Risk measures for a combination of quota-share and drop down excess-of-loss reinsurance treaties

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Abstract: We investigate the behavior of some common risk measures for the reinsured amount associated with a nonproportional reinsurance form defined as a combination of quota-share and drop down excess-of-loss reinsurance treaties. In particular, we consider the Value-at-Risk, the variance, the coefficient of variation, the dispersion and the reduction effect.

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1 Introduction

Let $\{Y_i;\ i \geq 1\}$ be a sequence of successive claim sizes that consists of independent and identically distributed (i.i.d.) random variables generated by the distribution F_Y of a generic nonnegative random variable Y. Let N be a nonnegative integer-valued random variable, independent of the Y_i's, representing the number of claims occurring in some fixed time interval. We denote by $(Y_1^*, Y_2^*, \ldots, Y_N^*)$ the order statistics, arranged in increasing order, of the random vector $(Y_1, Y_2, \ldots, Y_N)$ of successive claim sizes in the time interval.

Reinsurance can be considered as one way of risk sharing. The reinsurance forms all have in common the intention to diminish an excessive number of claims and/or the impact of the large claims. Of course, reinsurance diminishes the volatility in the portfolio as the risk is shared between the first line insurance and the reinsurance. The decision to involve other partners in the risk sharing depends on many factors, some of them have only marginal relation with reinsurance.

A first line insurance will always try to safeguard its position by subscribing itself to a variety of insurance contracts with an equally varied set of (re)insurance companies. As such, the first line insurance itself becomes an insured client by paying a specific premium to a reinsurance company in exchange for a policy covering the reinsured quantity. For the first line insurance company, it obviously does not make sense to sell the entire portfolio to a reinsurance company because it will then lose all premium income from that portfolio. The first line company has to ponder how it wants the portfolio to be split between itself and the reinsurer.

It is common to distinguish between two types of reinsurance: proportional and nonproportional. Within the area of proportional reinsurance treaties, we have two traditional forms. *Quota-share* or *proportional* reinsurance where the reinsurer accepts a proportion $a \in]0, 1]$ of the total portfolio. Further, *surplus* reinsurance where the reinsured amount is also determined by the value of the insured object as long as it exceeds a retention L. Within the framework of nonproportional reinsurance treaties, we cite four forms. An *excess-of-loss* reinsurance is determined by a retention M indicating that the reinsurer covers the part of the claims that overshoots M. In a *stop-loss* reinsurance contract the reinsured amount is equal to the part of the entire portfolio that overshoots a retention C. Note that excess-of-loss and stop-loss reinsurance treaties are equivalent when a single risk comes into play. Furthermore, there are reinsurance treaties classified as large claims reinsurance since they are defined in terms of the order statistics of the claims. A first form is *largest claims* reinsurance where the reinsured amount combines the values of the r largest claims in the portfolio. A second and slightly more popular form is *ECOMOR* reinsurance which is defined as an excess-of-loss treaty with the $(r + 1)$th largest claim as random retention. We refer to Embrechts et al. [4], Ladoucette and Teugels [5] and Teugels [10] for asymptotic problems pertaining to ECOMOR as well as to the largest claims reinsurance treaty. For an overview of most of the currently employed reinsurance forms with some of their properties, see Rolski et al. [7] and Teugels [11].

In accordance with current practice, quota-share and stop-loss treaties are popular if the number of claims is large. Also, excess-of-loss and surplus reinsurance are more traditional if the claim sizes are large. Furthermore, reinsurance based on the largest claims is almost never used. This fact is rather surprising if reinsurance is meant to protect the insurer against large claims. It looks almost necessary to use a reinsurance treaty that is based on these largest claims.

Combinations of different forms of treaties are easily constructed. Schmitter [8] combines quota-share and stop-loss treaties, where the reinsured amount is defined by:

$$R := \max\left(0, a\sum_{i=1}^{N} Y_i - C\right).$$

Quota-share and excess-of-loss treaties are combined in Centeno [3]. The reinsured amount then has the following expression:

$$R := \sum_{i=1}^{N} \max\left(0, aY_i - M\right).$$

Benktander and Ohlin [2] combine a surplus treaty with an excess-of-loss treaty. Then the reinsured amount is:

$$R := \sum_{i=1}^{N} \max\left\{0, \left(\frac{V_i - L}{V_i}\right) X_i I[V_i > L] - M\right\}$$

where V_i is the value insured for the ith claim policy. Here and throughout the paper, $I[.]$ stands for the indicator function.

