

# Tail asymptotics for exponential functionals of Lévy processes

Krishanu Maulik<sup>†</sup>  
Bert Zwart<sup>‡</sup>

October 26, 2004

**Abstract:** Motivated by recent studies in financial mathematics and other areas, we investigate the exponential functional  $Z = \int_0^\infty e^{-X(t)} dt$  of a Lévy process  $X(t), t \geq 0$ . In particular, we investigate its tail asymptotics. It is shown that, depending on the right tail of  $X(1)$ , the tail behavior of  $Z$  is exponential, Pareto, or extremely heavy-tailed.

*Keywords: Breiman's theorem. Perpetuities. Subexponential distributions. Mellin transforms. Tauberian theorems.*

---

<sup>†</sup>EURANDOM, P. O. Box 513, 5600 MB Eindhoven, The Netherlands, email: [maulik@eurandom.tue.nl](mailto:maulik@eurandom.tue.nl).

<sup>‡</sup>Department of Mathematics & Computer Science, Eindhoven University of Technology, P. O. Box 513, 5600 MB Eindhoven, The Netherlands, email: [zwart@win.tue.nl](mailto:zwart@win.tue.nl); also CWI, P. O. Box 94079, 1090 GB Amsterdam, The Netherlands.

Part of the work has been done while the first author was visiting Institut Mittag-Leffler in Stockholm, Sweden and grateful acknowledgement is made for hospitality and support.

# 1 Introduction and main results

Let  $\{X(t), t \geq 0\}$  be a Lévy process drifting to  $\infty$  and let  $Z$  be its associated *exponential functional*, i.e.  $Z = \int_0^\infty e^{-X(t)} dt$ . Random variables of this form have recently been analyzed in great detail; the reason being that such random variables appear in several applications, like mathematical finance ( $Z$  as the present value of a perpetuity [11], Asian options and COGARCH process [23]), Additive Increase Multiplicative Decrease (AIMD) algorithms [12, 20], order picking strategies in carousel systems [25], mathematical physics, and more. A recent monograph devoted to such exponential functionals is [30].

In this paper we analyze the behavior of  $\mathbb{P}\{Z > x\}$  as  $x$  becomes large, complementing the above-mentioned works, which largely focus on exact expressions for the distribution of  $Z$ . We analyze several classes of Lévy processes which give rise to qualitatively different tail behavior of  $Z$ , ranging from extremely heavy (of the form  $(\log x)^{-\alpha}$ ) to light tails ( $\exp\{-x^p\}, p \geq 1$ ).

Several results in this paper are derived using the following identity. Let  $\tau$  be a stopping time with respect to the filtration generated by  $X(t)$ . Then the following distributional identity (which is easily verified using the strong Markov property) holds:

$$Z \stackrel{d}{=} \int_0^\tau e^{-X(u)} du + e^{-X(\tau)} Z, \quad (1.1)$$

with  $Z$  on the right hand side independent of  $\int_0^\tau e^{-X(u)} du$  and  $e^{-X(\tau)}$ . Thus we have an equation of the form  $R \stackrel{d}{=} Q + MR$ . Such equations have been studied extensively. A classical result due to Kesten [21] and Goldie [17] states that, if there exists a solution  $\kappa > 0$  to the equation  $\mathbb{E}\{M^\kappa\} = 1$ , then under some further regularity conditions,

$$\mathbb{P}\{R > x\} \sim Cx^{-\kappa}. \quad (1.2)$$

The constant  $C$  is usually very hard to obtain, unless  $\kappa$  is integer-valued, see [17]. Recently, Rivero [26] applied these results to the setting of the present paper, in which  $\mathbb{E}\{M^\kappa\} = 1$  is equivalent to the Cramér condition  $\mathbb{E}\{e^{-\kappa X(1)}\} = 1$ . In Section 3 we give an extension of this result, by providing more explicit expressions for the prefactor  $C$ . These expressions partly rely on new expressions for the fractional moments (i.e. the Mellin transform) of  $Z$ , which are presented in Section 2 and could be of independent interest.

The main results of the paper are those concerning cases in which Cramér's condition, and hence the assumptions in [17, 26], are not satisfied. In Section 4, we consider the case in which  $X(1)$  does not have exponential moments. This yields a completely different tail behavior for  $Z$ : under the assumption that  $\bar{G}(x) := \min\{1, \int_x^\infty \mathbb{P}\{-X(1) > u\} du\}$  is subexponential, and  $\mu := \mathbb{E}\{X(1)\} \in (0, \infty)$ , we derive that

$$\mathbb{P}\left\{\log \int_0^\infty e^{-X(u)} du > x\right\} \sim \mathbb{P}\left\{\sup_{t>0}(-X(t)) > x\right\} \sim \frac{1}{\mu} \bar{G}(x). \quad (1.3)$$

We again use the embedding (1.1) but now choose a non-trivial stopping time. This choice is motivated by a recent study of Zachary [31]. Since we could not find the second equivalence in (1.3) in the literature, we present a short proof.

The third case we consider (apart from the Cramér case and the subexponential case) is when  $X(t)$  is a subordinator. Here we need to distinguish between a number of additional

cases. We assume first that  $X(t)$  is a compound Poisson process with rate  $\lambda$  and non-negative i.i.d. jumps  $B_i, i \geq 1$ . In this case we obtain light-tailed behavior. The following result is proven in Section 5:

$$\mathbb{P}\{Z > x\} \sim \mathbb{E}\left\{e^{\lambda e^{-B_1} Z}\right\} e^{-\lambda x}, \quad (1.4)$$

if and only if  $\mathbb{E}\{1/B_1\} < \infty$ . The proof of (1.4) is based on another variant of (1.1), namely the distributional identity  $Z \stackrel{d}{=} e^{-B_1} Z + \frac{1}{\lambda} E_1$ , where  $E_1$  is a unit exponential random variable. Multiplying by  $\lambda$  and taking exponents on both sides of this identity gives  $e^{\lambda Z} \stackrel{d}{=} e^{\lambda e^{-B_1} Z} e^{E_1}$ . Breiman's [8] theorem (which deals with the product of heavy-tailed random variables) now suggests the result of Theorem 1. We make this reasoning precise exploiting that  $e^{E_1}$  has a Pareto tail.

Section 6 considers the case in which the condition  $\mathbb{E}\{1/B_1\} < \infty$  does not hold. In this case, the tail asymptotics are of the form  $Cx^\mu e^{-\lambda x}$ , if  $\mathbb{P}\{B_1 < y\} \sim \mu y$  as  $y \downarrow 0$ . When the compound Poisson assumption is violated (i.e. the Lévy measure associated with the subordinator has infinite mass) then the asymptotics of  $Z$  are considerably lighter. We illustrate this with a certain class of subordinators, which includes completely right-skewed  $\alpha$ -stable processes,  $0 < \alpha < 1$ . The results in Section 6 are obtained with techniques that differ from the rest of the paper: We use explicit expressions for the moments  $\mathbb{E}\{Z^s\}$  to obtain the behavior of  $\mathbb{E}\{e^{sZ}\}$  around its abscissus of convergence (which may be at  $\infty$ ). We then relate this to the tail behavior of  $\mathbb{P}\{Z > x\}$  using Abelian and Tauberian theorems.

Before we present all these results in Sections 3–6, we introduce some notation and state some preliminary results in Section 2. In particular, we give some new explicit expressions for the fractional moments (i.e. the Mellin transform) of  $Z$ .

We would like to conclude this introduction by mentioning some related work. More results on the equation  $R \stackrel{d}{=} Q + MR$  leading to light-tail (exponential and Poissonian) behavior of  $R$  can be found in Goldie and Grübel [18]. Other recent results on this equation leading to Pareto tails can be found in Konstantinides and Mikosch [24] and references therein. When  $B_1 \equiv 1$ , equation (1.4) becomes a special case of a result of Rootzén [27]. The result (1.3) we obtain in the subexponential case is related to recent work on the tail behavior of various subadditive functionals of random walks and Lévy processes, see Braverman et al. [7] and Foss et al. [16]. An interesting problem, which is not discussed here, is the *lower* tail of  $Z$ . This tail is analyzed in [25] in the case that  $X(t)$  is a Poisson process. Finally, we would like to mention recent work of Blanchet and Glynn [6] who consider various asymptotic estimates for the distribution of  $Z$ , under an asymptotic regime which lets the drift of  $X(t)$  become small.

## 2 Finiteness and moments

In this section we develop some preliminary results. In particular, we give a criterion for a.s. boundedness of  $Z$ , and extend expressions for various integer and non-integer moments of  $Z$ . We first introduce some notation. Let  $X(t), t \geq 0$ , be a Lévy process with Laplace exponent  $\phi(s)$  determined by

$$\mathbb{E}\left\{e^{-sX(t)}\right\} = e^{-t\phi(s)}. \quad (2.1)$$

Using Hölder's inequality, it is easily checked that the function  $\phi(s)$  is concave. Moreover,  $\phi(s)$  is finite for  $s \geq 0$  when  $X(t)$  has no negative jumps, and for  $s \leq 0$  when  $X(t)$  has no positive jumps.

In several cases it is useful to consider the integrated tail distribution associated with  $-X(1)$ , which is given by  $\bar{G}(x) = \min\{1, \int_x^\infty \mathbb{P}\{-X(1) > u\} du\}$ . We often use the following infinite product representation of the Gamma function, which is due to Weierstrass:

$$\Gamma(s+1) = e^{-\gamma s} \prod_{k=1}^{\infty} e^{\frac{s}{k}} \frac{k}{s+k}. \quad (2.2)$$

In this expression,  $\gamma$  is Euler's constant.

## 2.1 Finiteness

The following result, which is not used in the sequel but included for completeness, gives a criterion for a.s. boundedness of  $Z$ .

**Proposition 2.1.**  *$Z < \infty$  a.s. if and only if  $X(t) \rightarrow \infty$  a.s.*

*Proof.* As in (1.1), write  $Z \stackrel{d}{=} \int_0^1 e^{-X(s)} ds + e^{-X(1)} Z$ . According to Theorem 2.1 of Goldie and Maller [19] (in particular Condition (2.3) of that result),  $Z$  is finite a.s. if

$$e^{-X(n)} \int_n^{n+1} e^{-(X(u)-X(n))} du = \int_n^{n+1} e^{-X(u)} du \rightarrow 0 \text{ a.s.}$$

The condition  $X(u) \rightarrow \infty$  a.s. is equivalent to the a.s. existence of some  $u_\varepsilon$  such that  $e^{-X(u)} < \varepsilon$  for  $u > u_\varepsilon$ ,  $\varepsilon > 0$ . Thus  $\int_n^{n+1} e^{-X(u)} du \leq \varepsilon$  if  $n > u_\varepsilon$ , implying that  $X(t) \rightarrow \infty$  a.s. is a sufficient condition for a.s. finiteness of  $Z$ .

