ASYMPTOTIC ANALYSIS OF MEASURES OF VARIATION

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Abstract. The coefficient of variation and the dispersion are two examples of widely used measures of variation. We show that their applicability in practice heavily depends on the existence of sufficiently many moments of the underlying distribution. In particular, we offer a set of results that illustrate the behavior of these measures of variation when such a moment condition is not satisfied.
Our analysis is based on an auxiliary statistic that is interesting in its own right. Let X_i, $i = 1, \ldots, n$ be a sequence of positive independent identically distributed random variables and define

$$T_n := \frac{X_1^2 + X_2^2 + \ldots + X_n^2}{(X_1 + X_2 + \ldots + X_n)^2}.$$

Utilizing Karamata theory of functions of regular variation, we determine the asymptotic behavior of arbitrary moments $\mathbb{E}(T_n^k)$ ($k \in \mathbb{N}$) for large n, given that X_1 satisfies a tail condition, akin to the domain of attraction condition from extreme value theory. Moreover, weak laws for T_n are proven. The methodology is then used to analyze asymptotic properties of both the sample coefficient of variation and the sample dispersion. As a side product, the paper offers a new method for estimating the extreme value index of Pareto-type tails.

1. Introduction

Let X_i, $i = 1, \ldots, n$ be a sequence of positive independent identically distributed (i.i.d.) random variables with distribution function F and define

$$T_n := \frac{X_1^2 + X_2^2 + \ldots + X_n^2}{(X_1 + X_2 + \ldots + X_n)^2}. \tag{1}$$

The asymptotic behavior of $\mathbb{E}\,(T_n)$ was investigated in [7], simplifying and generalizing earlier results in [4] and [11].
In this paper we extend several results of [7] and derive the limiting behavior of arbitrary moments $\mathbb{E}\,(T_n^k)$ ($k \in \mathbb{N}$). This is achieved by using an integral representation of $\mathbb{E}(T_n^k)$ in terms of the Laplace transform of X_1, which is derived in Section 2.
Most of our results will be derived under the condition that X_1 satisfies

$$1 - F(x) \sim x^{-\alpha}\ell(x), \quad x \uparrow \infty \tag{2}$$

where $\alpha > 0$ and $\ell(x)$ is slowly varying, i.e. $\lim_{x\to\infty} \ell(tx)/\ell(x) = 1\, \forall\, t > 0$, see e.g. [3]. It is well known that condition (2) appears as the essential condition in the domain of attraction problem of extreme value theory. For a recent treatment, see [1]. A distribution satisfying (2) is called of *Pareto-type* with *index* α. When $\alpha < 2$, then the condition coincides with

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1991 *Mathematics Subject Classification.* Primary 62G20; secondary 62G32.
Key words and phrases. Functions of regular variation, domain of attraction of a stable law, sample coefficient of variation, sample dispersion, extreme value theory.
* Supported by Fellowship F/04/009 of the Katholieke Universiteit Leuven and the Austrian Science Foundation Project S-8308-MAT.

the domain of attraction condition for weak convergence to a non-normal stable law. It is then obvious that for $\beta > 0$,

$$E(X_1^\beta) := \mu_\beta = \beta \int_0^\infty x^{\beta-1}(1 - F(x)\, dx \leq \infty \tag{3}$$

will be finite if $\beta < \alpha$ but infinite whenever $\beta > \alpha$. For convenience, we define $\mu_0 := 1$ and $\mu := \mu_1$.
The main results are given in Section 3 and are based on the theory of functions of regular variation (see e.g. [3]). Clearly, if $\mathbb{E}(X_1) = \infty$, both the numerator and the denominator in (1) will exhibit an erratic behavior, whereas for $\mathbb{E}(X_1) < \infty$ and $\mathbb{E}(X_1^2) = \infty$ this is the case only for the numerator. The results of Section 3 quantify this effect. Section 4 then provides weak laws for T_n.

The quantity T_n represents, up to scaling, the sample coefficient of variation of a given set of independent observations $X_1, \ldots, X_n$ from a random variable X. Although this is a frequently used risk measure in practical applications, the existence of moments of X is not always ensured. The results of Sections 3 and 4 allow us to analyze asymptotic properties of the sample coefficient of variation, viewed as a statistic, also in these cases (see Section 5.1). In Section 5.2, the asymptotic behavior of the sample dispersion is investigated.

As another by-product, the results of this paper suggest a new method for estimating the extreme value index of Pareto-type distributions from a data set of observations, which is discussed in Section 5.3.

2. Preliminaries

Let $\varphi(s) := \mathbb{E}(e^{-sX_1}) = \int_0^\infty e^{-sx} dF(x)$, $s \geq 0$ denote the Laplace transform of X_1. Then, following an idea of [7], one can use the identity

$$\frac{1}{x^\beta} = \frac{1}{\Gamma(\beta)} \int_0^\infty e^{-s\,x} s^{\beta-1}\, ds, \quad \beta > 0$$

and Fubini's theorem to deduce that

$$\mathbb{E}\frac{1}{X_1^\beta} = \frac{1}{\Gamma(\beta)} \int_0^\infty s^{\beta-1} \varphi(s)\, ds.$$

More generally, for i.i.d. random variables $X_1, \ldots, X_n$, one obtains the representation formula

$$\mathbb{E}\frac{\prod_{i=1}^n X_i^{k_i}}{(X_1 + X_2 + \ldots + X_n)^\beta} = \frac{(-1)^{k_1+\ldots+k_n}}{\Gamma(\beta)} \int_0^\infty s^{\beta-1} \prod_{i=1}^n \frac{\partial^{k_i}\varphi(s)}{\partial s^{k_i}}\, ds, \tag{4}$$

for nonnegative integers k_i $(i = 1, \ldots, n)$.
In particular, by symmetry

$$\mathbb{E}(T_n) = \mathbb{E}\frac{X_1^2 + X_2^2 + \ldots + X_n^2}{(X_1 + X_2 + \ldots + X_n)^2} = n \int_0^\infty s\varphi''(s)\varphi^{n-1}(s)\, ds, \tag{5}$$

which formed the basis for the analysis in [7]. The representation (5) can be generalized in the following way:

Lemma 2.1. *For an arbitrary positive integer k,*

$$\mathbb{E}\,(T_n^k) = \sum_{r=1}^{k} \sum_{\substack{k_1,\ldots,k_r \ge 1 \\ k_1+\ldots+k_r=k}} \frac{k!}{k_1!\cdots k_r!}\, B(n,k_1,\ldots,k_r) \tag{6}$$

with

$$B(n,k_1,\ldots,k_r) = \frac{\binom{n}{r}}{\Gamma(2k)} \int_0^\infty s^{2k-1}\varphi^{(2k_1)}(s)\cdots\varphi^{(2k_r)}(s)\,\varphi^{n-r}(s)\,ds.$$

Proof. For an arbitrary positive integer k we have

$$\mathbb{E}\,(T_n^k) = \mathbb{E}\,\frac{(X_1^2+X_2^2+\ldots+X_n^2)^k}{(X_1+X_2+\ldots+X_n)^{2k}} = \sum_{\substack{k_1,\ldots,k_n \ge 0 \\ k_1+\ldots+k_n=k}} \frac{k!}{k_1!\cdots k_n!}\;\mathbb{E}\,\frac{X_1^{2k_1}X_2^{2k_2}\cdots X_n^{2k_n}}{(X_1+X_2+\ldots+X_n)^{2k}},$$

where $k_i \le k$ are nonnegative integers. Choose an n-tuple $(k_1,\ldots,k_n)$ in the above sum and let r denote the number of its non-zero elements $(k_{i_1},\ldots,k_{i_r})$ (clearly $1 \le r \le k$). There are exactly $\binom{n}{r}$ possibilities of extending $(k_{i_1},\ldots,k_{i_r})$ to an n-tuple by filling in $n-r$ zeroes; each of the resulting n-tuples leads to the same summand in (6). Thus we can write

$$\mathbb{E}\,(T_n^k) = \sum_{r=1}^{k} \sum_{\substack{k_1,\ldots,k_r \ge 1 \\ k_1+\ldots+k_r=k}} \frac{k!}{k_1!\cdots k_r!}\; \underbrace{\binom{n}{r}\,\mathbb{E}\,\frac{X_1^{2k_1}X_2^{2k_2}\cdots X_n^{2k_r}}{(X_1+X_2+\ldots+X_n)^{2k}}}_{:=B(n,k_1,\ldots,k_r)}, \tag{7}$$

so that (6) holds in view of (4). ☐

3. Main Results

As promised, we will assume in the sequel that X_1 satisfies condition (2). Recall that when $\alpha > 1$ then $\mu < \infty$ while $\mu_2 < \infty$ as soon as $\alpha > 2$. The finiteness of μ and/or μ_2 has its influence on the asymptotic behavior of the summands that make up the statistic T_n. It is therefore not surprising that our results will be heavily depending on the range of α. We state a first and general result.

