A Note on Edwards' Hypothesis for Zero-Temperature Ising Dynamics

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Abstract

We give a simple criterion for checking Edwards' hypothesis in certain zero-temperature, ferromagnetic spin-flip dynamics and use it to invalidate the hypothesis in various examples in dimension one and higher.

Keywords: Edwards' hypothesis, stochastic Ising model, zero-temperature limit.

1 Introduction

In many physical systems, the dynamics at low temperature or high density is so slow that the system is out of equilibrium at all practical time scales, so that ordinary thermodynamics does not apply to those situations. As a consequence, because of their practical as well as theoretical interest, non-standard thermodynamics have been proposed to describe out-of-equilibrium systems.

In the context of the slow compaction dynamics of granular materials, Edwards [1, 2, 3] proposed to compute thermodynamic quantities by means of a flat ensemble average over all the blocked configurations of grains with prescribed density, leading to a natural definition of configurational temperature. Since the approach is not justified from first principles, its validity has to be tested with specific models and experiments (see, for example, [4] and references therein).

Extending the range of applicability of this idea to other systems with a large number of "metastable" states, the so-called Edwards' hypothesis consists in assuming that all the metastable states in which the system can be trapped are equivalent for the dynamics. This corresponds, as before, to computing thermodynamic quantities using a flat ensemble average over all the metastable states.

A situation where this prescription makes sense is represented by glassy dynamics, which are often described as a slow motion in a complex energy (or free energy) landscape, with many "valleys" separated by barriers. Several approaches have been proposed

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to make this heuristic picture more precise, and different notions of "valleys" have appeared, but no general and unambiguous definition of metastable states has appeared yet. Nonetheless, once these valleys are appropriately defined, one can estimate their number $\mathcal{N}_E(N)$ at fixed energy (or free energy) density E for a fixed size N of the system. $\mathcal{N}_E(N)$ generally grows exponentially with the size of the system:

$$\mathcal{N}_E(N) \sim \exp\left(N\, S_c(E)\right), \tag{1}$$

where $S_c(E)$ is a *configurational entropy* or *complexity*. The key question concerning the dynamics is whether all valleys play the same role, or whether they have different *dynamical* weights, according to their basin of attraction. This question arises, for instance, when a system is instantaneously quenched into the glassy (low temperature) phase, starting from a disordered (high temperature) configuration.

In this context, Edwards' hypothesis is valid for some mean-field models, where valleys are know to be explored with a flat measure [5]. One can therefore define a thermodynamics based on the flat ensemble over valleys and a *configurational temperature* T_c by

$$\frac{1}{T_c} = \frac{\mathrm{d}S_c}{\mathrm{d}E}. \tag{2}$$

Besides the mean-field case, another situation where valleys can be unambiguously defined is the zero-temperature limit, where no energy barrier can be crossed and the valleys correspond to the blocked configurations under the chosen dynamics. Models with constrained dynamics have been extensively studied in one dimension (see [6] and references therein), where Edwards' hypothesis has been tested using exact results on the statistics of blocked configurations, as well as numerical simulations. The results obtained so far seem to invalidate Edwards' hypothesis in the case of zero-temperature spin-flip dynamics [6].

In this paper, we introduce a general criterion for checking Edwards' hypothesis for *attractive* (this refers to the ferromagnetic nature of the interaction between spins – see Section 1.1) symmetric Ising dynamics with initial configuration chosen from a symmetric distribution that satisfies the FKG inequality (see 1.1 and [7, 8, 9]). As a first application (Section 2.1, we rigorously invalidate Edwards' hypothesis for the one-dimensional constrained Glauber dynamics analyzed in [6] (see also [10, 11, 12]), which we use as a prototype. Later (Section3.1), we show how the criterion can be easily applied to other dynamics in higher dimensions, where the analytic techniques of [6] cannot be used and exact results are not available.

1.1 FKG Inequality and Harris' Theorem

We give here the tools needed to check Edwards' hypothesis for attractive (see the next paragraph), zero-temperature Ising dynamics. In the context of an Ising spin model on a lattice $\mathbb{L}$, we will call *increasing* an event $\mathcal{E}$ such that its indicator function $I_{\mathcal{E}}(\sigma)$ is increasing in the number of plus spins present in the configuration $\sigma = \{\sigma_x\}_{x\in\mathbb{L}}$, $\sigma_x = \pm 1$.