To check the necessity of the condition, suppose that  $X(t) \rightarrow \infty$  a.s. is false. Hence there exists  $\varepsilon > 0$  and a real number  $A$  such that

$$\mathbb{P}\left\{\liminf_{t \rightarrow \infty} X(t) < A\right\} > \varepsilon.$$

Let us define the stopping times  $\tau_0 = 0$ , and  $\tau_{n+1} = \inf\{t > \tau_n + 1 : X(t) < A\}$ . Then, on  $\{\liminf_{t \rightarrow \infty} X(t) < A\}$ , we must have, for all  $n$ ,  $\tau_n < \infty$ . Further, on that set, we have,

$$Z = \sum_{n=0}^{\infty} \int_{\tau_n}^{\tau_{n+1}} e^{-X(t)} dt \geq e^{-A} \sum_{n=0}^{\infty} \int_0^1 e^{-(X(t+\tau_n)-X(\tau_n))} dt.$$

The sum on the right hand side has i.i.d. positive summands and hence the right hand side is infinite a.s. on  $\{\liminf_{t \rightarrow \infty} X(t) < A\}$ . So,

$$\begin{aligned} \mathbb{P}\{Z = \infty\} &\geq \mathbb{P}\left\{Z = \infty; \liminf_{t \rightarrow \infty} X(t) < A\right\} \\ &\geq \mathbb{P}\left\{\sum_{n=0}^{\infty} \int_0^1 e^{-(X(t+\tau_n)-X(\tau_n))} dt = \infty; \liminf_{t \rightarrow \infty} X(t) < A\right\} \\ &= \mathbb{P}\left\{\liminf_{t \rightarrow \infty} X(t) < A\right\} > \varepsilon. \end{aligned}$$

□

## 2.2 Explicit expressions for moments

We now turn to some expressions for moments; these expressions will be useful later on. The following recursion, valid as long as  $s > 0$  and  $\phi(s) > 0$ , can be found in, for example, Proposition 3.1 of Carmona et al. [9].

$$\mathbb{E}\{Z^{s-1}\} = \frac{\phi(s)}{s}\mathbb{E}\{Z^s\}.$$

However, they require a further condition of  $\mathbb{E}\{Z^s\}$  to be finite when  $0 < s < 1$ , which is unnecessary and restrictive in our discussion. So we give a new proof of the result removing this condition in the following lemma.

**Lemma 2.1.** *If  $s > 0$  and  $\phi(s) > 0$ , we have*

$$\mathbb{E}\{Z^{s-1}\} = \frac{\phi(s)}{s}\mathbb{E}\{Z^s\}. \quad (2.3)$$

*The equality is interpreted to mean that both sides can be  $\infty$ . If we further assume that  $\mu = \mathbb{E}\{X(1)\} \in (0, \infty)$ , then  $\mathbb{E}\{Z^s\} < \infty$  for all  $s \in [-1, 0]$  and all  $s > 0$  for which  $\phi(s) > 0$ .*

*Proof.* Define  $\zeta(t) = \int_0^t e^{-X(u)} du$ . Then

$$\begin{aligned} \zeta(t)^s &= s \int_0^t (\zeta(t) - \zeta(u))^{s-1} e^{-X(u)} du \\ &= s \int_0^t e^{-sX(u)} \left( \int_0^{t-u} e^{-(X(v+u)-X(u))} dv \right)^{s-1} du. \end{aligned}$$

Note that the two factors in the last integrand are independent and the second factor has the same distribution as  $\zeta(t-u)^{s-1}$  by the strong Markov property. So taking expectations of both sides we have,

$$\mathbb{E}\{\zeta(t)^s\} = s \int_0^t e^{-u\phi(s)} \mathbb{E}\{\zeta(t-u)^{s-1}\} du = s \frac{\int_0^t e^{u\phi(s)} \mathbb{E}\{\zeta(u)^{s-1}\} du}{e^{t\phi(s)}}.$$

Since  $\phi(s) > 0$ , both the numerator and denominator on the right hand side go to  $\infty$ . So we use L'Hôpital's rule to take the limit as  $t \rightarrow \infty$  and obtain

$$\mathbb{E}\{Z^s\} = s \lim_{t \rightarrow \infty} \frac{e^{t\phi(s)} \mathbb{E}\{\zeta(t)^{s-1}\}}{\phi(s) e^{t\phi(s)}} = \frac{s}{\phi(s)} \mathbb{E}\{Z^{s-1}\}.$$

Note that in L'Hôpital's rule,  $\infty$  is allowed as possible limit.

If  $\mu \in (0, \infty)$ , Proposition 2 of [4] implies that  $\mathbb{E}\{Z^{-1}\}$  is finite, and hence  $\mathbb{E}\{Z^s\}$  is finite for  $-1 \leq s \leq 0$ . Using (2.3), we then have  $\mathbb{E}\{Z^s\} < \infty$  for all  $s > 0$  with  $\phi(s) > 0$ .  $\square$

The recursion (2.3) above can be solved explicitly for integer values of  $s$ , yielding

$$\mathbb{E}\{Z^n\} = \frac{n!}{\prod_{k=1}^n \phi(k)}. \quad (2.4)$$

For non-integer values of  $s$ , it is much harder to obtain explicit results. In the remaining part of the present section, we analyze two classes of Lévy processes for which it is possible to obtain such expressions. In particular, we focus on subordinators and Lévy processes

with no positive jumps.

### Subordinators

First we consider the class of subordinators as possible choice of  $X(t)$ . In this case, the Laplace exponent can be written as

$$\phi(s) = \eta + ds + \int_0^\infty (1 - e^{-st})\nu(dt) \quad (2.5)$$

$$= \eta + ds + s \int_0^\infty e^{-st}\nu(t, \infty)dt, \quad (2.6)$$

with  $\nu$  a measure satisfying  $\int_0^\infty \min\{1, x\}\nu(dx) < \infty$  (cf. chapter III of [2]).  $d \geq 0$  is the drift of the subordinator and we allow the killing rate  $\eta$  to be strictly positive as well.

**Proposition 2.2.** *Suppose that  $X(t)$  is a subordinator with Laplace exponent  $\phi(s)$  given as in (2.5). Suppose further that there exists an  $\alpha \in [0, 1]$  such that the function  $\phi_\alpha(s) = \phi(s)/s^\alpha$  is eventually monotone and converges to a limit  $c_\alpha \in (0, \infty)$  as  $s \rightarrow \infty$ . Then for any  $s > -1$ ,*

$$\mathbb{E}\{Z^s\} = \Gamma(s+1)^{1-\alpha} c_\alpha^{-s} \prod_{k=1}^\infty \frac{\phi(s+k)}{\phi(k)} \left(\frac{k}{k+s}\right)^\alpha.$$

The assumptions of the above proposition seem restrictive, but are satisfied in a large number of cases. Examples are:

- Any subordinator with positive drift  $d$  in which case always  $\phi(s)/s \rightarrow d$ , cf. Theorem I.2.(ii) in [2]. Monotonicity of the function  $\phi(s)/s$  follows easily from (2.6).
- A (possibly terminating) compound Poisson process with rate  $\lambda$  and i.i.d. jumps  $B_i \geq 0$  with LST  $\beta(s)$ . In this case  $\phi(s) = \eta + \lambda(1 - \beta(s))$  and the assumption of the proposition is satisfied with  $\alpha = 0$ , since  $\phi(s)$  is increasing. The case  $\eta = 0$  and  $B_i \equiv 1$  has been treated before in [3] and was extended in [20] assuming a certain lower tail condition on  $B_i$ .
- An  $\alpha$ -stable subordinator,  $0 < \alpha < 1$ , in which case  $\phi(s) = s^\alpha$ , and hence  $\mathbb{E}\{Z^s\} = \Gamma(s+1)^{1-\alpha}$ . Also sums of independent stable subordinators (yielding  $\phi(s) = s^\alpha + s^\beta$ ) are admissible.

*Proof of Proposition 2.2.* Since  $X$  is a subordinator,  $\phi(s) > 0$  for  $s > 0$  and hence the product  $\prod_{k=1}^\infty \frac{\phi_\alpha(s+k)}{\phi_\alpha(k)}$  is well-defined for  $s > -1$ . We now show that it is strictly positive and finite.

For this, let  $m$  be such that  $\phi_\alpha(s)$  is monotone for  $s \geq m$  and set  $M_s = \prod_{k=1}^m \frac{\phi_\alpha(s+k)}{\phi_\alpha(k)}$ . Write

$$\prod_{k=1}^\infty \frac{\phi_\alpha(s+k)}{\phi_\alpha(k)} = M_s \lim_{N \rightarrow \infty} \prod_{k=m+1}^N \frac{\phi_\alpha(s+k)}{\phi_\alpha(k)}$$

Assume now that  $\phi_\alpha(s)$  is increasing for  $s \geq m$ . Let  $n$  be the smallest integer larger than  $s$ . Then  $\prod_{k=m+1}^N \frac{\phi_\alpha(s+k)}{\phi_\alpha(k)}$  increases in  $N$  and is bounded above by

$$\prod_{k=m+1}^N \frac{\phi_\alpha(n+k)}{\phi_\alpha(k)} = \prod_{l=1}^n \frac{\phi_\alpha(N+l)}{\phi_\alpha(m+l)},$$

cancelling the common factors when  $n + m < N$ , and the right hand side converges to a finite limit as  $N \rightarrow \infty$ . The case in which  $\phi_\alpha(s)$  is decreasing for  $s \geq m$  is similar.

After these preliminaries, we now turn to (2.3). That recursion is indeed valid since  $\phi(s) > 0$  for all  $s > 0$ . Moreover Proposition 3.3 of [9] states that  $Z$  has some exponential moments; hence all moments are finite.

Thus, we are allowed to apply (2.3) and write, for  $s > 0$ ,

$$\Gamma(s+1)^\alpha \mathbb{E}\{Z^s\} = \frac{s}{\phi_\alpha(s)} \Gamma(s)^\alpha \mathbb{E}\{Z^{s-1}\}. \quad (2.7)$$

Define now the function, for  $s > 0$ ,

$$\psi(s) := \Gamma(s)^\alpha c_\alpha^{s-1} \mathbb{E}\{Z^{s-1}\} \prod_{k=1}^{\infty} \frac{\phi_\alpha(k)}{\phi_\alpha(s+k-1)} \quad (2.8)$$

$$= \mathbb{E}\{Z^{s-1}\} c_\alpha^{s-1} e^{-\gamma\alpha(s-1)} \prod_{k=1}^{\infty} e^{\frac{\alpha(s-1)}{k}} \frac{\phi(k)}{\phi(s+k-1)} \quad (2.9)$$

using Weierstrass' representation (2.2) of the Gamma function. According to (2.7),  $\psi$  satisfies

$$\begin{aligned} \psi(s+1) &= \Gamma(s+1)^\alpha \mathbb{E}\{Z^s\} c_\alpha^s \lim_{N \rightarrow \infty} \prod_{k=1}^N \frac{\phi_\alpha(k)}{\phi_\alpha(s+k)} \\ &= s \Gamma(s)^\alpha \mathbb{E}\{Z^{s-1}\} c_\alpha^s \lim_{N \rightarrow \infty} \prod_{k=1}^N \frac{\phi_\alpha(k)}{\phi_\alpha(s+k-1)} \lim_{N \rightarrow \infty} \frac{1}{\phi_\alpha(s+N)} \\ &= s \Gamma(s)^\alpha \mathbb{E}\{Z^{s-1}\} c_\alpha^{s-1} \lim_{N \rightarrow \infty} \prod_{k=1}^N \frac{\phi_\alpha(k)}{\phi_\alpha(s+k-1)} = s\psi(s), \end{aligned}$$

for any  $s > 0$ , and  $\psi(1) = 1$ . It suffices to prove that  $\psi(s) = \Gamma(s)$ . Bohr-Mollerup's theorem implies that it is sufficient to prove that  $\log(\psi(s))$  is convex. From (2.9) we have that

$$\log \psi(s) = (s-1)(\log c_\alpha - \gamma\alpha) + \log \mathbb{E}\{Z^{s-1}\} + \sum_{k=1}^{\infty} \left( \frac{\alpha}{k}(s-1) + \log \phi(k) - \log \phi(s+k-1) \right).$$

The first term is linear and always convex. As in [20] we can conclude that  $\log \mathbb{E}\{Z^{s-1}\}$  is convex: Since  $Z$  has some exponential moments, the second derivative of  $\log \mathbb{E}\{Z^{s-1}\}$  exists and is equal to

$$\frac{\mathbb{E}\{Z^{s-1}\} \mathbb{E}\{Z^{s-1}(\log Z)^2\} - (\mathbb{E}\{Z^{s-1}(\log Z)\})^2}{\mathbb{E}\{Z^{s-1}\}^2},$$

which is nonnegative by Cauchy-Schwarz's inequality.

