

In the following section, we will use the previous remark to build our new switching process: the periodic switching process.

3 A new model: the periodic switching process

This section is divided in two subsections. In the first one, we define our new process: the periodic switching process. In the second subsection, we study a simple example of such a process.

3.1 The general model

In the previous section, we have recalled that the distribution of the sojourn time random variable is geometric in the case of Markov-switching processes. In this paper, we are interested in the case where this distribution is not geometric. For that let $(S_t)_t$ be an unobservable process. We assume that this unobservable process can be in r different states: $\mathbb{E}_1, \dots, \mathbb{E}_r$. To simplify the notations, we use sometimes the following notation: we write $\{S_t = \alpha\}$ for the event $\{S_t \in \mathbb{E}_\alpha\}$, for $\alpha \in \{1,\dots,r\}$.

We define the sojourn time U_t^α of S_t as follows:

$$\{U_t^\alpha = 0\} := \{S_{t+1} = \alpha + 1\} \Big| \{S_t = \alpha\} \cap \{S_{t-1} = \alpha - 1\} \tag{2}$$

$$\{U_t^\alpha = u_{t,\alpha}\} := \{S_{t+u_{t,\alpha}+1} = \alpha + 1\} \cap \left\{\bigcap_{s=1}^{u_{t,\alpha}} S_{t+s} = \alpha\right\} \Big| \{S_t = \alpha\} \cap \{S_{t-1} = \alpha - 1\} \tag{3}$$

where $u_{t,\alpha} \in \mathbb{N}^*$ and $\alpha \in \{1, \dots, r\}$ (if $\alpha = 1$ or $\alpha = r$, then we respectively have $\alpha + 1 = 1$ and $\alpha - 1 = r$).

Here, it is important to note that the random variables U_t^α, $\alpha \in \{1, \dots, r\}$, are not defined for all $t \in \mathbb{Z}$. We denote by $\mathbb{L}_\alpha$ the range of t-values of U_t^α, $\alpha \in \{1, \dots, r\}$. Then:

$$\mathbb{L}_\alpha \subset \mathbb{Z} \quad \text{and} \quad \mathbb{L}_\alpha \cap \mathbb{L}_\beta = \emptyset$$

for $\alpha, \beta \in \{1, \dots, r\}$ and $\alpha \neq \beta$. The second condition is obvious and it means that the process cannot be in two different states in the same time.

Some relationship exists between the elements of $\mathbb{L}_\alpha$. Indeed, let $l_{j,\alpha} \in \mathbb{L}_\alpha = \{l_{1,\alpha}, \dots, l_{n,\alpha}, \dots\}$ and assume that $U^i_{l_{j,\alpha}} = u_{l_{j,\alpha}}$ with probability 1. Then:

$$l_{j,\alpha} + (u_{l_{j,\alpha}} + 1) = l_{j,\alpha+1} \quad \text{if} \quad \alpha < r \quad \text{with} \quad l_{j,\alpha+1} \in \mathbb{L}_{\alpha+1}$$

and

$$l_{j,r} + (u_{l_{j,r}} + 1) = l_{j+1,1} \quad \text{if} \quad \alpha = r \quad \text{with} \quad l_{j+1,1} \in \mathbb{L}_1.$$

To define the periodic switching process, we need two assumptions on the unobservable process $(S_t)_t$ that we specify now. The first one concerns the necessary information for the unobservable process to be well defined. The second assumption concerns the possible state changes.

In the case of the Markov-switching process $(X_t)_t$, the unobservable process $(S_t)_t$ depends only on the last state. This means that we have:

$$\mathbb{P}[S_t | S_{t-1}, S_{t-2}, S_{t-3}, \dots] = \mathbb{P}[S_t | S_{t-1}].$$

In the case of the periodic switching process $(X_t)_t$, the unobservable process $(S_t)_t$ depends only on the last change of state. The following assumption illustrates this.

Assumption 1

$$\mathbb{P}\Big[S_{t+1} = \alpha + 1 \Big| S_t = \alpha, S_{t-1} = \alpha - 1, \bigcap_{s=1}^{m} \{S_{t-1-s} = l_s\}\Big] = \mathbb{P}[U_t^\alpha = 0]$$

$$\mathbb{P}\Big[S_{t+u_{t,\alpha}+1} = \alpha + 1, \bigcap_{s=1}^{u_{t,\alpha}} \{S_{t+s} = \alpha\} \Big| S_t = \alpha, S_{t-1} = \alpha - 1, \bigcap_{s=1}^{m} \{S_{t-1-s} = l_s\}\Big] = \mathbb{P}[U_t^\alpha = u_{t,\alpha}], \quad n \in \mathbb{N}^*$$

where $m \in \mathbb{N}^*$, $\alpha \in \{2, \dots, r-1\}$, and if $\alpha = 1$ or $\alpha = r$, then $\alpha - 1 = r$ or $\alpha + 1 = 1$ respectively, and $l_s \in \{1, \dots, r\}$ for all $1 \leq s \leq m$.

The following assumption concerns the possible changes of states.