If $\mathcal{E}_1$ and $\mathcal{E}_2$ are two increasing events, the FKG inequality (see, for instance, [8, 9]) states that, roughly speaking, the occurrence of $\mathcal{E}_2$ makes $\mathcal{E}_1$ more likely, or more precisely, the conditional probability of $\mathcal{E}_1$ given $\mathcal{E}_2$ is larger than or equal to the probability of $\mathcal{E}_1$:

$$P(\mathcal{E}_1 \,|\, \mathcal{E}_2) \geq P(\mathcal{E}_1). \tag{3}$$

Many interesting distributions satisfy the FKG inequality (3), among them are Gibbs measures (with some restrictions) and product measures, and in particular the symmetric Bernoulli product measure from which the initial configuration of the constrained Glauber dynamics of Section 2 is chosen (corresponding to a spin system prepared at "infinite" temperature).

We will say that a spin-flip dynamics is *attractive* if, for all vertices x of $\mathbb{L}$, the rate for the spin flip $\sigma_x = -1 \to \sigma_x = +1$ is non-decreasing in the number of plus spins in σ (we will consider only symmetric dynamics, so the roles of plus and minus spins can be interchanged). Stochastic Ising models with a ferromagnetic interaction are examples of attractive dynamics. A theorem of Harris [13, 14] states that attractive dynamics preserve the FKG property, i.e., if one starts with a measure P_0 that satisfies the FKG inequality and applies to the spin system an attractive dynamics, the measure P_t describing the spin system at time t still satisfies the FKG inequality.

In particular, this result can be applied to the constrained Glauber dynamics of Section 2 to deduce that the limiting (as $t \to \infty$) measure P_∞ satisfies the FKG inequality.

2 A Constrained Glauber Dynamics in 1D

We consider the following ferromagnetic Ising chain with single spin-flip (Glauber) dynamics (see [6] for more details) with (formal) Hamiltonian

$$\mathcal{H}(\sigma) = -\sum_{n\in\mathbb{Z}} \sigma_n \sigma_{n+1} \tag{4}$$

and flipping rates W determined by the energy difference between the configurations after and before the proposed move, i.e.,

$$W(\sigma_n \to -\sigma_n) = \mathcal{W}_{\delta\mathcal{H}}, \tag{5}$$

with

$$\delta\mathcal{H} = 2(\sigma_{n-1} + \sigma_{n+1})\sigma_n \in \{-4, 0, 4\}. \tag{6}$$

Detailed balance with respect to (4) at inverse temperature β imposes the condition:

$$\frac{\mathcal{W}_4}{\mathcal{W}_{-4}} = e^{-4\beta}. \tag{7}$$

Choosing time units so that $\mathcal{W}_{-4} = 1$, we have $\mathcal{W}_4 = e^{-4\beta}$. As in [6], we are interested in the zero-temperature case, so that $\mathcal{W}_4 = 0$. The rate $\mathcal{W}_0$ remains a free parameter, which we choose to be zero (corresponding to the zero-temperature limit of the Glauber dynamics). Setting $\mathcal{W}_0 = 0$ corresponds to allowing only spin flips that lower the energy, therefore the only possible moves, happening with rate 1, are flips of plus spins surrounded by minus spins or minus spins surrounded by plus spins:

$$- + - \to - - -, \quad + - + \to + + +. \tag{8}$$

The blocked configurations (i.e., the absorbing states of the dynamics) are those where the unsatisfied bonds (i.e., bonds between spins of opposite sign) are isolated.

We consider the *deep-quench* situation, where the system is prepared at infinite temperature and the temperature is then decreased to zero instantaneously. This corresponds to an initial configuration chosen randomly from a symmetric Bernoulli product measure, i.e., with

$$\begin{cases} \sigma_n(0) = +1 & \text{with probability } 1/2 \\ \sigma_n(0) = -1 & \text{with probability } 1/2 \end{cases} \tag{9}$$

where $\sigma_n(t)$ is the value of the spin σ_n at time t. We call P_t the distribution of the spin configuration $\sigma(t) = \{\sigma_n(t)\}_{n\in\mathbb{Z}}$ at time t, and denote by P_∞ the limiting distribution obtained as $t \to \infty$.

2.1 Checking Edwards' Hypothesis

In [6], P_∞ is compared to the uniform distribution P_{unif} on blocked configurations, corresponding to an ensemble where *all* blocked configurations have the *same* weight. The conclusions reached there, using exact results on the statistics of the blocked configurations reached by the system, invalidate Edwards' hypothesis for this particular model.