Further, note that  $\phi(s)$  is concave since from (2.5) its derivative is  $d + \int_0^\infty te^{-st} \nu(dt)$ , which is decreasing. This implies that  $-\log \phi(s+k-1)$  is convex for any  $k$  and hence each term in the infinite sum is convex. Since sums of convex functions are convex as well, we can conclude that  $\log \psi(s)$  is indeed convex.  $\square$

#### *Lévy processes with no positive jumps*

The second case which allows an explicit moment analysis arises when  $X(t)$  has no positive jumps. We exploit a certain identity for the distribution of  $Z$  in terms of the exponential functional of a certain subordinator, which satisfies the assumptions of Proposition 2.2.

**Proposition 2.3.** *Suppose that  $X(t)$  has no positive jumps with drift  $d > 0$  and diffusion coefficient  $\sigma$ . Suppose further that  $\mu = \mathbb{E}\{X(1)\} \in (0, \infty)$  is finite. Define  $\bar{s} = \sup\{s : \phi(s) > 0\}$  and define  $\alpha = 1$  if  $X(t)$  has a Brownian component and 0 otherwise. Then, for all  $s < \bar{s}$ , except the non-negative integers,*

$$\mathbb{E}\{Z^s\} = \mu c_\alpha^{-(s+1)} e^{\gamma\alpha(s+1)} \prod_{k=1}^{\infty} e^{-\frac{(s+1)\alpha}{k}} \frac{\phi(-k)}{\phi(s+1-k)} \frac{k-s-1}{k}, \quad (2.10)$$

with  $c_\alpha$  specified by  $c_0 = d$ , and  $c_1 = \sigma^2/2$ .

The above formula fails for the non-negative integers as  $\frac{0}{0}$  appears as a factor. However, in that case, (2.4) gives the required formula.

*Proof of Proposition 2.3.* Let  $H(t)$  be the ladder height process associated with  $-X(t)$ . Then  $H(t)$  is a subordinator with Laplace exponent  $\theta(s) = \phi(-s)/(-s)$ . Let  $Z_H$  be the exponential functional associated with  $H(t)$ . Let finally  $M = \sup_{t>0}(-X(t))$  and  $E_1$  an exponentially distributed random variable with mean 1. Note that  $M$  is a finite random variable since  $X$  drifts off to  $\infty$ . Then the following remarkable identity, due to Bertoin & Yor [4] holds:

$$Z/Z_H \stackrel{d}{=} e^M/E_1, \quad (2.11)$$

where the random variables on both sides are independent pairs.

Let  $s > 0$ . Then we have

$$\mathbb{E}\{Z^{-s}\} = \frac{\mathbb{E}\{e^{-sM}\} \Gamma(s+1)}{\mathbb{E}\{Z_H^s\}} = \frac{\mu \Gamma(s+1)}{\theta(s) \mathbb{E}\{Z_H^s\}},$$

since  $\mathbb{E}\{e^{sM}\} = \mu/\theta(s)$ , cf. Equation (VII.3) and Theorem VII.8 in [2]. So it suffices to compute  $\mathbb{E}\{Z_H^s\}$  only. For this we use Proposition 2.2. We first check whether the assumptions are satisfied. Since  $X(t)$  has no positive jumps, its exponent defined as in (2.1) satisfies

$$\phi(s) = ds - \frac{\sigma^2}{2} s^2 - \int_0^\infty (e^{sx} - 1) \nu(dx) = ds - \frac{\sigma^2}{2} s^2 - s \int_0^\infty e^{sx} \nu(x, \infty) dx \quad (2.12)$$

with the measure  $\nu$  satisfying  $\int_0^\infty (1 \wedge x) \nu(dx) < \infty$ . Moreover, since  $X(t) \rightarrow \infty$  and  $X(t)$  has no positive jumps, we also have  $\int_0^\infty x \nu(dx) < d$ . Then we can write

$$\theta(s) = \bar{d} + \frac{1}{2} \sigma^2 s + \int_0^\infty (1 - e^{-sx}) \nu(x, \infty) dx$$

with  $\bar{d} = d - \int_0^\infty x \nu(dx) > 0$ . Thus  $H$  is a subordinator killed at rate  $\bar{d}$  with drift  $\sigma^2/2$  and Lévy measure with density  $\nu(x, \infty)$ . So the assumptions of Proposition 2.2 hold (where  $\phi$  is replaced by  $\theta$ ) with  $\alpha$  and  $c_\alpha$  as defined. We then obtain from Proposition 2.2, for  $s > 0$ ,

$$\mathbb{E}\{Z_H^s\} = \Gamma(s+1)^{1-\alpha} c_\alpha^{-s} \prod_{k=1}^{\infty} \frac{\theta(s+k)}{\theta(k)} \left(\frac{k}{s+k}\right)^\alpha.$$

Consequently, we get for  $s > 0$ ,

$$\mathbb{E}\{Z^{-s}\} = \frac{\mu}{\theta(s)} \Gamma(s+1)^\alpha c_\alpha^s \prod_{k=1}^{\infty} \frac{\theta(k)}{\theta(s+k)} \left(\frac{s+k}{k}\right)^\alpha$$



$$\begin{aligned}
&= \mu \Gamma(s)^\alpha c_\alpha^s \prod_{k=1}^{\infty} \frac{\theta(k)}{\theta(s+k-1)} \left( \frac{s+k-1}{k} \right)^\alpha \\
&= \mu c_\alpha^{s-1} e^{-\gamma\alpha(s-1)} \prod_{k=1}^{\infty} e^{\frac{(s-1)\alpha}{k}} \frac{\theta(k)}{\theta(k+s-1)}
\end{aligned}$$

using Weierstrass' representation (2.2) for the Gamma function. This proves Proposition 2.3 for  $s < 0$ , and it is extended to all  $s < \bar{s}$  using the recursion Lemma 2.1.  $\square$

We can also recover the formula for negative moments from (2.10), also obtained by Bertoin and Yor [4] in Proposition 2:

$$\mathbb{E} \{ Z^{-n} \} = \mu \frac{\prod_{k=1}^{n-1} (-\phi(-k))}{(n-1)!}.$$

Assume first that the diffusion component of  $X$  is absent. Then  $\alpha = 0$  and

$$\mathbb{E} \{ Z^{-n} \} = \mu d^{n-1} \lim_{N \rightarrow \infty} \prod_{k=1}^N \frac{\theta(k)}{\theta(n+k-1)} = \mu \frac{d^{n-1} \prod_{k=1}^{n-1} \theta(k)}{\prod_{k=1}^{n-1} \lim_{N \rightarrow \infty} \theta(N+k)} = \mu \frac{\prod_{k=1}^{n-1} (-\phi(-k))}{(n-1)!},$$

since  $\theta(N) \rightarrow c_0 = d$  as  $N \rightarrow \infty$  in absence of diffusion component. Next assume the diffusion component is present with diffusion coefficient  $\sigma$ . Then, using Weierstrass' representation (2.2), we have,

$$\begin{aligned}
\mathbb{E} \{ Z^{-n} \} &= \mu \left( \frac{\sigma^2}{2} \right)^{n-1} (n-1)! \lim_{N \rightarrow \infty} \prod_{k=1}^N \frac{\phi_2(-k)}{\phi_2(-(n+k-1))} \\
&= \mu (n-1)! \prod_{k=1}^{n-1} \phi_2(-k) \prod_{k=1}^{n-1} \frac{\frac{\sigma^2}{2}}{\lim_{N \rightarrow \infty} \phi_2(-N-k)} \\
&= \mu (n-1)! \prod_{k=1}^{n-1} (-\phi_2(-k)),
\end{aligned}$$

since from (2.12), we have, as  $s \rightarrow \infty$ ,

$$\phi_2(-s) = \frac{\phi(-s)}{s^2} = -\frac{d}{s} - \frac{\sigma^2}{2} - \frac{1}{s} \int_0^\infty e^{-sx} \nu(x, \infty) dx \rightarrow -\frac{\sigma^2}{2},$$

and then

$$\mathbb{E} \{ Z^{-n} \} = \mu \frac{\prod_{k=1}^{n-1} (-\phi(-k))}{(n-1)!}.$$

### 3 The Cramér case and related results

As mentioned in Section 1, tail asymptotics for  $Z$  under Cramér's condition have been studied by Rivero [26]. The focus of this section is to obtain more appealing expressions for the prefactor in the tail asymptotics, which are of Pareto type. We first relate the prefactor to a possibly fractional moment of  $Z$  and then, using results from the previous section, give explicit expressions in terms of  $\phi(\cdot)$  in the case that  $X(t)$  has no negative jumps.

**Theorem 3.1.** *Suppose that the distribution of  $X(1)$  is non-arithmetic and suppose there exists a solution  $\kappa > 0$  to the equation  $\phi(s) = 0$ , such that  $\phi'(\kappa) \in (-\infty, 0)$ . Also assume that  $\mathbb{E}\{X(1)\}$  is positive and finite. Then*

$$\mathbb{P}\{Z > x\} \sim \frac{\mathbb{E}\{Z^{\kappa-1}\}}{-\phi'(\kappa)} x^{-\kappa}.$$

The form of the prefactor was also given in [26] under the assumption  $0 < \kappa < 1$ . Our proof below allows for all values of  $\kappa$ .

*Proof.* For completeness we give the full argument. By Hölder's inequality,  $\phi$  is concave and  $\kappa > 0$  is unique.

As in [26], we apply Theorem 4.1 of [17]. According to that result, it suffices to show that  $\mathbb{E}\{\zeta(\varepsilon)^\kappa\} < \infty$ , where  $\zeta(\varepsilon) = \int_0^\varepsilon e^{-X(u)} du$ .

Since  $\phi'(\kappa)$  is negative and  $\phi(\kappa) = 0$ , choose  $c > 1$  such that  $\phi(\kappa/c) > 0$ . Hence

$$\begin{aligned} \mathbb{E}\{\zeta(\varepsilon)^\kappa\} &\leq \varepsilon^\kappa \mathbb{E}\left\{\sup_{0 \leq u \leq \varepsilon} e^{-\kappa X(u)}\right\} = \varepsilon^\kappa \mathbb{E}\left\{\left(\sup_{0 \leq u \leq \varepsilon} e^{-\frac{\kappa}{c} X(u)}\right)^c\right\} \\ &\leq \varepsilon^\kappa \mathbb{E}\left\{\left(\sup_{0 \leq u \leq \varepsilon} e^{-\left(\frac{\kappa}{c} X(u) - u\phi\left(\frac{\kappa}{c}\right)\right)}\right)^c\right\} \\ &\leq \varepsilon^\kappa \left(\frac{c}{c-1}\right)^c \mathbb{E}\left\{e^{-\kappa X(\varepsilon)}\right\} e^{c\varepsilon\phi\left(\frac{\kappa}{c}\right)} = \varepsilon^\kappa \left(\frac{c}{c-1}\right)^c e^{c\varepsilon\phi\left(\frac{\kappa}{c}\right)} < \infty. \end{aligned}$$

The last inequality holds by virtue of Doob's  $L_p$  inequality, as  $\{e^{-\frac{\kappa}{c} X(u) + u\phi\left(\frac{\kappa}{c}\right)}\}$  is a martingale.