Assumption 2

$$\mathbb{P}[S_{t+1} = \beta | S_t = \alpha] = 0 \quad \textit{if} \quad \beta \neq \alpha \quad \textit{and} \quad ((\beta \neq \alpha + 1 \quad \textit{if} \quad \alpha < r) \quad \textit{or} \quad (\beta \neq 1 \quad \textit{if} \quad \alpha = r)).$$

This assumption forces the process to have always the same state changes. This leads the process to have some periodicity. If the process is in the state $\mathbb{E}_\alpha$, then the next state of the process is either $\mathbb{E}_\alpha$ or $\mathbb{E}_{\alpha+1}$ if $\alpha < r$, $\mathbb{E}_1$ if $\alpha = r$, with probability 1.

We can now define our new process.

Definition 3.1 *Let $(X_t)_t$ be a process defined as:*

$$X_t = \begin{cases} m_1 + a_{0,1} + a_{1,1}X_{t-1} + \cdots + a_{p,1}X_{t-p} + \varepsilon_{t,1} & \text{for} \quad S_t \in \mathbb{E}_1 \\ \vdots \qquad\qquad\qquad \vdots & \qquad\qquad \vdots \\ m_r + a_{0,r} + a_{1,r}X_{t-1} + \cdots + a_{p,r}X_{t-p} + \varepsilon_{t,r} & \text{for} \quad S_t \in \mathbb{E}_r \end{cases} \tag{4}$$

where $a_{i,\alpha} \in \mathbb{R}$ for all $i \in \{0, \ldots, p\}$ and $\alpha \in \{1, \ldots, r\}$, $m_\alpha \in \mathbb{R}$ and $(\varepsilon_{t,\alpha})_t$ is a white noise $WN(0, \sigma_\alpha^2)$ for all $\alpha \in \{1, \ldots, r\}$, and $(S_t)_t$ is an unobservable process. Assume that $(S_t)_t$ follows Assumption 1 and Assumption 2, such that for each $\alpha \in \{1, \ldots, r\}$, the sojourn time process $(U_t^\alpha)_{t \in \mathbb{L}_\alpha}$ defined by (2) and (3) has an integer valued distribution $\mathcal{L}(\theta_\alpha)$. Then, the process $(X_t)_t$ is called a periodic switching process.

A important consequence is that the periodic switching model encompasses the classical class of Markov-switching processes. Indeed, if the distributions $\mathcal{L}(\theta_\alpha)$ of the sojourn times $(U_t^\alpha)_t$ are geometric for all $\alpha \in \{1, \ldots, r\}$, then we retrieve the Markov-switching model.

In the following proposition, we derive the transition matrix of the unobservable process $(S_t)_t$.

Proposition 3.1 *Let $(X_t)_t$ be a periodic switching process. Let the sojourn times $(U_t^\alpha)_{t \in \mathbb{L}_\alpha}$ of the process $(X_t)_t$ have the distribution $\mathcal{L}(\theta_\alpha)$, whose distribution function is denoted by F_α, for $\alpha \in \{1, \ldots, r\}$. Then, the transition matrix $P(n)$, $n \geq 1$, has the following expression:*

$$P(n) = \begin{bmatrix} p_1(n) & q_1(n) & 0 & . & . & 0 \\ 0 & p_2(n) & q_2(n) & 0 & . & . \\ . & . & . & . & . & . \\ . & . & 0 & p_{r-2}(n) & q_{r-2}(n) & 0 \\ 0 & . & . & 0 & p_{r-1}(n) & q_{r-1}(n) \\ q_r(n) & 0 & . & . & 0 & p_r(n) \end{bmatrix} \tag{5}$$

where the $p_\alpha(n)$'s and $q_\alpha(n)$'s are given for $\alpha \in \{1, \ldots, r\}$ by:

$$p_\alpha(n) := \mathbb{P}[S_{t+n+1} = \alpha \Big| \bigcap_{s=0}^{n} \{S_{t+s} = \alpha\}, S_{t-1} = \alpha - 1] = 1 - \frac{\mathbb{P}[U_t^\alpha = n]}{1 - F_\alpha(n-1)}$$

$$q_\alpha(n) := \mathbb{P}[S_{t+n+1} = \alpha + 1 \Big| \bigcap_{s=0}^{n} \{S_{t+s} = \alpha\}, S_{t-1} = \alpha - 1] = 1 - p_\alpha(n)$$

with $\alpha + 1 = 1$ when $\alpha = r$ and $\alpha - 1 = r$ when $\alpha = 1$.

Proof :
Postponed to Subsection 6.1. ■

In the case of the Markov-switching process, the transition matrix is constant while in the case of the periodic switching process, it is not. The matrix depends on n which corresponds to the time of the last change of state.

Remark 3.1 *We can easily check that we get the transition matrix of the Markov-switching process if the distributions $\mathcal{L}(\theta_\alpha)$ of the sojourn times $(U_t^\alpha)_t$ are assumed to be geometric for all $\alpha \in \{1, \ldots, r\}$. Let us do it for a 2-states Markov-switching process. The distributions of the sojourn times U_t^1 and U_t^2 being geometric, we denote their respective parameters $(1 - \theta_1)$ and $(1 - \theta_2)$, with $0 < |\theta_1|, |\theta_2| < 1$. We then have for all $n \in \mathbb{N}$:*

$$\mathbb{P}[U_t^1 = n] = \theta_1^n (1 - \theta_1) \quad \text{and} \quad \mathbb{P}[U_t^2 = n] = \theta_2^n (1 - \theta_2).$$

Thus, we get:

$$F_1(n-1) = \sum_{i=0}^{n-1} \mathbb{P}[U_t^1 = i] = (1-\theta_1) \sum_{i=0}^{n-1} \theta_1^i = 1 - \theta_1^n.$$

Using the same method, we easily get $F_2(n-1) = 1 - \theta_2^n$. As a consequence and using (5), we get the following transition matrix:

$$P = \begin{bmatrix} \theta_1 & 1-\theta_1 \\ 1-\theta_2 & \theta_2 \end{bmatrix}$$

which exactly corresponds to the transition matrix of a Markov-switching process. Here, the transition matrix is constant.