Such combinations are referred to as *partial* reinsurance. For example, see Steenackers and Goovaerts [9]. Even more popular is the combination of a stop-loss treaty on top of an excess-of-loss treaty. In this case, one has:

$$R := \max\left\{0, \sum_{i=1}^{N} \max\left(0, Y_i - M\right) - C\right\}.$$

A further generalization is called *drop down excess-of-loss* reinsurance. In this type of setting, the claim is curtailed at both ends, both of them possibly depending on the order of the claim. The reinsured amount has a form of the kind:

$$R := \sum_{i=1}^{N} \min\left\{L_i, \max\left(0, Y^*_{N-i+1} - M_i\right)\right\}$$

where the M_i's stand for the lower retentions while the L_i's determine the drop down upper retentions $M_i + L_i$.

In the present paper, we investigate a nonproportional reinsurance form defined as a combination of quota-share and drop down excess-of-loss treaties in the following way. Consider a nonnegative random variable X, called *risk*, with distribution function F_X. As examples, the risk X may represent an individual claim Y_i, the ith largest claim Y^*_{N-i+1} or even the total claim amount $\sum_{i=1}^{N} Y_i$, where N is the number of claims occurred over a given period of time. Choose drop down excess-of-loss lower and upper retentions which we respectively denote by u and $u + v$. The range for which the treaty is used is then indicated by v. Secondly, choose a quota-share retention $a \in \,]0, 1]$, so that the reinsured amount $X_{a,u,v}$ of the risk X is given by:

$$X_{a,u,v} := a \min\left\{v, \max\left(0, X - u\right)\right\}. \tag{1}$$

For what follows, we assume that u and v are such that $0 \leq u < \infty$ and $0 < v \leq \infty$.

It is easy to see that $X_{a,u,v}$ may be rewritten in the more explicit form:

$$X_{a,u,v} = \begin{cases} 0, & X \leq u \\ a(X - u), & u < X \leq u + v \\ av, & X > u + v. \end{cases}$$

From its definition, it immediately follows that $0 \leq X_{a,u,v} \leq av$, i.e. that $X_{a,u,v}$ is a nonnegative random variable which is bounded when $v \neq \infty$. Moreover, the distribution function $F_{X_{a,u,v}}$ of $X_{a,u,v}$ is simply given by:

$$F_{X_{a,u,v}}(x) := \mathbb{P}[X_{a,u,v} \leq x] = \begin{cases} 0, & x < 0 \\ F_X(u + x/a), & 0 \leq x < av \\ 1, & x \geq av. \end{cases} \tag{2}$$

In its most general setting, (1) defines a combination of quota-share and drop down excess-of-loss reinsurance treaties. In the particular case $v = \infty$, one arrives at a quota-share treaty when $u = 0$, a combination of quota-share and excess-of-loss treaties for an individual claim or the ith largest claim when $u \neq 0$, or a combination of quota-share and stop-loss treaties when X is used as the total claim amount and $u \neq 0$. The reinsurance form defined in (1) makes the classical treaties a bit more flexible by also allowing an upper retention. This seems worthwhile in view of the different situations between first and second line insurance. For a first line insurer, $u = 0$ while $v = M < \infty$. For the first reinsurer however, $u = M$ while v may take any nonnegative value. If the first line reinsurer does not shift part of the risk to a second reinsurer, then $v = \infty$. Otherwise $v < \infty$ and so on. By not specifying u and v, our results apply to any company in a reinsurance chain of this type. Of course, any reinsurer can apply the type of reinsurance of his choice, irrespective of what a former insurer has been doing.

In the following, we are interested in the behavior of some risk measures that are frequently used in practical situations, in particular in reinsurance. Risk measures are useful in evaluating or estimating the risk associated with a random quantity. In the context of (re)insurance, such measures also permit to illustrate and quantify the effect of a premium scheme. Look in Venter [12] where it is advocated to use a variety of risk measures.

