Phase transitions for the long-time behaviour of interacting diffusions

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Abstract

Let $(\{X_i(t)\}_{i\in\mathbb{Z}^d})_{t\geq 0}$ be the system of interacting diffusions on $[0,\infty)$ defined by the following collection of coupled stochastic differential equations:

$$dX_i(t) = \sum_{j\in\mathbb{Z}^d} a(i,j)[X_j(t) - X_i(t)]\,dt + \sqrt{bX_i^2(t)}\,dW_i(t), \qquad i\in\mathbb{Z}^d, t\geq 0.$$

Here, $a(\cdot,\cdot)$ is an irreducible random walk transition kernel on $\mathbb{Z}^d\times\mathbb{Z}^d$, $b\in(0,\infty)$ is a diffusion parameter, and $(\{W_i(t)\}_{i\in\mathbb{Z}^d})_{t\geq 0}$ is a collection of independent standard Brownian motions on $\mathbb{R}$. The initial condition is chosen such that $\{X_i(0)\}_{i\in\mathbb{Z}^d}$ is a shift-invariant and shift-ergodic random field on $[0,\infty)$ with 1-st moment $\Theta\in(0,\infty)$ (during the evolution, the 1-st moment is preserved). We show that the long-time behaviour of this system is the result of a delicate interplay between $a(\cdot,\cdot)$ and b, in contrast to systems where the diffusion function is subquadratic. In particular, let $\widehat{a}(i,j) = \frac{1}{2}[a(i,j) + a(j,i)]$, $i,j\in\mathbb{Z}^d$, denote the symmetrised transition kernel. We show that:

(A) If $\widehat{a}(\cdot,\cdot)$ is recurrent, then for any $b>0$ the system locally dies out.

(B) If $\widehat{a}(\cdot,\cdot)$ is transient, then there exist $b_* > b_2 > 0$ such that:
- (B1) The system converges to an equilibrium ν_Θ if $0<b<b_*$.
- (B2) The system locally dies out if $b>b_*$.
- (B3) ν_Θ has a finite 2-nd moment if and only if $0<b<b_2$.
- (B4) The 2-nd moment diverges exponentially fast if and only if $b>b_2$.

For the case where $a(\cdot,\cdot)$ is symmetric and transient we further show that:

(C) There exists a sequence $b_2\geq b_3\geq b_4\geq\cdots>0$ such that:
- (C1) ν_Θ has a finite m-th moment if and only if $0<b<b_m$.
- (C2) The m-th moment diverges exponentially fast if and only if $b>b_m$.
- (C3) $b_2\leq(m-1)b_m<2$.
- (C4) $\lim_{m\to\infty}(m-1)b_m$ exists.

The proof of these results is based on self-duality and on a representation formula through which the moments of the components are related to exponential moments of the intersection local time of random walks. Via large deviation theory, the latter lead to variational

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expressions for b_* and the b_m's, from which sharp bounds are deduced. The critical value b_* arises from a representation formula for the Palm distribution of the system. The equilibrium ν_Θ is shown to be associated and mixing for all $0 < b < b_*$.

The special case where $a(\cdot,\cdot)$ is simple random walk is commonly referred to as the parabolic Anderson model with Brownian noise. This case was studied in the memoir by Carmona and Molchanov [6], where some of our results were already established.

Keywords: Interacting diffusions, phase transitions, large deviations, intersection local time of random walks, self-duality, representation formula, quasi-stationary distribution, Palm distribution.
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Contents

1 Introduction and main results

1.1 Motivation and background

This paper is concerned with the long-time behaviour of a particular class of systems with interacting components. In this class, the components are interacting diffusions that take values in $[0, \infty)$ and that are labelled by a countably infinite Abelian group I. The reason for studying these systems is two-fold: *new phenomena* occur, and a number of methodological problems can be tackled that are unresolved in the broader context of interacting systems with non-compact components. We begin by describing in more detail the background of the questions to be addressed.

A large class of interacting systems has the property that single components change according to a certain random evolution, while the interaction between the components is linear and can be interpreted as migration of mass, charge or particles. Examples are:

(1) *interacting particle systems*, e.g. voter model (Holley and Liggett [33]), branching random walk (Kallenberg [35], Durrett [21]), generalised potlatch and smoothing process (Holley and Liggett [34]), binary path process (Griffeath [30]), coupled branching process (Greven [26], [27]), locally dependent branching process (Birkner [3]), catalytic branching (Kesten and Sidoravicius [36], Gärtner and den Hollander [25]).

(2) *interacting diffusions*, e.g. Fisher-Wright diffusion (Shiga [40], [41], Dawson and Greven [13], [14], Cox and Greven [10], Fleischmann and Greven [23], [24], Cox, Fleischmann and Greven [9], den Hollander and Swart [32], den Hollander [31], Swart [44]), critical Ornstein-Uhlenbeck process (Deuschel [18], [19]), Feller's branching diffusion (Shiga [41], Dawson and Greven [15]), parabolic Anderson model with Brownian noise (Carmona and Molchanov [6]).

(3) *interacting measure-valued diffusions*, e.g. Fleming-Viot process (Dawson, Greven and Vaillancourt [16]), mutually catalytic diffusions (Dawson and Perkins [17]), catalytic interacting diffusions (Greven, Klenke and Wakolbinger [29]).

Most of these systems display the following *universality*: independently of the nature of the random evolution of single components, the ergodic behaviour of the system depends only on recurrence vs. transience of the migration mechanism. More precisely, if the symmetrised migration kernel is recurrent then the system approaches trivial equilibria (concentrated on the "traps" of the system), whereas if the symmetrised migration kernel is transient then nontrivial extremal equilibria exist that can be parametrised by the spatial density of the components.

In this paper we study an example in a *different universality class*, one where the nature of the random evolution of single components does influence in a crucial way the long-time behaviour of the system. In particular, we consider a system where the components evolve as diffusions on $[0,\infty)$ with diffusion function bx^2 and interact linearly according to a random walk transition kernel. Such a system is called the *parabolic Anderson model with Brownian noise* in the special case where the random walk is simple. In the recurrent case the system, as before, approaches a trivial equilibrium (concentrated on the "trap" with all components 0), so local extinction prevails. However, in the transient case we find three regimes, separated by critical thresholds $b_* > b_2 > 0$ (see Fig. 1):

(I) ("low noise") $0 < b < b_2$: equilibria with finite 2-nd moment.

(II) ("moderate noise") $b_2 \leq b < b_*$: equilibria with finite 1-st moment and infinite 2-nd moment.

(III) ("high noise") $b > b_*$: local extinction.

For the case where the random walk transition kernel is *symmetric* we do a finer analysis. We show that in regime (I) there exists a sequence $b_2 \geq b_3 \geq b_4 \geq \cdots > 0$ such that the equilibria have a finite m-th moment if and only if $0 < b < b_m$, while the m-th moment diverges exponentially fast if and only if $b > b_m$ (see Fig. 1). Moreover, we show that $b_2 \leq (m-1)b_m < 2$ and that $\lim_{m\to\infty}(m-1)b_m$ exists. We show that in regimes (I) and (II) the equilibria are associated and mixing. We show that the critical value b_* separating regimes (II) and (III) is linked to the Palm distribution of the system.

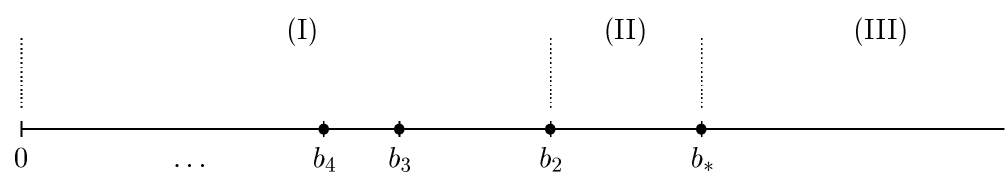

Fig. 1: Phase diagram for the transient case.

The reason for the above phase diagram is that there are two *competing* mechanisms: the migration pushes the components towards the mean value of the initial configuration, while the diffusion pushes them towards the boundary of the state space. Hence, there is a dichotomy in that either the migration dominates (giving nontrivial equilibria) or the diffusion dominates (giving local extinction). In the class of interacting diffusions we are concerned with here, *the migration and the diffusion have a strength of the same order of magnitude* and therefore the precise value of the diffusion parameter in relation to the migration kernel is crucial for the ergodic behaviour of the system.

Our results are a completion and a generalization of the results in the memoir of Carmona and Molchanov [6]. In [6], Chapter III, the focus is on the *annealed Lyapunov exponents* for simple random walk, i.e., on $\chi_m(b)$, the exponential growth rate of the m-th moment of $X_0(t)$, for successive m. It is shown that for each m there is a critical value b_m where $\chi_m(b)$ changes from being zero to being positive (see Fig. 2), and that the sequence (b_m) has the qualitative properties established here, i.e., $b_m = 0$ for all m in $d = 1, 2$ (recurrent case) and $b_2 \geq b_3 \geq b_4 \geq \cdots > 0$ in $d \geq 3$ (transient case). No existence of and convergence to equilibria is established below b_2, nor is any information on the equilibria obtained. There is also no analysis of what happens at the critical values. In our paper we are able to handle these issues due to the fact that we have variational expressions for $\chi_m(b)$ and b_m, which give us better control. In addition, we are able to get sharp bounds on b_m that are valid for arbitrary symmetric random walk.

In [6], Chapter IV, an analysis is given of the *quenched Lyapunov exponent* for simple random walk, i.e., on $\chi_*(b)$, the a.s. exponential growth rate of $X_0(t)$. It is shown $\chi_*(b)$ is negative for all $b > 0$ in $d = 1, 2$ (recurrent case), negative for $b > b_*$ and zero for $0 < b \leq b_*$ in $d \geq 3$ (transient case) for some $b_* \geq b_2$ (see Fig. 2). This corresponds to the crossover at b_*, except for the proof that $b_* > b_2$, which we establish here with the help of a variational expression for b_*. In [6], Chapter IV, it is further shown that $\chi_*(b)$ has a singular asymptotics for $b \to \infty$. This asymptotics has been sharpened in a sequence of subsequent papers by Carmona, Molchanov and Viens [7], Carmona, Koralov and Molchanov [5] and Cranston, Mountford and Shiga [12].