In the following proposition we give the probability for a periodic switching process to be in a given state.

Proposition 3.2 *Let $(X_t)_t$ be a periodic switching process with transition matrix given by (5). For $\alpha \in \{1, \dots, r\}$, the expectation of the sojourn time random variable U_t^α is denoted by θ_α. We denote by $(S_t)_t$ the unobservable process. Then, for $\alpha \in \{1, \dots, r\}$:*

$$\mathbb{P}[S_t = \alpha] = \frac{\theta_\alpha}{\theta_1 + \cdots + \theta_r}.$$

Proof :
Postponed to Subsection 6.2. ■

The result of the proposition permits one to determine some statistical characteristics of the periodic switching process. In the following subsection we now study a particular case of such a process.

3.2 Example of periodic switching process

We consider a particular case of periodic switching process, which is usually called *mean-plus-noise*. We consider the following r-states process:

$$X_t = \begin{cases} m_1 + \varepsilon_{t,1} & \text{for} \quad S_t = 1 \\ \vdots & \qquad\quad \vdots \\ m_r + \varepsilon_{t,r} & \text{for} \quad S_t = r \end{cases} \tag{6}$$

where $(\varepsilon_{t,\alpha})_t$ is a Gaussian white noise $GWN(0, \sigma_\alpha^2)$ for $\alpha \in \{1, \dots, r\}$ and m_α is a constant such that $m_\alpha \neq m_\beta$ for all $\alpha \neq \beta \in \{1, \dots, r\}$.

For each $\alpha \in \{1, \dots, r\}$ and time t, we denote $\xi_{t,\alpha} := m_\alpha + \varepsilon_{t,\alpha}$. Obviously, we remark that $(\xi_{t,\alpha})_t$ is a Gaussian white noise process $GWN(m_\alpha, \sigma_\alpha^2)$. We denote by θ_α the expectation of the random variable $(U_t^\alpha)_t$, $\alpha \in \{1, \dots, r\}$.

Corollary 3.1 *The distribution of the process $(X_t)_t$ defined in (6) is a mixture of Gaussian distributions and its density function f_X is given by:*

$$f_X(x) = \sum_{\alpha=1}^{r} \frac{\theta_\alpha}{\theta_1 + \cdots + \theta_r} f_\alpha(x), \quad x \in \mathbb{R} \tag{7}$$

where f_α is the density function of the white noise process $(\xi_{t,\alpha})_t := (m_\alpha + \varepsilon_{t,\alpha})_t$, $\alpha \in \{1, \dots, r\}$.

Proof :
First of all, using Proposition 3.2, we get $\mathbb{P}[S_t = \alpha] = \frac{\theta_\alpha}{\theta_1+\cdots+\theta_r}$ for each $\alpha \in \{1,\ldots,r\}$. Now, if $S_t = \alpha$ with probability 1, we thus have $X_t = \xi_{t,\alpha} = m_\alpha + \varepsilon_{t,\alpha}$, whose density is equal to the density f_α of $(\xi_{t,\alpha})_t$. As a consequence, the density f_X of $(X_t)_t$ is given by (7). ■

From Corollary 3.1, we easily derive that if the process $(X_t)_t$ is defined by (6), then:

$$\mathbb{E}[X_t] = \frac{1}{\theta_1+\cdots+\theta_r}\sum_{\alpha=1}^{r}\theta_\alpha m_\alpha$$

and

$$\mathbb{V}[X_t] = \frac{1}{\theta_1+\cdots+\theta_r}\sum_{\alpha=1}^{r}\theta_\alpha\sigma_\alpha^2 + \frac{1}{(\theta_1+\cdots+\theta_r)^2}\sum_{\alpha=1}^{r-1}\sum_{h=\alpha+1}^{r}\theta_\alpha\theta_h(m_\alpha-m_h)^2.$$

The interest of exhibiting the distribution of such a process is the possibility to use a parametric estimator for the expectation of the sojourn time in each state.

In the following section, we start by proposing a procedure to simulate periodic switching processes. Then, we study an example in considering a 2-states periodic switching process.

4 Simulation

This section is composed by three parts. In the first subsection, we propose a method to simulate a periodic switching process. In the second subsection, we illustrate this method by simulating an example and we study it. We point out that, under some conditions, the periodic switching process has some similar characteristics as the long memory process. Then, in the last subsection, we propose to illustrate empirically the long memory behavior of the periodic switching process.