In Section 2, we investigate a couple of risk measures for the quantity $X_{a,u,v}$. We successively deal with the *Value-at-Risk* (Subsection 2.1), the *variance* (Subsection 2.2) and the *coefficient of variation* (Subsection 2.3). Section 3 is devoted to the computation of risk measures over a portfolio with N claims, each of them reinsured under (1). We successively deal with the variance (Subsection 3.1), the *dispersion* (Subsection 3.2), the coefficient of variation (Subsection 3.3) and the *reduction effect* (Subsection 3.4). We conclude in Section 4.

For theoretical and practical results pertaining to the coefficient of variation within the context of (re)insurance, see Mack [6]. For the asymptotic behavior of the sample coefficient of variation as well as for the sample dispersion, see the paper by Albrecher and Teugels [1].

2 Risk Measures for a Single Amount Reinsured

In this section, we investigate some usual risk measures (Value-at-Risk, variance and coefficient of variation) for the reinsured amount $X_{a,u,v}$ defined by (1).

We start with some relations that will prove useful later on. As the random variable $X_{a,u,v}$ is nonnegative, we can define for any integral $k > 0$:

$$\tilde{\mu}_k := \mathbb{E}X^k_{a,u,v} = k\int_0^\infty x^{k-1}\left(1 - F_{X_{a,u,v}}(x)\right)\, dx.$$

Recalling (2), it easily follows that:

$$\tilde{\mu}_k = k\int_0^{av} x^{k-1}\left(1 - F_X(u + x/a)\right)\, dx. \tag{3}$$

This quickly leads to the obvious inequalities:

$$(av)^k(1 - F_X(u+v)) \leq \tilde{\mu}_k \leq (av)^k(1 - F_X(u)).$$

On the other hand, we also have $\mathbb{E}X^k_{a,u,v} \leq av\mathbb{E}X^{k-1}_{a,u,v}$ so that:

$$\tilde{\mu}_k \leq av\tilde{\mu}_{k-1}. \tag{4}$$

Using (3), we calculate the partial derivative of $\tilde{\mu}_k$ with respect to v to be:

$$\frac{\partial\tilde{\mu}_k}{\partial v} = ak(av)^{k-1}\left(1 - F_X(u+v)\right). \tag{5}$$

We also deduce the following relation between the partial derivatives of two successive moments:

$$\frac{\partial\tilde{\mu}_{k+1}}{\partial v} = \frac{k+1}{k}av\frac{\partial\tilde{\mu}_k}{\partial v}. \tag{6}$$

2.1 Value-at-Risk

We define and denote the Value-at-Risk of a random variable Z with the help of the *tail quantile function* U_Z by:

$$VaR_q(Z) := U_Z(q), \quad q \geq 1$$

recalling that $U_Z(y) := \inf\{x : F_Z(x) \geq 1 - 1/y\}$, where F_Z is the distribution function of Z.

If Z represents a claim size, it is rather clear that when q is large, this is only defined for the reinsurer that carries the ultimate tail of the claim size.

For the random variable $X_{a,u,v}$, we are naturally led to the following Value-at-Risk:

$$VaR_q(X_{a,u,v}) = \begin{cases} 0, & 1 \leq q < U_X^{\leftarrow}(u) \\ a(U_X(q) - u), & U_X^{\leftarrow}(u) \leq q < U_X^{\leftarrow}(u+v) \\ av, & q \geq U_X^{\leftarrow}(u+v) \end{cases}$$

where U_X is the tail quantile function of F_X and $U_X^{\leftarrow}$ is its left-continuous inverse function defined by $U_X^{\leftarrow}(x) := \inf\{y : U_X(y) \geq x\}$.

Moreover, we note that:

$$VaR_q(X_{a,u,v}) \leq VaR_q(aX) \leq VaR_q(X), \quad q \geq 1.$$

2.2 Variance

Now, we deal with the variance $\mathbb{V}X_{a,u,v}$ of the reinsured amount $X_{a,u,v}$. From (3), it is immediate that:

$$\mathbb{V}X_{a,u,v} = 2\int_0^{av} x\,(1 - F_X(u + x/a))\;dx - \left(\int_0^{av} (1 - F_X(u + x/a))\;dx\right)^2. \tag{7}$$