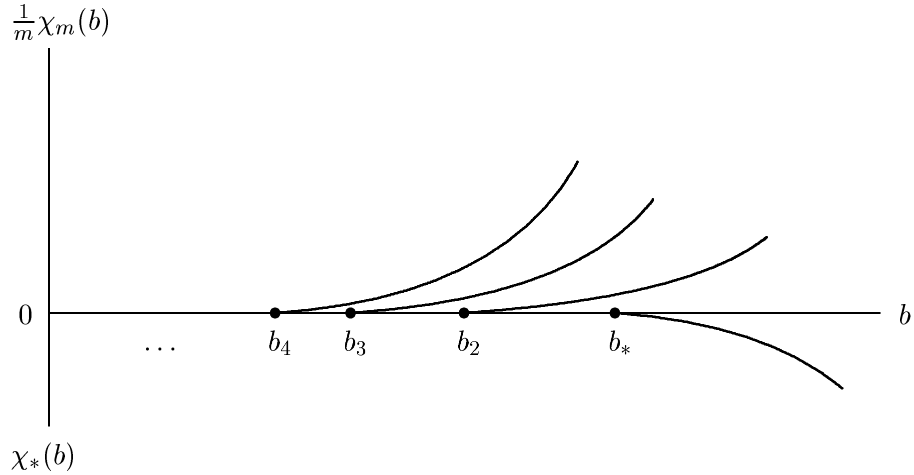

Fig. 2: Qualitative picture of $b \mapsto \frac{1}{m}\chi_m(b)$ and $b \mapsto \chi_*(b)$ for the transient case.

A scenario as described above is expected to hold for a number of interacting systems where the components take values in a non-compact state space, e.g. generalised potlatch and smoothing (Holley and Liggett [34]) and coupled branching (Greven [26], [27]). But for none of these systems has the scenario actually been fully proved.

1.2 Open problems

We formulate a number of open problems that are not addressed in the present paper:

(A) Show that $b_2 > b_3 > b_4 > \ldots$ This property is claimed in [6], Chapter III, Section 1.6, but no proof is provided. We are able to show that $b_2 > b_3 > \ldots > b_m$ for an arbitrary symmetric random walk for which the average number of returns to the origin is $\leq 1/(m-2)$. For $m = 3$, this includes simple random walk in $d \geq 3$.

(B) Show that the system locally dies out at the critical value b_*.

(C) Show that $\chi_*(b) < 0$ for $b > b_*$, i.e., show that there is no intermediate regime where the system locally dies out but only subexponentially fast. Shiga [41] has shown that the system locally dies out exponentially fast for b sufficiently large.

(D) Find out whether there exists a characterisation of b_* in terms of the intersection local time of random walks. This turns out to be a subtle problem, which has analogues in other models (see Birkner [3]). We find that such a characterisation does exist for b_m and for a certain b_{**} with $b_* \geq b_{**}$. We have a characterisation of b_* in terms of the Palm distribution of our process, but this is relatively inaccessible.

1.3 Outline

In Section 1.4 we define the model, formulate a theorem by Shiga and Shimizu [42] stating that our system of interacting diffusions has a unique strong solution with the Feller property, and introduce some key notions. In Section 1.5 we formulate two more theorems, due to Shiga [41] and to Cox and Greven [10], stating that our system locally dies out in the recurrent case and has associated mixing equilibria with finite 2-nd moment in the transient case in regime (I). We complement these two theorems with two new results, stating that our system has associated mixing equilibria with finite 1-st moment in the transient case in regime (II) and no equilibria in the transient case in regime (III). In Section 1.6 we present our finer results for regime (I), and have a closer look at regimes (II) and (III) as well, although much less detailed information is obtained for these regimes.

Sections 2–4 contain the proofs. Section 2 is devoted to moment calculations, which are based on a (Feynman-Kac type) representation formula for the solution of our system due to Shiga [41]. Through this representation formula, we express the moments of the components of our system in terms of exponential moments of the intersection local time of random walks. Through the latter we are able to establish convergence to a (possibly trivial) equilibrium and to prove that this equilibrium is shift-invariant, ergodic and associated. In Section 3 we study the exponential moments of the intersection local time with the help of large deviation theory, which leads to a detailed analysis of the critical thresholds b_m as a function of m in regime (I), as well as to a description of the behaviour of the system at b_m. Section 4 looks at survival versus extinction and relates the critical threshold b_* between regimes (II) and (III) to the so-called Palm distribution of our system, where the law of the process is changed by size-biasing with the value of the coordinate at the origin. There it is also shown that $b_* > b_2$, the proof of which relies on an explicit representation formula for the Palm distribution.

1.4 The model

The models that we consider are systems of interacting diffusions $X = (X(t))_{t \geq 0}$, where

$$X(t) = \{X_i(t)\}_{i \in I} \in [0, \infty)^I, \tag{1.4.1}$$

with I a countably infinite Abelian group. The evolution is defined by the following system of stochastic differential equations (SSDE):

$$dX_i(t) = \sum_{j\in I} a(i,j)[X_j(t) - X_i(t)]\, dt + \sqrt{bX_i^2(t)}\, dW_i(t), \qquad i \in I, t \geq 0. \tag{1.4.2}$$

Here

(i) $a(\cdot,\cdot)$ is a Markov transition kernel on $I \times I$.

(ii) $b \in (0,\infty)$ is a parameter.

(iii) $W = (\{W_i(t)\}_{i\in I})_{t\geq 0}$ is a collection of independent standard Brownian motions on $\mathbb{R}$.

As initial condition we take

$$X(0) \in \mathcal{E}_1, \tag{1.4.3}$$

where

$$\mathcal{E}_1 = \Big\{x = (x_i)_{i\in I} \in [0,\infty)^I : \sum_{i\in I} \gamma_i x_i < \infty\Big\} \subset L^1(\gamma) \tag{1.4.4}$$

for any $\gamma = (\gamma_i)_{i\in I}$ satisfying the requirements

$$\begin{aligned} &\gamma_i > 0 \quad \forall i \in I, \\ &\sum_{i\in I} \gamma_i < \infty, \\ &\exists M < \infty : \sum_{i\in I} \gamma_i a(i,j) \leq M\gamma_j \quad \forall j \in I. \end{aligned} \tag{1.4.5}$$

We endow $\mathcal{E}_1$ with the product topology of $[0,\infty)^I$.

Since $|I| = \infty$, it is not possible to define the process uniquely in the strong sense on $[0,\infty)^I$ without putting growth conditions on the initial configuration, as in (1.4.4). However, the dependence of $\mathcal{E}_1$ on γ is not very serious. For example, every probability measure ρ on $[0,\infty)^I$ satisfying $\sup_{i\in I} E^\rho(X_i) < \infty$ is concentrated on $\mathcal{E}_1$ regardless of the γ chosen (E^ρ denotes expectation with respect to ρ). We also need the space $\mathcal{E}_2 \subset L^2(\gamma)$, which is defined as in (1.4.4) but with the condition $\sum_{i\in I}\gamma_i x_i < \infty$ replaced by $\sum_{i\in I}\gamma_i (x_i)^2 < \infty$.

The most basic facts about the process $(X(t))_{t\geq 0}$ are summarized in the following result.

Theorem 1.4.1 (Shiga and Shimizu [42])

(a) The SSDE *in* (1.4.2) *has a unique strong solution $(X(t))_{t\geq 0}$ on $\mathcal{E}_1$ with continuous paths.*

(b) $(X(t))_{t\geq 0}$ is the unique Markov process on $\mathcal{E}_1$ whose semigroup $(S(t))_{t\geq 0}$ satisfies

$$S(t)f - f = \int_0^t S(s)Lf\, ds, \qquad f \in C_0^2(\mathcal{E}_1), \tag{1.4.6}$$

where $C_0^2(\mathcal{E}_1)$ is the space of functions on $\mathcal{E}_1$ depending on finitely many components and twice continuously differentiable in each component, and L is the pregenerator

$$(Lf)(x) = \sum_{i\in I}\Big\{\sum_{j\in I} a(i,j)[x_j - x_i]\Big\}\frac{\partial f}{\partial x_i} + \frac{1}{2}\sum_{i\in I} bx_i^2\, \frac{\partial^2 f}{\partial x_i^2}, \qquad x \in \mathcal{E}_1. \tag{1.4.7}$$

(c) Restricted to $\mathcal{E}_2$, $(X(t))_{t\geq 0}$ is a diffusion process with the Feller property.

The model defined by (1.4.2) represents a special case of the SSDE

$$dX_i(t) = \sum_{j\in I} a(i,j)[X_j(t) - X_i(t)]\; dt + \sqrt{g(X_i(t))}\; dW_i(t), \qquad i \in I, t \geq 0, \tag{1.4.8}$$

with $g\colon (-\infty,\infty) \to [0,\infty)$ some locally Lipschitz continuous function. This SSDE has, as far as its long-time behaviour is concerned, four important classes:

(i) $g(x) > 0$ on $(0,1)$.
Examples: $g(x) = x(1-x)$ Fisher-Wright, $g(x) = (x(1-x))^2$ Ohta-Kimura.

(ii) $g(x) > 0$ on $(-\infty,\infty)$ and $g(x) = o(x^2)$ as $x \to \pm\infty$.
Example: $g(x) \equiv \sigma^2$ critical Ornstein-Uhlenbeck.

(iii) $g(x) > 0$ on $(0,\infty)$ and $g(x) = o(x^2)$ as $x \to \infty$.
Example: $g(x) = x$ Feller's continuous-state branching diffusion.

(iv) $g(x) > 0$ on $(0,\infty)$ and $g(x) \sim bx^2$ as $x \to \infty$.