4.1 Methods of simulation

Here we are interested in a procedure to simulate a switching process $(X_t)_t$, with r states, defined by:

$$X_t = \begin{cases} m_1 + a_{0,1} + a_{1,1}X_{t-1} + \cdots + a_{p,1}X_{t-p} + \varepsilon_{t,1} & \text{for } S_t = 1 \\ \vdots \qquad\qquad\qquad \vdots & \vdots \\ m_r + a_{0,r} + a_{1,r}X_{t-1} + \cdots + a_{p,r}X_{t-p} + \varepsilon_{t,r} & \text{for } S_t = r \end{cases}$$

where for $\alpha \in \{1,\ldots,r\}$, $(\varepsilon_{t,\alpha})_t$ is a white noise $WN(0,\sigma_\alpha^2)$ and $(a_{i,\alpha})_{i=1,\ldots,p}$ are sequences of reals.

For $\alpha \in \{1,\ldots,r\}$, we assume that the sojourn time process $(U_t^\alpha)_{t\in\mathbb{L}_\alpha}$ has the following distribution:

$$U_t^\alpha \sim \mathcal{L}(\theta_\alpha)$$

with $\mathbb{E}[U_t^\alpha] = \theta_\alpha$.

To simulate a sample path of $(X_t)_t$ with length M, we propose the following four-steps procedure.

(*i*) For each $\alpha \in \{1,\ldots,r\}$, simulate a sample path of the random variable U_t^α from the distribution $\mathcal{L}(\theta_\alpha)$, with length $N = [\frac{M}{\theta_1+\cdots+\theta_r}]$ (where [.] denotes the integer part). This sample is denoted by $(u_{l_{1,\alpha}}^\alpha,\ldots,u_{l_{N,\alpha}}^\alpha)$, where for each $i \in \{1,\ldots,N\}$ and $\alpha \in \{1,\ldots,r\}$, $l_{i,\alpha} \in \mathbb{L}_\alpha$ ($\mathbb{L}_\alpha$ being the range of t-values of U_t^α).

If $\sum_{\alpha=1}^{r}\sum_{i=1}^{N} u_{l_{i,\alpha}}^\alpha < M$, then consider sample paths of the random variable U_t^α, with length $N+1$ or more, until $\sum_{\alpha=1}^{r}\sum_{i=1}^{N+j} u_{l_{i,\alpha}}^\alpha \geq M$, with $j \geq 1$. In any case, the length of the sample path of U_t^α will be denoted by N in what follows, for $\alpha \in \{1,\ldots,r\}$.

(*ii*) For each $\alpha \in \{1, \ldots, r\}$, simulate a sample path of the random variable $\varepsilon_{t,\alpha}$. The length of this sample has to be equal to $\sum_{i=1}^{N} u^{\alpha}_{l_{i,\alpha}}$.

(*iii*) Fix $i \in \{1, \ldots, N\}$ and $\alpha \in \{1, \ldots, r-1\}$. Consider the interval $[l_{i,\alpha},\ l_{i,\alpha+1} - 1]$, where $l_{i,\alpha+1} = l_{i,\alpha} + (u_{l_{i,\alpha}} + 1)$, with $u_{l_{i,\alpha}}$ the value of $U^{\alpha}_{l_{i,\alpha}}$. For this interval, simulate X_t using the following relation:

$$X_s = m_\alpha + a_{0,\alpha} + a_{1,\alpha} X_{s-1} + \cdots + a_{p,\alpha} X_{s-p} + \varepsilon_{\alpha,s}, \qquad \forall s \in [l_{i,\alpha},\ l_{i,\alpha+1} - 1].$$

If $\alpha = r$, the interval becomes $[l_{i,r},\ l_{i,1} - 1]$ and we have

$$X_s = m_r + a_{0,r} + a_{1,r} X_{s-1} + \cdots + a_{p,r} X_{s-p} + \varepsilon_{r,s}, \qquad \forall s \in [l_{i,r},\ l_{i,1} - 1].$$

(*iv*) A sample path of $(X_t)_t$ is then obtained, but with length greater than M. To get a sample of length M, just consider the first M terms.

Using this procedure, we now propose to simulate an example of a periodic switching process. We then make some remarks on its long memory behavior.

4.2 Simulation of a periodic switching process

In this section, we propose to simulate a 2-states periodic switching process $(X_t)_t$ defined by:

$$X_t = \begin{cases} m_1 + \varepsilon_{t,1} & \text{for } S_t = 1 \\ m_2 + \varepsilon_{t,2} & \text{for } S_t = 2 \end{cases} \tag{8}$$

where $m_1 = 0$, $m_2 = 2$, and $(\varepsilon_{t,1})_t$ and $(\varepsilon_{t,2})_t$ are centered Gaussian white noise with respective variances equal to 1 and 0.16. We choose for the sojourn times $(U^1_t)_t$ and $(U^2_t)_t$, Poisson distributions with the same parameter $\theta = 25$. For all t, we denote $\xi_{t,1} := m_1 + \varepsilon_{t,1}$ and $\xi_{t,2} := m_2 + \varepsilon_{t,2}$. The transition matrix of this process is given by Proposition 3.1.

The statistical characteristics of the process $(X_t)_t$ are the following.

- The distribution of X_t is a mixture of two Gaussian distributions. Since the distribution of the sojourn times are identical, the density function of X_t is given by (7), so it has the following form:

$$f_X(x) = \frac{1}{2} f_1(x) + \frac{1}{2} f_2(x)$$

  where f_1 and f_2 are the respective density functions of the random variables $(\xi_{t,1})_t$ and $(\xi_{t,2})_t$.