By (5), we therefore get:

$$\begin{aligned} \frac{\partial \mathbb{V}X_{a,u,v}}{\partial v} &= \frac{\partial \tilde{\mu}_2}{\partial v} - 2\tilde{\mu}_1 \frac{\partial \tilde{\mu}_1}{\partial v} \\ &= 2a^2 v\,(1 - F_X(u+v)) - 2a\,(1 - F_X(u+v)) \int_0^{av} (1 - F_X(u + x/a))\;dx \\ &= 2a\,(1 - F_X(u+v)) \int_0^{av} F_X(u + x/a)\;dx \geq 0. \end{aligned}$$

The partial derivative with respect to v being nonnegative, the variance is nondecreasing in v so that:

$$\begin{aligned} \mathbb{V}X_{a,u,v} &\leq 2\int_0^{\infty} x\,(1 - F_X(u + x/a))\;dx - \left(\int_0^{\infty} (1 - F_X(u + x/a))\;dx\right)^2 \\ &= 2a^2 \int_u^{\infty} (y - u)\,(1 - F_X(y))\;dy - a^2 \left(\int_u^{\infty} (1 - F_X(y))\;dy\right)^2 =: g_a(u). \end{aligned}$$

Now, we compute:

$$\begin{aligned} \frac{\partial g_a(u)}{\partial u} &= -2a^2 \int_u^{\infty} (1 - F_X(y))\;dy + 2a^2\,(1 - F_X(y)) \int_u^{\infty} (1 - F_X(y))\;dy \\ &= -2a^2 F_X(y) \int_u^{\infty} (1 - F_X(y))\;dy \leq 0. \end{aligned}$$

Thus, the function $g_a(u)$ is nonincreasing in u and therefore smaller that the same expression where we put $u = 0$, that is $g_a(0)$. However, that quantity is equal to $a^2\mathbb{V}X$. Hence, we get the following result for the variance of the reinsured amount $X_{a,u,v}$:

$$\mathbb{V}X_{a,u,v} \leq a^2 \mathbb{V}X.$$

Since $a \in]0,1]$, we also remark that $\mathbb{V}X_{a,u,v} \leq \mathbb{V}X$ as it should be.

2.3 Coefficient of Variation

Another risk measure that is popular among actuaries is the coefficient of variation of a positive random variable Z, defined and denoted by:

$$CoVar(Z) := \frac{\sqrt{\mathbb{V}Z}}{\mathbb{E}Z}.$$

Note that the dimensionless coefficient of variation is a parameter of any distribution with finite variance and that it is a relative measure of dispersion around the mean.

For the reinsured amount $X_{a,u,v}$, we get:

$$CoVar(X_{a,u,v}) = \sqrt{\frac{\tilde{\mu}_2}{\tilde{\mu}_1^2} - 1}$$

and hence, the coefficient will depend monotonically on the ratio under the square root. Incorporating both retentions into the notation, we write $CoVar(u,v) := CoVar(X_{a,u,v})$ for arbitrary u and v.

Inequality (4) together with (6) leads to the following:

$$\begin{aligned} \frac{\partial}{\partial v}\frac{\tilde{\mu}_2}{\tilde{\mu}_1^2} &= (\tilde{\mu}_1)^{-3}\left(\tilde{\mu}_1\frac{\partial\tilde{\mu}_2}{\partial v} - 2\tilde{\mu}_2\frac{\partial\tilde{\mu}_1}{\partial v}\right) \\ &\geq (\tilde{\mu}_1)^{-2}\left(\frac{\partial\tilde{\mu}_2}{\partial v} - 2av\frac{\partial\tilde{\mu}_1}{\partial v}\right) = 0. \end{aligned}$$

The above then shows that $CoVar(u,v)$ is nondecreasing in v. Hence, we get:

$$CoVar(u,v) \leq CoVar(u,\infty). \tag{8}$$

The dependence on u is more intricate. Let us introduce the *retention distribution*:

$$G_{a,u,v}(x) := \begin{cases} \frac{F_X(u+x/a)-F_X(u)}{F_X(u+v)-F_X(u)}, & 0 \leq x < av \\ 1, & x \geq av \end{cases}$$

with moments $\nu_k := k\int_0^\infty x^{k-1}\left(1 - G_{a,u,v}(x)\right)\, dx$.