Lemma 3.1. *If X_1 has a regularly varying tail with index $\alpha > 0$ (i.e. $1-F(x) \sim x^{-\alpha}\ell(x)$), then the asymptotic behavior of the m-th derivative of the Laplace transform $\varphi(s)$ as $s \downarrow 0$ is given by*

$$\varphi^{(m)}(s) \sim (-1)^m \alpha\,\Gamma(m-\alpha)\,s^{\alpha-m}\ell(1/s), \quad m > \alpha. \tag{8}$$

Proof. Let $\chi(s) := \int_0^\infty e^{-sx}(1-F(x))\,dx$. Since $1-F(x) \sim x^{-\alpha}\ell(x)$, it follows that for $k > \alpha - 1$

$$(-1)^k\chi^{(k)}(s) = \int_0^\infty x^k e^{-sx}(1-F(x))\,dx \sim \Gamma(k+1-\alpha)s^{-k-1}(1-F(\frac{1}{s})) \quad \text{as } s \to 0.$$

Since $\varphi(s) = 1 - s\,\chi(s)$, we have for $m \ge 1$

$$\varphi^{(m)}(s) = -m\chi^{(m-1)}(s) - s\,\chi^{(m)}(s),$$

so that for $m > \alpha$

$$\begin{aligned}\frac{s^m \varphi^{(m)}(s)}{1-F(\frac{1}{s})} &= -m\,\frac{s^m \chi^{(m-1)}(s)}{1-F(\frac{1}{s})} - \frac{s^{m+1}\,\chi^{(m)}(s)}{1-F(\frac{1}{s})}\\ &\sim (-1)^m (m\,\Gamma(m-\alpha) - \Gamma(m+1-\alpha)) = (-1)^m\,\alpha\,\Gamma(m-\alpha),\end{aligned}$$

from which the assertion follows. □

Theorem 3.1. *If X_1 belongs to the domain of attraction of a stable law with index α, $0<\alpha<1$, then for all $k \geq 1$*

$$\lim_{n\to\infty} \mathbb{E}(T_n^k) = \frac{k!}{\Gamma(2k)} \sum_{r=1}^{k} \frac{\alpha^{r-1}}{r\,\Gamma(1-\alpha)^r}\; G(r,k), \tag{9}$$

where $G(r,k)$ is the coefficient of x^k in the polynomial

$$\left(\sum_{j=1}^{k-r+1} \frac{\Gamma(2j-\alpha)}{j!}\, x^j \right)^r .$$

Proof. From $1-F(x) \sim x^{-\alpha}\ell(x)$ it follows that $1-\varphi(s) \sim \Gamma(1-\alpha)\, s^\alpha \ell(\frac{1}{s})$ (see e.g. Corollary 8.1.7 in [3]). Moreover, for any sequence $(a_n)_{n\geq 1}$ with $a_n \to \infty$ we have

$$\varphi^n(\frac{s}{a_n}) = e^{n\log\varphi(s/a_n)} \sim e^{-n\,(1-\varphi(s/a_n))} \sim \exp[-n\left(\frac{s}{a_n}\right)^\alpha \ell\left(\frac{a_n}{s}\right)\Gamma(1-\alpha)].$$

Choose $(a_n)_{n\geq 1}$ such that

$$n\, a_n^{-\alpha}\, \ell(a_n)\Gamma(1-\alpha) \to 1 \quad \text{for } n\to\infty. \tag{10}$$

Then for all $s \geq 0$

$$\lim_{n\to\infty} \varphi^n(\frac{s}{a_n}) = e^{-s^\alpha}.$$

We will now make use of the representation (6) for $\mathbb{E}(T_n^k)$. We have to investigate the asymptotic behavior of $B(n,k_1,\ldots,k_r)$. The change of variables $s = t/a_n$ together with an application of Potter's Theorem [3, Th.1.5.6], Lebesgue's dominated convergence theorem and Lemma 3.1 leads to

$$\begin{aligned} B(n,k_1,\ldots,k_r) &= \frac{\binom{n}{r}}{a_n\Gamma(2k)} \int_0^\infty \left(\frac{t}{a_n}\right)^{2k-1} \varphi^{(2k_1)}\left(\frac{t}{a_n}\right)\cdots\varphi^{(2k_r)}\left(\frac{t}{a_n}\right) \underbrace{\varphi^{n-r}\left(\frac{t}{a_n}\right)}_{\to\, e^{-t^\alpha}} dt\\ &\sim \frac{\alpha^r\binom{n}{r}}{a_n\Gamma(2k)} \int_0^\infty \left(\frac{t}{a_n}\right)^{2k-1} \left(\frac{t}{a_n}\right)^{r\,\alpha-2k} \ell^r\left(\frac{a_n}{t}\right) \left(\prod_{j=1}^r \Gamma(2k_j-\alpha)\right) e^{-t^\alpha}\, dt\\ &\sim \frac{\alpha^r \prod_{j=1}^r \Gamma(2k_j-\alpha)}{\Gamma(2k)} \underbrace{\frac{\binom{n}{r}\,\ell^r(a_n)}{{a_n}^{r\,\alpha}}}_{\to\,\Gamma(1-\alpha)^{-r}/r!} \underbrace{\int_0^\infty t^{r\,\alpha-1}\, e^{-t^\alpha}\, dt}_{=\,(r-1)!/\alpha}\\ &\sim \frac{\alpha^{r-1}\prod_{j=1}^r \Gamma(2k_j-\alpha)}{r\,\Gamma(1-\alpha)^r\,\Gamma(2k)}. \end{aligned}$$

Summing up over all $r = 1, \dots, k$ in (6), we arrive at

$$\lim_{n\to\infty} \mathbb{E}\,(T_n^k) = \frac{k!}{(2k-1)!} \sum_{r=1}^{k} \frac{\alpha^{r-1}}{r\,\Gamma(1-\alpha)^r} \sum_{\substack{k_1,\dots,k_r\geq 1\\ k_1+\dots+k_r=k}} \prod_{j=1}^{r} \frac{\Gamma(2k_j-\alpha)}{k_j!}. \tag{11}$$

Now observe that

$$G(r,k) := \sum_{\substack{k_1,\dots,k_r\geq 1\\ k_1+\dots+k_r=k}} \prod_{j=1}^{r} \frac{\Gamma(2k_j-\alpha)}{k_j!}$$

can be determined by generating functions. Concretely, if we look at the r-fold product

$$\left(\Gamma(2-\alpha)\,x + \frac{\Gamma(4-\alpha)}{2!}\,x^2 + \dots + \frac{\Gamma(2m-\alpha)}{m!}\,x^m\right)^r$$

for m sufficiently large, then $G(r,k)$ can be read off as its coefficient of x^k, since the kth power exactly comprises all contributions of combinations $k_1, \dots, k_r \geq 1$ with $k_1 + \dots + k_r = k$ in the above sum. It suffices to choose $m = k - r + 1$, since larger powers do not contribute to the coefficient of x^k any more. Hence Theorem 3.1 follows from (11). □

Remark 3.1. For $k = 1$, we obtain $\lim_{n\to\infty} \mathbb{E}(T_n) = 1 - \alpha$, which is Theorem 5.3 of [7]. The limit of moments of higher order can now be calculated from (9):

$$\begin{aligned}
\lim_{n\to\infty} \mathbb{E}(T_n^2) &= \tfrac{1}{3}(1-\alpha)(2\,\alpha-3),\\
\lim_{n\to\infty} \mathbb{E}(T_n^3) &= \tfrac{1}{15}(1-\alpha)(15-17\,\alpha+5\,\alpha^2),\\
\lim_{n\to\infty} \mathbb{E}(T_n^4) &= \tfrac{1}{105}(1-\alpha)(105-155\,\alpha+79\,\alpha^2-14\,\alpha^3),\\
\lim_{n\to\infty} \mathbb{E}(T_n^5) &= \tfrac{1}{945}(1-\alpha)(945-1644\,\alpha+1106\,\alpha^2-344\,\alpha^3+42\,\alpha^4).
\end{aligned}$$

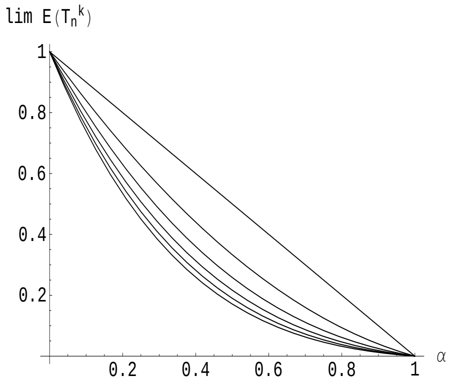

FIGURE 1. $\lim_{n\to\infty} \mathbb{E}(T_n^k)$ as a function of α ($k = 1, \dots, 5$ from top to bottom)

The following result generalizes Theorem 5.5 of [7], where the case $k = 1$ was covered:

Theorem 3.2. *If X_1 belongs to the domain of attraction of a stable law with index $\alpha = 1$ and $\mathbb{E}(X_1) = \infty$, then for all $k \geq 1$*

$$\mathbb{E}(T_n^k) \sim \frac{1}{2k-1}\,\frac{\ell(a_n)}{\tilde{\ell}(a_n)}, \tag{12}$$

where $\tilde{\ell}(x) = \int^x (\ell(t)/t)\,dt$ and $(a_n)_{n\geq 1}$ is a sequence satisfying $a_n \sim n\,\tilde{\ell}(a_n)$.