- The expectation and the variance of X_t are given by:

$$\mathbb{E}[X_t] = 1 \qquad \text{and} \qquad \mathbb{V}[X_t] = 1.58.$$

Using the procedure given in Subsection 4.1, we simulate a sample path $(X_1, \ldots, X_M)$ of the process $(X_t)_t$ with $M = 1000$.

We show in Figure 1 some characteristics of the simulated sample path $(X_1, \ldots, X_{1000})$ of the process $(X_t)_t$: its trajectory (a), its empirical autocorrelation function (b), its periodogram (c) which estimates the spectral density and its empirical distribution (d).

First, we can easily remark from the trajectory shown in Figure 1(a) the presence of state changes. In Figure 1(d), the theoretical distribution of the process (dashed line) has been superimposed on the empirical distribution (black line). However, the most interesting features are the empirical autocovariance function and the periodogram respectively given in Figure 1(b) and Figure 1(c). Indeed, these two functions are similar to those of a long memory process, since there is a slow decay of the empirical autocovariance function and an explosion of the periodogram at a (non zero) frequency. We refer to Beran (1994) and references therein for details about long memory processes.

In the next subsection, we then propose to study through an empirical way, some link between long memory behavior and periodic switching processes.

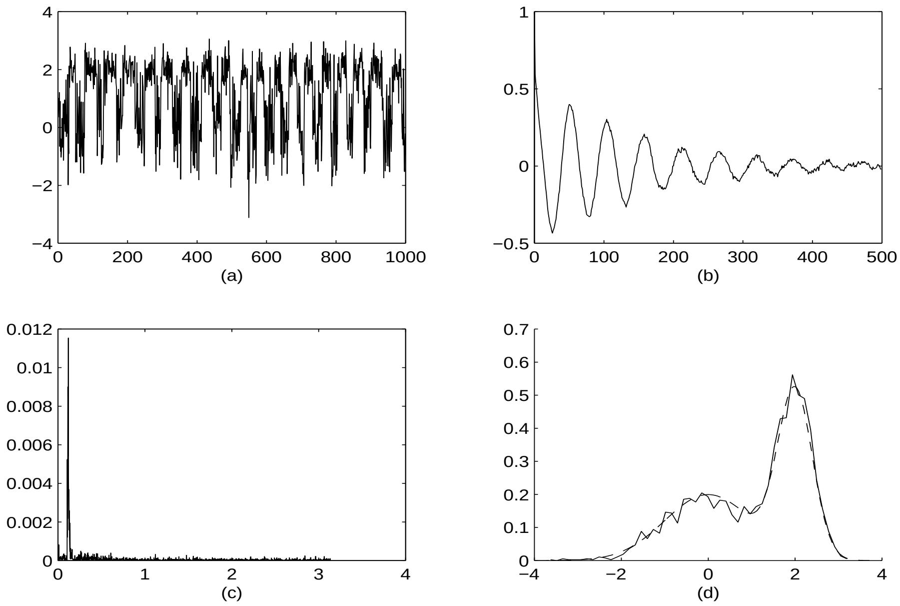

Figure 1: Some empirical characteristics of a periodic switching process: trajectory (a), empirical autocorrelation function (b), periodogram (c) and empirical distribution (d).

4.3 Periodic switching process and long memory behavior

As we have noted previously the periodic switching process may present some characteristics of long memory behavior, for instance when the sojourn times follow Poisson distributions. Of course many other discrete distributions can be used. For example, we can cite the Poisson-Inverse Gaussian distribution, the discrete uniform distribution and the binomial distribution.

In this subsection, we propose several simulations of periodic switching processes to quantify empirically this long memory behavior. To do that, we consider the 2-states periodic switching model specified in (8).

To measure the long memory behavior of a process, one assumes generally that the spectral density of the process has the following behavior close to a singularity λ_0:

$$f_X(\lambda) \sim C(\lambda)|\lambda - \lambda_0|^{-2d}, \quad \lambda \to \lambda_0$$

where $C(\lambda)$ is a slowly varying function at λ_0 and d is called the *long memory parameter*.

As in Diebold and Inoue (2001), we use the log-periodogram regression estimator proposed by Geweke and Porter-Hudak (1983), refined by Robinson (1994, 1995) and adapted by Ferrara (2000) to the case of an unbounded spectral density at a non zero frequency, to estimate the long memory parameter d. We recall that the Robinson estimator is a non parametric estimator which lies on the regression of the periodogram near the singularity (for more details, see Robinson (1995)). The location of the singularity is estimated as being the Fourier frequency for which the periodogram, denoted I_T, is a local maximum (this estimation lies on results given by Yajima (1996)). Suppose that we have a time series $X_1, \ldots, X_T$ of length T and denote λ_{i_0} the Fourier frequency for which the periodogram is maximum. The Robinson estimator of d is then given for $0 \leq l < m < T$ by:

$$\hat{d}_{R,i_0}(l,m) = \frac{\sum_{j=i_0+l+1}^{i_0+m}(Y_j - \overline{Y}) \log I_T(\lambda_j)}{\sum_{j=i_0+l+1}^{i_0+m}(Y_j - \overline{Y})^2}$$

where $Y_j = -2\log|\lambda_j - \lambda_{i_0}|$, $I_T(\lambda_j)$ is the periodogram evaluated at the frequency λ_j and $\overline{Y}$ is the empirical mean of the Y_j's for $l+1 \leq j \leq m$. The parameters m and l determine the domain of regression and they are integer constants. For a review on the choice of such parameters, we refer to Ferrara (2000). For the following, we choose $l = 3$ and $m \simeq T^{0.6}$.