Then, using Theorem 4.1 of [17], it easily follows that

$$\mathbb{P}\{Z > x\} \sim Cx^{-\kappa}, \tag{3.1}$$

where

$$C = \frac{\mathbb{E}\{Z^\kappa - (Z - \zeta(\varepsilon))^\kappa\}}{-\kappa \mathbb{E}\{e^{-\kappa X(\varepsilon)} X(\varepsilon)\}} = \frac{\mathbb{E}\{Z^\kappa - (Z - \zeta(\varepsilon))^\kappa\}}{-\kappa \varepsilon \phi'(\kappa)}$$

is independent of  $\varepsilon$ .

We now continue by simplifying the constant  $C$ . Since  $\zeta'(t) = e^{-X(t)}$ , we can write

$$\begin{aligned} -C\phi'(\kappa) &= \frac{1}{\kappa\varepsilon} \mathbb{E}\{(Z - \zeta(0))^\kappa - (Z - \zeta(\varepsilon))^\kappa\} \\ &= \frac{1}{\varepsilon} \mathbb{E}\left\{\int_0^\varepsilon e^{-X(u)} (Z - \zeta(u))^{\kappa-1} du\right\} \\ &= \frac{1}{\varepsilon} \int_0^\varepsilon \mathbb{E}\left\{e^{-X(u)} (Z - \zeta(u))^{\kappa-1}\right\} du. \end{aligned} \tag{3.2}$$

Now observe that

$$Z - \zeta(u) = e^{-X(u)} \int_u^\infty e^{-(X(s)-X(u))} ds =: e^{-X(u)} \bar{Z},$$

where  $\bar{Z}$  is a copy of  $Z$ , which is also independent of  $e^{-X(u)}$ . Since also  $\mathbb{E}\{e^{-\kappa X(u)}\} = e^{-u\phi(\kappa)} = 1$ , we have from (3.2),

$$-C\phi'(\kappa) = \frac{1}{\varepsilon} \int_0^\varepsilon \mathbb{E}\left\{e^{-\kappa X(u)} \bar{Z}^{\kappa-1}\right\} du = \mathbb{E}\{Z^{\kappa-1}\}.$$

Observe that, using (3.1),  $Z$  has  $(\kappa - 1)$ -st moment finite, if  $\kappa \geq 1$ . If  $0 < \kappa < 1$ , since  $X(1)$  has finite and positive mean, we use Proposition 2 of [4] which states that  $\mathbb{E}\{Z^{-1}\}$  is finite and hence  $\mathbb{E}\{(1/Z)^{1-\kappa}\}$  must be finite, as  $0 < 1 - \kappa < 1$ .  $\square$

Using the moment formulae from the previous section, the prefactor  $C$  in the above theorem can be simplified further by using Proposition 2.3 to express  $\mathbb{E}\{Z^{\kappa-1}\}$  in terms of the Laplace exponent  $\phi$  when  $X(t)$  has no negative jumps.

**Corollary 3.1.** *Suppose  $X(t)$  has no negative jumps with drift  $d$  and diffusion coefficient  $\sigma$ . Let  $\alpha$  be 1 if  $X$  has a Brownian part and 0 otherwise. Define  $c_0 = d$ , and  $c_1 = \sigma^2/2$ . Assume furthermore that there exists a solution  $\kappa > 0$  to the equation  $\phi(s) = 0$  with  $\phi'(\kappa) \in (-\infty, 0)$ . Then*

$$\mathbb{P}\{Z > x\} \sim Cx^{-\kappa},$$

with

$$C = \frac{\mu c_\alpha^{-\kappa} e^{\gamma \alpha \kappa}}{-\phi'(\kappa)} \prod_{k=1}^{\infty} e^{-\frac{\kappa \alpha}{k}} \frac{k - \kappa}{k} \frac{\phi(-k)}{\phi(\kappa - k)},$$

if  $\kappa$  is not a positive integer, and

$$C = \frac{1}{-\phi'(\kappa)} \frac{(\kappa - 1)!}{\prod_{k=1}^{\kappa-1} \phi(k)}$$

otherwise.

## 4 Subexponential jumps

Throughout this section we set  $\bar{X}(t) = -X(t)$ . We assume that  $\mathbb{E}\{\bar{X}(t)\} = -\mu t$ ,  $\mu \in (0, \infty)$  and that  $\bar{G}(x) = \min\{1, \int_x^\infty \mathbb{P}\{\bar{X}(1) > u\} du\}$  is subexponential. The latter condition is equivalent to the requirement that  $\min\{1, \int_x^\infty \bar{\nu}(u, \infty) du\}$  is subexponential, with  $\bar{\nu}$  the Lévy measure of  $\bar{X}$ , cf. [13]. For further background on heavy-tailed distributions we refer to Embrechts et al. [14].

The main result of this section is the following theorem:

**Theorem 4.1.** *If  $\bar{G}(x)$  is subexponential and if  $\mathbb{E}\{\bar{X}(1)\} = -\mu$ ,  $\mu \in (0, \infty)$ , then*

$$\mathbb{P}\left\{\log \int_0^\infty e^{\bar{X}(u)} du > x\right\} \sim \mathbb{P}\left\{\sup_{t>0} \bar{X}(t) > x\right\} \sim \frac{1}{\mu} \bar{G}(x). \quad (4.1)$$

This theorem is proven in a number of steps. We first derive an asymptotic upper bound for the tail behavior of

$$\log Z = \log \int_0^\infty e^{\bar{X}(u)} du.$$

Our proof is inspired by a recent study of Zachary [31], who gave a proof of Veraverbeke's [28] theorem without the use of Wiener-Hopf factorization identities. Like in [31], we define a sequence of stopping times  $\{\sigma_n, n \geq 1\}$  as follows. Choose  $\varepsilon \in (0, \mu)$  and let  $A$  be some large constant. Let furthermore  $\sigma_0 = 0$  and

$$\sigma_n = \inf\{t > \sigma_{n-1} : \bar{X}(t) - \bar{X}(\sigma_{n-1}) \geq -(\mu - \varepsilon)(t - \sigma_{n-1}) + A\},$$

with  $\sigma_n = \infty$  if  $\sigma_{n-1} = \infty$ . Define further  $N = \max\{n : \sigma_n < \infty\}$  and let for  $n \geq 1$ ,  $Y_n$  have the same distribution as the conditional distribution of  $\bar{X}(\sigma_n) - \bar{X}(\sigma_{n-1})$  given the event  $\{\sigma_n < \infty\}$ . Finally  $C = \log \frac{e^A}{\mu - \varepsilon}$ . We can now present the following important distributional inequality.

**Lemma 4.1.**

$$\log Z \stackrel{d}{\leq} C + \sum_{i=1}^N [C + Y_i^+].$$

*Proof.* Write, as in (1.1),

$$Z \stackrel{d}{=} \int_0^{\sigma_1} e^{\bar{X}(u)} du + e^{\bar{X}(\sigma_1)} Z,$$

and observe that the first term on the right hand side is less than  $e^A/(\mu - \varepsilon)$ . Since  $e^{\bar{X}(\sigma_1)} = 0$  if  $\sigma_1 = \infty$  we obtain the upper bound

$$Z \stackrel{d}{\leq} e^C + e^{\bar{X}(\sigma_1)} Z e^C + e^{\bar{X}(\sigma_1)} Z I(\sigma_1 < \infty).$$

This implies,

$$\log Z \stackrel{d}{\leq} C + [\bar{X}(\sigma_1)^+ + \log^+ Z] I(\sigma_1 < \infty)$$

and since the right hand side is positive, we have

$$\log^+ Z \stackrel{d}{\leq} C + [\bar{X}(\sigma_1)^+ + \log^+ Z] I(\sigma_1 < \infty),$$

where  $\log^+ x = \max(0, \log x)$ . Iterating this inequality then yields

$$\log Z \leq \log^+ Z \leq C + \sum_{n=1}^{\infty} [(\bar{X}(\sigma_n) - \bar{X}(\sigma_{n-1}))^+ + C] I(\sigma_n < \infty)$$

implying the assertion of the lemma.  $\square$

The second step of our analysis is to investigate the tail behavior of  $Y_1$ . Define  $q = \mathbb{P}\{\sigma_1 < \infty\}$ . Note that  $\mathbb{P}\{N = n\} = q^n(1 - q)$ , for  $n \geq 0$ . Recall that  $Y_1$  depends on  $\varepsilon$ .

**Lemma 4.2.** *If  $\bar{G}$  is long-tailed, then*

$$\limsup_{x \rightarrow \infty} \frac{\mathbb{P}\{Y_1 > x\}}{\bar{G}(x)} \leq \frac{1}{q(\mu - \varepsilon)}.$$

*Proof.* Write

$$\begin{aligned} \mathbb{P}\{Y_1 > x\} &= \frac{1}{q} \sum_{n=0}^{\infty} \mathbb{P}\{\bar{X}(\sigma_1) > x; n < \sigma_1 \leq n+1\} \\ &\leq \frac{1}{q} \sum_{n=0}^{\infty} \mathbb{P}\left\{ \sup_{n < u \leq n+1} \bar{X}(u) > x; \bar{X}(n) < -(\mu - \varepsilon)n + A \right\} \\ &\leq \frac{1}{q} \sum_{n=0}^{\infty} \mathbb{P}\left\{ \sup_{0 < u \leq 1} \bar{X}(u) > x + (\mu - \varepsilon)n - A \right\}. \end{aligned}$$

We now invoke the following result which is stated as Lemma 1 in Willekens [29]. For  $u > 0$  and any  $u_0 \in (0, u)$ ,

$$\mathbb{P} \left\{ \sup_{0 < s \leq 1} \bar{X}(s) > u \right\} \mathbb{P} \left\{ \inf_{0 < s \leq 1} \bar{X}(s) \geq -u_0 \right\} \leq \mathbb{P} \{ \bar{X}(1) > u - u_0 \}. \quad (4.2)$$

Thus, setting  $H(u_0) = 1/\mathbb{P} \{ \inf_{0 < s < 1} \bar{X}(s) \geq -u_0 \}$  we obtain, for  $0 < u_0 < x - B$ ,

$$\begin{aligned} \mathbb{P} \{ Y_1 > x \} &\leq \frac{H(u_0)}{q} \sum_{n=0}^{\infty} \mathbb{P} \{ \bar{X}(1) > x + (\mu - \varepsilon)n - A - u_0 \} \\ &\sim \frac{H(u_0)}{q(\mu - \varepsilon)} \bar{G}(x). \end{aligned}$$

Since this holds for any  $u_0$  as  $x \rightarrow \infty$ , and  $H(u_0) \rightarrow 1$  as  $u_0 \rightarrow \infty$ , we are done.  $\square$

We are now ready to prove the desired upper bound.