Proof. Since X_1 belongs to the domain of attraction of a stable law with index $\alpha = 1$, we have $1 - F(x) \sim x^{-1}\ell(x)$ for some slowly varying function $\ell(x)$. Moreover $1 - \varphi(s) \sim s\,\tilde{\ell}(\frac{1}{s})$ with $\tilde{\ell}(x) = \int^x (\ell(t)/t)\,dt$ (see e.g. [3]). Note that $\tilde{\ell}(x)$ is again a slowly varying function. For any sequence $(a_n)_{n\geq 1}$ with $a_n \to \infty$ we have

$$\varphi^n(\frac{s}{a_n}) = e^{n\log\varphi(s/a_n)} \sim e^{-n\,(1-\varphi(s/a_n))} \sim \exp[-n\left(\frac{s}{a_n}\right)\tilde{\ell}\left(\frac{a_n}{s}\right)].$$

If we choose a_n such that

$$n\,a_n^{-1}\,\tilde{\ell}(a_n) \to 1 \quad \text{for } n \to \infty, \tag{13}$$

then

$$\lim_{n\to\infty} \varphi^n(\frac{s}{a_n}) = e^{-s}.$$

Take a_n as in (13) and replace s by t/a_n in the representation (6). An application of Potter's Theorem, Lebesgue's dominated convergence theorem and Lemma 3.1 yields

$$\begin{aligned}
B(n,k_1,\ldots,k_r) &= \frac{\binom{n}{r}}{a_n\Gamma(2k)}\int_0^\infty \left(\frac{t}{a_n}\right)^{2k-1}\varphi^{(2k_1)}\left(\frac{t}{a_n}\right)\cdots\varphi^{(2k_r)}\left(\frac{t}{a_n}\right)\underbrace{\varphi^{n-r}\left(\frac{t}{a_n}\right)}_{\to e^{-t}}dt \\
&\sim \frac{\binom{n}{r}}{a_n\Gamma(2k)}\int_0^\infty \left(\frac{t}{a_n}\right)^{2k-1}\left(\frac{t}{a_n}\right)^{r-2k}\ell^r\left(\frac{a_n}{t}\right)\left(\prod_{j=1}^r \Gamma(2k_j-1)\right)e^{-t}\,dt \\
&\sim \frac{\prod_{j=1}^r\Gamma(2k_j-1)}{r!\,\Gamma(2k)}\,\frac{n^r\,\ell^r(a_n)}{a_n{}^r}\underbrace{\int_0^\infty t^{r-1}\,e^{-t}\,dt}_{=(r-1)!} \\
&\sim \frac{\prod_{j=1}^r\Gamma(2k_j-1)}{r\,\Gamma(2k)}\left(\frac{\ell(a_n)}{\tilde{\ell}(a_n)}\right)^r
\end{aligned}$$

Note that $\ell(a_n)/\tilde{\ell}(a_n) \to 0$ for $n \to \infty$ and thus, opposed to the case $\alpha < 1$, only the summand with $r = 1$ contributes to the dominating asymptotic term of (6). Therefore we obtain

$$\mathbb{E}\,(T_n^k) \sim \frac{1}{2k-1}\,\frac{\ell(a_n)}{\tilde{\ell}(a_n)}.$$

□

Theorem 3.3. *Let X_1 belong to the domain of attraction of a stable law with index α, $1 \leq \alpha < 2$ and $\mu := \mathbb{E}(X_1) < \infty$. Then for all $k \geq 1$*

$$\mathbb{E}(T_n^k) \sim \frac{\Gamma(2k-\alpha)\Gamma(1+\alpha)}{\Gamma(2k)\,\mu^\alpha}\,n^{1-\alpha}\ell(n). \tag{14}$$

Proof. Since μ is finite, it follows that

$$\lim_{n\to\infty}\varphi^n(t/n) = e^{-\mu t} \quad \text{for all } t \geq 0. \tag{15}$$

However, in view of (15), we will use the change of variables $s = t/n$ in the representation (6). By virtue of Potter's Theorem, Lebesgue's dominated convergence theorem and Lemma 3.1 we then obtain

$$\begin{aligned}
B(n,k_1,\dots,k_r) &= \frac{\binom{n}{r}}{n\Gamma(2k)} \int_0^\infty \left(\frac{t}{n}\right)^{2k-1} \varphi^{(2k_1)}\left(\frac{t}{n}\right)\cdots\varphi^{(2k_r)}\left(\frac{t}{n}\right) \underbrace{\varphi^{n-r}\left(\frac{t}{n}\right)}_{\to e^{-\mu t}} dt \\
&\sim \frac{\alpha^r\binom{n}{r}}{n\Gamma(2k)} \int_0^\infty \left(\frac{t}{n}\right)^{2k-1} \left(\frac{t}{n}\right)^{r\alpha-2k} \ell^r\left(\frac{n}{t}\right) \left(\prod_{j=1}^r \Gamma(2k_j-\alpha)\right) e^{-\mu t}\, dt \\
&\sim \frac{\alpha^r \prod_{j=1}^r \Gamma(2k_j-\alpha)}{\Gamma(2k)} \underbrace{\frac{\binom{n}{r}\ell^r(n)}{n^{r\alpha}}}_{\sim n^{r(1-\alpha)}\ell^r(n)/r!} \underbrace{\int_0^\infty t^{r\alpha-1} e^{-\mu t}\, dt}_{=\Gamma(r\alpha)/\mu^{r\alpha}} \\
&\sim \frac{\alpha^r \Gamma(r\alpha)\prod_{j=1}^r \Gamma(2k_j-\alpha)}{r!\,\mu^{r\alpha}\,\Gamma(2k)}\, n^{r(1-\alpha)}\ell^r(n).
\end{aligned}$$

Hence the first-order asymptotic behavior of (6) is solely determined by the term with $r=1$ and we obtain

$$\mathbb{E}(T_n^k) \sim \frac{\Gamma(2k-\alpha)\Gamma(1+\alpha)}{\Gamma(2k)\,\mu^\alpha}\, n^{1-\alpha}\ell(n).$$

□

Remark 3.2. For the special case $k=1$, (14) yields $\mathbb{E}(T_n) \sim \frac{\Gamma(2-\alpha)\Gamma(1+\alpha)}{\mu^\alpha}\, n^{1-\alpha}\ell(n)$, which is Theorem 5.1. of [7].

We pass to the case $\alpha > 2$.