Here, we confirm those results by rigorously proving that $P_\infty \neq P_{unif}$, but the main goal of this section is to introduce, via a simple specific example, a general criterion for comparing the limiting distribution P_∞ of a spin system subjected to an attractive dynamics to the uniform distribution P_{unif} on the absorbing states. The general strategy is described in Section 3, where we also give further applications.

Consider all blocked configurations of the spin chain such that $\sigma_{\pm 2} = \sigma_{\pm 3} = +1$. It is easy to see that such blocked configurations are of only four different types:

A	$\sigma_{-1} = \sigma_0 = \sigma_1 = +1$	$\ldots + + + + + + + \ldots$
B	$\sigma_{-1} = +1,\ \sigma_0 = \sigma_1 = -1$	$\ldots + + + - - + + \ldots$
C	$\sigma_{-1} = \sigma_0 = -1,\ \sigma_1 = +1$	$\ldots + + - - + + + \ldots$
D	$\sigma_{-1} = \sigma_0 = \sigma_1 = -1$	$\ldots + + - - - + + \ldots$

Under the uniform distribution on blocked configuration, the occurrence of each type has equal probability; therefore, conditioned on having $\sigma_{\pm 2} = \sigma_{\pm 3} = +1$,

$$P_{unif}(\sigma_0 = +1 \,|\, \sigma_{\pm 2} = \sigma_{\pm 3} = +1) = P_{unif}(A \,|\, \sigma_{\pm 2} = \sigma_{\pm 3} = +1) = 1/4. \tag{10}$$

On the other hand, Harris' Theorem (see Section 1.1) applied to this specific dynamics implies that P_∞ satisfies the FKG inequality, so that we have

$$P_\infty(\sigma_0 = +1 \,|\, \sigma_{\pm 2} = \sigma_{\pm 3} = +1) \geq P_\infty(\sigma_0 = +1) = 1/2, \tag{11}$$

where the equality follows from the $\pm$ symmetry of the dynamics and the initial distribution. The last two equations show that P_∞ cannot be the uniform distribution P_{unif}.

3 The General Strategy

The strategy we used for the constrained Glauber dynamics of the previous section can be generalized to any attractive, symmetric Ising dynamics with locally stable configurations (later, we will also give an application where there are no locally stable configurations – see Example 4 in Section 3.1), with initial configuration chosen from a symmetric distribution that satisfies the FKG inequality. For simplicity, we restrict our attention to nearest neighbor models; in this context, by the existence of locally stable configurations we mean that there are *finite* subsets G of the lattice $\mathbb{L}$ such that, if $\sigma_x(t_0) = +1\ (-1)\ \forall x \in G$, then $\sigma_x(t) = +1\ (-1)\ \forall x \in G,\ \forall t > t_0$. When this is the case, we say that the spins in G are *stable* and we call G a *stable set.* If G is a smallest set with this property (there could be more than one, with different shapes), we call it a *minimal stable set.*

Some more notation is needed before we can proceed with the general strategy and further applications. Given a subset Λ of $\mathbb{L}$, we call *exterior boundary* $\partial_e \Lambda$ of Λ the set of vertices $x \notin \Lambda$ that are adjacent to a vertex in Λ, and *interior boundary* $\partial_i \Lambda$ of Λ the set of vertices $x \in \Lambda$ that are adjacent to a vertex not in Λ.

We are now ready to explain the general strategy; in the next section we will illustrate it with some examples. Let G_1 and G_2 be two distinct minimal stable sets both containing the origin ($0 \in G_1 \cap G_2$) and denote by $G = G_1 \cup G_2$ their union. Let L be a (finite) stable set such that $G \cap L = \emptyset$ and $\partial_e G \subset L$ (in words, G is "surrounded" by L). G_1, G_2 and L should be chosen so that $\{G \setminus G_1\} \cup L$ and $\{G \setminus G_2\} \cup L$ are stable sets. Notice that, since G_1 and G_2 are minimal stable sets, $G \setminus G_1$ and $G \setminus G_2$ are smaller than any minimal stable set and therefore are not stable sets.

Now it is easy to convince oneself that, conditioned on the spins in L all being plus, there are only four possible types of blocked configurations:

1. All the spins in G are plus.

2. All the spins in G are minus.

3. The spins in G_1 are minus and those in $G \setminus G_1$ are plus.

4. The spins in G_2 are minus and those in $G \setminus G_2$ are plus.

This implies that, conditioned on all the spins in L being plus, the uniform distribution on stable configurations assigns probability $1/4$ to the event that the spin at the origin is plus (corresponding to case 1 above).