**Proposition 4.1.** *If  $\bar{G}(x)$  is subexponential, then*

$$\limsup_{x \rightarrow \infty} \frac{\mathbb{P} \{ \log Z > x \}}{\bar{G}(x)} \leq \frac{1}{\mu}.$$

*Proof.* By Lemma 4.2 and long-tailedness of  $\bar{G}(x)$ ,  $Y_i^+ + C$  is stochastically dominated by a subexponential random variable which has tail  $\bar{G}(x)/q(\mu - \varepsilon)$ . Combining Lemmas 4.1 and 4.2 with a well-known result for geometric random sums with subexponential summands (see e.g. Corollary A3.21 in [14]), we obtain

$$\begin{aligned} \limsup_{x \rightarrow \infty} \frac{\mathbb{P} \{ \log Z > x \}}{\bar{G}(x)} &\leq \limsup_{x \rightarrow \infty} \frac{\mathbb{P} \left\{ C + \sum_{i=1}^N (C + Y_i^+) > x \right\}}{\bar{G}(x)} \\ &\leq \frac{1}{q(\mu - \varepsilon)} \mathbb{E} \{ N \} = \frac{1}{\mu - \varepsilon} \frac{1}{1 - q}. \end{aligned}$$

Now first let  $A \rightarrow \infty$  (so that  $q \rightarrow 0$ ) and then  $\varepsilon \rightarrow 0$ .  $\square$

This concludes the proof of the asymptotic upper bound. We now continue with a lower bound. The proof of the lower bound relies on the following result, which seems to be new in the present setting.

**Lemma 4.3.** *Define  $\tau_d(x) = \inf \{ n \in \mathbb{N} : \bar{X}(n) \geq x \}$ . For every  $y \in \mathbb{R}$ ,*

$$\lim_{x \rightarrow \infty} \mathbb{P} \{ \bar{X}(\tau_d(x)) - x > y \mid \tau_d(x) < \infty \} = 1$$

*if  $\mathbb{P} \{ \tau_d(x) < \infty \}$  is long-tailed (as function of  $x$ ).*

*Proof.* The result is obvious for  $y \leq 0$ . So assume  $y > 0$ . Observe that

$$\begin{aligned} \mathbb{P} \{ \tau_d(x + y) < \infty \mid \tau_d(x) < \infty \} &= \mathbb{P} \{ \tau_d(x + y) = \tau_d(x) \mid \tau_d(x) < \infty \} + \\ &\quad \mathbb{P} \{ \tau_d(x) < \tau_d(x + y) < \infty \mid \tau_d(x) < \infty \}. \end{aligned}$$

Since

$$\mathbb{P} \{ \tau_d(x + y) = \tau_d(x) \mid \tau_d(x) < \infty \} = \mathbb{P} \{ \bar{X}(\tau_d(x)) - x > y \mid \tau_d(x) < \infty \}$$

and  $\mathbb{P}\{\tau_d(x) < \infty\}$  is long-tailed, it suffices to show that the second term converges to 0 as  $x \rightarrow \infty$ . Thus, write

$$\begin{aligned}
& \mathbb{P}\{\tau_d(x) < \tau_d(x+y) < \infty \mid \tau_d(x) < \infty\} \\
&= \mathbb{P}\{\tau_d(x+y) < \infty; \bar{X}(\tau_d(x)) < x+y \mid \tau_d(x) < \infty\} \\
&= \int_0^{y^-} \mathbb{P}\{\tau_d(x+y) < \infty \mid \bar{X}(\tau_d(x)) - x = u; \\
&\quad \tau_d(x) < \infty\} \mathbb{P}\{\bar{X}(\tau_d(x)) - x \in du \mid \tau_d(x) < \infty\} \\
&= \int_0^{y^-} \mathbb{P}\{\tau_d(y-u) < \infty\} \mathbb{P}\{\bar{X}(\tau_d(x)) - x \in du \mid \tau_d(x) < \infty\} \\
&\leq \int_0^{y^-} \mathbb{P}\{\tau_d(0) < \infty\} \mathbb{P}\{\bar{X}(\tau_d(x)) - x \in du \mid \tau_d(x) < \infty\} \\
&= \mathbb{P}\{\tau_d(0) < \infty\} \mathbb{P}\{\bar{X}(\tau_d(x)) - x < y \mid \tau_d(x) < \infty\} \\
&= \mathbb{P}\{\tau_d(0) < \infty\} \mathbb{P}\{\tau_d(x) < \tau_d(x+y) < \infty \mid \tau_d(x) < \infty\} + \\
&\quad \mathbb{P}\{\tau_d(0) < \infty\} \mathbb{P}\{\tau_d(x+y) = \infty \mid \tau_d(x) < \infty\}.
\end{aligned}$$

Hence,

$$\mathbb{P}\{\tau_d(x) < \tau_d(x+y) < \infty \mid \tau_d(x) < \infty\} \leq \frac{\mathbb{P}\{\tau_d(0) < \infty\}}{\mathbb{P}\{\tau_d(0) = \infty\}} \mathbb{P}\{\tau_d(x+y) = \infty \mid \tau_d(x) < \infty\}.$$

Observe now that  $\mathbb{P}\{\tau_d(0) < \infty\} \in (0, 1)$  and that  $\mathbb{P}\{\tau_d(x+y) = \infty \mid \tau_d(x) < \infty\} \rightarrow 0$  since  $\mathbb{P}\{\tau_d(x) < \infty\}$  is long-tailed, which completes the proof.  $\square$

The above Lemma states that the overshoot  $\bar{X}(\tau_d(x)) - x$  converges to  $\infty$  as  $x \rightarrow \infty$ . This is exactly what is needed in the proof of the lower bound:

**Proposition 4.2.** *Let  $\bar{G}(x)$  and  $\mathbb{P}\{\tau_d(x) < \infty\}$  be long-tailed. Then*

$$\liminf_{x \rightarrow \infty} \frac{\mathbb{P}\{\log Z > x\}}{\bar{G}(x)} \geq \frac{1}{\mu}.$$

It is not known to us whether long-tailedness of  $\bar{G}(x)$  implies long-tailedness of  $\mathbb{P}\{\tau_d(x) < \infty\}$ , but both conditions are satisfied if  $\bar{G}(x)$  is subexponential, cf. Veraverbeke's [28] theorem.

*Proof.* Let  $\hat{Z}$  be an independent copy of  $Z$ , independent of  $\bar{X}(t), t \geq 0$ . Write

$$\begin{aligned}
\mathbb{P}\{\log Z > x\} &\geq \mathbb{P}\{\log Z > x; \tau_d(x) < \infty\} \\
&\geq \mathbb{P}\left\{\log \int_{\tau_d(x)}^{\infty} e^{\bar{X}(t)} dt > x; \tau_d(x) < \infty\right\}.
\end{aligned}$$

Using the strong Markov property, this is equal to

$$\mathbb{P}\left\{\bar{X}(\tau_d(x)) + \log \hat{Z} > x; \tau_d(x) < \infty\right\} \sim \mathbb{P}\{\tau_d(x) < \infty\},$$

using the previous Lemma, since  $\mathbb{P}\{\tau_d(x) < \infty\}$  is long-tailed. Finally, since  $\mathbb{P}\{\tau_d(x) < \infty\} = \mathbb{P}\{\sup_{n \in \mathbb{N}} \bar{X}(n) \geq x\}$ . We conclude, using Veraverbeke's Theorem (see e.g. Theorem 1(i) of [31]),

$$\liminf_{x \rightarrow \infty} \frac{\mathbb{P}\{\log Z > x\}}{\bar{G}(x)} \geq \liminf_{x \rightarrow \infty} \frac{\mathbb{P}\{\sup_{n \geq 1} \bar{X}(n) > x\}}{\bar{G}(x)} = \frac{1}{\mu}.$$

proving our assertion.  $\square$

The above results imply that  $\mathbb{P}\{\log Z > x\} \sim (1/\mu)\bar{G}(x)$ . To conclude the proof of Theorem 4.1, we need to show the appealing asymptotic form  $\mathbb{P}\{\log Z > x\} \sim \mathbb{P}\{\sup_{t>0} \bar{X}(t) > x\}$ . For this, it suffices to show that

$$\mathbb{P}\left\{\sup_{t>0} \bar{X}(t) > x\right\} \sim \mathbb{P}\left\{\sup_{n\geq 1} \bar{X}(n) > x\right\}, \quad (4.3)$$

since, due to Veraverbeke's theorem, the latter supremum is tail-equivalent to  $(1/\mu)\bar{G}(x)$ . Surprisingly enough, we could not find this result in the literature. Asmussen [1, Corollary 2.5] only proves a version of Veraverbeke's theorem for continuous time under the assumption that the jump process associated to the Lévy process has bounded variation. A recent paper [22] relates the tail of the supremum to that of the ladder height process. The following result settles the issue in complete generality, since subexponentiality of  $\bar{G}$  implies subexponentiality of  $\sup_{n\geq 1} \bar{X}(n)$ , using Veraverbeke's [28] theorem. For a more general discussion, see the forthcoming paper [15].

**Proposition 4.3.** *The following are equivalent:*

1.  $\sup_{t>0} \bar{X}(t)$  is long-tailed,
2.  $\sup_{n\geq 1} \bar{X}(n)$  is long-tailed.

Moreover, both imply (4.3).

*Proof.* We use an argument similar to that in Willekens [29]. Set  $\tau(x) = \inf\{t : \bar{X}(t) \geq x\}$ . Note that, for any  $x_0 > 0$ ,

$$\mathbb{P}\left\{\sup_{t>0} \bar{X}(t) > x\right\} \leq \mathbb{P}\left\{\sup_{n\geq 1} \bar{X}(n) > x - x_0\right\} + \mathbb{P}\left\{\sup_{t>0} \bar{X}(t) > x; \sup_{n\geq 1} \bar{X}(n) \leq x - x_0\right\}.$$

The second term on the right hand side is clearly smaller than

$$\begin{aligned} \mathbb{P}\left\{\tau(x) < \infty; \inf_{s \in [\tau(x), \tau(x)+1]} \bar{X}(s) - \bar{X}(\tau(x)) \leq -x_0\right\} \\ = \mathbb{P}\{\tau(x) < \infty\} \mathbb{P}\left\{\inf_{0 < s < 1} \bar{X}(s) \leq -x_0\right\}, \end{aligned}$$

where we used the strong Markov property in the last step. Combining the two formulas and noting that  $\sup_{t>0} \bar{X}(t) \geq \sup_{n\geq 1} \bar{X}(n)$  we obtain for any  $x_0 > 0$ ,

$$\begin{aligned} \mathbb{P}\left\{\sup_{t>0} \bar{X}(t) > x - x_0\right\} &\geq \mathbb{P}\left\{\sup_{n\geq 1} \bar{X}(n) > x - x_0\right\} \\ &\geq \mathbb{P}\left\{\sup_{t>0} \bar{X}(t) > x\right\} \mathbb{P}\left\{\inf_{0 < s < 1} \bar{X}(s) > -x_0\right\}. \end{aligned}$$

With this result, it is easy to see that both 1. and 2. imply (4.3) and hence 1. and 2. imply each other.  $\square$

We would like to remark that the equivalence (4.3) does not require that the mean of  $\bar{X}(1)$  exists. Thus the explicit results of [10] can be combined with Proposition 4.3 to obtain tail asymptotics for  $\sup_{t>0} \bar{X}(t)$  when  $\mathbb{E}\{\bar{X}(1)\}$  is not finite.

## 5 A compound Poisson process

In this section we assume that  $X(t)$  is a type of subordinator, in particular, a compound Poisson process with positive jumps, and prove the following result:

**Theorem 5.1.** *Assume that  $X(t) = \sum_{i=1}^{N(t)} B_i$ , where  $N(t)$  is a Poisson process with rate  $\lambda$  and  $\{B_i, i \geq 1\}$  is an i.i.d. sequence of non-negative random variables. Then*

$$\mathbb{P}\{Z > x\} \sim \mathbb{E}\left\{e^{\lambda e^{-B_1} Z}\right\} e^{-\lambda x} \quad (5.1)$$

if and only if  $\mathbb{E}\left\{e^{\lambda e^{-B_1} Z}\right\} < \infty$ , which is the case if and only if  $\mathbb{E}\{1/B_1\} < \infty$ .