Theorem 3.4. *Let $1-F(x) \sim x^{-\alpha}\,\ell(x)$ for some slowly varying function $\ell(x)$ and $\alpha > 2$. Then for all integers $k < \alpha - 1$*

$$\mathbb{E}(T_n^k) \sim \left(\frac{\mu_2}{\mu^2}\right)^k n^{-k} \tag{16}$$

and for $k > \alpha - 1$

$$\mathbb{E}(T_n^k) \sim \frac{\Gamma(2k-\alpha)\Gamma(1+\alpha)}{\Gamma(2k)\,\mu^\alpha}\, n^{1-\alpha}\ell(n). \tag{17}$$

If $k = \alpha - 1$, then

(i) (16) *holds if* $\mathbb{E}(X_1^{k+1}) < \infty$,

(ii) $\mathbb{E}(T_n^k) \sim \left(\left(\frac{\mu_2}{\mu^2}\right)^k + \frac{\Gamma(k-1)\Gamma(k+2)}{\Gamma(2k)\,\mu^{1+k}}\right) n^{-k}$ *holds if* $\ell(x) \sim$ *const.*,

(iii) *and else* (17) *holds.*

Proof. Let us look at the quantity $B(n,k_1,\dots,k_r)$. By Lemma 3.1 and the Bingham-Doney Lemma (see e.g. [3, Th.8.1.6]) the asymptotic behavior of $\varphi^{(m)}(s)$ at the origin is given by

$$(-1)^m\,\varphi^{(m)}(s) \sim \begin{cases} \alpha\,\Gamma(m-\alpha)\, s^{\alpha-m}\ell(1/s) & \text{if } m > \alpha \\ \alpha\,\tilde{\ell}(1/s) & \text{if } m = \alpha \text{ and } \mathbb{E}(X_1^m) = \infty \\ \mu_m & \text{if } m \le \alpha \text{ and } \mathbb{E}(X_1^m) < \infty \end{cases},$$

where $\tilde{\ell}(x) = \int_0^x (\ell(u)/u)du$ is itself a slowly varying function. For simplicity, let us first assume that $\alpha \notin \mathbb{N}$. Then one can conclude in an analogous way as in the proof of Theorem 3.3 that the asymptotic behavior of $B(n, k_1, \ldots, k_r)$ is given by

$$B(n, k_1, \ldots, k_r) \sim C\, n^{r-\alpha\, r_1 - 2(k-u_1)} \ell^{r_1}(n),$$

where r_1 is the number of integers among $k_1, \ldots, k_r$ that are greater than $\alpha/2$, u_1 is the sum of these and C is some constant. It remains to determine the dominating asymptotic term among all possible $B(n, k_1, \ldots, k_r)$: If $r_1 > 0$, then $r_1 = 1$, $u_1 = k$ and thus $r = 1$ yields the largest exponent, so that the asymptotic order is $n^{1-\alpha}\ell(n)$. Note that $r_1 > 0$ is possible for $2k > \alpha$ only. For $r_1 = 0$, on the other hand, $r = k$ and thus $k_1 = \ldots = k_r = 1$ dominates leading to asymptotic order n^{-k}. Hence the asymptotically dominating power among $B(n, k_1, \ldots, k_r)$ is given by $\max(1-\alpha, -k)$. From this we see that for $k < \alpha - 1$, $r = k$ dominates and we obtain from (6)

$$\mathbb{E}(T_n^k) \sim k! \frac{n^k\, \mu_2^k\, \Gamma(2k)}{k!\, \Gamma(2k)\, n^{2k}\, \mu^{2k}} \sim \left(\frac{\mu_2}{\mu^2}\right)^k n^{-k}.$$

Alternatively, if $k > \alpha - 1$, the term with $r = 1$ dominates and we obtain (17) in just the same way as in Theorem 3.3.
Finally, the above conclusions also hold for $\alpha \in \mathbb{N}$ except when $k = \alpha - 1$. In the latter case the slowly varying function $\ell(x)$ determines which of the two terms $n^{1-\alpha}\ell(n)$ (corresponding to $r = 1$) and n^{-k} (corresponding to $r = k$) dominates the asymptotic behavior: if $\ell(x) = o(1)$ (which due to $\mathbb{E}(X_1^{k+1}) \sim (k+1)\int_0^n x^{-1}\ell(x)\, dx$ is equivalent to $\mathbb{E}(X_1) < \infty$), the second one dominates. If $\ell(x) \sim const.$, then both terms matter and the assertion of the theorem follows. □

Corollary 3.1. *If $1 - F(x) \sim x^{-2}\,\ell(x)$, then for $k \geq 2$*

$$\mathbb{E}(T_n^k) \sim \frac{1}{(k-1)(2k-1)\mu^2}\, \frac{\ell(n)}{n}$$

and

$$\mathbb{E}(T_n) \sim \begin{cases} \frac{\mu_2}{\mu^2\, n} & \text{if } \ \mathbb{E}(X_1^2) < \infty \\ \frac{2}{\mu^2}\,\frac{\tilde{\ell}(n)}{n} & \text{if } \ \mathbb{E}(X_1^2) = \infty. \end{cases}$$

Proof. One can easily verify that Theorem 3.4 remains true for $\alpha = 2$ except for $k = 1$ in the case $\mathbb{E}(X_1^2) = \infty$. In the latter case obviously $r = 1$ and one obtains (using $\varphi''(s) \sim 2\,\tilde{\ell}(1/s)$)

$$\mathbb{E}(T_n) \sim B(n, 1) \sim \frac{2\, n\, \tilde{\ell}(n)}{n^2} \int_0^\infty t e^{-\mu\, t}\, dt \sim \frac{2}{\mu^2}\, \frac{\tilde{\ell}(n)}{n},$$

which is already contained in [7, Theorem 5.2]. □

Remark 3.3. One might wonder whether a general limit result for $\mathbb{E}(T_n^k)$ for X_1 in the domain of attraction of a normal law (in the spirit of Theorem 5.2 of [7] for $k = 1$) can be obtained with the integral representation approach used in this paper. This is however not the case: From $\int_0^x y^2\, dF(y) \sim \ell_2(x)$ (where $\ell_2(x)$ is a slowly varying function) it follows by partial integration that $\varphi^{(2k)}(s)/\ell_2(1/s) = o(s^{2-2k})$ for $k > 1$ as $s \to 0$, but the latter is not strong enough to identify the dominating term among the $B(n, k_1, \ldots, k_r)$ without any further assumptions on the distribution of X_1.

	$\mathbb{E}(T_n)$	$\mathrm{Var}\,(T_n)$	$\mathrm{Var}\,(T_n)/\mathbb{E}(T_n)$
$0<\alpha<1$	$1-\alpha$	$\frac{\alpha(1-\alpha)}{3}$	$\frac{\alpha}{3}$
$\alpha=1$	$\frac{\ell(a_n)}{\tilde\ell(a_n)}\ (\to 0)$	$\frac{1}{3}\,\frac{\ell(a_n)}{\tilde\ell(a_n)}\ (\to 0)$	$\frac{1}{3}$
$1<\alpha<2$	$\frac{\Gamma(2-\alpha)\Gamma(1+\alpha)}{\mu^2}\ n^{1-\alpha}\ell(n)$	$\frac{\Gamma(4-\alpha)\Gamma(1+\alpha)}{6\mu^\alpha}\ n^{1-\alpha}\ell(n)$	$\frac{(3-\alpha)(2-\alpha)}{6}$
$\alpha=2$	$\frac{2}{\mu^2}\ \frac{\tilde\ell(n)}{n}$	$\frac{\ell(n)}{3\,n\,\mu^2}$	$\frac{1}{6}\ \frac{\ell(n)}{\tilde\ell(n)}\ (\to 0)$
$2<\alpha<4$	$\frac{\mu_2}{\mu^2\,n}$	$\frac{\Gamma(4-\alpha)\Gamma(1+\alpha)}{6\,\mu^\alpha}\ n^{1-\alpha}\ell(n)$	$\frac{\Gamma(4-\alpha)\Gamma(1+\alpha)}{6\,\mu^{\alpha-2}\,\mu_2}\ n^{2-\alpha}\ell(n)$
$\alpha\geq 4$	$\frac{\mu_2}{\mu^2\,n}$	$\frac{\mu_4\mu^2-\mu_2^2\mu^2+4\,\mu_2^3-4\mu\mu_2\mu_3}{\mu^6}\ \frac{1}{n^3}$	$\frac{\mu_4\mu^2/\mu_2-\mu_2\mu^2+4\,\mu_2^2-4\mu\mu_3}{\mu^4}\ \frac{1}{n^2}$

TABLE 1. First order asymptotic terms of $\mathbb{E}(T_n)$, $\mathrm{Var}\,(T_n)$ and $\mathrm{Var}\,(T_n)/\mathbb{E}(T_n)$ for $1-F(x)\sim x^{-\alpha}\ell(x)$ as a function of α

As an illustration of the results of this section, Table 1 gives the first order asymptotic terms of $\mathbb{E}(T_n)$, $\mathrm{Var}\,(T_n)$ and the dispersion $\mathrm{Var}\,(T_n)/\mathbb{E}(T_n)$ as a function of α. Note that the entries for $\alpha>2$ have been obtained by calculating second-order asymptotic terms. The result for $\alpha>4$ in the table actually holds whenever $\mu_4<\infty$, since in this case the derivation of second-order terms does not rely on the assumption of regular variation and one obtains $\mathbb{E}(T_n^2)\sim\frac{\mu_2^2}{\mu^4}\frac{1}{n^2}+(\frac{10\mu_2^3-3\mu_2^2\mu^2-8\mu\mu_2\mu_3+\mu^2\mu_4}{\mu^6})\frac{1}{n^3}+O(\frac{1}{n^4})$ and $\mathbb{E}^2(T_n)\sim\frac{\mu_2^2}{\mu^4}\frac{1}{n^2}+(\frac{6\mu_2^3-4\mu\mu_2\mu_3-2\mu^2\mu_2^2}{\mu^6})\frac{1}{n^3}+O(\frac{1}{n^4}))$.
From Table 1 we see that the dispersion of T_n is a continuous function in α with its maximum in $\alpha=1$ (see Figure 2).