Classes (i–iii) are well understood (Shiga [40], Deuschel [18], Shiga [41], Cox and Greven [10], Deuschel [19], Fleischmann and Greven [23], Cox, Fleischmann and Greven [9]). The qualitative properties of the process defined by (1.4.8) are similar for these three classes, and the universality of the long-time behaviour as a function of g has been systematically investigated via renormalisation methods (Dawson and Greven [13], [14], Baillon, Clément, Greven and den Hollander [1], Dawson and Greven [15], Baillon, Clément, Greven and den Hollander [2]). Class (iv), which is the subject of the current paper, is very different. For the case where $a(\cdot,\cdot)$ is simple random walk, this class was investigated in Shiga [41] and in the memoir by Carmona and Molchanov [6], where some of our results were already established.

The long-time behaviour of the process defined by (1.4.2) is fairly complex. In order to keep the exposition transparent, we restrict our analysis to a subclass of models given by the following additional requirements:

$$\begin{aligned} &I = \mathbb{Z}^d, \quad d \geq 1, \\ &a(\cdot,\cdot) \text{ is homogeneous: } a(i,j) = a(0,j-i) \quad \forall i,j \in I, \\ &a(\cdot,\cdot) \text{ is irreducible: } \sum_{n=0}^{\infty} [a^n(i,j) + a^n(j,i)] > 0 \quad \forall i,j \in I. \end{aligned} \tag{1.4.9}$$

Moreover, we put $a(0,0) = 0$.

Before we start, let us fix some notation. We write $\mathcal{P}(\mathcal{E}_1)$ for the set of probability measures on $(\mathcal{E}_1, \mathcal{B}(\mathcal{E}_1))$, with $\mathcal{B}$ the Borel σ-algebra. For $\rho \in \mathcal{P}(\mathcal{E}_1)$, we write E^ρ to denote expectation with respect to ρ. A measure $\rho \in \mathcal{P}(\mathcal{E}_1)$ is called *shift-invariant* if

$$\rho\Big((X_i)_{i\in I} \in A\Big) = \rho\Big((X_{i+j})_{i\in I} \in A\Big) \quad \forall j \in I \quad \forall A \in \mathcal{B}(\mathcal{E}_1), \tag{1.4.10}$$

is called *mixing* if

$$\lim_{\|k\|\to\infty} E^\rho(\, f\,[g \circ \sigma_k]\,) = E^\rho(f)\, E^\rho(g) \tag{1.4.11}$$

for all bounded $f, g\colon \mathcal{E}_1 \to \mathbb{R}$ that depend on finitely many coordinates, where σ_k is the k-shift acting on I, and is called *associated* if

$$E^\rho(f_1 f_2) \geq E^\rho(f_1)\, E^\rho(f_2) \tag{1.4.12}$$

for all bounded $f_1, f_2\colon \mathcal{E}_1 \to \mathbb{R}$ that depend on finitely many coordinates and that are non-decreasing in each coordinate.

We further need

$$\begin{aligned} \mathcal{T} &= \big\{\rho \in \mathcal{P}(\mathcal{E}_1)\colon\ \rho \text{ is shift-invariant}\big\}, \\ \mathcal{T}^1 &= \{\rho \in \mathcal{T}\colon\ E^\rho(X_0) < \infty\}, \end{aligned} \tag{1.4.13}$$

and

$$\mathcal{T}^1_\Theta = \big\{\rho \in \mathcal{T}^1\colon\ \rho \text{ is shift-ergodic}, E^\rho(X_0) = \Theta\big\}, \qquad \Theta \in [0,\infty). \tag{1.4.14}$$

The set of extreme points of a convex set C is written C_e. The element $(x_i)_{i\in I}$ with $x_i = \Theta$ for all $i \in I$ is denoted by $\underline{\Theta}$. The initial distribution of our system is denoted by $\mu = \mathcal{L}(X(0))$ and is assumed to be concentrated on $\mathcal{E}_1$. The symbols P, E without index denote probability and expectation with respect to μ and the Brownian motion driving (1.4.2). The notation $w - \lim$ means weak limit.

1.5 Phase transitions

In Theorems 1.5.1–1.5.4 below we state our main results on the long-time behaviour of $(X(t))_{t\geq 0}$ and on the properties of its equilibria. Let

$$\mathcal{I} = \big\{\rho \in \mathcal{P}(\mathcal{E}_1)\colon\ \rho \text{ is invariant}\big\} \tag{1.5.1}$$

be the set of all equilibrium measures ρ of (1.4.2), i.e., $\rho S(t) = \rho$ for all $t \geq 0$. This set of course depends on $a(\cdot,\cdot)$ and b.

1.5.1 Recurrent case

The ergodic behaviour of our system is simple when $\widehat{a}(\cdot,\cdot)$ defined by

$$\widehat{a}(i,j) = \frac{1}{2}[a(i,j) + a(j,i)], \qquad i,j \in I, \tag{1.5.2}$$

is recurrent. Namely, the process becomes extinct independently of the value of b.

Theorem 1.5.1 (Shiga [41]) *If $\widehat{a}(\cdot,\cdot)$ is recurrent, then for every $b > 0$ and every initial distribution $\mu \in \mathcal{T}^1$:*

$$w - \lim_{t\to\infty} \mathcal{L}(X(t)) = \delta_{\underline{0}}. \tag{1.5.3}$$

Consequently, there exists no equilibrium in $\mathcal{T}^1$ other than $\delta_{\underline{0}}$, i.e.,

$$\mathcal{I} \cap \mathcal{T}^1 = \delta_{\underline{0}}. \tag{1.5.4}$$

Using the fact that if $\mu \in \mathcal{T}^1_\Theta$ then $E(X_i(t)) = \Theta$ for all $i \in I$ and $t \geq 0$, we conclude from Theorem 1.5.1 that the system *clusters*, i.e., on only few sites there is a nontrivial mass but at these sites the mass is very large (for t large).

1.5.2 Transient case: regimes (I), (II) and (III)

In the case where $\widehat{a}(\cdot,\cdot)$ is transient, the ergodic behaviour of our system depends on the parameter b and we observe interesting phase transitions. There are three regimes, separated by two critical values.

(I) Small b**.** Define the Green function

$$\widehat{G}(i,j) = \sum_{n=0}^{\infty} \widehat{a}^n(i,j), \qquad i,j \in I, \tag{1.5.5}$$

and put

$$b_2 = \frac{2}{\widehat{G}(0,0)}. \tag{1.5.6}$$

We first consider the regime

$$\text{(I)} \qquad \widehat{a}(\cdot,\cdot) \text{ transient}, \; b \in (0, b_2). \tag{1.5.7}$$

Theorem 1.5.2 (Shiga [41], Cox and Greven [10]) *In regime (I):*

(a) For $\mu = \delta_{\underline{\Theta}}$ with $\Theta \in [0,\infty)$ the following limit exists:

$$\nu_\Theta = w - \lim_{t\to\infty} \mathcal{L}(X(t)). \tag{1.5.8}$$

(b) The measure ν_Θ satisfies

$$\begin{aligned} &\nu_\Theta \in (\mathcal{I}\cap\mathcal{T}^1)_e, \\ &\nu_\Theta \;\; \textit{is shift-invariant, mixing and associated}, \\ &E^{\nu_\Theta}(X_0) = \Theta; \; \nu_\Theta \; \textit{is not a point mass if } \Theta > 0, \\ &E^{\nu_\Theta}([X_0]^2) < \infty. \end{aligned} \tag{1.5.9}$$

(c) The set of shift-invariant extremal equilibria is given by

$$(\mathcal{I}\cap\mathcal{T}^1)_e = \{\nu_\Theta\}_{\Theta\in[0,\infty)}. \tag{1.5.10}$$

(d) For every $\mu \in \mathcal{T}^1_\Theta$ with $\Theta \in [0,\infty)$:

$$w - \lim_{t\to\infty} \mathcal{L}(X(t)) = \nu_\Theta. \tag{1.5.11}$$

(e) For every $\mu \in \mathcal{T}_e$ with $E^\mu(X_0) = \infty$:

$$w - \lim_{t\to\infty} \mathcal{L}(X(t)) = \delta_{\underline{\infty}}. \tag{1.5.12}$$

Consequently,

$$(\mathcal{I}\cap\mathcal{T})_e = (\mathcal{I}\cap\mathcal{T}^1)_e. \tag{1.5.13}$$

Theorem 1.5.2 tells us that if b remains below an $a(\cdot,\cdot)$-dependent threshold, then the process $(X(t))_{t\geq 0}$ exhibits persistent behaviour, in the sense that an equilibrium is approached with a spatial density equal to the initial spatial density and with a one-dimensional marginal that has a *finite* 2-nd moment. This equilibrium is nontrivial unless the initial state is identically 0. If, on the other hand, the initial spatial density is infinite, then every component diverges in probability.

(II) Moderate b**.** We next consider the regime

$$\text{(II)} \qquad \widehat{a}(\cdot,\cdot) \text{ transient}, \;\; b \in [b_2, b_*). \tag{1.5.14}$$

In Section 4 we will obtain a variational expression for b_*. This expression will turn out to be somewhat delicate to analyse.

Theorem 1.5.3 *In regime (II):*

(a) The same properties hold as in Theorem 1.5.2*(a) and (c–e).*

(b) The measure ν_Θ *satisfies*

$$\begin{aligned} &\nu_\Theta \in (\mathcal{I} \cap \mathcal{T}^1)_e, \\ &\nu_\Theta \textit{ is shift-invariant, mixing and associated}, \\ &E^{\nu_\Theta}(X_0) = \Theta; \;\; \nu_\Theta \textit{ is not a point mass if } \Theta > 0, \\ &E^{\nu_\Theta}([X_0]^2) = \infty. \end{aligned} \tag{1.5.15}$$

Theorem 1.5.3, which will be proved in Section 2, says that for moderate b the equilibria continue to exist and to be well-behaved, but with a one-dimensional marginal having *infinite* 2-nd moment. The latter has important consequences for the fluctuations of the equilibrium in large blocks. Indeed, in regime (I) we may expect Gaussian limits after suitable scaling (see e.g. Zähle [46], [47] in a different context), while in regime (II) we may expect non-Gaussian limits. In regime (II), the tail of X_0 under ν_Θ is likely to be of stable law type, but a closer investigation of this question is beyond the scope of the present paper.