On the other hand, if we consider a symmetric, attractive dynamics with initial configuration chosen from a symmetric Bernoulli product measure, conditioned on the same event (all the spins in L being plus), the limiting distribution P_∞ must assign probability at least $1/2$ to the event that the spin at the origin is plus, which clearly shows that P_∞ cannot be the uniform distribution.

3.1 Higher Dimensional Examples

Here we present some examples in dimension higher than one where we can use the method described above to rule out the uniform distribution. All we have to do is choose the sets G_1, G_2, and L appropriately. We will consider zero-temperature dynamics such that a spin flips at rate 1 if it disagrees with a strict majority of its neighbors and at rate 0 otherwise. As in the example of Section 2, we will always start with a Bernoulli symmetric product measure (see (9)), corresponding to the deep-quench situation. We note that exact results are usually not available for higher-dimensional models, which limits the range of applicability of the methods used in [6] to one-dimensional models.

Example 1: Zero-temperature dynamics on the ladder $\mathbb{Z} \times \{0, 1\}$.
The blocked configurations are such that each spin has at least two neighbors of the same sign; squares are minimal stable sets. We choose G_1 and G_2 to be the sets of vertices of the two squares containing the origin $\{0\} \times \{0\}$ (the shaded squares in Figure 1) and L to be the set of vertices $\{\pm 2, \pm 3\} \times \{0, 1\}$.

Conditioning on the increasing event $\mathcal{E} = \{\sigma_{\pm 2} = \sigma_{\pm 3} = \sigma'_{\pm 2} = \sigma'_{\pm 3} = +1\}$ (see Figure 1), it is easy to see that $P_{unif}(\sigma_0 = +1 \,|\, \mathcal{E}) = 1/4$, because the fact that $\sigma_0 = +1$ implies that $\sigma_{\pm 1} = \sigma'_{\pm 1} = +1$, while there are three possible local blocked configurations with $\sigma_0 = -1$. On the other hand, $P_{unif}(\sigma_0 = +1) = 1/2$ by symmetry, so that P_{unif} does not satisfy the FKG inequality.

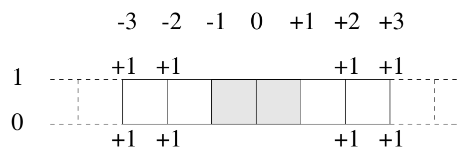

Figure 1: The increasing event $\mathcal{E}$ used in Example 1; σ_n are the spins in the lower row $\mathbb{Z} \times \{0\}$ and σ'_n those in the upper one $\mathbb{Z} \times \{1\}$.

Example 2: Zero-temperature dynamics on the hexagonal lattice.
The blocked configurations are again such that each spin has at least two neighbors of the same sign; hexagons are minimal stable sets. Let $\Lambda = G_1 \cup G_2 \cup L = G \cup L$ be the set of vertices of the portion of hexagonal lattice shown in Figure 2, where G_1 and G_2 are the sets of vertices of the two shaded hexagons containing the origin and $\partial_e G \subset L = \Lambda \setminus G = \partial_i \Lambda$.

Conditioning on the increasing event $\mathcal{E} = \{\sigma_y = +1, \forall y \in L = \partial_i \Lambda\}$ that all the spins in $L = \partial_i \Lambda$ are $+1$, it is easy to see that $P_{unif}(\sigma_0 = +1 \,|\, \mathcal{E}) = 1/4$, because the fact that

$\sigma_0 = +1$ implies that $\sigma_y = +1$ for all $y \in G$, while there are three possible local blocked configurations with $\sigma_0 = -1$. On the other hand, $P_{unif}(\sigma_0 = +1) = 1/2$ by symmetry, so that P_{unif} does not satisfy the FKG inequality.

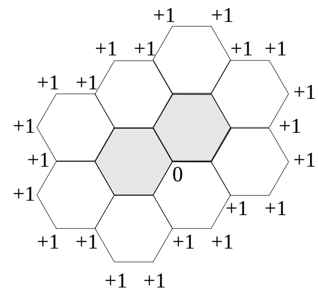

Figure 2: The increasing event $\mathcal{E}$ used in Example 2. As indicated, the vertices of the interior boundary $\partial_i\Lambda$ of Λ all have spin $+1$.