If we abbreviate $\Delta_X := F_X(u+v) - F_X(u)$, then we get:

$$\begin{aligned} \Delta_X\nu_k &= k\left(F_X(u+v) - F_X(u)\right)\int_0^{av} x^{k-1}\left(1 - \frac{F_X(u+x/a) - F_X(u)}{F_X(u+v) - F_X(u)}\right)\, dx \\ &= -k\int_0^{av} x^{k-1}\left\{1 - F_X(u+v) - \left(1 - F_X(u+x/a)\right)\right\}\, dx \\ &= -(av)^k\left(1 - F_X(u+v)\right) + k\int_0^{av} x^{k-1}\left(1 - F_X(u+x/a)\right)\, dx \\ &= \tilde{\mu}_k - (av)^k\left(1 - F_X(u+v)\right). \end{aligned} \tag{9}$$

The latter relation is handy in rewriting the partial derivative of $CoVar(u,v)$ with respect to u. Indeed, it follows that:

$$\frac{\partial\tilde{\mu}_1}{\partial u} = -\int_0^{av}\frac{\partial}{\partial u}F_X(u+x/a)\, dx = -a\left(F_X(u+v) - F_X(u)\right) = -a\Delta_X$$

and:

$$\begin{aligned} \frac{\partial\tilde{\mu}_2}{\partial u} &= -2a\int_0^{av} x\, d_xF_X(u+x/a) \\ &= \Big[-2axF_X(u+x/a)\Big]_0^{av} + 2a\int_0^{av} F_X(u+x/a)\, dx \end{aligned}$$

$$
\begin{aligned}
&= -2a^2 v F_X(u+v) + 2a \int_0^{av} F_X(u+x/a)\ dx \\
&= -2a \int_0^{av} (1 - F_X(u+x/a))\ dx + 2a^2 v - 2a^2 v F_X(u+v) \\
&= 2a^2 v\,(1 - F_X(u+v)) - 2a \int_0^{av} (1 - F_X(u+x/a))\ dx = -2a\Delta_X \nu_1.
\end{aligned}
$$

But then, we have:

$$
\tilde{\mu}_1^3 \frac{\partial}{\partial u} \frac{\tilde{\mu}_2}{\tilde{\mu}_1^2} = \tilde{\mu}_1 \frac{\partial \tilde{\mu}_2}{\partial u} - 2\tilde{\mu}_2 \frac{\partial \tilde{\mu}_1}{\partial u} = 2a\Delta_X\left(\tilde{\mu}_2 - \tilde{\mu}_1 \nu_1\right).
$$

Replacing in the last expression the moments $\tilde{\mu}_k$ by their analogues ν_k from (9), we get:

$$
\begin{aligned}
\tilde{\mu}_1^3 \frac{\partial}{\partial u} \frac{\tilde{\mu}_2}{\tilde{\mu}_1^2} &= 2a\Delta_X \left\{ \Delta_X \nu_2 + (av)^2 (1 - F_X(u+v)) - \nu_1 \left(\Delta_X \nu_1 + av\,(1 - F_X(u+v))\right)\right\} \\
&= 2a\Delta_X^2 \left(\nu_2 - \nu_1^2\right) + 2a^2 \Delta_X v\,(av - \nu_1)\,(1 - F_X(u+v)) \geq 0.
\end{aligned}
$$

Indeed, the quantity $\nu_2 - \nu_1^2$ is the variance of the distribution $G_{a,u,v}$ while by definition $\nu_1 \leq av$. This then shows that the requested partial derivative is nonnegative and hence that $CoVar(u,v)$ is also nondecreasing in u. For fixed $a \in]0,1]$, $CoVar(u,v)$ is thus nondecreasing both in u and v. In particular, we find that:

$$
CoVar(u,v) \geq CoVar(0,v). \tag{10}
$$

Applying both inequalities (8) and (10), we get the following lower and upper bounds for the coefficient of variation of the reinsured quantity $X_{a,u,v}$:

$$
CoVar(0,v) \leq CoVar(u,v) \leq CoVar(u,\infty)
$$

comparing the risk measure for three of the partners in a reinsurance chain for any value of $a \in]0,1]$. Applying the inequality on the left for $v = \infty$ and that on the right for $u = 0$, we also get the following inequalities:

$$
CoVar(0,v) \leq CoVar(0,\infty) = CoVar(X) \leq CoVar(u,\infty)
$$

where the quantity in the middle is the coefficient of variation of the initial risk X. The latter inequality has been obtained by Mack [6] for $a = 1$.