(III) Large b**.** Finally, we consider the regime

$$\text{(III)} \qquad \widehat{a}(\cdot,\cdot) \text{ transient}, \;\; b \in [b_*, \infty). \tag{1.5.16}$$

Theorem 1.5.4 *In regime (III), for every* $\mu \in \mathcal{T}^1$*:*

$$w - \lim_{t\to\infty} \mathcal{L}(X(t)) = \delta_{\underline{0}}. \tag{1.5.17}$$

Consequently, $\mathcal{I} \cap \mathcal{T}^1 = \delta_{\underline{0}}$.

Theorem 1.5.4, which will be proved in Section 4, shows that for large b again *clustering* occurs, i.e., the same situation as described in Theorem 1.5.1 for the case where $\widehat{a}(\cdot,\cdot)$ is recurrent.

1.6 Finer analysis of the transient case

In Section 1.5 we saw that different values of b lead to qualitatively different behaviour of the process $(X(t))_{t\geq 0}$. Therefore the question arises in which way the value of b influences the properties of the process within one regime. For part of this finer analysis we need to assume that $a(\cdot,\cdot)$ is *symmetric*:

$$a(i,j) = a(j,i) \quad \forall i,j \in I. \tag{1.6.1}$$

1.6.1 Regime (I)

Let $\xi = (\xi(t))_{t\geq 0}$ be the random walk on I with transition kernel $a(\cdot,\cdot)$ and jump rate 1, starting at 0. For $m \geq 2$, let $\xi^{(m)} = (\xi_1,\ldots,\xi_m)$ be m independent copies of ξ, and define the *differences random walk* $\eta^{(m)} = (\eta^{(m)}(t))_{t\geq 0}$ by putting

$$\eta^{(m)}(t) = \big(\xi_p(t) - \xi_q(t)\big)_{1\leq p<q\leq m}. \tag{1.6.2}$$

This is a random walk on $I^{(m)}$, the subgroup of $I^{\frac{1}{2}m(m-1)}$ generated by all the possible pairwise differences of m elements of I, with jump rate m and transition kernel $a^{(m)}(\cdot,\cdot)$ that can be formally written out as

$$\begin{aligned} a^{(m)}(x,y) &= a^{(m)}(0,y-x) \\ &= \sum_{j\in I} a(0,j)\Big[\frac{1}{m}\sum_{r=1}^{m} 1\{jD_r = y-x\}\Big], \qquad x,y\in I^{(m)}, \end{aligned} \tag{1.6.3}$$

where D_r is the triangular array of $-1,0,+1$'s given by

$$D_r = (\delta_{pr} - \delta_{qr})_{1\leq p<q\leq m} \tag{1.6.4}$$

and jD_r denotes the triangular array obtained from D_r by multiplying all its elements with the vector j. Note that $a^{(m)}(\cdot,\cdot)$ is symmetric because of our assumption in (1.6.1). Note that $a^{(2)}(\cdot,\cdot) = \widehat{a}(\cdot,\cdot)$, the symmetrised transition kernel defined in (1.5.2), which is symmetric even without (1.6.1). The differences random walk is to be seen as the evolution of the random walks "relative to their center of mass". This will serve us later on.

Define the Green function

$$G^{(m)}(x,y) = \sum_{n=0}^{\infty}[a^{(m)}]^n(x,y), \qquad x,y\in I^{(m)}. \tag{1.6.5}$$

Also define the *intersection function* $\sharp^{(m)}\colon\ I^{(m)} \to \mathbb{N}_0$ as

$$\sharp^{(m)}(z) = \sum_{1\leq p<q\leq m} 1_{\{z_p - z_q = 0\}}, \qquad z = (z_p - z_q)_{1\leq p<q\leq m}, \quad z_p, z_q\in I, \tag{1.6.6}$$

and put

$$S^{(m)} = \mathrm{supp}(\sharp^{(m)}) \subset I^{(m)}. \tag{1.6.7}$$

Define

$$K^{(m)}(x,y) = \sqrt{\sharp^{(m)}(x)}\, G^{(m)}(x,y)\, \sqrt{\sharp^{(m)}(y)}, \qquad x,y\in S^{(m)}. \tag{1.6.8}$$

Viewed as an operator acting on $\ell^2(S^{(m)})$, $K^{(m)}$ is self-adjoint, positive and bounded. The latter two properties will be proved in Section 2.

The following result shows that in regime (I) there is an infinite sequence of critical values characterising the convergence of successive moments.

Theorem 1.6.1 *Suppose that $a(\cdot,\cdot)$ is symmetric. Then, in regime (I), there exists a sequence $b_2 \geq b_3 \geq b_4 \geq \ldots$ such that:*

(a) *If* $\mu = \delta_{\underline{\Theta}}$ *with* $\Theta \in (0,\infty)$, *then*

$$\lim_{t\to\infty} E([X_0(t)]^m) = E^{\nu_\Theta}([X_0]^m) \begin{cases} < \infty & \text{for} \quad b < b_m, \\ = \infty & \text{for} \quad b \geq b_m. \end{cases} \tag{1.6.9}$$

(b) *If* $\mu = \delta_{\underline{\Theta}}$ *with* $\Theta \in (0,\infty)$, *then*

$$\lim_{t\to\infty} \frac{1}{t} \log E([X_0(t)]^m) = \chi_m(b) \tag{1.6.10}$$

exists with

$$\chi_m(b) \begin{cases} = 0 & \text{for} \quad b \leq b_m, \\ > 0 & \text{for} \quad b > b_m. \end{cases} \tag{1.6.11}$$

(c) *The critical value* b_m *has the representation*

$$b_m = \frac{m}{\lambda_m} \tag{1.6.12}$$

with $\lambda_m \in (0,\infty)$ *the spectral radius of* $K^{(m)}$ *in* $\ell^2(S^{(m)})$. *This spectral radius is an eigenvalue if and only if* $b_{m-1} > b_m$.

(d) *The critical value* b_2 *is given by* (1.5.6), *and*

$$b_2 \geq b_3 \geq b_4 \cdots > 0. \tag{1.6.13}$$

Moreover,

$$\frac{2}{G^{(2)}(0,0)} = b_2 \leq (m-1)b_m \leq \frac{2}{G^{(m)}(0,0)} < 2, \tag{1.6.14}$$

and $\lim_{m\to\infty}(m-1)b_m$ *exists.*

(e) *The function* $b \mapsto \frac{1}{m}\chi_m(b)$ *is convex on* $[0,\infty)$ *and strictly increasing on* $[b_m,\infty)$, *with*

$$\lim_{b\to\infty} \frac{1}{bm}\, \chi_m(b) = \frac{1}{2}(m-1). \tag{1.6.15}$$

Theorem 1.6.1 will be proved in Section 3. Part (a) tells us that equilibria with finite m-th moment exist if and only if $0 < b < b_m$. Part (b) tells us that the m-th moment diverges exponentially fast if and only if $b > b_m$. The limit $\chi_m(b)$ is the *m-th annealed Lyapunov exponent.* Part (c) gives a variational representation for b_m. Part (d) gives sharp bounds for b_m and shows that the tail of the one-dimensional marginal of ν_Θ decays algebraically with a power that is a non-increasing function of b when b is small. It identifies the asymptotic behaviour of this power as $\sim Cst/b$ for $b \downarrow 0$. Part (e) shows that for large b the curve $b \mapsto \frac{1}{m}\chi_m(b)$ has slope $\frac{1}{2}(m-1)$.

By Hölder's inequality, $m \mapsto \frac{1}{m}\chi_m(b)$ is non-decreasing. The system is called *intermittent of order n* if

$$\frac{1}{n}\,\chi_n(b) < \frac{1}{n+1}\,\chi_{n+1}(b) < \frac{1}{n+2}\,\chi_{n+2}(b) < \ldots \tag{1.6.16}$$

(It is shown in Carmona and Molchanov [6], Chapter III, that the first of these inequalities implies all the subsequent ones.) Thus, for all $n \geq 2$ our system is intermittent of order n precisely when $b \in (b_{n+1}, b_n]$ (see also Fig. 2 in Section 1.1.)

We conjecture that $b_2 > b_3 > b_4 > \ldots$ (see open problem (A) in Section 1.2). A partial result in this direction is the following:

Corollary 1.6.2 *Suppose that $a(\cdot,\cdot)$ is symmetric.*

(a) $(m-1)b_m \to 2$ uniformly in m as $G^{(2)}(0,0) \to 1$.

(b) $b_2 > b_3 > \ldots > b_m$ when $G^{(2)}(0,0) \le (m-1)/(m-2)$.

Proof. (a) Obvious from (1.6.14).
(b) This follows from (1.6.14) and $G^{(m)}(0,0) > 1$. ∎

Claim (a) follows from (1.5.6) and (1.6.14), and corresponds to the limit when the random walk becomes more and more transient. This includes simple random walk on $\mathbb{Z}^d$ with $d \to \infty$. Thus, in this limit all inequalities in (1.6.13) become strict. Claim (b) follows from (1.6.14). This includes simple random walk on $\mathbb{Z}^d$ with $d \ge 3$.

As we will see in Sections 2–3, the representation for b_m in (1.6.12) comes from a link with intersection local time of random walks. Indeed, let

$$T(\xi^{(m)}) = \int_0^\infty \sum_{1\le p<q\le m} 1_{\{\xi_p(t)=\xi_q(t)\}}\,dt \tag{1.6.17}$$

be the total intersection local time (in pairs) of the m independent copies of the random walk ξ. Then we will show that

$$b_m = \sup\Big\{b>0:\ E^{\xi^{(m)}}\Big(\exp[bT(\xi^{(m)})]\Big) < \infty\Big\}. \tag{1.6.18}$$

1.6.2 Regime (II)

Our next result ensures that regime (II) is non-empty and may be seen as an extension of Theorem 1.6.1(d).

Theorem 1.6.3 *$b_* > b_2$ when $\widehat{a}(\cdot,\cdot)$ is transient.*

Theorem 1.6.3, which will be proved in Section 4, tells us that *equilibria with stable law tails* indeed occur in our system for moderate b (recall (1.5.15)). We conjecture that the system locally dies out at b_* (see open problem (B) in Section 1.2).