Example 3: Zero-temperature dynamics on $\mathbb{Z}^d$.
For simplicity, we consider the two-dimensional case $d = 2$, but the same reasoning works for all $d \geq 2$. In two dimensions the blocked configurations are again such that each spin has at least two neighbors of the same sign; squares are minimal stable sets. Let $\Lambda = G_1 \cup G_2 \cup L = G \cup L$ be the set of vertices of the portion of square lattice shown in Figure 3, where G_1 and G_2 are the sets of vertices of the two shaded squares containing the origin and $\partial_e G \subset L = \Lambda \setminus G = \partial_i\Lambda$.

Conditioning on the increasing event $\mathcal{E} = \{\sigma_y = +1, \forall y \in L = \partial_i\Lambda\}$ that all the spins in $L = \partial_i\Lambda$ are $+1$, it is easy to see that $P_{unif}(\sigma_0 = +1 \,|\, \mathcal{E}) = 1/4$, because the fact that $\sigma_0 = +1$ implies that $\sigma_y = +1$ for all $y \in G$, while there are three possible local blocked configurations with $\sigma_0 = -1$. On the other hand, $P_{unif}(\sigma_0 = +1) = 1/2$ by symmetry, so that P_{unif} does not satisfy the FKG inequality.

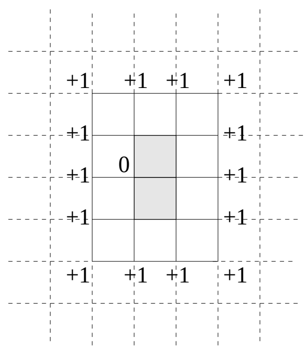

Figure 3: The increasing event $\mathcal{E}$ used in Example 3. As indicated, the vertices of the interior boundary $\partial_i\Lambda$ of Λ all have spin $+1$.

Example 4: Zero-temperature dynamics on the Cayley tree of degree 3.

This last example is interesting because, contrary to all the previous ones, there are no locally stable configurations (the only stable structures are doubly-infinite plus or minus paths). Nonetheless, the criterion described in this paper can still be used.

With reference to Figure 4, conditioning on the increasing event $\mathcal{E} = \{\sigma_{x_1} = \sigma_{x_2} = \sigma_{x_3} = +1$ and x_1, x_2, x_3 belong to doubly-infinite $+1$ paths that do not contain $y_1, y_2, y_3\}$, it is easy to see that, while $P_{unif}(\sigma_0 = +1) = 1/2$ by symmetry, $P_{unif}(\sigma_0 = +1 \,|\, \mathcal{E}) = 1/4$, because if $\sigma_0 = +1$, then σ_{y_1}, σ_{y_2} and σ_{y_3} are all forced to be $+1$. Therefore, once again P_{unif} does not satisfy the FKG inequality.

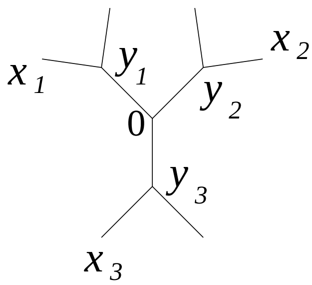

Figure 4: A portion of the Cayley tree of degree three.

4 More on the Constrained Glauber Dynamics

In [6], P_∞ is compared to two distributions: the uniform distribution P_{unif} on all blocked configurations, already discussed in Section 2.1, and an ensemble P_{Ed} where all blocked configurations *with a given energy density* are taken with equal weight. Formally, we can write

$$P_{Ed}(\sigma) \propto e^{-\beta_c E(\sigma)}, \tag{12}$$

where σ is a blocked configuration and $E(\sigma)$ its energy density, and $\beta_c = 1/T_c$ is the inverse of the configurational temperature (2).