Various cases in which the assumptions of this theorem fail are treated in the next section. Under certain additional assumptions, the prefactor in the above theorem may be expressed in terms of  $q$ -hypergeometric functions using techniques as in [20]; we omit the details.

To prove Theorem 5.1, we consider the following set-up: we use the random equation  $R \stackrel{d}{=} Q + MR$ . If we use the stopping time  $\bar{\tau}$ , which is the first jump time of the compound Poisson process  $X(t)$ , then the above random equation becomes

$$\lambda Z \stackrel{d}{=} \lambda \bar{\tau} + e^{-B_1}(\lambda Z), \quad (5.2)$$

where  $Z$  on the right hand side is independent of  $\bar{\tau}$  and  $B_1$ . Thus we may assume that  $\mathbb{P}\{0 < M < 1\} = 1$  and that  $Q$  has an exponential distribution such that the tail of  $e^Q$  has unit Pareto distribution.

We aim to find conditions under which the following analogue of Breiman's Theorem holds:

$$\mathbb{P}\{R > x\} \sim \mathbb{E}\{e^{MR}\} \mathbb{P}\{Q > x\}. \quad (5.3)$$

Breiman's [8] theorem states that, if  $U$  is regularly varying of index  $-\nu, \nu > 0$ , and  $V$  is independent of  $U$  such that  $\mathbb{E}\{V^{\nu+\delta}\} < \infty$  for some  $\delta > 0$ , then  $\mathbb{P}\{UV > x\} \sim \mathbb{E}\{V^\nu\} \mathbb{P}\{U > x\}$ . As mentioned in Section 1, this result becomes relevant after writing  $e^R \stackrel{d}{=} e^Q e^{MR}$ . Fortunately, the 'extra'  $\delta$  is not necessary when  $U$  has a Pareto distribution. We can even get a necessary and sufficient condition for the equivalence. This is summarized in the next Lemma:

**Lemma 5.1.** *Let  $U$  be independent of the non-negative random variable  $V$  with  $\mathbb{P}\{U > x\} = x^{-\alpha}, x \geq 1$ , where  $\alpha > 0$ . Then  $\mathbb{P}\{UV > x\} = O(x^{-\alpha})$  if and only if  $\mathbb{E}\{V^\alpha\} < \infty$ , in which case*

$$\mathbb{P}\{UV > x\} \sim \mathbb{E}\{V^\alpha\} x^{-\alpha}.$$

*Proof.* Clearly, we have,

$$\mathbb{P}\{UV > x\} = \int_0^\infty \mathbb{P}\left\{U > \frac{x}{y}\right\} d\mathbb{P}\{V \leq y\} = \frac{1}{x^\alpha} \int_0^x y^\alpha d\mathbb{P}\{V \leq y\} + \mathbb{P}\{V > x\}.$$

Since, the first term on the right side is increasing and positive,  $x^\alpha \mathbb{P}\{UV > x\}$  is bounded if and only if  $\mathbb{E}\{V^\alpha\}$  is finite and  $x^\alpha \mathbb{P}\{V > x\}$  is bounded. This proves the "only if" part. Now if  $\mathbb{E}\{V^\alpha\}$  is finite, then  $x^\alpha \mathbb{P}\{V \geq x\} \rightarrow 0$  and hence  $\mathbb{P}\{UV > x\} \sim \mathbb{E}\{V^\alpha\} x^{-\alpha}$ .  $\square$



We apply this to our random equation (5.2). The first step is to derive a criterion for  $\mathbb{E}\{e^{MR}\}$  to be finite. This is provided by the following proposition.

**Proposition 5.1.** *Assume  $\mathbb{P}\{M = 1\} = 0$ . The following are equivalent:*

1.  $\mathbb{E}\{e^{MR}\} < \infty$ ,
2.  $\mathbb{E}\{e^{MQ}\} < \infty$ .

*Proof.* That 1. implies 2. is trivial. Assume now that  $\mathbb{E}\{e^{MQ}\} < \infty$ . Let  $M_n, n \geq 0$  and  $Q_n, n \geq 0$  be mutually independent i.i.d. copies of  $M$  and  $Q$  respectively. Then we can write

$$MR \stackrel{d}{=} \sum_{k=1}^{\infty} Q_k \prod_{i=1}^k M_i.$$

Further, since  $\mathbb{P}\{M = 1\} < 1$ , there exists an  $\eta \in (0, 1)$  such that  $\mathbb{P}\{M > \eta\} \in (0, 1)$  and  $\mathbb{E}\{e^{MQ}; M > \eta\} < 1$ . Define the sequence of random times  $\bar{\tau}_k, k \geq 0$ , as follows. Let  $\bar{\tau}_0 = 0$ , and, for  $k \geq 1$ ,

$$\bar{\tau}_k = \inf\{n > \bar{\tau}_{k-1} : M_n \leq \eta\}.$$

Then, write

$$MR \stackrel{d}{=} \sum_{k=1}^{\infty} Q_k \prod_{i=1}^k M_i = \sum_{k=1}^{\infty} \sum_{n=\bar{\tau}_{k-1}+1}^{\bar{\tau}_k} Q_n \prod_{i=1}^n M_i \leq \sum_{k=1}^{\infty} \eta^{k-1} \sum_{n=\bar{\tau}_{k-1}+1}^{\bar{\tau}_k} Q_n M_n.$$

Set  $C_k = \sum_{n=\bar{\tau}_{k-1}+1}^{\bar{\tau}_k} Q_n M_n$ . The sequence  $\{C_k, k \geq 1\}$  is i.i.d. Note that

$$\mathbb{E}\{e^{C_k}\} = \sum_{m=1}^{\infty} \mathbb{E}\{e^{MQ}; M > \eta\}^{m-1} \mathbb{E}\{e^{MQ}; M \leq \eta\} = \frac{\mathbb{E}\{e^{MQ}; M \leq \eta\}}{1 - \mathbb{E}\{e^{MQ}; M > \eta\}} < \infty.$$

Since  $\mathbb{E}\{e^{C_1}\} < \infty$ , there exists a finite constant  $K$  such that  $\mathbb{P}\{C_1 > x\} \leq Ke^{-x}$ . Now, define a random variable  $C'$  such that  $\mathbb{P}\{C' > x\} = \min\{1, Ke^{-x}\}$ . Note that, for  $s \in (0, 1)$ ,  $\mathbb{E}\{e^{sC'}\} = \frac{K^s}{1-s}$ . Since we have the stochastic ordering  $C_1 \leq C'$ , and  $x \rightarrow e^{sx}$  is a convex function for any  $s > 0$ , we have  $\mathbb{E}\{e^{sC_1}\} \leq \frac{K^s}{1-s}$ . Thus, it follows that

$$\mathbb{E}\{e^{MR}\} \leq \mathbb{E}\left\{e^{\sum_{k=1}^{\infty} \eta^{k-1} C_k}\right\} = \prod_{k=1}^{\infty} \mathbb{E}\left\{e^{\eta^{k-1} C_k}\right\}.$$

Since  $\mathbb{E}\left\{e^{\eta^{k-1} C_k}\right\} \leq \frac{K\eta^{k-1}}{1-\eta^{k-1}}$ , we obtain for  $k \geq 2$ ,

$$\begin{aligned} \log \mathbb{E}\{e^{MR}\} &\leq \log \mathbb{E}\{e^{C_1}\} + \sum_{k=1}^{\infty} \log \left( \frac{K\eta^k}{1-\eta^k} \right) \\ &= \log \mathbb{E}\{e^{C_1}\} + \sum_{k=1}^{\infty} [\eta^k \log K - \log(1-\eta^k)]. \end{aligned}$$

This sum clearly converges. □

If  $Q$  has an exponential distribution with rate 1, we have

**Lemma 5.2.** *Assume  $\mathbb{P}\{Q > x\} = e^{-x}$  and independent of  $M$ . Then  $\mathbb{E}\{e^{MQ}\} < \infty$  if and only if  $\mathbb{E}\{1/(1-M)\} < \infty$  if and only if  $\mathbb{E}\{1/(-\log M)\} < \infty$ .*

*Proof.* Write

$$\mathbb{E}\{e^{MQ}\} = \int_0^1 \frac{1}{1-m} d\mathbb{P}\{M \leq m\}.$$

The second equivalence is plain from the asymptotic equivalence  $-\log x \sim 1-x$  as  $x \rightarrow 1$ .  $\square$

Theorem 5.1 now follows by combining the results in this section:

*Proof of Theorem 5.1.* We use the above results with  $R = \lambda Z$ ,  $Q = \lambda \bar{\tau}$  and  $M = e^{-B_1}$ . Exponentiating both sides of (5.2), we have

$$e^{\lambda Z} \stackrel{d}{=} e^{\lambda \bar{\tau}} e^{\lambda e^{-B_1} Z}.$$

Since  $\bar{\tau}$  is exponential with rate  $\lambda$ ,  $e^{\lambda \bar{\tau}}$  has a unit Pareto tail. So, using Lemma 5.1, we have the required result if and only if  $\mathbb{E}\{e^{\lambda e^{-B_1} Z}\} < \infty$ . And, finally, since  $\bar{\tau}$  and  $B_1$  are independent, using Proposition 5.1 and Lemma 5.2,  $\mathbb{E}\{e^{\lambda e^{-B_1} Z}\} < \infty$  holds if and only if  $\mathbb{E}\{1/B_1\} < \infty$ .  $\square$

## 6 Other subordinators

The previous section showed that the tail behavior of  $Z$  is exponential if  $X(t)$  is a compound Poisson process with positive jumps  $B_i$  such that  $\mathbb{E}\{1/B_1\} < \infty$ . The goal of the present section is to consider what may happen when these assumptions do not hold.

Consider the case  $\mathbb{E}\{1/B_1\} = \infty$ . This is not an unreasonable assumption, since it is satisfied when  $B_1$  is exponentially distributed with rate  $b$ . This is the same as Example B of Carmona et al. [9] with  $a = \lambda$ ,  $b = b$  and  $c = 0$ . Hence  $Z$  has Gamma distribution with scale parameter  $\lambda$  and shape parameter  $b+1$  and we have

$$\mathbb{P}\{Z > x\} \sim \frac{\lambda^b}{\Gamma(b+1)} x^b e^{-\lambda x}. \quad (6.1)$$

This example and the result in the previous section lead us to conjecture that the tail behavior of  $Z$  may be influenced by the left tail behavior of  $B_1$ . The following Theorem confirms this.

**Theorem 6.1.** *Let  $X(t)$  be a compound Poisson process with rate  $\lambda$  and positive jumps  $\{B_i, i \geq 1\}$  with Laplace-Stieltjes transform  $\beta$ . Suppose that  $\mathbb{P}\{B_1 < x\} \sim bx$  as  $x \downarrow 0$ . Suppose furthermore that*

$$K = \prod_{k=1}^{\infty} (1 - \beta(k)) e^{b/k} \in (0, \infty).$$

*Then*

$$\mathbb{P}\{Z > x\} \sim \frac{1}{K e^{-b\gamma}} (\lambda x)^b e^{-\lambda x}$$

*as  $x \rightarrow \infty$ , where  $\gamma$  is Eulers constant.*

A sufficient condition for  $K \in (0, \infty)$  is that  $\mathbb{P}\{B < x\} - bx = o(x^{1+\delta})$  for some  $\delta > 0$ . As expected from (6.1), the prefactor  $Ke^{-b\gamma}$  indeed reduces to  $\Gamma(b+1)$  by (2.2), if  $\beta(s) = b/(b+s)$ .