In view of (1.6.18), we may ask whether it is possible to obtain a variational characterisation for b_*. To that end, let $\xi = (\xi(t))_{t\ge0}$ and $\xi' = (\xi'(t))_{t\ge0}$ be two independent copies of the random walk on I with transition kernel $a(\cdot,\cdot)$ and jump rate 1, both starting at 0. Let

$$T(\xi,\xi') = \int_0^\infty 1_{\{\xi(t)=\xi'(t)\}}dt \tag{1.6.19}$$

be their total intersection local time. Define

$$b_{**} = \sup\Big\{b>0:\ E^{\xi'}\Big(\exp[bT(\xi,\xi')]\Big) < \infty \ \ \xi-a.s.\Big\}, \tag{1.6.20}$$

where we note that $\{E^{\xi'}(\exp[bT(\xi,\xi')]) < \infty\}$ is a tail event for ξ. Since b_2 *is given by the same formula as* (1.6.20) *but with the average taken over both ξ and ξ'* (recall (1.6.18)), we have $b_{**} \ge b_2$. The proof of Theorem 1.6.3 will be achieved by showing that

$$b_* \ge b_{**} \qquad \text{and} \qquad b_{**} > b_2. \tag{1.6.21}$$

1.6.3 Regime (III)

The last result shows that in regime (III) the system gets extinct very rapidly.

Theorem 1.6.4 (Cranston, Mountford and Shiga [12]) *In regime (III):*

(a) If $\mu = \delta_{\underline{\Theta}}$ with $\Theta \in (0,\infty)$, then

$$\lim_{t\to\infty} \frac{1}{t} \log X_0(t) = \chi_*(b) \tag{1.6.22}$$

exists and is constant a.s.

(b) There exists a $b_{***} \in [b_2, \infty)$ such that

$$\chi_*(b) \begin{cases} = 0 & \text{for} \quad b \le b_{***}, \\ < 0 & \text{for} \quad b > b_{***}. \end{cases} \tag{1.6.23}$$

(c)

$$\lim_{b\to\infty} \frac{\log b}{b} \left[\chi_*(b) + \frac{1}{2} b\right] \tag{1.6.24}$$

exists in $(0,\infty)$.

The limit $\chi_*(b)$ is the *quenched Lyapunov exponent.* Theorem 1.6.4, states that the speed of extinction is exponentially fast above a critical threshold b_{***}. Trivially,

$$b_{***} \ge b_*. \tag{1.6.25}$$

We conjecture that equality holds (see open problem (C) in Section 1.2). See also Figs. 1 and 2 in Section 1.1.

1.6.4 Separation between regimes (II) and (III)

The key tool in the identification of b_* is the notion of *Palm distribution* of our process X at time t. This is the law of the process seen from a "randomly chosen mass" drawn at time t. This concept was introduced by Kallenberg [35] in the study of branching particle systems with migration. There the idea is to take a large box at time t, pick a particle at random from this box (the "tagged particle"), shift the origin to the location of this particle, consider the law of the shifted configuration, and let the box tend to infinity. Under suitable conditions, a limiting law is obtained, which is called the Palm distribution. Similarly, in our system the Palm distribution is a size-biasing of the original distribution according to the "mass" at the origin. The criterion for *survival vs. extinction of the original distribution translates into tightness vs. divergence of the Palm distribution.*

This criterion is useful for two reasons. First, the size-biasing is an easy operation. Second, often it is possible to obtain a representation formula for the Palm distribution in terms of a nice Markov process. For instance, for branching particle systems the Palm distribution is obtained from an independent superposition of the original distribution and a realisation of the so-called Palm canonical distribution. The latter can be identified as a branching random walk with immigration of particles at rate 1 along the path of the tagged particle. Fortunately, we can give an explicit representation of the Palm distribution of our process X as well, namely, as the solution of a system of biased stochastic differential equations (see Section 4 for details). It turns out that the latter again has a (Feynman-Kac type) representation formula for the single components as an expectation over an exponential functional of the Brownian motions, the random walk, and an additional tagged random walk, with the expectation running over the two random walks.

We will use the Palm distribution to identify b_*. We will see that, within the interval (b_2, b_*), we can distinguish between a regime where the average of the Palm distribution over the Brownian motions (i.e., the Palm distribution conditioned on the tagged path) is tight as $t \to \infty$ and a regime where it diverges. The separation between these two regimes is b_{**}. Within the interval $[b_{**}, b_*)$, we can separate further by conditioning the Palm distribution also on the Brownian increments along the tagged path. However, we will not pursue this point further, even though it is of interest for a better insight into what controls our system. See Birkner [3], [4] for more background.

We will see in Section 4.1.2 that (1.6.19) plays an important role in the description of the Palm distribution. Equation (4.1.38) in Section 4.1.2 identifies b_*. However, this formula is much harder than the one for b_{**} in (1.6.20). It would be interesting to know whether there exists a characterisation of b_* in terms of the intersection local time of random walks (see open problem (D) in Section 1.2), in the same way as for b_m in (1.6.18) and for b_{**} in (1.6.20).

2 Moment calculations

2.1 Definition of b_m, $\bar{b}_m$ and $\widetilde{b}_m$

For $\Theta > 0$ and $m \geq 2$, let

$$\begin{aligned} b_m &= \sup\{b > 0\colon\ E^{\nu_\Theta}([X_0]^m) < \infty\}, \\ \bar{b}_m &= \sup\Big\{b > 0\colon\ \sup_{t\geq 0} E\Big([X_0(t)]^m \mid X(0) = \underline{\Theta}\Big) < \infty\Big\}, \\ \widetilde{b}_m &= \sup\Big\{b > 0\colon\ \lim_{t\to\infty} \tfrac{1}{t}\log E\Big([X_0(t)]^m \mid X(0) = \underline{\Theta}\Big) = 0\Big\}. \end{aligned} \tag{2.1.1}$$

In these definitions the choice of Θ is irrelevant as long as $\Theta > 0$, as is evident from Lemma 2.2.1 below.

In Section 2.2 we derive a representation formula for the solution of (1.4.2), which is due to Shiga [41] and which plays a key role in the present paper. We also derive as a self-duality property, which is needed to obtain convergence to equilibrium. In Section 2.3 we express the m-th moment of a single component of our system, at time t, in terms of the intersection local time, up to time t, of m independent copies of our random walk. In Section 2.4 we prove that ν_Θ exists and that $b_m = \bar{b}_m$. In Section 2.5 we prove some basic properties of $G^{(m)}$ and $K^{(m)}$ defined in (1.6.5) and (1.6.8). The results in this section will be used in Sections 3–4 to prove Theorems 1.5.3–1.5.4, 1.6.1 and 1.6.3–1.6.4.

2.2 Representation formula and self-duality

If our process starts in a constant initial configuration, then a nice representation formula is available. This formula will play a key role throughout the paper.

Lemma 2.2.1 *The process* $(X(t))_{t\geq 0}$ *starting in* $X(0)=\underline{\Theta}$ *can be represented as the following functional of the Brownian motions:*

$$X_i(t) = \Theta e^{-\frac{1}{2}bt} E_i^{\xi}\Big(\exp\Big[\sqrt{b}\int_0^t \sum_{j\in I} 1_{\{\xi(t-s)=j\}} dW_j(s)\Big]\Big), \qquad i\in I, t\geq 0, \tag{2.2.1}$$

where $\xi = (\xi(t))_{t\geq 0}$ *is the random walk on* I *with transition kernel* $a(\cdot,\cdot)$ *and jump rate 1, and the expectation is over* ξ *conditioned on* $\xi(0)=i$ *(*ξ *and* W *are independent).*

Proof. This lemma appears in Shiga [41] without proof. We write out the proof here, because it will serve us later on. The symbol $\mathbb{1}$ denotes the identity matrix. Note that Θ enters into (2.2.1) only as a front factor.

<u>Step 1</u>: For all i and t:

$$X_i(t) = \Theta + \sqrt{b}\int_0^t \sum_j a_{t-s}(i,j)\, X_j(s)\, dW_j(s) \qquad \text{with } a_t = \exp[t(a-\mathbb{1})]. \tag{2.2.2}$$

Proof. Fix i and t. For $0\leq s\leq t$, let $Y_i(s) = \sum_j a_{t-s}(i,j)X_j(s)$. Then

$$dY_i(s) = \Big[\sum_j (\mathbb{1}-a)a_{t-s}(i,j)\, X_j(s)\Big] ds + \sum_j a_{t-s}(i,j)\, dX_j(s). \tag{2.2.3}$$

From (1.4.2) we have

$$dX_i(s) = \sum_j (a-\mathbb{1})(i,j)\, X_j(s)\, ds + \sqrt{b}\, X_i(s)\, dW_i(s), \tag{2.2.4}$$

which after substitution into (2.2.3) and cancellation of two terms gives

$$dY_i(s) = \sqrt{b}\sum_j a_{t-s}(i,j)\, X_j(s)\, dW_j(s). \tag{2.2.5}$$

Integrate both sides from 0 to t, and note that $Y_i(0) = \sum_j a_t(i,j)X_j(0) = \Theta$ and $Y_i(t) = \sum_j a_0(i,j)X_j(0) = X_i(t)$, to get the claim. ∎

<u>Step 2</u>: For all t:

$$\exp\Big[\sqrt{b}\, Z_t(t) - \frac{1}{2}bt\Big] = 1 + \sqrt{b}\int_0^t \exp\Big[\sqrt{b}\, Z_t(s) - \frac{1}{2}bs\Big]\, dZ_t(s) \tag{2.2.6}$$

with $Z_t(s) = \int_0^s \sum_j 1_{\{\xi(t-r)=j\}} dW_j(r)$.