In our study, we choose the same Poisson distribution Poi(θ) with parameter θ for the sojourn times. We consider three cases for this parameter: $\theta \in \{10, 20, 50\}$. We also consider that the parameters (m_1, σ_1^2) and (m_2, σ_2^2) of the Gaussian distributions of the white noise processes $(\varepsilon_{t,1})_t$ and $(\varepsilon_{t,2})_t$ are different: $(m_1, \sigma_1^2) \in \{(3, 1), (-1, 1)\}$ and $(m_2, \sigma_2^2) \in \{(-3, 1/4), (-3, 1), (-3, 4)\}$.

For each case, we simulate 100 samples of periodic switching process with length $T = 1500$, and we estimate the long memory parameter d and the location of the singularity λ_0. This method permits to get the mean and the standard deviation of the estimated long memory parameter and singularity's location.

In Table 1, we give the estimated values $\hat{d}$ of the long memory parameter computed at the estimated frequency $\hat{\lambda}_0$. Their respective standard deviations are also given in brackets.

	$m_1 = 3 \quad \sigma_1^2 = 1 \quad m_2 = -3 \quad \sigma_2^2 = 1/4$		$m_1 = -1 \quad \sigma_1^2 = 1 \quad m_2 = 1 \quad \sigma_2^2 = 1/4$	
	$\hat{d}$	$\hat{\lambda}_0$	$\hat{d}$	$\hat{\lambda}_0$
$\theta = 10$	0.5569	0.2968	0.4281	0.2984
	(0.0852)	(0.0239)	(0.1042)	(0.0235)
$\theta = 20$	0.4166	0.1527	0.3479	0.1521
	(0.0842)	(0.0072)	(0.1092)	(0.0088)
$\theta = 50$	0.3332	0.0605	0.2242	0.0600
	(0.0761)	(0.0019)	(0.0875)	(0.0018)
	$m_1 = 3 \quad \sigma_1^2 = 1 \quad m_2 = -3 \quad \sigma_2^2 = 1$		$m_1 = -1 \quad \sigma_1^2 = 1 \quad m_2 = 1 \quad \sigma_2^2 = 1$	
	$\hat{d}$	$\hat{\lambda}_0$	$\hat{d}$	$\hat{\lambda}_0$
$\theta = 10$	0.5228	0.3019	0.3928	0.3004
	(0.0957)	(0.0225)	(0.0876)	(0.0249)
$\theta = 20$	0.4276	0.1518	0.3139	0.1540
	(0.1092)	(0.0088)	(0.0996)	(0.0083)
$\theta = 50$	0.2866	0.0605	0.2076	0.0609
	(0.0929)	(0.0021)	(0.0858)	(0.0020)
	$m_1 = 3 \quad \sigma_1^2 = 1 \quad m_2 = -3 \quad \sigma_2^2 = 4$		$m_1 = -1 \quad \sigma_1^2 = 1 \quad m_2 = 1 \quad \sigma_2^2 = 4$	
	$\hat{d}$	$\hat{\lambda}_0$	$\hat{d}$	$\hat{\lambda}_0$
$\theta = 10$	0.4710	0.2990	0.2910	0.3005
	(0.1099)	(0.0260)	(0.0854)	(0.0258)
$\theta = 20$	0.3808	0.1518	0.2291	0.1513
	(0.0882)	(0.0073)	(0.0927)	(0.0071)
$\theta = 50$	0.0832	0.0620	0.1654	0.0603
	(0.0929)	(0.0021)	(0.0834)	(0.0019)

Table 1: Estimation of the long memory parameter d and the location of the singularity λ_0, and standard deviations in brakets.

There exists a relationship between the value of the parameter θ of the Poisson distribution and the location of the singularity λ_0, given by:

$$\theta = \frac{\pi}{\lambda_0}.$$

This relation is explained by the fact that the "periodicity" of the process may be defined as the expectation of the necessary time for the process to come back to the initial state, thus, in the particular case of a 2-states periodic switching process, to change state twice. This expectation is equal to the sum of the expectations of the sojourn times. Here, this expectation is equal to 2θ because the sojourn times have the same Poisson distribution Poi(θ) and their expectations are then identical with value θ. Therefore, the theoretical expectation of this necessary time corresponds in the spectral domain to a frequency $\lambda_0 = \pi/\theta$. For each $\theta \in \{10, 20, 50\}$, the estimations of the location of the singularity λ_0 given in Table 1 are always good regarding to their standard deviations if we compare them to their corresponding theoretical values. Also, the standard deviations of $\hat{\lambda}_0$ being always close to 0, we can conclude that the state changes are relatively regular.