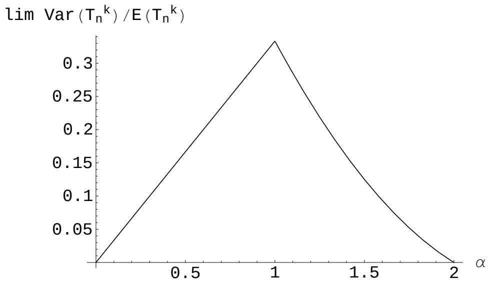

FIGURE 2. Limit of the dispersion of T_n as a function of α

4. WEAK LAWS FOR T_n

We will now complement the results of Section 3 by providing weak laws for the sequence of random variables $(T_n)_{n\geq 1}$. This is done by distinguishing four cases depending on the value of α:

Proposition 4.1. *Let X_1 belong to the domain of attraction of a stable law with index α, $0<\alpha<1$ (including $\alpha=1$ if $\mathbb{E}(X_1)=\infty$). Then*

$$T_n \xrightarrow{d} \frac{U}{V^2}, \tag{18}$$

where the joint distribution of the random variables (U,V) is given by the Laplace transform

$$\mathbb{E}(e^{-sU-tV}) = \exp\left(-2\,e^{t^2/(4\,s)}\int_0^\infty e^{-\left(u+\frac{t}{2\sqrt{s}}\right)^2}\left(u+\frac{t}{2\sqrt{s}}\right)\left(\frac{t}{\sqrt{s}}\right)^{-\alpha}du\right). \tag{19}$$

Proof. For $\theta > 0$, $\psi \geq 0$ we have

$$\begin{aligned} 1-\mathbb{E}\,(e^{-\theta X_1^2-\psi X_1}) &= \int_0^\infty (1-e^{-\theta x^2-\psi x})\,dF(x) \\ &= \int_0^\infty (1-F(x))\,e^{-\theta x^2-\psi x}(2\theta\,x+\psi)\,dx \\ &= 2e^{\psi^2/(4\theta)}\int_{\psi/(2\sqrt{\theta})}^\infty e^{-y^2}\,y\left(1-F(\frac{y}{\sqrt{\theta}}-\frac{\psi}{2\theta})\right)dy \end{aligned} \tag{20}$$

where the last equality is obtained by the change of variables $y=\sqrt{\theta}\,x+\frac{\psi}{2\sqrt{\theta}}$.
Define a sequence $(a_n)_{n\geq 1}$ by $1-F(a_n)\sim\frac{1}{n}$, i.e. $\ell(a_n)/a_n^\alpha \sim \frac{1}{n}$. Now, from the independence of the random variables X_i one gets

$$\begin{aligned} \mathbb{E}\left(e^{-s\,\frac{1}{a_n^2}\sum_{i=1}^n X_i^2 - t\,\frac{1}{a_n}\sum_{i=1}^n X_i}\right) &= \exp\left(n\,\log\mathbb{E}\,(-s\,\frac{1}{a_n^2}\,X_1^2-t\,\frac{1}{a_n}\,X_1)\right) \\ &\sim \exp\left(-n\,(1-\mathbb{E}\,(-s\,\frac{1}{a_n^2}\,X_1^2-t\,\frac{1}{a_n}\,X_1))\right) \end{aligned}$$

and by choosing $\theta = s/a_n^2$ and $\psi = t/a_n$, we obtain from (20) and dominated convergence

$$\mathbb{E}\left(e^{-\frac{s}{a_n^2}\sum_{i=1}^n X_i^2-\frac{t}{a_n}\sum_{i=1}^n X_i}\right)\sim\exp\left(-\,2\,e^{t^2/(4\,s)}\int_{\frac{t}{2\sqrt{s}}}^\infty y\,e^{-y^2}\,n\left(1-F(a_n(\frac{y}{\sqrt{s}}-\frac{t}{2\,s}))\right)dy\right)$$

$$\to\exp\left(-2\,e^{t^2/(4\,s)}\int_{\frac{t}{2\sqrt{s}}}^\infty y\,e^{-y^2}\left(\frac{y}{\sqrt{s}}-\frac{t}{2\,s}\right)^{-\alpha}dy\right) := e^{-\psi_\alpha(s,t)}.$$

An additional change of variables $y = u+\frac{t}{2\sqrt{s}}$ leads to

$$\psi_\alpha(s,t) = 2\,e^{t^2/(4\,s)}\int_0^\infty e^{-\left(u+\frac{t}{2\sqrt{s}}\right)^2}\left(u+\frac{t}{2\sqrt{s}}\right)\left(\frac{t}{\sqrt{s}}\right)^{-\alpha}du.$$

Thus we have shown

$$\Big(\frac{1}{a_n^2}\sum_{i=1}^n X_i^2,\ \frac{1}{a_n}\sum_{i=1}^n X_i\Big)\xrightarrow{d}(U,V),$$

where the joint distribution of U and V is given by (19). The continuous mapping theorem finally gives

$$T_n = \frac{X_1^2+X_2^2+\ldots+X_n^2}{(X_1+X_2+\ldots+X_n)^2}\xrightarrow{d}\frac{U}{V^2}\,.$$

□

Remark 4.1. The marginal distribution of U is the weak limit of $\frac{1}{a_n^2}\sum_{i=1}^n X_i^2$ and hence is determined by taking $t=0$ in (19). This leads to

$$\mathbb{E}(e^{-s\,U}) = \exp(-s^{\alpha/2}\,\Gamma(1-\alpha/2)),$$

so that U is a stable distribution with index $\alpha/2$. For the marginal distribution of V, which is the weak limit of $\frac{1}{a_n}\sum_{i=1}^n X_i$, a little more care is needed, but following the same line of arguments in the proof above with $s=0$, one obtains

$$\mathbb{E}(e^{-tV}) = \exp(-t^\alpha\,\Gamma(1-\alpha)),$$

so V is stable with index α, as it should be.

Proposition 4.2. *Let X_1 belong to the domain of attraction of a stable law with index α, $1<\alpha<2$ (including $\alpha=1$ if $\mathbb{E}(X_1)<\infty$ and $\alpha=2$ if $\mathbb{E}(X_1^2)=\infty$). Then*

$$\left(\frac{n}{a_n}\right)^2 T_n \xrightarrow{d} \frac{U}{\mu^2}, \tag{21}$$

where U is a stable random variable with Laplace transform

$$\mathbb{E}(e^{-sU}) = \exp(-s^{\alpha/2}\,\Gamma(1-\alpha/2)) \tag{22}$$

and $(a_n)_{n\geq 1}$ is defined by $1-F(a_n)\sim\frac{1}{n}$.

Proof. Repeating the arguments of the proof of Proposition 4.1 and Remark 4.1, one obtains $\frac{1}{a_n^2}\sum_{i=1}^n X_i^2 \xrightarrow{d} U$ with (22) (note that X_1^2 is in the domain of attraction of a stable law with index $\alpha/2$). Moreover, since $\mathbb{E}(X_1)<\infty$, it follows that $\frac{1}{n}\sum_{i=1}^n X_i \xrightarrow{p} \mu$. But then, Slutsky's theorem and the continuous mapping theorem can be used to deduce that

$$\frac{\frac{1}{a_n^2}\sum_{i=1}^n X_i^2}{(\frac{1}{n}\sum_{i=1}^n X_i)^2} \xrightarrow{d} \frac{U}{\mu^2}$$

which is equivalent to (21). □

Proposition 4.3. *Let X_1 be regularly varying with index α, $2<\alpha<4$ (including $\alpha=2$ if $\mathbb{E}(X_1^2)<\infty$ and $\alpha=4$ if $\mathbb{E}(X_1^4)=\infty$). Then*

$$\frac{n^{1-2/\alpha}}{\ell_1(n)}\left(n\,T_n - \frac{\mu_2}{\mu^2}\right) \xrightarrow{d} \frac{W}{\mu^2}, \tag{23}$$

where W is a stable random variable with index $\alpha/2$, $\ell_1(n)$ is a slowly varying function given by $\ell_1(n)\sim c_n\,n^{-2/\alpha}$ and where in turn the sequence $(c_n)_{n\geq 1}$ is defined by $1-F(\sqrt{c_n})\sim\frac{1}{n}$.