Proof. Fix t. For $z\in\mathbb{R}$ and $s\geq 0$, let $f(z,s) = e^{\sqrt{b}z-\frac{1}{2}bs}$ and put $g(s) = f(Z_t(s),s)$. Itô's formula gives

$$dg(s) = f_z(Z_t(s),s)\, dZ_t(s) + \frac{1}{2}f_{zz}(Z_t(s),s)\, (dZ_t(s))^2 + f_s(Z_t(s),s)\, ds, \tag{2.2.7}$$

which after cancellation of two terms (because $\frac{1}{2}f_{zz}+f_s=0$ and $(dZ_t(s))^2 = ds$) gives

$$dg(s) = g(s)\sqrt{b}\, dZ_t(s). \tag{2.2.8}$$

Integrate both sides from 0 to t and use that $g(0) = f(Z_t(0), 0) = 1$, to get the claim. ∎

Step 3: The proof of the representation formula in Lemma 2.2.1 is now completed as follows. Let $\widetilde{X}_i(t)$ denote the right-hand side of (2.2.1). Taking the expectation over ξ conditioned on $\xi(0) = i$ on both sides of (2.2.6), we get

$$\begin{aligned} \Theta^{-1}\widetilde{X}_i(t) &= 1 + \sqrt{b}\int_0^t E_i^{\xi}\Big(\sum_j 1_{\{\xi(t-s)=j\}}\Theta^{-1}\widetilde{X}_j(s)\,dW_j(s)\Big) \\ &= 1 + \sqrt{b}\int_0^t \sum_j a_{t-s}(i,j)\,\Theta^{-1}\widetilde{X}_j(s)\,dW_j(s), \end{aligned} \tag{2.2.9}$$

where the first equality uses the Markov property of ξ at time $t - s$. Thus we see that $\widetilde{X}_i(t)$ satisfies (2.2.2). Since $\widetilde{X}_i(0) = \Theta = X_i(0)$ for all i, we may therefore conclude that $\widetilde{X}_i(t) = X_i(t)$ for all i and t, by the strong uniqueness of the solution of our system (1.4.2) (recall Theorem 1.4.1(a)). ∎

In addition to the representation formula in Lemma 2.2.1, we have another nice property: our process is self-dual. Let

$$a^*(i,j) = a(j,i), \quad i,j \in I, \tag{2.2.10}$$

be the reflected transition kernel. Let (1.4.2*) denote (1.4.2) with $a(\cdot,\cdot)$ replaced by $a^*(\cdot,\cdot)$. Abbreviate $\langle x, x^*\rangle = \sum_{i\in I} x_i x_i^*$.

Lemma 2.2.2 *Let $X = (X(t))_{t\geq 0}$ be the solution of* (1.4.2) *starting from any $X(0) \in \mathcal{E}_1$. Let $X^* = (X^*(t))_{t\geq 0}$ be the solution of* (1.4.2*) *starting from any $X^*(0) \in \mathcal{E}_1$ such that $\langle \underline{1}, X^*(0)\rangle < \infty$. Then*

$$E^X\left(e^{-\langle X(t), X^*(0)\rangle}\right) = E^{X^*}\left(e^{-\langle X(0), X^*(t)\rangle}\right) \quad \forall t \geq 0. \tag{2.2.11}$$

Proof. See Cox, Klenke and Perkins [11]. ∎

2.3 Representation of the m-th moment in terms of intersection local time

Let us abbreviate

$$W = (\{W_i(t)\}_{i\in I})_{t\geq 0} \tag{2.3.1}$$

and write

$$E\Big([X_0(t)]^m \mid X(0) = \underline{\Theta}\Big) = E^w\Big([X_0(t)]^m \mid X(0) = \underline{\Theta}\Big) \tag{2.3.2}$$

to display that (1.4.2) is driven by W. This subsection contains a moment calculation in which we use the representation formula of Lemma 2.2.1 to express the right-hand side of (2.3.2) as the expectation of the exponential of b times the *intersection local time* of m independent copies of the random walk with transition kernel $a(\cdot,\cdot)$ and jump rate 1, all starting at 0.

We begin by checking that the evolution is mean-preserving. This property is evident from (2.2.2), but its proof will serve as a preparation for the calculation of the higher moments.

Lemma 2.3.1 $E^W\Big(X_0(t) \mid X(0) = \underline{\Theta}\Big) = \Theta$ *for all* $t \geq 0$.

Proof. Taking the expectation over W in (2.2.1) and using Fubini's theorem, we have

$$E^W\Big(X_0(t) \mid X(0) = \underline{\Theta}\Big) = \Theta e^{-\frac{1}{2}bt} E_0^\xi\Big(E^W\Big(\exp\Big[\sqrt{b}\int_0^t \sum_{i\in I} 1_{\{\xi(t-s)=i\}} dW_i(s)\Big]\Big)\Big). \tag{2.3.3}$$

Since the Brownian motions W are i.i.d. and have independent increments, it follows that for any ξ:

$$\int_0^t \sum_{i\in I} 1_{\{\xi(t-s)=i\}} dW_i(s) \triangleq W'(t), \tag{2.3.4}$$

where $W' = (W'(t))_{t\geq 0}$ is a single Brownian motion and $\triangleq$ denotes equality in distribution. Combining (2.3.3) and (2.3.4) we arrive at (the expectation over ξ being irrelevant)

$$E^W\Big(X_0(t) \mid X(0) = \underline{\Theta}\Big) = \Theta e^{-\frac{1}{2}bt} E^{W'}\Big(\exp[\sqrt{b}W'(t)]\Big). \tag{2.3.5}$$

Now use that, by Itô's formula, $\exp[\sqrt{b}W'(t) - \frac{1}{2}bt]$ is a martingale, to get that the r.h.s. of (2.3.5) equals Θ. ∎

A version of the above argument will produce the following expression for the moments of order $m \geq 2$.

Lemma 2.3.2 *Let $\xi^{(m)} = (\xi_1, \ldots, \xi_m)$ be m independent copies of the random walk with transition kernel $a(\cdot,\cdot)$ and jump rate 1, all starting at 0. Then*

$$E^W\Big([X_0(t)]^m \mid X(0) = \underline{\Theta}\Big) = \Theta^m E^{\xi^{(m)}}\Big(\exp[bT^{(m)}(t)]\Big), \tag{2.3.6}$$

where

$$\begin{aligned} T^{(m)}(t) &= \sum_{1\leq k<l\leq m} T_{kl}(t), \\ T_{kl}(t) &= \textstyle\int_0^t ds\, 1_{\{\xi_k(s)=\xi_l(s)\}} ds, \end{aligned} \tag{2.3.7}$$

is the intersection local time (in pairs) up to time t.

Proof. Similarly as in (2.3.3) we may use (2.2.1) to write

$$\begin{aligned} &E^W\Big([X_0(t)]^m \mid X(0) = \underline{\Theta}\Big) \\ &= \Theta^m e^{-\frac{m}{2}bt} E^{\xi^{(m)}}\Big(E^W\Big(\exp\Big[\sqrt{b}\int_0^t \sum_{k=1}^m \sum_{i\in I} 1_{\{\xi_k(t-s)=i\}} dW_i(s)\Big]\Big)\Big). \end{aligned} \tag{2.3.8}$$

Next, let $W^{(m)} = (W'_1, \ldots, W'_m)$ be m independent Brownian motions. Then the analogue of (2.3.4) reads

$$\int_0^t \sum_{k=1}^m \sum_{i\in I} 1_{\{\xi_k(t-s)=i\}} dW_i(s) \triangleq \int_0^t \sum_{p=1}^m \sum_{j^{(p)}} 1_{j^{(p)}}(t-s) \sum_{q=1}^p j_q dW'_q(s). \tag{2.3.9}$$

Here $1_{j^{(p)}}(t-s)$ denotes the indicator of the event that at time $t-s$ the components of $\xi^{(m)} = (\xi_1, \ldots, \xi_m)$ coincide in p subgroups of sizes $j^{(p)} = (j_1, \ldots, j_p)$ with $j_1 + \ldots + j_p = m$. The equality in (2.3.9) again follows from the fact that the Brownian motions W are i.i.d. and have independent increments. The point to note here is that all j_q random walks in the q-th coincidence group pick up the same increment of the Brownian motion in W at the site where they coincide at time s, and this increment has the same distribution as $dW'_q(s)$. Next, define

$$T_{j^{(p)}}(t) = \int_0^t 1_{j^{(p)}}(t-s)ds. \tag{2.3.10}$$

Then clearly we have

$$\int_0^t 1_{j^{(p)}}(t-s)dW'_q(s) \triangleq W'_q(T_{j^{(p)}}(t)). \tag{2.3.11}$$

Now combine (2.3.8–2.3.11) to get

$$\begin{aligned} &E^W\Big([X_0(t)]^m \mid X(0) = \underline{\Theta}\Big) \\ &= \Theta^m e^{-\frac{m}{2}bt} E^{\xi^{(m)}}\Big(E^{W^{(m)}}\Big(\exp\Big[\sqrt{b}\sum_{p=1}^m \sum_{j^{(p)}} \sum_{q=1}^p j_q W'_q(T_{j^{(p)}}(t))\Big]\Big)\Big) \\ &= \Theta^m e^{-\frac{m}{2}bt} E^{\xi^{(m)}}\Big(\exp\Big[b\sum_{p=1}^m \sum_{j^{(p)}} T_{j^{(p)}}(t)\Big\{\sum_{q=1}^p \tfrac{1}{2}j_q^2\Big\}\Big]\Big). \end{aligned} \tag{2.3.12}$$

Finally, absorb the term $-\frac{m}{2}bt$ into the sum by writing $\frac{1}{2}j_q(j_q-1)$ instead of $\frac{1}{2}j_q^2$ (use that $\sum_{p=1}^m \sum_{j^{(p)}} T_{j^{(p)}}(t) = t$ and $\sum_{q=1}^p j_q = m$). The resulting exponent is the same as b times the intersection local time in (2.3.7). ∎

2.4 Convergence to equilibrium and $b_m = \bar{b}_m$

The following important facts will be needed later on and will be derived via the representation formula in Lemma 2.2.1 and the self-duality in Lemma 2.2.2.

Proposition 2.4.1 *For all $\widehat{a}(\cdot,\cdot)$ transient and all $b > 0$:*

(a) $\nu_\Theta = w - \lim_{t\to\infty} \mathcal{L}(X(t) \mid X(0) = \underline{\Theta})$ *exists, is shift-invariant and associated for all* $\Theta \in [0,\infty)$.

(b) $b_m = \bar{b}_m$ *for all* $m \geq 2$.