The proof of the above proposition is based on an Abelian-Tauberian approach. In particular, we first determine the rate of growth of  $\mathbb{E}\{Z^n\}/n!$  as  $n \rightarrow \infty$ , then apply an Abelian theorem to obtain the behavior of the moment generating function around  $s = \lambda$  and finally relate this to the tail behavior of  $Z$  using a Tauberian argument.

This type of argument seems perfectly fit for the present problem, since explicit expressions for all moments are available, cf. Section 2. Furthermore, a probabilistic technique based on, for example, a change of measure argument seems far from obvious.

*Proof of Theorem 6.1.* Note that  $\phi(s) = \lambda(1 - \beta(s))$ . Using the Abelian theorem for Laplace-Stieltjes transforms, we get  $\beta(s) \sim b/s$  as  $s \rightarrow \infty$ . Consequently, we have  $\log(1 - \beta(s)) \sim -b/s$  as  $s \rightarrow \infty$ . Using (2.4), we also have

$$\mathbb{E}\{Z^n\} = \frac{n!}{\lambda^n \prod_{k=1}^n \frac{\phi(k)}{\lambda}} = \frac{n!}{\lambda^n \prod_{k=1}^n (1 - \beta(k))} =: \frac{n!}{\lambda^n p(n)}.$$

Now observe

$$\begin{aligned} \log p(n) &= \sum_{k=1}^n \log(1 - \beta(k)) \\ &= -b \log n + \sum_{k=1}^n [\log(1 - \beta(k)) + b/k] - b\gamma + o(1) \\ &= -b \log n + \log K - b\gamma + o(1) \end{aligned}$$

as  $n \rightarrow \infty$ . This implies

$$p(n) \sim Ke^{-b\gamma} n^{-b} (1 + o(1)).$$

Consequently, using the direct half of Karamata's theorem (Proposition 1.5.8 in [5], which also applies to sums),

$$\sum_{k=1}^n \frac{1}{p(k)} \sim \frac{1}{b+1} \frac{1}{Ke^{-b\gamma}} n^{b+1}.$$

Now use Corollary 1.7.3 of [5] to conclude that

$$\mathbb{E}\{e^{sZ}\} = \sum_{n=0}^{\infty} \frac{(\frac{s}{\lambda})^n}{p(n)} \sim \frac{\Gamma(b+1)}{Ke^{-b\gamma}} \left(1 - \frac{s}{\lambda}\right)^{-(b+1)}, \quad \text{as } s \uparrow \lambda. \quad (6.2)$$

Define  $\psi(x) = e^{\lambda x} \mathbb{P}\{Z > x\}$ . From (6.2), the Laplace transform of  $\psi$  is given by

$$\frac{1}{\lambda - s} \mathbb{E}\left\{e^{(\lambda-s)Z} - 1\right\} \sim \lambda^b \frac{\Gamma(b+1)}{Ke^{-b\gamma}} s^{-(b+1)}, \quad \text{as } s \downarrow 0.$$

Hence, the Laplace-Stieltjes transform of  $\psi(x)$  behaves like  $\lambda^b \frac{\Gamma(b+1)}{Ke^{-b\gamma}} s^{-b}$  as  $s \downarrow 0$ . Again, from Carmona et al. [9], the density  $k$  of  $Z$  exists everywhere and satisfies the differential equation

$$k(x) = \lambda \int_x^{\infty} k(u) \mathbb{P}\{B_1 > \log(u/x)\} du.$$

This implies that  $k(x) \leq \lambda \mathbb{P}\{Z > x\}$ . Thus,  $\psi'(x) = e^{\lambda x}(\lambda \mathbb{P}\{Z > x\} - k(x)) \geq 0$ , implying that  $\psi(x)$  is a monotone function. This implies, by Karamata's Tauberian theorem (see e.g. [5], Theorem 1.7.1),

$$\psi(x) \sim \frac{(\lambda x)^b}{K e^{-b\gamma}} \quad \text{as } x \rightarrow \infty.$$

□

We finally treat a class of subordinators which are not compound Poisson processes. In particular, we consider subordinators which have Laplace exponents which are regularly varying at infinity:  $\phi(s) = s^\alpha L(s)$ , where  $0 < \alpha < 1$  and  $L$  is a slowly varying function. A special case of this class are the completely right-skewed stable Lévy processes. Since in particular  $\phi(s) \rightarrow \infty$ , the Levy measure of  $X(t)$  has infinite mass, and thus  $X(t)$  leaves 0 immediately. Thus, one can expect tail asymptotics for  $\mathbb{P}\{Z > x\}$  which are considerably lighter than exponential. This is confirmed by the following Theorem, which provides logarithmic (rather than precise) asymptotics.

**Theorem 6.2.** *Suppose  $X(t)$  is a subordinator with Laplace exponent  $\phi(s) = s^\alpha L(s)$ ,  $0 < \alpha < 1$ , with  $L(\cdot)$  slowly varying at infinity. Then, as  $x \rightarrow \infty$ ,*

$$-\log \mathbb{P}\{Z > x\} \sim (1 - \alpha)g^\leftarrow(x), \quad (6.3)$$

with  $g^\leftarrow(x)$  the right-inverse of  $g(x) = x/\phi(x)$ .

*Proof.* The first step is again to obtain the asymptotic behavior of  $\mathbb{E}\{Z^n\}/n!$ . Using the representation of  $\phi(s)$  and (2.4) we can write

$$\mathbb{E}\{Z^n\} = (n!)^{1-\alpha} / \prod_{k=1}^n L(k).$$

Using Karamata's representation theorem for slowly varying functions, we can write  $L(x) = c(x) \exp\{\int_1^x (h(u)/u) du\}$  with  $c(x) \rightarrow c > 0$ , and  $h(u) \rightarrow 0$ . Hence, we obtain,

$$\begin{aligned} \prod_{k=1}^n L(k) &= \left( \prod_{k=1}^n c(k) \right) e^{\sum_{k=1}^n \int_1^k \frac{h(u)}{u} du} = \left( \prod_{k=1}^n c(k) \right) e^{\int_1^n \sum_{k=1}^n \frac{h(u)}{u} I(u < k) du} \\ &= \left( \prod_{k=1}^n c(k) \right) e^{n \int_1^n \frac{h(u)}{u} du} e^{-\int_1^n \frac{h(u)[u]}{u} du} \\ &= (L(n))^n \left( \prod_{k=1}^n \frac{c(k)}{c(n)} \right) e^{-\int_1^n \frac{h(u)[u]}{u} du} \end{aligned}$$

with  $[u]$  the integer part of  $u$ . Using Stirling's formula, we have,

$$\frac{\mathbb{E}\{Z^n\}}{n!} = \frac{(2\pi)^{-\frac{\alpha}{2}} e^{n\alpha} n^{-\alpha(n+\frac{1}{2})}}{(L(n))^n \prod_{k=1}^n \left( \frac{c(k)}{c(n)} \right) e^{-\int_1^n \frac{h(u)[u]}{u} du}} (1 + o(1)),$$

and, since  $h(u) \rightarrow 0$ , we can conclude that

$$\log \frac{\mathbb{E}\{Z^n\}}{n!} = -n(\alpha \log n + \log L(n) - \alpha) + o(n) = -n(\log \phi(n) - \alpha) + o(n). \quad (6.4)$$

We are now ready to obtain the behavior of  $\log \mathbb{E} \{e^{sZ}\}$  as  $s \rightarrow \infty$ . For, any  $m \geq 0$ ,

$$\mathbb{E} \{e^{sZ}\} = \sum_{n=0}^{\infty} s^n \mathbb{E} \{Z^n\} / n! \geq s^m \mathbb{E} \{Z^m\} / m!$$

Now take  $m = \lceil \phi^{\leftarrow}(s) \rceil$ . Then it follows from (6.4), using a straightforward computation, that

$$\liminf_{s \rightarrow \infty} \frac{\log \mathbb{E} \{e^{sZ}\}}{\phi^{\leftarrow}(s)} \geq \alpha + \lim_{s \rightarrow \infty} \log \frac{s}{\phi(\lceil \phi^{\leftarrow}(s) \rceil)} = \alpha.$$

We need to prove a corresponding upper bound to be able to conclude that

$$\log \mathbb{E} \{e^{sZ}\} \sim \alpha \phi^{\leftarrow}(s). \quad (6.5)$$

For this, write

$$\mathbb{E} \{e^{sZ}\} = \sum_{n=0}^{k-1} s^n \mathbb{E} \{Z^n\} / n! + \sum_{n=k}^{\infty} s^n \mathbb{E} \{Z^n\} / n!, \quad (6.6)$$

for  $k = \lceil e^s \rceil$ . Note first that, from (6.4), given  $\varepsilon > 0$ , for all large enough  $n$ , we have

$$\frac{\mathbb{E} \{Z^n\}}{n!} \leq \phi(n)^{-n} e^{(\alpha+\varepsilon)n}. \quad (6.7)$$

Then, given  $\varepsilon > 0$ , for all large enough  $s$ , and hence all large enough  $k$ , the second term on the right hand side of (6.6) is bounded above by

$$\begin{aligned} \sum_{n=k}^{\infty} s^n \phi(n)^{-n} e^{(\alpha+\varepsilon)n} &= \frac{e^{(\alpha+\varepsilon)k} s^k}{\phi(k)^k} \sum_{n=k}^{\infty} \frac{\phi(k)^k}{\phi(n)^n} e^{(\alpha+\varepsilon)(n-k)} s^{(n-k)} \\ &\leq \left( \frac{e^{(\alpha+\varepsilon)} s}{\phi(k)} \right)^k \sum_{n=0}^{\infty} \left( \frac{e^{\alpha+\varepsilon}}{\phi(k)} \right)^n, \end{aligned}$$

since,  $X(t)$  being subordinator,  $\phi$  is increasing. Since  $\phi$  is regularly varying of index  $\alpha > 0$ , we have  $\frac{s}{\phi(k)} \leq \frac{\log(k+1)}{\phi(k)} \rightarrow 0$ . Hence the second term on the right hand side of (6.6) is bounded in  $s$ .