The inverse temperature β_c that appears in (12) is itself a function of the energy density E, and can be computed using an explicit expression for the configurational entropy S_c (see [6] and references therein)

$$S_c(E) = E \ln(-2E) + \frac{1-E}{2} \ln(1-E) - \frac{1+E}{2} \ln(1+E), \tag{13}$$

from which we obtain

$$\beta_c(E) = \frac{\mathrm{d}S_c}{\mathrm{d}E} = \ln\left(\frac{-2E}{\sqrt{1-E^2}}\right). \tag{14}$$

To derive (13), consider a finite chain of N spins with periodic boundary conditions. The blocked configurations such that exactly n bonds are unsatisfied (i.e., are between spins of opposite sign) have energy density $E(\nu) = -(N-2n)/N = -(1-2\nu)$, where $\nu = n/N$, and their number is

$$\mathcal{N}(N,n) = \binom{N-n}{n}. \tag{15}$$

Indeed, this is the number of ways of inserting n unsatisfied bonds between the $N-n$ satisfied ones, in such a way that the unsatisfied bonds are isolated. It is now easy to see that $\frac{1}{N}\ln(\mathcal{N}(N,n)) \to S_c(E(\nu))$ as $N, n \to \infty$ and $n/N = \nu$ is kept constant.

The translation-ergodicity of the model and the translation-invariance of the energy density imply that, with P_0-probability 1, in the thermodynamic limit, the energy density of the final blocked configuration is a deterministic constant, which can be computed by solving exactly a dynamical equation for the densities of clusters of consecutive unsatisfied bonds. This is done in [6] (see also [10, 11, 12]), where the energy density for the deep-quench situation that we are considering in this paper (see (9)) is shown to be $E = -1 + e^{-1} \approx -0.63212$. Therefore, using (14), the appropriate value of the inverse temperature in (12) for the deep-quench situation is $\beta_c \approx 0.4895$.

Having an exact calculation for the configurational inverse temperature β_c, we can use our general criterion to check whether $P_\infty = P_{Ed}$ or not. A simple calculation is sufficient to rule out this possibility. Let $\hat{P}_{Ed}(\cdot) = P_{Ed}(\cdot \,|\, \sigma_{\pm 2} = \sigma_{\pm 3} = +1)$ be the distribution P_{Ed} conditioned on having $\sigma_{\pm 2} = \sigma_{\pm 3} = +1$. Then, we have the following straightforward relations (with reference to the events A, B, C, and D of Section 2.1)

$$\hat{P}_{Ed}(B) = e^{-2\beta_c}\hat{P}_{Ed}(A), \tag{16}$$

$$\hat{P}_{Ed}(B) = \hat{P}_{Ed}(C) = \hat{P}_{Ed}(D), \tag{17}$$

$$\hat{P}_{Ed}(A) + \hat{P}_{Ed}(B) + \hat{P}_{Ed}(C) + \hat{P}_{Ed}(D) = 1, \tag{18}$$

from which it follows that

$$\hat{P}_{Ed}(A) = \frac{1}{1+3e^{-2\beta_c}}. \tag{19}$$

Identifying P_∞ with P_{Ed} would imply, using (11),

$$1 + 3e^{-2\beta_c} \leq 2 \tag{20}$$

and thus

$$\beta_c \geq \frac{1}{2}\ln 3 \approx 0.5493, \tag{21}$$

which contradicts the value $\beta_c \approx 0.4895$ corresponding to the deep-quench situation that we are considering.

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References

[1] S. F. Edwards, in *Granular Matter: An Interdisciplinary Approach* (ed. A. Mehta), Springer, New York (1994).

[2] S. F. Edwards, in *Disorder in Condensed Matter Physics* (eds. J. Blackman, J. Taguena), Oxford Univ. Press, Oxford (1991).

[3] A. Mehta, S. F. Edwards, *Physica A* **157**, 1091 (1989).

[4] H. A. Makse, J. Kurchan, *Nature* **451**, 614 (2002).

[5] S. Franz, M. A. Virasoro, *J. Phys. A* **33**, 891 (2000).

[6] G. De Smedt, C. Godrèche, and J. M. Luck, *Eur. Phys. J. B* **27**, 363 (2002).

[7] T. E. Harris, *Proceedings of the Cambridge Phil. Soc.* **56**, 13 (1960).

[8] C. M. Fortuin, P. W. Kasteleyn, J. Ginibre, *Comm. Math. Phys.* **22**, 89 (1971).

[9] G. R. Grimmett, *Percolation*, Second edition, Springer, Berlin (1999).

[10] D. S. Dean, A. Lefévre, *Phys. Rev. Lett.* **86**, 5639 (2001).

[11] A. Lefévre, D. S. Dean, *J. Phys. A* **34**, L213 (2001).

[12] A. Prados, J. J. Brey, *J. Phys. A* **34**, L453 (2001).

[13] T. E. Harris, *Ann. Probab.* **5**, 451 (1975).

[14] T. M. Liggett, *Interacting Particle Systems*, Springer, New York (1985).