(c) $\lim_{t\to\infty} E([X_0(t)]^m \mid X(0) = \underline{\Theta}) = E^{\nu_\Theta}([X_0]^m)$ *for all* $\Theta \in [0,\infty)$ *and all* $m \geq 2$.

(d) ν_Θ *is mixing for all* $\Theta \in [0,\infty)$.

Proof. (a) The proof of existence uses Lemma 2.2.2. If $X(0) = \underline{\Theta}$ and $X^*(0) = f$, then (2.2.11) reads

$$E^X\Big(e^{-\langle X(t),f\rangle} \mid X(0) = \underline{\Theta}\Big) = E^{X^*}\Big(e^{-\Theta M_t} \mid M(0) = \langle \underline{1}, f\rangle\Big) \tag{2.4.1}$$

with

$$M(t) = \langle \underline{1}, X^*(t)\rangle. \tag{2.4.2}$$

Since $(M(t))_{t\geq 0}$ is a non-negative martingale (as is obvious from (2.2.2) with the reflected transition kernel), we have that $\lim_{t\to\infty} M(t) = M(\infty) < \infty$ W-a.s. for every f (use that

$E^{X^*}(M(\infty)) \leq M(0) = \langle \underline{1}, f \rangle < \infty)$. Hence we conclude that $X(t)$ converges in law to a limit, which we call $\nu_{\underline{\Theta}}$, given by

$$\int e^{-\langle x,f\rangle} \nu_{\underline{\Theta}}(dx) = E^{X^*}\left(e^{-\Theta M_\infty} \mid M(0) = \langle \underline{1}, f \rangle\right) \text{ for all } f \text{ such that } \langle \underline{1}, f \rangle < \infty. \tag{2.4.3}$$

Because $\delta_{\underline{\Theta}}$ is shift-invariant, so is ν_Θ. The fact that ν_Θ is associated follows from Cox and Greven [10]. There it is shown that for systems of the type in (1.4.2) – even with a general diffusion term – the evolution preserves the associatedness. Since $\delta_{\underline{\Theta}}$ is associated, the system is associated at time zero and hence at all later times, and the equilibrium inherits this property.

(b) Fatou's lemma in combination with part (a) shows that

$$\liminf_{t\to\infty} E\Big([X_0(t)]^m \mid X(0) = \underline{\Theta}\Big) \geq E^{\nu_\Theta}([X_0]^m). \tag{2.4.4}$$

Hence $\bar{b}_m \leq b_m$. To prove the converse, we need to make an excursion into potential theory.

Step 1: Suppose that $b < b_m$. Define the m-point correlation function in equilibrium:

$$f(j_1, \ldots, j_m) = E^{\nu_\Theta}\left(\prod_{p=1}^m X_{j_p}\right), \qquad j_1, \ldots, j_m \in I \tag{2.4.5}$$

(the indices need not be distinct). Since ν_Θ is shift-invariant, it follows from Hölder's inequality that

$$\sup_{j_1,\ldots,j_m \in I} f(j_1, \ldots, j_m) \leq E^{\nu_\Theta}([X_0]^m) = C < \infty. \tag{2.4.6}$$

Let $a^{(m),\otimes}$ be the transition kernel on I^m for m independent random walks, i.e.,

$$a^{(m),\otimes}\Big((j_1, \ldots, j_m), (j_1', \ldots, j_m')\Big) = \frac{1}{m} \sum_{p=1}^m a(j_p, j_p'), \tag{2.4.7}$$

where $a(\cdot,\cdot)$ is the transition kernel on I for a single random walk, and $\frac{1}{m}$ comes from the fact that the m random walks jump one at a time. Also let $\sharp^{(m),\otimes}\colon I^m \to \mathbb{N}_0$ be the intersection function

$$\sharp^{(m),\otimes}(j_1, \ldots, j_m) = \sum_{1\leq p<q\leq m} 1_{\{j_p = j_q\}}. \tag{2.4.8}$$

(The upper index $\otimes$ is used in (2.4.7–2.4.8) to distinguish the quantities from those in (1.6.3) and (1.6.6); in the latter the random walks are viewed "relative to their center of mass", while in the former they are not.)

Fact 2.4.2 *f is excessive, i.e., $a^{(m),\otimes} f \leq f$.*

Proof. Consider our SSDE in (1.4.2):

$$dX_i(t) = \sum_{j\in I} a(i,j)[X_j(t) - X_i(t)]\, dt + \sqrt{bX_i^2(t)}\, dW_i(t), \qquad i \in I,\, t \geq 0. \tag{2.4.9}$$

By Ito's formula we have

$$d\left(\prod_{p=1}^{m} X_{j_p}(t)\right) = \sum_{p=1}^{m} dX_{j_p}(t)\left[\prod_{\substack{r=1\\r\neq p}}^{m} X_{j_r}(t)\right] + \frac{1}{2}\sum_{\substack{p,q=1\\p\neq q}}^{m} dX_{j_p}(t)dX_{j_q}(t)\left[\prod_{\substack{r=1\\r\neq p,q}}^{m} X_{j_r}(t)\right]. \quad (2.4.10)$$

If we start the system in the equilibrium ν_Θ, then the expectation of the l.h.s. is zero. For the first term in the right-hand side we get, after substituting (2.4.9) and taking the expectation,

$$[m(a^{(m),\otimes} - \mathbb{1})f](j_1,\ldots,j_m)\,dt, \quad (2.4.11)$$

For the second term in the r.h.s., on the other hand, we get

$$[b\,\sharp^{(m),\otimes} f](j_1,\ldots,j_m)\,dt. \quad (2.4.12)$$

Thus, we have

$$0 = [m(a^{(m),\otimes} - \mathbb{1}) + b\,\sharp^{(m),\otimes}]\,f. \quad (2.4.13)$$

Since $\sharp^{(m),\otimes} \geq 0$, this proves the claim. ∎

Step 2: Because f is excessive, the Riesz decomposition theorem (see [43] Section 27) tells us that

$$f = c + G^{(m),\otimes}\psi \quad (2.4.14)$$

for some constant $c \geq 0$ and some function $\psi \geq 0$, where $G^{(m),\otimes}$ is the Green function on I^m associated with $a^{(m),\otimes}$, i.e.,

$$G^{(m),\otimes} = (\mathbb{1} - a^{(m),\otimes})^{-1}, \quad (2.4.15)$$

and we use that bounded harmonic functions are constant (see [43] Section 24). With (2.4.14) we have a representation for the m-point correlation function in equilibrium. We want to compare the latter with the m-point correlation function at time t when we start the system from the constant configuration $X(0) = \underline{\Theta}$:

$$f_t(j_1,\ldots,j_m) = E\left(\prod_{p=1}^{m} X_{j_p}(t) \mid X(0) = \underline{\Theta}\right), \qquad j_1,\ldots,j_m \in I. \quad (2.4.16)$$

Fact 2.4.3 $\frac{\partial f_t}{\partial t} = \left[m(a^{(m),\otimes} - \mathbb{1}) + b\,\sharp^{(m),\otimes}\right] f_t.$

Proof. The claim follows from the same argument as in the proof of Fact 2.4.2 leading to (2.4.13). Note that f is the stationary solution. ∎

Step 3: Let $\xi^{(m)} = (\xi_1,\ldots,\xi_m)$ be m independent copies of the random walk with transition kernel $a(\cdot,\cdot)$ and jump rate 1. The solution of the equation in Fact 2.4.3 in terms of the Feynman-Kac formula reads

$$f_t(j_1,\ldots,j_m) = E^{\xi^{(m)}}\Big(\exp\Big[b\int_0^t \sharp^{(m),\otimes}(\xi^{(m)}(s))ds\Big] f_0(\xi^{(m)}(t)) \mid \xi^{(m)}(0) = (j_1,\ldots,j_m)\Big), \quad (2.4.17)$$

which is order preserving:

$$f_0 \leq f_0' \qquad \Longrightarrow \qquad f_t \leq f_t' \quad \forall t \geq 0. \quad (2.4.18)$$

We will exploit (2.4.18) to show that $b > \bar{b}_m$ leads to a contradiction, thereby obtaining that $\bar{b}_m \geq b_m$ (because we assumed that $b < b_m$). Indeed, if $b > \bar{b}_m$, then $f_t \to \infty$ as $t \to \infty$ on the diagonal. Suppose that $c > 0$. Pick $\Theta = c^{1/m}$. Then $f_0 \equiv c$, while $f \geq c$ by (2.4.14). It follows from (2.4.18) with $f_0' = f$ that

$$f_t \leq f \quad \forall t \geq 0. \tag{2.4.19}$$

Thus $f = \infty$ on the diagonal, which is a contradiction with (2.4.6). Therefore $c = 0$. However, if $|j_p - j_q| \to \infty$ for all $1 \leq p < q \leq m$, then $(G^{(m),\otimes}\psi)(j_1, \ldots, j_m) \to 0$ by the transience of $\widehat{a}(\cdot,\cdot)$ (see [43], Section 24), while $f(j_1, \ldots, j_m) \to \Theta^m$ by the mixing property of ν_Θ. This is a contradiction with $\Theta > 0$, as desired.

(c) We need to show that (2.4.4) is an equality. This is trivial when the right-hand side of (2.4.4) is infinite. Recall (2.4.5). We need only consider the case where the right-hand side of (2.4.4) is finite. In that case, f is bounded by (2.4.6). Using (2.4.17), but this time starting from the equilibrium ν_Θ, we therefore have the relation

$$f(j) = E^{\xi^{(m)}}\Big(\exp\Big[b\int_0^t \sharp^{(m),\otimes}(\xi^{(m)}(s))ds\Big] f(\xi^{(m)}(t)) \mid \xi^{(m)}(0) = j\Big). \tag{2.4.20}$$

Because ν_Θ is associated, we have $f(j) \geq \Theta^m$ for all j. Hence

$$f(j) \geq \Theta^m \limsup_{t\to\infty} E^{\xi^{(m)}}\Big(\exp\Big[b\int_0^t \sharp^{(m),\otimes}(\xi^{(m)}(s)ds\Big] \mid \xi^{(m)}(0)) = j\Big). \tag{2.4.21}$$

For $j = 0$ the l.h.s. equals $E^{\nu_\Theta}([X_0]^m)$, while the expectation under the limit equals $E([X_0(t)]^m \mid X(0) = \underline{\Theta})$ by (2.4.17). Therefore we have the reverse of (2.4.4).