Proof. For a deterministic sequence $(b_n)_{n\geq 1}$ to be chosen later on, consider

$$b_n\left(n\,T_n-\frac{\mu_2}{\mu^2}\right) = \underbrace{\frac{n^2\,b_n}{(X_1+\cdots+X_n)^2}\left(\frac{1}{n}\sum_{i=1}^n X_i^2-\mu_2\right)}_{:=A_n} + \underbrace{\frac{\mu_2\,n^2\,b_n}{(X_1+\cdots+X_n)^2}-\frac{b_n\,\mu_2}{\mu^2}}_{:=B_n}.$$

Since $\mathbb{P}(X_1^2>x)=\mathbb{P}(X_1>\sqrt{x})\sim x^{-\alpha/2}\ell(\sqrt{x})$, the tail of X_1^2 is regularly varying with index $\alpha/2$. Thus we have $\frac{n}{c_n}\,(\frac{1}{n}\sum_{i=1}^n X_i^2-\mu_2)\xrightarrow{d} W$ for a stable law W with index $\alpha/2$ and a sequence of normalizing constants $(c_n)_{n\geq 1}$ defined by $1-F(\sqrt{c_n})\sim\frac{1}{n}$. The latter implies $c_n\sim n^{2/\alpha}\,\ell^{-2/\alpha}(\sqrt{c_n})$ so that for $\ell_1(n):=\ell^{-2/\alpha}(\sqrt{c_n})$ we obtain

$$\frac{n^{1-2/\alpha}}{\ell_1(n)}\left(\frac{1}{n}\sum_{i=1}^n X_i^2-\mu_2\right)\xrightarrow{d} W.$$

Clearly, $\frac{1}{n}\sum_{i=1}^{n} X_i \xrightarrow{p} \mu$ and thus Slutsky's theorem and the continuous mapping theorem imply $A_n \xrightarrow{d} \frac{\ell_1(n)\, b_n}{\mu^2\, n^{1-2/\alpha}}\, W$. At the same time, we have

$$B_n = -\underbrace{\frac{n^2\,\mu_2}{(X_1+\cdots+X_n)^2}}_{\xrightarrow{p}\frac{\mu_2}{\mu^2}}\,\frac{b_n}{\mu}\underbrace{\left(\frac{1}{n}\sum_{i=1}^{n} X_i-\mu\right)}_{\xrightarrow{d}(\sigma/\sqrt{n})\,N(0,1)}\underbrace{\left(\frac{1}{n\,\mu}\sum_{i=1}^{n} X_i+1\right)}_{\xrightarrow{p}2},$$

where $\sigma^2 := \operatorname{Var} X_i$. Now, the choice $b_n = \frac{n^{1-2/\alpha}}{\ell_1(n)}$ yields $A_n \xrightarrow{d} \frac{W}{\mu^2}$ and $B_n \xrightarrow{d} 0$, so that the result follows by another application of Slutsky's theorem. □

Finally, if the fourth moment of X_1 exists, then $n\,T_n$ is asymptotically normal:

Proposition 4.4. *Let* $\mathbb{E}(X_1^4) = \mu_4 < \infty$. *Then*

$$\sqrt{n}\left(n\,T_n - \frac{\mu_2}{\mu^2}\right) \xrightarrow{d} N\left(0, \sigma_*^2\right), \tag{24}$$

where

$$\sigma_*^2 = \frac{\mu_4}{\mu^4} - \left(\frac{\mu_2}{\mu^2}\right)^2 + 4\left(\frac{\mu_2}{\mu^2}\right)^3 - \frac{4\mu_2\mu_3}{\mu^5}. \tag{25}$$

Proof. This result is contained in classical statistical theory and the proof is just sketched here for completeness: From the two-dimensional Lindeberg-Lévy central limit theorem one deduces that the random vector $(\frac{1}{n}\sum_{i=1}^{n} X_i, \frac{1}{n}\sum_{i=1}^{n} X_i^2)$ converges weakly to a bivariate normal distribution with mean vector (μ, μ_2) and covariance matrix

$$\begin{pmatrix} \mu_2-\mu^2 & \mu_3-\mu\,\mu_2 \\ \mu_3-\mu\,\mu_2 & \mu_4-\mu_2^2 \end{pmatrix}.$$

The asymptotic normality then carries over to any function $H(\frac{1}{n}\sum_{i=1}^{n} X_i, \frac{1}{n}\sum_{i=1}^{n} X_i^2)$ that is twice continuously differentiable in the neighborhood of (μ, μ_2) (see for instance [5]). The limiting normal distribution has mean $H(\mu, \mu_2)$ and variance

$$H_1^2 \operatorname{Var}(\frac{1}{n}\sum_{i=1}^{n} X_i) + 2H_1H_2 \operatorname{Cov}(\frac{1}{n}\sum_{i=1}^{n} X_i, \frac{1}{n}\sum_{i=1}^{n} X_i^2) + H_2^2 \operatorname{Var}(\frac{1}{n}\sum_{i=1}^{n} X_i^2)$$

with

$$H_1 = \frac{\partial H}{\partial a_1}\Big|_{a_1=\mu, a_2=\mu_2} \text{ and } H_2 = \frac{\partial H}{\partial a_2}\Big|_{a_1=\mu, a_2=\mu_2}.$$

The choice $H(a_1, a_2) = \frac{a_2}{a_1^2}$ finally yields (24). □

Remark 4.2. Given the uniform integrability of $n(n\,T_n - \mu_2/\mu^2)^2$, one can obtain the asymptotic behavior of $\operatorname{Var}(n\,T_n)$ as the corresponding variance of the limiting normal distribution which leads to $\operatorname{Var}(n\,T_n) \sim \sigma_*^2/n$ (this was also obtained directly from Theorem 3.4 by employing a second-order asymptotic analysis of the integral representation, cf. Table 1).

5. Applications

5.1. Implications for the asymptotic behavior of the sample coefficient of variation.

In various practical situations (including applications in finance and insurance, in particular reinsurance), the *coefficient of variation* of a positive random variable X (with distribution function F) defined by

$$CoV(X) = \frac{\sqrt{\operatorname{Var} X}}{\mathbb{E}X}$$

is used as a measure to assess the risk associated with X. From a given set of independent observations $X_1, \ldots, X_n$ of X with sample mean $\bar{X} := \frac{X_1+\cdots+X_n}{n}$ and sample variance $S^2 = \frac{1}{n}\sum_{i=1}^n (X_i - \bar{X})^2$, $CoV(X)$ is then typically estimated by

$$\widehat{CoV(X)} := \frac{S}{\bar{X}},$$

which is called the *sample coefficient of variation*. (Note that there is a slight abuse of notation, since S is usually estimated with norming factor $n-1$ instead of n, but for the asymptotic considerations in the sequel this does not matter). The analysis of properties of $\widehat{CoV(X)}$ is typically based on the assumption that sufficiently many moments of X exist (see e.g. [6, 10]). The special case of normally distributed X, where the exact distribution of $\widehat{CoV(X)}$ is available, has received considerable attention in the literature (see e.g. [2, 8, 12]). For statistical inference for the coefficient of variation based on small sample size n, we refer to [13]. For applications within the context of insurance and especially reinsurance, see [9].

Although $\widehat{CoV(X)}$ is a widely-used risk measure, the existence of moments of X is not always ensured in practical applications and there is a need to study asymptotic properties of $\widehat{CoV(X)}$, viewed as a statistic, also in these cases. Due to

$$n\,T_n = \widehat{CoV(X)}^{\,2} + 1,$$

this can be done by using the results of Sections 3 and 4:

<u>(i) $0 < \alpha < 1$</u>: If X is in the domain of attraction of a stable law with index $0 < \alpha < 1$ (including $\alpha = 1$ if $\mathbb{E}(X) = \infty$), then it follows from Proposition 4.1, Slutsky's theorem and the continuous mapping theorem that $\sqrt{T_n - 1/n} \xrightarrow{d} \sqrt{U}/V$ so that

$$\frac{\widehat{CoV(X)}}{\sqrt{n}} \xrightarrow{d} \frac{\sqrt{U}}{V},$$

where the joint distribution of U and V is given by (19). Thus $\widehat{CoV(X)} \nearrow \infty$ at rate $n^{1/2}$ as $n \to \infty$ and the estimator $\widehat{CoV(X)}$ is useless in this case. Note that from Theorem 3.1 it follows that $\mathbb{E}(n\,T_n) \sim (1-\alpha)\,n$ and $\operatorname{Var}(n\,T_n) \sim \alpha(1-\alpha)n^2/3$.