3 Risk Measures for the Total Amount Reinsured over the Portfolio

After a combination of a drop down excess-of-loss treaty with a quota-share treaty, the reinsured amount over a portfolio with N claims is the quantity given by:

$$
R_{a,u,v} := \sum_{i=1}^{N} {}_iY_{a,u,v} \tag{11}
$$

where ${}_iY_{a,u,v}$, which is used as the reinsured part of the original claim size Y_i, has the same distribution as the random variable $Y_{a,u,v} := a \min\{v, \max(0, Y - u)\}$ corresponding to the reinsured part defined by (1) of the generic claim size Y whose distribution function is F_Y.

When $v = \infty$ in (11), we get a combination of quota-share and excess-of-loss reinsurance treaties. Furthermore, if also $u = 0$, we get a quota-share reinsurance for the aggregate claim amount in that then $R_{a,0,\infty} = a \sum_{i=1}^{N} Y_i$.

Using the so-called Wald identities, we easily derive the following expressions for the mean and the variance of the quantity $R_{a,u,v}$:

$$\begin{aligned} \mathbb{E}R_{a,u,v} &= \tilde{\mu}_1 \, \mathbb{E}N \\ \mathbb{V}R_{a,u,v} &= \mathbb{V}Y_{a,u,v} \, \mathbb{E}N + \tilde{\mu}_1^2 \, \mathbb{V}N \end{aligned}$$

where $\tilde{\mu}_1$ and $\mathbb{V}Y_{a,u,v}$ are respectively given by (3) and (7), replacing F_X by F_Y in these formulas.

By way of example and without loosing too much of generality, we assume from now on that there are exactly three partners involved. For the first line there is a deductible determined by a retention u. The first reinsurer is responsible for the part of the claims above the level u and up to the retention $u + v$. Finally, a last reinsurer deals with all the claims above the level $u + v$. Consequently, we assume that $F_Y(u)\Delta_Y(1 - F_Y(u + v)) > 0$. To investigate the effect of reinsurance on the partners involved, let us introduce some notation.

1. For the first insurer, the total claim amount is $S_L := \sum_{i=1}^N L_i$ where L_i has the same distribution as the random variable:

$$L := aYI[Y \leq u] + auI[Y > u] = a \min(u, Y).$$

   This corresponds to the choice $u = 0$ and $v = u$ in the general approach (11). We use the letter L to point at the lower part of the claims.

2. For the first reinsurer, the total claim amount is $S_M := \sum_{i=1}^N M_i$ where M_i has the same distribution as the random variable:

$$M := a(Y - u)I[u < Y \leq u + v] + avI[Y > u + v] = a \min\{v, \max(0, Y - u)\}.$$

   This case corresponds to the general approach (11) with u and v. The letter M points to the middle part of the claims.

3. Finally, the top layer is carried by the second and last reinsurer. Its total claim amount is given by: $S_U := \sum_{i=1}^N U_i$ where U_i has the same distribution as the random variable:

$$U := a(Y - (u + v))I[Y > u + v] = a \max\{0, Y - (u + v)\}.$$

   In (11) we need to replace u by $u + v$ while $v = \infty$. We use the letter U to indicate that the upper part of the claims is now at stake.

Of course, the sum of the three total claim amounts S_L, S_M and S_U adds up to $R_{a,0,\infty} = a \sum_{i=1}^N Y_i$. The remaining part $(1 - a) \sum_{i=1}^N Y_i$ should be considered as taken out, prior to the risk sharing of the part $a \sum_{i=1}^N Y_i$ by the three partners.