Now, the estimated values of the long memory parameter are very interesting. First of all, these values are significantly different from 0. For fixed parameters of the white noise processes' distributions, we remark that the estimated values of the long memory parameter decrease with the increasing of the parameter θ. Indeed, the expectation and the variance of the sojourn times being equal, the variance increases with the expectation and then with θ. Thus, if the variance of the necessary time for the process to change state twice, and then to come back to its initial state, increases then the periodicity of the process is less "regular" and the long memory parameter d decreases.

Also, it is important to remark that the estimated values of d are highly dependent on the parameters of the distributions of the white noise processes $(\varepsilon_{t,1})_t$ and $(\varepsilon_{t,2})_t$. When the expectations and the variances of the white noises change, the states of the simulated process are more or less distinct. For instance, the two states are more distinct when the white noises have respective expectations equal to $m_1 = 3$ and $m_2 = -3$, and respective variances equal to $\sigma_1^2 = 1$ and $\sigma_2^2 = 1/4$ than when the expectations are equal to $m_1 = 1$ and $m_2 = -1$, and the variances equal to $\sigma_1^2 = 1$ and $\sigma_2^2 = 4$. This more or less good distinction may affect the estimation of the long memory parameter. This further problem has still to be investigated.

5 Conclusion

The new switching process that we have proposed here, the periodic switching process, has some very interesting characteristics. Indeed, it gives the possibility to control the state changes. On the one hand, this feature is really interesting when one wants to model and forecast dependent time series. On the other hand and contrary to the case of the Markov-switching process, the possibility to control the distribution of the sojourn time process permits to simulate a switching process with very regular state changes. Also, this regularity implies the presence of peaks in the spectral density of the process. Then, we have empirically investigated this feature and we have shown via some estimations that such a process can be considered as a long memory process.

6 Proofs

6.1 Proof of Proposition 3.1

Without loss of generality we assume that $k = 2$. Then we have to compute the elements of the following matrix:

$$P(n) = \begin{bmatrix} p_1(n) & q_1(n) \\ q_2(n) & p_2(n) \end{bmatrix}.$$

First, we compute the term $q_1(n)$, for $n \geq 0$:

$$\mathbb{P}\left[S_{t+n+1} = 2 \Big| \bigcap_{i=0}^{n} \{S_{t+i} = 1\}, S_{t-1} = 2\right] = \frac{\mathbb{P}\left[S_{t+n+1} = 2, \bigcap_{i=0}^{n} \{S_{t+i} = 1\}, S_{t-1} = 2\right]}{\mathbb{P}\left[\bigcap_{i=0}^{n} \{S_{t+i} = 1\}, S_{t-1} = 2\right]}.$$

We study the two terms of the ratio:

$$\mathbb{P}\left[S_{t+n+1} = 2, \bigcap_{i=0}^{n} \{S_{t+i} = 1\}, S_{t-1} = 2\right] = \mathbb{P}\left[S_{t+n+1} = 2, \bigcap_{i=1}^{n} \{S_{t+i} = 1\} \ |S_t = 1, S_{t-1} = 2\right] \mathbb{P}\left[S_t = 1, S_{t-1} = 2\right].$$

$$\mathbb{P}\left[\bigcap_{i=0}^{n} \{S_{t+i} = 1\}, S_{t-1} = 2\right] = \mathbb{P}\left[\bigcap_{i=1}^{n} S_{t+i} = 1 \ |S_t = 1, S_{t-1} = 2\right] \mathbb{P}\left[S_t = 1, S_{t-1} = 2\right].$$

Thus, we get:

$$\mathbb{P}\left[S_{t+n+1} = 2 \Big| \bigcap_{i=0}^{n} \{S_{t+i} = 1\}, S_{t-1} = 2\right] = \frac{\mathbb{P}\left[S_{t+n+1} = 2, \bigcap_{i=1}^{n} \{S_{t+i} = 1\} \Big| S_t = 1, S_{t-1} = 2\right]}{\mathbb{P}\left[\bigcap_{i=1}^{n} \{S_{t+i} = 1\} \Big| S_t = 1, S_{t-1} = 2\right]}.$$

The numerator of the ratio is nothing else but $\mathbb{P}\left[U_t^1 = n\right]$. To compute the denominator, we use the relation (9) given in Appendix A and we obtain:

$$\begin{aligned}
\mathbb{P}\left[\bigcap_{i=1}^{n}\{S_{t+i}=1\}\Big|S_t=1,S_{t-1}=2\right] &= \mathbb{P}\left[\bigcap_{i=1}^{n+1}\{S_{t+i}=1\}\Big|S_t=1,S_{t-1}=2\right]+\mathbb{P}\left[U_t^1=n\right]\\
&= \mathbb{P}\left[\bigcap_{i=1}^{n+2}\{S_{t+i}=1\}\Big|S_t=1,S_{t-1}=2\right]+\mathbb{P}\left[U_t^1=n\right]+\mathbb{P}\left[U_t^1=n+1\right]\\
&= \dots\\
&= \sum_{i=n}^{\infty}\mathbb{P}\left[U_t^1=i\right]=1-F_1(n-1).
\end{aligned}$$