<u>(ii) $1 < \alpha < 2$</u>: If at least the mean exists and X is in the domain of attraction of a stable law with $1 < \alpha < 2$ (including $\alpha = 1$ if $\mathbb{E}(X) < \infty$ and $\alpha = 2$ if $\mathbb{E}(X^2) = \infty$), Proposition 4.2, Slutsky's theorem and the continuous mapping theorem lead to

$$\frac{\sqrt{n}}{a_n}\,\widehat{CoV(X)} \xrightarrow{d} \frac{\sqrt{U}}{\mu},$$

where the distribution of U is given by (22) and $(a_n)_{n\geq 1}$ is defined by $1 - F(a_n) \sim \frac{1}{n}$. This implies $\widehat{CoV(X)} \nearrow \infty$ at rate $a_n/\sqrt{n}$ as $n \to \infty$. Theorem 3.3 shows that $\mathbb{E}(n\,T_n) \sim \frac{\Gamma(2-\alpha)\,\Gamma(1+\alpha)}{\mu^\alpha}\, n^{2-\alpha}\ell(n)$ and $\mathrm{Var}\,(n\,T_n) \sim \frac{\Gamma(4-\alpha)\,\Gamma(1+\alpha)}{6\,\mu^\alpha}\, n^{3-\alpha}\ell(n)$.

(iii) $2 < \alpha < 4$: (including $\alpha = 2$ if $\mathbb{E}(X^2) < \infty$ and $\alpha = 4$ if $\mathbb{E}(X^4) = \infty$)
Here $\mathrm{Var}\,(X) < \infty$ and from Proposition 4.3 we see that $n\,T_n \xrightarrow{p} \frac{\mu_2}{\mu^2} + o\left(\frac{\ell_1(n)}{n^{1-2/\alpha}}\right)$. Thus by virtue of the continuous mapping theorem

$$\widehat{CoV(X)} \xrightarrow{p} \frac{\sqrt{\mathrm{Var}\, X}}{\mathbb{E}X} + o\left(\frac{\ell_1(n)}{n^{1-2/\alpha}}\right). \tag{26}$$

Moreover, using the identity

$$b_n\left(\widehat{CoV(X)} - \frac{\sqrt{\mathrm{Var}\, X}}{\mathbb{E}X}\right) = \underbrace{\frac{b_n\,(\widehat{CoV(X)}^2 - \frac{\mathrm{Var}\, X}{\mathbb{E}^2 X})}{2\frac{\sqrt{\mathrm{Var}\, X}}{\mathbb{E}X}}}_{:=A_n} - \underbrace{\frac{b_n\left(\widehat{CoV(X)}^2 - \frac{\mathrm{Var}\, X}{\mathbb{E}^2 X}\right)^2}{2\frac{\sqrt{\mathrm{Var}\, X}}{\mathbb{E}X}\left(\widehat{CoV(X)} + \frac{\sqrt{\mathrm{Var}\, X}}{\mathbb{E}X}\right)^2}}_{:=B_n}$$

one also observes from Proposition 4.3 that for $b_n = \frac{n^{1-2/\alpha}}{\ell_1(n)}$ we have $A_n \xrightarrow{d} \frac{1}{2\,\mathbb{E}X\,\sqrt{\mathrm{Var}\, X}}\, W$ and $B_n \xrightarrow{d} 0$ leading to

$$\frac{n^{1-2/\alpha}}{\ell_1(n)}\left(\widehat{CoV(X)} - CoV(X)\right) \xrightarrow{d} \frac{1}{2\,\mathbb{E}X\,\sqrt{\mathrm{Var}\, X}}\; W,$$

where W is a stable law with index $\alpha/2$.
Now, a detailed study of the proof of Theorem 3.4 gives $\mathbb{E}(n\,T_n) \to \frac{\mu_2}{\mu^2} + O\left(n^{\max(2-\alpha,-1)}\right)$ which implies (by virtue of the above identity and $B_n \geq 0$) that $b_n(\widehat{CoV(X)} - CoV(X))$ is uniformly integrable and subsequently $\mathbb{E}(\widehat{CoV(X)}) \to CoV(X)$ as $n \to \infty$. Together with (26), it follows that $\widehat{CoV(X)}$ is a consistent and asymptotically unbiased estimator for $CoV(X)$.
From Theorem 3.4 we also see that for $\alpha < 3$, $\mathrm{Var}\,(n\,T_n) \sim \frac{\Gamma(4-\alpha)\,\Gamma(1+\alpha)}{6\,\mu^\alpha}\, n^{3-\alpha}\ell(n) \nearrow \infty$. Only in the case $\alpha > 3$ we have $\mathrm{Var}\,(n\,T_n) \searrow 0$. However,

$$\mathrm{Var}\,(\widehat{CoV(X)}) = \mathbb{E}(\widehat{CoV(X)}^2) - \mathbb{E}^2(\widehat{CoV(X)}) = \mathbb{E}(nT_n - 1) - \mathbb{E}^2(\widehat{CoV(X)}) \to 0.$$

(iv) $\mu_4 = \mathbb{E}(X^4) < \infty$: If the first four moments of X exist, then Proposition 4.4 shows that $n\,T_n$ is asymptotically normal. Moreover, the corresponding proof can be repeated using the function $H(a_1, a_2) = \sqrt{a_2 - a_1^2}/a_1$, leading to

$$\sqrt{n}\left(\widehat{CoV(X)} - CoV(X)\right) \xrightarrow{d} N\left(0, \frac{\sigma_*^2\,\mu^2}{4\,\sigma^2}\right), \tag{27}$$

where σ_*^2 is given by (25) and $\sigma^2 := \mathrm{Var}\, X$ (alternatively, this result can also be obtained by using the decomposition under (iii)). The weak law (27) can now be used to set up confidence intervals for the estimation procedure of $CoV(X)$.
If $n(\widehat{CoV(X)} - CoV(X))^2$ is uniformly integrable, then one can obtain the limit of $\mathrm{Var}\,(n\widehat{CoV(X)})$ as the variance of the limiting normal distribution, which by (27) implies that $\mathrm{Var}\,(\widehat{CoV(X)}) \sim \sigma_*^2\mu^2/(4\sigma^2\, n)$. Thus "the coefficient of variation of the coefficient

of variation" behaves like $CoV(\widehat{CoV(X)}) \sim \sigma_* \mu^2/(2\sigma^2\sqrt{n})$.
Another approach to determine the limiting behavior of $\mathbb{E}(\widehat{CoV(X)}^k)$ based on k-statistics can be found in [6].

Summarizing, $\widehat{CoV(X)}$ should be used as an estimator for $CoV(X)$ with great care, and only for $\mathbb{E}(X^4) < \infty$ practical confidence intervals are available.

5.2. **The asymptotic behavior of the sample dispersion.** Another measure of variation of a random variable X that is frequently used in practice is the *dispersion*

$$D(X) = \frac{\mathrm{Var}\,(X)}{\mathbb{E}(X)}.$$

For instance, in insurance the value of the dispersion allows to determine whether a given portfolio has a Poissonian character or not. From a given set of independent observations $X_1, \ldots, X_n$ of X with sample mean $\bar{X} := \frac{X_1 + \cdots + X_n}{n}$ and sample variance $S^2 = \frac{1}{n}\sum_{i=1}^n (X_i - \bar{X})^2$, $D(X)$ is typically estimated by

$$\widehat{D(X)} := \frac{S^2}{\bar{X}},$$

which is called the *sample dispersion.* (Again, the norming factor n instead of $n-1$ in S^2 does not matter asymptotically). If we introduce

$$C_n := \frac{X_1^2 + \cdots + X_n^2}{X_1 + \cdots + X_n},$$

then

$$\widehat{D(X)} = C_n - \bar{X},$$

and we can use results from Sections 3 and 4 to investigate asymptotic properties of the statistic $\widehat{D(X)}$ also in cases where X is in the domain of attraction of a stable law:

<u>(i) $0 < \alpha < 1$</u>: If X is in the domain of attraction of a stable law with index $0 < \alpha < 1$ (including $\alpha = 1$ if $\mathbb{E}(X) = \infty$), then it follows from Proposition 4.1 and the continuous mapping theorem that $\frac{1}{a_n}\, C_n \xrightarrow{d} U/V$, where $(a_n)_{n \geq 1}$ is defined by $1 - F(a_n) \sim \frac{1}{n}$ and the joint distribution of the random variables (U, V) is given by (19). Slutsky's theorem yields

$$\frac{1}{a_n}\,\widehat{D(X)} \xrightarrow{d} \frac{U}{V},$$

so $\widehat{D(X)}$ goes to infinity at rate a_n as $n \to \infty$.
It is easy to verify that, since $\alpha < 1$, $\mathbb{E}(C_n/a_n)$ and subsequently $\mathbb{E}(\widehat{D(X)})$ goes to ∞.