As an additional risk measure, let us also look at the *correlations*. From the definitions of the three variables L, M and U, it follows that $\mathbb{E}\{LM\} = au \; \mathbb{E}M$, $\mathbb{E}\{LU\} = au \; \mathbb{E}U$ and $\mathbb{E}\{MU\} = av \; \mathbb{E}U$. Therefore, the three covariances $Cov(L, M)$, $Cov(L, U)$ and $Cov(M, U)$ are all positive. To see the influence on the three separate total claim amounts, we calculate the covariances between the quantities S_L, S_M and S_U. An easy calculation learns us that:

$$\mathbb{E}\{S_L S_M\} = \sum_{n=0}^{\infty} p_n \sum_{i=1}^{n} \mathbb{E}\{L_i M_i\} + \sum_{n=0}^{\infty} p_n \sum_{i=1}^{n} \sum_{\substack{j=1 \\ j \neq i}}^{n} \mathbb{E}\{L_i M_j\}$$

where $p_n := \mathbb{P}[N = n]$ for $n \in \mathbb{N}$, and hence:

$$Cov(S_L, S_M) = Cov(L, M) \, \mathbb{E}N + \mathbb{E}L \, \mathbb{E}M \, \mathbb{V}N. \tag{12}$$

In equation (12), one can replace the pair (L, M) by (L, U) and (M, U) without any problem. Using the expressions for the covariances between the variables L, M and U, one finds the expressions:

$$\begin{aligned}
Cov(S_L, S_M) &= \mathbb{E}L\,\mathbb{E}M\,(\mathbb{V}N - \mathbb{E}N) + au\,\mathbb{E}M\,\mathbb{E}N \\
Cov(S_L, S_U) &= \mathbb{E}L\,\mathbb{E}U\,(\mathbb{V}N - \mathbb{E}N) + au\,\mathbb{E}U\,\mathbb{E}N \\
Cov(S_M, S_U) &= \mathbb{E}M\,\mathbb{E}U\,(\mathbb{V}N - \mathbb{E}N) + av\,\mathbb{E}U\,\mathbb{E}N.
\end{aligned}$$

In the following, we compare some risk measures for the different partners in the chain, namely variance, dispersion, coefficient of variation and reduction effect. Let A and B be any two distinct letters from the set $\{L, M, U\}$.

3.1 Variance

First, we deal with the variances. If one uses the variance as a risk measure then one should compare the variance of a combined risk with the sum of the individual variances. It is clear that:

$$\mathbb{V}\{S_A + S_B\} = \mathbb{V}\{A + B\}\,\mathbb{E}N + (\mathbb{E}A + \mathbb{E}B)^2\,\mathbb{V}N.$$

Since (12) learns us that $Cov(S_A, S_B) > 0$, we get by a simple calculation:

$$\mathbb{V}\{S_A + S_B\} = \mathbb{V}S_A + \mathbb{V}S_B + 2Cov(S_A, S_B) > \mathbb{V}S_A + \mathbb{V}S_B.$$

A similar relation holds between the two other pairs. Hence, there is an increase in variance if one combines different layers in the reinsurance chain.

3.2 Dispersion

The situation is different for the *dispersions*. We recall that the dispersion of a positive random variable Z is defined and denoted by:

$$D(Z) := \frac{\mathbb{V}Z}{\mathbb{E}Z}.$$

This risk measure is frequently used in (re)insurance and its value allows to compare the volatility with respect to the Poisson case.

The correlation coefficient between S_A and S_B is at most 1 in absolute value. Hence, we see that $Cov(S_A, S_B) < \sqrt{\mathbb{V}S_A \mathbb{V}S_B}$ and as a result:

$$\sqrt{\mathbb{V}\{S_A + S_B\}} < \sqrt{\mathbb{V}S_A} + \sqrt{\mathbb{V}S_B}. \tag{13}$$

Using (13), we calculate:

$$\begin{aligned}
D(S_A + S_B) &= \frac{\mathbb{V}\{S_A + S_B\}}{\mathbb{E}\{S_A + S_B\}} \\
&< \frac{\mathbb{V}S_A + \mathbb{V}S_B + 2\sqrt{\mathbb{V}S_A \mathbb{V}S_B}}{\mathbb{E}\{S_A + S_B\}} \\
&= D(S_A) + D(S_B) - \left\{ \frac{\mathbb{V}S_A \mathbb{E}S_B}{\mathbb{E}S_A \mathbb{E}\{S_A + S_B\}} + \frac{\mathbb{V}S_B \mathbb{E}S_A}{\mathbb{E}S_B \mathbb{E}\{S_A + S_B\}} - 2\frac{\sqrt{\mathbb{V}S_A \mathbb{V}S_B}}{\mathbb{E}\{S_A + S_B\}} \right\} \\
&= D(S_A) + D(S_B) - \left\{ \frac{\mathbb{V}S_A(\mathbb{E}S_B)^2 + \mathbb{V}S_B(\mathbb{E}S_A)^2 - 2\sqrt{\mathbb{V}S_A}\mathbb{E}S_B\sqrt{\mathbb{V}S_B}\mathbb{E}S_A}{\mathbb{E}S_A \mathbb{E}S_B \mathbb{E}\{S_A + S_B\}} \right\} \\
&\leq D(S_A) + D(S_B).
\end{aligned}$$

Therefore, the dispersion decreases by combining different layers.