Therefore, we have:

$$q_1(n)=\frac{\mathbb{P}\left[U_t^1=n\right]}{1-F_1(n-1)}.$$

Generalizing to the case of r states, we obtain for $\alpha\in\{1,\dots,r\}$:

$$q_\alpha(n)=\frac{\mathbb{P}\left[U_t^\alpha=n\right]}{1-F_\alpha(n-1)}\qquad\text{and}\qquad p_\alpha(n)=1-\frac{\mathbb{P}\left[U_t^\alpha=n\right]}{1-F_\alpha(n-1)}.$$

As a consequence, the transition matrix (5) follows. ■

6.2 Proof of Proposition 3.2

Let the random variable T_n be built as follows:

$$T_n=\sum_{\alpha=1}^{r}\sum_{i=1}^{[n/r]}U_{l_{i,\alpha}}^\alpha$$

where [.] represents the integer part. This random variable has the following expectation:

$$\mathbb{E}[T_n]=\left[\frac{n}{r}\right]\sum_{\alpha=1}^{r}\theta_\alpha.$$

The quantity $\mathbb{E}[T_n]$ represents the expected time for $r[n/r]$ state changes.
Now, if we denote $T_n^\alpha=\sum_{i=1}^{[n/r]}U_{l_{i,\alpha}}^\alpha$, we get:

$$T_n=\sum_{\alpha=1}^{r}T_{n,\alpha}.$$

The quantity $\mathbb{E}[T_{n,\alpha}]$ is the expected time spent in the state α after $r[n/r]$ state changes. On the interval $[0,\mathbb{E}[T_n]]$, we get for the probability of the process to be in the state α the following expression:

$$\begin{aligned}
\mathbb{P}\left[S_t=\alpha;t\in\{1,\dots,\mathbb{E}[T_n]\}\right] &= \frac{\mathbb{E}[T_{n,\alpha}]}{\mathbb{E}[T_n]}\\
&= \frac{\theta_\alpha}{\sum_{\alpha=1}^{r}\theta_\alpha}.
\end{aligned}$$

Using the σ-additivity of the measure $\mathbb{P}$ and a sequence of well chosen disjoint sets, then when $n\to\infty$, we easily get the following probability:

$$\mathbb{P}\left[S_t=\alpha\right]=\frac{\theta_\alpha}{\sum_{\alpha=1}^{r}\theta_\alpha}$$

which ends the proof. ■

6.3 Appendix A

$$\begin{aligned}
\mathbb{P}\left[\bigcap_{i=1}^{n}\{S_{t+i}=1\}\Big|S_t=1,S_{t-1}=2\right] &= \frac{\mathbb{P}\left[\bigcap_{i=1}^{n}\{S_{t+i}=1\},S_t=1,S_{t-1}=2\right]}{P\left[S_t=1,S_{t-1}=2\right]}\\
&= \frac{\mathbb{P}\left[S_{t+n}=1|\bigcap_{i=1}^{n-1}\{S_{t+i}=1\},S_t=1,S_{t-1}=2\right]\mathbb{P}\left[\bigcap_{i=1}^{n-1}\{S_{t+i}=1\},S_t=1,S_{t-1}=2\right]}{\mathbb{P}\left[S_t=1,S_{t-1}=2\right]}\\
&= \mathbb{P}\left[S_{t+n}=1\Big|\bigcap_{i=1}^{n-1}\{S_{t+i}=1\},S_t=1,S_{t-1}=2\right]\mathbb{P}\left[\bigcap_{i=1}^{n-1}\{S_{t+i}=1\}\Big|S_t=1,S_{t-1}=2\right].
\end{aligned}$$

Using the same argument, we get:

$$\begin{aligned}\mathbb{P}\left[S_{t+n}=2,\bigcap_{i=1}^{n-1}\{S_{t+i}=1\},\Big|S_t=1,S_{t-1}=2\right] &= \mathbb{P}\left[S_{t+n}=2\Big|\bigcap_{i=1}^{n-1}\{S_{t+i}=1\},S_t=1,S_{t-1}=2\right]\\ &\times\ \mathbb{P}\left[\bigcap_{i=1}^{n-1}\{S_{t+i}=1\}\Big|S_t=1,S_{t-1}=2\right].\end{aligned}$$

Using the equality:

$$\mathbb{P}\left[S_{t+n}=1\Big|\bigcap_{i=1}^{n-1}\{S_{t+i}=1\},S_t=1,S_{t-1}=2\right]+\mathbb{P}\left[S_{t+n}=2\Big|\bigcap_{i=1}^{n-1}\{S_{t+i}=1\},S_t=1,S_{t-1}=2\right]=1$$

we then deduce the following relation that is used in Subsection 6.1:

$$\begin{aligned}\mathbb{P}\left[\bigcap_{i=1}^{n-1}\{S_{t+i}=1\}\Big|S_t=1,S_{t-1}=2\right] &= \mathbb{P}\left[\bigcap_{i=1}^{n}\{S_{t+i}=1\}\Big|S_t=1,S_{t-1}=2\right]+\mathbb{P}\left[S_{t+n}=2,\bigcap_{i=1}^{n-1}\{S_{t+i}=1\}\Big|S_t=1,S_{t-1}=2\right]\\ &= \mathbb{P}\left[\bigcap_{i=1}^{n}\{S_{t+i}=1\}\Big|S_t=1,S_{t-1}=2\right]+\mathbb{P}\left[U_t^1=n-1\right].\ \blacksquare \qquad (9)\end{aligned}$$

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