<u>(ii) $1 < \alpha < 2$</u>: If at least the mean exists and X is in the domain of attraction of a stable law with $1 < \alpha < 2$ (including $\alpha = 1$ if $\mathbb{E}(X) < \infty$ and $\alpha = 2$ if $\mathbb{E}(X^2) = \infty$), Proposition 4.2, Slutsky's theorem and the continuous mapping theorem lead to

$$\frac{n}{a_n^2}\,\widehat{D(X)} \xrightarrow{d} \frac{U}{\mu},$$

where the distribution of U is given by (22) and again $(a_n)_{n\geq 1}$ is defined by $1-F(a_n)\sim \frac{1}{n}$. So in this case $\widehat{D(X)}\nearrow\infty$ at rate a_n^2/n as $n\to\infty$. Moreover, along the line of arguments developed in Section 3 it follows that

$$\begin{aligned}\mathbb{E}(C_n) &= n\int_0^\infty \varphi''(s)\varphi^{n-1}(s)\,ds=\int_0^\infty \varphi''\left(\frac{t}{n}\right)\underbrace{\varphi^{n-1}\left(\frac{t}{n}\right)}_{\to e^{-\mu t}}\,dt\\ &\sim \alpha\,\Gamma(2-\alpha)\,n^{2-\alpha}\ell(n)\int_0^\infty t^{\alpha-2}\,e^{-\mu t}\,dt=\frac{\alpha\,\pi}{\sin((\alpha-1)\pi)\,\mu^{\alpha-1}}\,n^{2-\alpha}\ell(n).\end{aligned}$$

Since $\mathbb{E}(\bar X)\sim\mu$, this term is negligible and we obtain

$$\mathbb{E}(\widehat{D(X)})\sim\frac{\alpha\,\pi}{\sin((\alpha-1)\pi)\,\mu^{\alpha-1}}\,n^{2-\alpha}\ell(n).$$

<u>(iii) $2<\alpha<4$:</u> (including $\alpha=2$ if $\mathbb{E}(X^2)<\infty$ and $\alpha=4$ if $\mathbb{E}(X^4)=\infty$)
Here $\mathrm{Var}\,(X)<\infty$ and analogous to Proposition 4.3 we obtain for $b_n=\frac{n^{1-2/\alpha}}{\ell_1(n)}$

$$\begin{aligned}b_n\left(\widehat{D(X)}-D(X)\right) &= b_n(C_n-\frac{\mu_2}{\mu}-\bar X+\mu)\\ &= \underbrace{\frac{b_n}{\bar X}\left(\frac{1}{n}\sum_{i=1}^n X_i^2-\mu_2\right)}_{\xrightarrow{d}\frac{W}{\mu}}+\underbrace{\frac{\mu_2\,b_n}{\bar X\,\mu}(\mu-\bar X)}_{\xrightarrow{d}0}-\underbrace{b_n(\bar X-\mu)}_{\xrightarrow{d}0},\end{aligned}$$

where W is a stable law with index $\alpha/2$ and the slowly varying function $\ell_1(n)$ is defined as in Proposition 4.3. Slutsky's theorem now leads to

$$\frac{n^{1-2/\alpha}}{\ell_1(n)}\left(\widehat{D(X)}-\frac{\mathrm{Var}\,(X)}{\mathbb{E}(X)}\right)\xrightarrow{d}\frac{W}{\mu}.$$

Analogous to Theorem 3.4, one can derive $\mathbb{E}(C_n)\to\frac{\mu_2}{\mu}+O(n^{\max(2-\alpha,-1)})$ and subsequently $\mathbb{E}(\widehat{D(X)})\to D(X)+O(n^{\max(2-\alpha,-1)})$. It follows that $\widehat{D(X)}$ is a consistent and asymptotically unbiased estimator for the dispersion of X. Furthermore, by adaptation of the proof of Theorem 3.4,

$$\begin{aligned}\mathbb{E}(C_n^2) &= n\int_0^\infty s\varphi^{(4)}(s)\varphi^{n-1}(s)\,ds+n(n-1)\int_0^\infty s(\varphi''(s))^2\varphi^{n-2}(s)\,ds\\ &\sim \frac{\alpha\,\Gamma(4-\alpha)\,\Gamma(\alpha-2)}{\mu^{\alpha-2}}\,n^{3-\alpha}\,\ell(n)+\frac{\mu_2^2}{\mu^2},\end{aligned}$$

where the first term asymptotically dominates for $\alpha<3$ and the second for $\alpha\geq 3$ with $\mu_3<\infty$. Now,

$$\mathbb{E}(\widehat{D(X)}^2)=\mathbb{E}(C_n^2)-2\,\mathbb{E}(\frac{1}{n}\sum_{i=1}^n X_i^2)+\mathbb{E}(\bar X^2)\sim\mathbb{E}(C_n^2)-2\mu_2+\mu^2.$$

Thus $\mathrm{Var}\,(\widehat{D(X)})=\mathbb{E}\widehat{D(X)}^2-\mathbb{E}^2(\widehat{D(X)})$ is bounded only if the third moment exists and since the constant terms in $\mathrm{Var}\,(\widehat{D(X)})$ cancel out, we obtain in general

$$\mathrm{Var}\,(\widehat{D(X)})=O(n^{3-\alpha}\ell(n)).$$

(iv) $\mu_4 = \mathbb{E}(X^4) < \infty$: If the first four moments of X exist, then using the function $H(a_1, a_2) = a_2/a_1 - a_1$ in the proof of Proposition 4.4 yields

$$\sqrt{n}\left(\widehat{D(X)} - D(X)\right) \xrightarrow{d} N(0, \tilde{\sigma}^2), \tag{28}$$

where $\tilde{\sigma}^2 := (\mu_2\mu^4 - \mu^6 + \mu_2^3 - 2\mu^3\mu_3 - 2\mu\mu_2\mu_3 + 2\mu_2^2\mu^2 + \mu_4\mu^2)/\mu^4$. This weak law can be used to set up confidence intervals for the estimation procedure of $D(X)$.
If sufficiently many moments exist, then it follows from an adaptation of Theorem 3.4 that $\mathbb{E}(C_n^k) \to (\frac{\mu_2}{\mu^2})^k + O(\frac{1}{n})$. In particular $\operatorname{Var}(\widehat{D(X)}) = O(\frac{1}{n})$ and, more specifically, from (28) we have $\operatorname{Var}(\widehat{D(X)}) \sim \tilde{\sigma}^2/n$. Thus "the dispersion of the sample dispersion" behaves like $D(\widehat{D(X)}) \sim \tilde{\sigma}^2\mu/(\sigma^2\, n)$.

5.3. **Estimation of the extreme value index for Pareto-type tails.** The results of Section 3 also give rise to an alternative and seemingly new method for estimating the extreme value index $1/\alpha$ for Pareto-type tails $1 - F(x) \sim x^{-\alpha}\ell(n)$ with $0 < \alpha < 2$ from a given data set of independent observations (see e.g. [1] for other estimators of the extreme value index). In fact, plotting $n\,T_n$ against n will tend to a line with slope $1 - \alpha$, if $0 < \alpha < 1$ and plotting $\log(n\,T_n)$ against $\log n$ will tend to a line with slope $2 - \alpha$, if $1 < \alpha < 2$. The asymptotic behavior of higher order moments of $n\,T_n$ available from Section 3 can then be used to increase the efficiency of the estimation procedure.
At the same time, this provides a technique to test the finiteness of the mean of a distribution in the domain of attraction of a stable law.

6. Conclusion

In practice, the unconscientious use of some measures of variation can lead to wrong conclusions. As a general statement one needs to emphasize that the norming sequence $(a_n)_{n\geq 1}$ is unknown in practice and thus needs to be estimated.
The sample dispersion $\widehat{D(X)}$ is perhaps the most vulnerable among the statistics we considered. In the case $0 < \alpha < 1$, $\mathbb{E}(\widehat{D(X)})$ tends to infinity at a rate faster than a_n. The same applies to the variance whenever $\alpha < 2$. Only when $\alpha > 4$ the asymptotic properties of $\widehat{D(X)}$ guarantee a reassuring conclusion. The sample coefficient of variation $\widehat{CoV(X)}$ seems to have slightly better properties. For both statistics, the best results are obtained in terms of weak convergence. Results on expectations and/or variances are seemingly intricate.
A more promising approach seems to consider statistics like T_n and C_n directly. Apart from the weak limits, we have been able to derive rather explicit results about expectations and variances, even about arbitrary moments. For $0 < \alpha < 1$, a plot of the sample values of T_n will show a constant pattern from which the parameter α can be estimated. For $1 < \alpha < 2$, a similar procedure can be used on the logarithms of the sample values. Moreover, the approach introduced in Section 2 offers the additional possibility to involve the parameter β.

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