3.3 Coefficient of Variation

Let us look at the coefficient of variation of S_A. From general principles, we derive:

$$\begin{aligned}\frac{\mathbb{V}S_A}{(\mathbb{E}S_A)^2} &= \frac{\mathbb{V}A\,\mathbb{E}N + (\mathbb{E}A)^2\,\mathbb{V}N}{(\mathbb{E}A\,\mathbb{E}N)^2} \\ &= \frac{\mathbb{V}A}{(\mathbb{E}A)^2\mathbb{E}N} + \frac{\mathbb{V}N}{(\mathbb{E}N)^2}.\end{aligned}$$

Hence, we obtain the interesting formula:

$$CoVar^2(S_A) = \frac{CoVar^2(A)}{\mathbb{E}N} + CoVar^2(N).$$

The second term on the right depends on the number of claims and is the same for all partners involved. Also the factor $(\mathbb{E}N)^{-1}$ is the same. Therefore, the differences in the risk measure solely depend on the coefficient of variation for the retained risk. However, the comparison of this risk measure or the quantity $\tilde{\mu}_2/\tilde{\mu}_1^2$ for the different values of u and v is by no means simple.

Moreover, using (13), we easily deduce:

$$CoVar(S_A + S_B) = \frac{\sqrt{\mathbb{V}\{S_A + S_B\}}}{\mathbb{E}\{S_A + S_B\}} < \frac{\sqrt{\mathbb{V}S_A}}{\mathbb{E}S_A} + \frac{\sqrt{\mathbb{V}S_B}}{\mathbb{E}S_B} = CoVar(S_A) + CoVar(S_B).$$

Thus, by combining different layers in the reinsurance chain, the coefficient of variation decreases.

3.4 Reduction Effect

As another risk measure, we finally look at the *reduction effect*. There are at least two ways of writing out a reduction effect depending on the partner one is looking at. For the first insurer however it does hardly matter how the second reinsurance retention v is decided. Therefore it seems natural to calculate the reduction by the quantity:

$$r_L := \frac{\mathbb{E}S_L}{\mathbb{E}\{S_L + S_M + S_U\}}.$$

However, this is easily calculated since:

$$r_L = \frac{\mathbb{E}L}{a\mathbb{E}Y} = \frac{1}{\mathbb{E}Y}\int_0^u (1 - F_Y(x))\,dx.$$

The latter quantity is well known from renewal theory where it is called the *equilibrium distribution* of F_Y. Properties of this distribution can be found in standard treatises of renewal theory.

For the first reinsurer, the natural reduction effect is:

$$r_M := \frac{\mathbb{E}S_M}{\mathbb{E}\{S_M + S_U\}}$$

as only the reduction with respect to the last reinsurer needs to be taken into account. Then we can write:

$$r_M = \frac{\mathbb{E}M}{\mathbb{E}M + \mathbb{E}U} = \frac{\int_0^v (1 - F_Y(u+x))\,dx}{\int_0^\infty (1 - F_Y(u+x))\,dx} =: F(v).$$

Note that the quantity on the right is indeed a proper distribution which for every fixed value of u can be considered as an equilibrium distribution.

4 Conclusion

In this paper, we have been concerned with some common risk measures for the reinsured amount associated with a nonproportional reinsurance form defined as a combination of quota-share and drop down excess-of-loss treaties. In particular, we have considered Value-at-Risk, variance, coefficient of variation, dispersion and reduction effect.

A forthcoming paper will be devoted to the study of the reinsured quantity defined in (1) when the risk X is coming from an ordered claim $X^*_{N(t)-i+1}$, u and v being replaced by functions both depending on time t and rank i. In particular, asymptotic properties of this quantity will be derived.

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