Next we consider the first term on the right hand side of (6.6). We split this term into two further terms with the first having finitely many terms. Fix an arbitrary  $\varepsilon > 0$ . Choose, using (6.4),  $k_\varepsilon$  such that for all  $n \geq k_\varepsilon$ , we have

$$\frac{\mathbb{E} \{Z^n\}}{n!} \leq \phi(n)^{-n} e^{(\alpha+\varepsilon)n}.$$

Then, we have, for any fixed  $K > k_\varepsilon$ , ( $K$  may depend on  $\varepsilon$ , but not on  $s$ )

$$\begin{aligned} \sum_{n=0}^{k-1} \frac{s^n \mathbb{E} \{Z^n\}}{n!} &\leq \sum_{n=0}^{K-1} \frac{s^n \mathbb{E} \{Z^n\}}{n!} + \sum_{n=K}^{k-1} \left( \frac{se^{\alpha+\varepsilon}}{\phi(n)} \right)^n \\ &\leq \sup_{n < K} \frac{\mathbb{E} \{Z^n\}}{n!} s^K + e^s \sup_{x \geq K} \left( \frac{se^{\alpha+\varepsilon}}{\phi(x)} \right)^x \end{aligned}$$

and hence, as  $\phi^\leftarrow$  is regularly varying of index  $1/\alpha > 1$ ,

$$\log \sum_{n=0}^{k-1} \frac{s^n \mathbb{E}\{Z^n\}}{n!} \leq o(\phi^\leftarrow(s)) + \sup_{x \geq K} x(\log s + \alpha + \varepsilon - \log \phi(x)). \quad (6.8)$$

Define the functions  $F_s(x) = x(\log s + \alpha + \varepsilon - \log \phi(x))$  for each  $s > 0$ . Observe that  $F'_s(x) = \log s + \alpha + \varepsilon - \left(\log \phi(x) + \frac{x\phi'(x)}{\phi(x)}\right)$  and  $F''_s(x) = -\left[2\frac{\phi'(x)}{\phi(x)} + x\left(\frac{\phi''(x)}{\phi(x)} - \frac{\phi'(x)^2}{\phi(x)^2}\right)\right]$ . Since  $\phi$  is regularly varying with index  $\alpha, 0 < \alpha < 1$ , it must be of the form (up to a constant term)

$$\int_0^\infty (1 - e^{-st})\nu(dt)$$

and hence  $\phi'(s) = \int_0^\infty te^{-st}\nu(dt)$  is monotone decreasing in  $s$  and thus, using the monotone density theorem, we have  $x\phi'(x)/\phi(x) \rightarrow \alpha$  as  $x \rightarrow \infty$ . Furthermore, since  $\phi''(s) = -\int_0^\infty t^2 e^{-st}\nu(dt)$  is increasing, we also have  $x\phi''(x)/\phi'(x) \rightarrow \alpha - 1$  and hence  $F''_s(x) \sim -\frac{\alpha}{x}$ . So  $F''_s$  is eventually negative and hence  $F_s$  is eventually concave. For the above given  $\varepsilon > 0$ , choose  $K > k_\varepsilon$ , such that

1.  $F_s$  is concave on  $[K, \infty)$ ,
2. for  $x > K$ , we have  $\left|\frac{x\phi'(x)}{\phi(x)} - \alpha\right| < \frac{\varepsilon}{2}$ .

Since  $F''_s$  is independent of  $s$  for all  $x$ , we can choose the above  $K$  independent of  $s$  (but depending on  $\varepsilon$ ). Observe that  $F'_s(x)$  goes to  $-\infty$  as  $x \rightarrow \infty$ . Also, for large enough  $s$ , we have  $F'_s(K) = \log s + \alpha + \varepsilon - \left(\log \phi(K) + \frac{K\phi'(K)}{\phi(K)}\right) > 0$ . So there exists  $x(s) > K$ , such that  $F'_s(x(s)) = 0$ . Since  $F_s$  is concave on  $[K, \infty)$ ,  $x(s)$  will be the unique maximizer on that interval. Then, from (6.8), and since  $F'_s(x(s)) = 0$ , we have

$$\log \sum_{n=0}^{k-1} \frac{s^n \mathbb{E}\{Z^n\}}{n!} \leq o(\phi^\leftarrow(s)) + x(s) \frac{x(s)\phi'(x(s))}{\phi(x(s))} \leq o(\phi^\leftarrow(s)) + x(s)(\alpha + \varepsilon/2)$$

since  $x(s) > K$ . Furthermore, we have, from the defining equation of  $x(s)$ , that

$$\log \phi(x(s)) = \log s + \alpha + \varepsilon - \frac{x(s)\phi'(x(s))}{\phi(x(s))} < \log s + \frac{3}{2}\varepsilon$$

and hence  $x(s) \leq \phi^\leftarrow(se^{1.5\varepsilon})$ . So

$$\log \sum_{n=0}^{k-1} \frac{s^n \mathbb{E}\{Z^n\}}{n!} \leq o(\phi^\leftarrow(s)) + \phi^\leftarrow(se^{1.5\varepsilon})(\alpha + \varepsilon/2)$$

and hence

$$\limsup_{s \rightarrow \infty} \frac{1}{\phi^\leftarrow(s)} \log \sum_{n=0}^{k-1} \frac{s^n \mathbb{E}\{Z^n\}}{n!} \leq e^{1.5\varepsilon/\alpha}(\alpha + \varepsilon/2).$$

Letting  $\varepsilon \rightarrow 0$ , we have,

$$\limsup_{s \rightarrow \infty} \frac{1}{\phi^\leftarrow(s)} \log \sum_{n=0}^{k-1} \frac{s^n \mathbb{E}\{Z^n\}}{n!} \leq \alpha.$$

Combining with the fact that the second term of (6.6) is bounded, we have (6.5).

We are now in a position to apply Kasahara's Tauberian theorem, cf. Theorem 4.7.12 in [5]. Set  $g(s) = s/\phi(s)$ . Then (6.5) implies

$$\log \mathbb{P}\{Z > x\} \sim (1 - \alpha)g^{\leftarrow}(x)$$

which completes the proof. □

## References

- [1] S. Asmussen. Subexponential asymptotics for stochastic processes: extremal behavior, stationary distributions and first passage probabilities. *Ann. Appl. Probab.*, 8(2):354–374, 1998. ISSN 1050-5164.
- [2] J. Bertoin. *Lévy processes*, volume 121 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 1996. ISBN 0-521-56243-0.
- [3] J. Bertoin, P. Biane, and M. Yor. Poissonian exponential functionals, q-series, q-integrals and the moment problem for log-normal distributions. Preprint PMA-705, Laboratoire de Probabilités, 2002.
- [4] J. Bertoin and M. Yor. On the entire moments of self-similar Markov processes and exponential functionals of Lévy processes. *Ann. Fac. Sci. Toulouse Math. (6)*, 11(1):33–45, 2002. ISSN 0240-2963.
- [5] N. H. Bingham, C. M. Goldie, and J. L. Teugels. *Regular variation*, volume 27 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 1987. ISBN 0-521-30787-2.
- [6] J. Blanchet and P. Glynn. Approximations for the distribution of infinite horizon discounted rewards. Technical report. Department of Management Science and Engineering. Stanford University, 2004.
- [7] M. Braverman, T. Mikosch, and G. Samorodnitsky. Tail probabilities of subadditive functionals of Lévy processes. *Ann. Appl. Probab.*, 12(1):69–100, 2002. ISSN 1050-5164.
- [8] L. Breiman. On some limit theorems similar to the arc-sin law. *Teor. Veroyatnost. i Primenen.*, 10:351–360, 1965.
- [9] P. Carmona, F. Petit, and M. Yor. On the distribution and asymptotic results for exponential functionals of Lévy processes. In *Exponential functionals and principal values related to Brownian motion*, Bibl. Rev. Mat. Iberoamericana, pages 73–130. Rev. Mat. Iberoamericana, Madrid, 1997.
- [10] D. Denisov, S. Foss, and D. Korshunov. Tail asymptotics for the supremum of a random walk when the mean is not finite. *Queueing Syst.*, 46(1-2):15–33, 2004. ISSN 0257-0130.
- [11] D. Dufresne. The distribution of a perpetuity, with applications to risk theory and pension funding. *Scand. Actuar. J.*, (1-2):39–79, 1990. ISSN 0346-1238.
- [12] V. Dumas, F. Guillemin, and P. Robert. A Markovian analysis of additive-increase multiplicative-decrease algorithms. *Adv. in Appl. Probab.*, 34(1):85–111, 2002. ISSN 0001-8678.
- [13] P. Embrechts, C. M. Goldie, and N. Veraverbeke. Subexponentiality and infinite divisibility. *Z. Wahrsch. Verw. Gebiete*, 49(3):335–347, 1979. ISSN 0044-3719.

- [14] P. Embrechts, C. Klüppelberg, and T. Mikosch. *Modelling extremal events for insurance and finance*, volume 33 of *Applications of Mathematics (New York)*. Springer-Verlag, Berlin, 1997. ISBN 3-540-60931-8.
- [15] S. Foss, T. Konstantopoulos, and S. Zachary. Lévy processes with subexponential Lévy measures. In preparation, 2004.
- [16] S. Foss, Z. Palmowski, and S. Zachary. The probability of exceeding a high boundary on a random time interval for a heavy-tailed random walk. Technical report. Department of Actuarial Mathematics and Statistics. Heriot-Watt University. Available at <http://www.ma.hw.ac.uk/~stan/papers/fpz.pdf>, 2004.
- [17] C. M. Goldie. Implicit renewal theory and tails of solutions of random equations. *Ann. Appl. Probab.*, 1(1):126–166, 1991. ISSN 1050-5164.
- [18] C. M. Goldie and R. Grübel. Perpetuities with thin tails. *Adv. in Appl. Probab.*, 28(2):463–480, 1996. ISSN 0001-8678.
- [19] C. M. Goldie and R. A. Maller. Stability of perpetuities. *Ann. Probab.*, 28(3):1195–1218, 2000. ISSN 0091-1798.
- [20] F. Guillemin, P. Robert, and B. Zwart. AIMD algorithms and exponential functionals. *Ann. Appl. Probab.*, 14(1):90–117, 2004. ISSN 1050-5164.
- [21] H. Kesten. Random difference equations and renewal theory for products of random matrices. *Acta Math.*, 131:207–248, 1973.
- [22] C. Klüppelberg, A. E. Kyprianou, and R. Maller. Ruin probabilities and overshoots for general Lévy insurance risk processes. To appear in *Ann. Appl. Probab.*, 14(4). Available at <http://www-lit.ma.tum.de/veroeff/quel/039.60002.pdf>, 2004.
- [23] C. Klüppelberg, A. Lindner, and R. Maller. Continuous time volatility modelling: COGARCH versus Ornstein-Uhlenbeck models. Invited paper for Bachelier Colloquium 2005. Available at <http://www-m4.ma.tum.de/m4/Papers/Klueppelberg/cog-spr.pdf>, 2004.
- [24] D. G. Konstantinides and T. Mikosch. Large deviations and ruin probabilities for solutions to stochastic recurrence equations with heavy-tailed innovations. Working paper no. 192. Laboratory of Actuarial Mathematics. Institute for Mathematical Sciences. University of Copenhagen. Available at <http://www.act.ku.dk/wop/2004/wop192.pdf>, 2004.
- [25] N. Litvak and W. R. van Zwet. On the minimal travel time needed to collect  $n$  items on circle. *Ann. Appl. Probab.*, 14(2):881–902, 2004. ISSN 1050-5164.
- [26] V. Rivero. Recurrent extensions of self-similar markov processes and Cramér’s condition. To appear in *Bernoulli*, 2004.
- [27] H. Rootzén. Extreme value theory for moving average processes. *Ann. Probab.*, 14(2):612–652, 1986. ISSN 0091-1798.
- [28] N. Veraverbeke. Asymptotic behaviour of Wiener-Hopf factors of a random walk. *Stochastic Processes Appl.*, 5(1):27–37, 1977.
- [29] E. Willekens. On the supremum of an infinitely divisible process. *Stochastic Process. Appl.*, 26(1):173–175, 1987. ISSN 0304-4149.
- [30] M. Yor. *Exponential functionals of Brownian motion and related processes*. Springer Finance. Springer-Verlag, Berlin, 2001. ISBN 3-540-65943-9. With an introductory chapter by Hélyette Geman, Chapters 1, 3, 4, 8 translated from the French by Stephen S. Wilson.



- [31] S. Zachary. A note on Veraverbeke's theorem. *Queueing Syst.*, 46(1-2):9–14, 2004. ISSN 0257-0130.