REMARK: Note that (2.4.17) implies Lemma 2.3.2. We nevertheless find the proof of Lemma 2.3.2 via the representation formula in Lemma 2.2.1 as given in Section 2.3 instructive.

(d) For $0 < b < b_2$ the mixing property of the equilibrium ν_Θ was proved in Cox and Greven [10] via a covariance argument. However, for $b_2 \leq b < b_*$ covariances are infinite, and so we must follow a different route.

The proof uses the exponential duality in Lemma 2.2.2. We will prove that, for all $f, g \in \mathcal{E}_1$ (recall (1.4.4)) with finite support,

$$\lim_{\|k\|\to\infty} E^{\nu_\Theta}\left(e^{-\langle X,f\rangle}e^{-\langle X,\sigma_k g\rangle}\right) = E^{\nu_\Theta}\left(e^{-\langle X,f\rangle}\right) E^{\nu_\Theta}\left(e^{-\langle X,g\rangle}\right), \tag{2.4.22}$$

where $\sigma_k g = g \circ \sigma_k$ with σ_k the k-shift acting on I. This implies the mixing property, because the Laplace functional determines the distribution.

Step 1: In order to prove (2.4.22), we use the self-duality of our process and the fact that $\nu_\Theta = w - \lim_{t\to\infty} \delta_{\underline{\Theta}} S(t)$, as follows. Denote by

$$X^{*,h} = (X^{*,h}(t))_{t\geq 0} = (\{X_i^{*,h}(t)\}_{i\in\mathbb{Z}^d})_{t\geq 0} \tag{2.4.23}$$

our process with reflected transition kernel $a^*(\cdot,\cdot)$ (recall (2.2.10)) starting from initial configuration $h \in \mathcal{E}_1$ with finite support. Then

$$\begin{aligned} E^{\nu_\Theta}\left(e^{-\langle X,f\rangle}e^{-\langle X,g\rangle}\right) &= E^{\nu_\Theta}\left(e^{-\langle X,f+g\rangle}\right) \\ &= \lim_{t\to\infty} E\left(e^{-\langle X(t),f+g\rangle} \mid X(0) = \underline{\Theta}\right) = \lim_{t\to\infty} E\left(e^{-\langle \underline{\Theta}, X^{*,f+g}(t)\rangle}\right). \end{aligned} \tag{2.4.24}$$

Observe that, by the linearity of the system, we may use the same Brownian motions for $X^{*,f}$ and $X^{*,g}$, which gives us in addition

$$X^{*,f+g} \triangleq X^{*,f} + X^{*,g}. \tag{2.4.25}$$

Hence, in order to verify (2.4.22), we must investigate the quantity

$$\lim_{t\to\infty} E\left(e^{-\langle \underline{\Theta}\,,\, X^{*,f}(t)+X^{*,\sigma_k g}(t)\rangle}\right) \tag{2.4.26}$$

and show that it factorizes in the limit as $\|k\| \to \infty$.

Next, note that

$$\big(\langle \underline{\Theta}\,,\, X^{*,f}(t)\rangle\big)_{t\geq 0} \qquad \text{and} \qquad \big(\langle \underline{\Theta}\,,\, X^{*,\sigma_k g}(t)\rangle\big)_{t\geq 0} \tag{2.4.27}$$

are (continuous-path square-integrable) non-negative martingales. In particular, their limit as $t \to \infty$ exists by the martingale convergence theorem. Their covariation over the time interval $[0, \infty)$ is given by

$$C(f, \sigma_k g) = \int_0^\infty ds\ \langle X^{*,f}(s), X^{*,\sigma_k g}(s)\rangle = \int_0^\infty ds \sum_{i\in I} X_i^{*,f}(s) X_i^{*,\sigma_k g}(s). \tag{2.4.28}$$

Due to these structural properties, we know that if

$$C(f, \sigma_k g) \to 0 \quad \text{ in probability as } \|k\| \to \infty, \tag{2.4.29}$$

then the two martingales in (2.4.27) become independent as $\|k\| \to \infty$. Consequently, the two random variables

$$\lim_{t\to\infty} \langle \underline{\Theta}\,,\, X^{*,f}(t)\rangle \qquad \text{and} \qquad \lim_{t\to\infty} \langle \underline{\Theta}\,,\, X^{*,\sigma_k g}(t)\rangle \tag{2.4.30}$$

also become independent as $\|k\| \to \infty$, which proves (2.4.22) via (2.4.24–2.4.25). In order to prove (2.4.29), observe that, by the linearity of the system, it suffices to verify (2.4.22) for the special case where f and g are indicators of a single site in I, say, p and q, respectively.

Step 2: Let ξ and ξ' be two independent random walks with transition kernel $a^*(\cdot,\cdot)$ and jump rate 1, both starting in $i \in I$. Then, for $f = 1_{\{p\}}$ and $g = 1_{\{q\}}$, it follows from Lemma 2.2.1 that

$$\begin{aligned} X_i^{*,f}(s) X_i^{*,\sigma_k g}(s) &= e^{-bs}\, E_{i,i}^{\xi,\xi'}\Bigg(1\{\xi(s) = p,\, \xi'(s) = q + k\} \\ &\times \exp\left[\sqrt{b}\int_0^s du \sum_{m\in I} [1\{\xi(s-u) = m\} + 1\{\xi'(s-u) = m\}]\, dW_m(u)\right]\Bigg). \end{aligned} \tag{2.4.31}$$

Reversing time, we may start ξ in p and ξ' in $q + k$, and give them transition kernel $a(\cdot,\cdot)$ and jump rate 1. Then

$$\begin{aligned} X_i^{*,f}(s) X_i^{*,\sigma_k g}(s) &= e^{-bs}\, E_{p,q+k}^{\xi,\xi'}\Bigg(1\{\xi(s) = \xi'(s) = i\} \\ &\times \exp\left[\sqrt{b}\int_0^s du \sum_{m\in I} [1\{\xi(u) = m\} + 1\{\xi'(u) = m\}]\, dW_m(u)\right]\Bigg) \end{aligned} \tag{2.4.32}$$

and so

$$
\begin{aligned}
\langle X^{*,f}(s), X^{*,\sigma_k g}(s)\rangle &= \sum_{i\in I} X_i^{*,f}(s) X_i^{*,\sigma_k g}(s) \\
&= e^{-bs}\, E_{p,q+k}^{\xi,\xi'}\Bigg(1\{\xi(s)=\xi'(s)\} \\
&\qquad \times \exp\left[\sqrt{b}\int_0^s du \sum_{m\in I}[1\{\xi(u)=m\}+1\{\xi'(u)=m\}]\, dW_m(u)\right]\Bigg).
\end{aligned} \tag{2.4.33}
$$

Next, for any s we have

$$
\begin{aligned}
\langle X^{*,f}(s), X^{*,\sigma_k g}(s)\rangle &= \sum_{i\in I} X_i^{*,f}(s) X_i^{*,\sigma_k g}(s) \\
&\le \frac{1}{2}\sum_{i\in I}[X_i^{*,f}(s)]^2 + \frac{1}{2}\sum_{i\in I}[X_i^{*,\sigma_k g}(s)]^2 \\
&= \frac{1}{2}M_1(s) + \frac{1}{2}M_2(s),
\end{aligned} \tag{2.4.34}
$$

where

$$
\begin{aligned}
M_1(s) &= e^{-bs}E_p^{\xi}\left(\exp\left[2\sqrt{b}\int_0^s du \sum_{m\in I} 1\{\xi(u)=m\} dW_m(u)\right]\right), \\
M_2(s) &= e^{-bs}E_{q+k}^{\xi}\left(\exp\left[2\sqrt{b}\int_0^s du \sum_{m\in I} 1\{\xi(u)=m\} dW_m(u)\right]\right).
\end{aligned} \tag{2.4.35}
$$

Below we will show that

$$
\int_0^\infty ds\, M_i(s) < \infty \quad W-a.s. \quad \text{for } i=1,2. \tag{2.4.36}
$$

Assuming (2.4.36), we pick $T>0$ and estimate, with the help of (2.4.34),

$$
\begin{aligned}
C(f,\sigma_k g) &= \int_0^\infty ds\ \langle X^{*,f}(s), X^{*,\sigma_k g}(s)\rangle \\
&\le \int_0^T ds\, \langle X^{*,f}(s), X^{*,\sigma_k g}(s)\rangle + \frac{1}{2}\int_T^\infty [M_1(s)+M_2(s)].
\end{aligned} \tag{2.4.37}
$$

By (2.4.36) and the fact that the law of $(M_2(s))_{s\ge 0}$ is independent of k, it now suffices to show that

$$
\int_0^T ds\ \langle X^{*,f}(s), X^{*,\sigma_k g}(s)\rangle \to 0 \quad \text{in probability as } \|k\|\to\infty \text{ for any } T>0. \tag{2.4.38}
$$

Step 3: To prove (2.4.38), we return to (2.4.33). Write

$$\begin{aligned}
&\sum_{m\in I}[1\{\xi(u)=m\}+1\{\xi'(u)=m\}]\,dW_m(u)\\
&\quad=\sum_{m\in I}1\{\xi(u)=\xi'(u)=m\}\,2dW_m(u)\\
&\qquad+\sum_{m\in I}1\{\xi(u)=m\neq\xi'(u)\}\,dW_m(u)\\
&\qquad+\sum_{m\in I}1\{\xi(u)\neq m=\xi'(u)\}\,dW_m(u).
\end{aligned}\tag{2.4.39}$$

Since the three terms in the right-hand side of (2.4.39) involve disjoint time intervals and the W_m's have independent increments, it follows from (2.4.39) that, for any ξ,

$$\begin{aligned}
&\sqrt{b}\int_0^s du\sum_{m\in I}[1\{\xi(u)=m\}+1\{\xi'(u)=m\}]\,dW_m(u)\\
&\quad\triangleq W_1\big(4bT_s(\xi,\xi')\big)+W_2\big(b[s-T_s(\xi,\xi')]\big)+W_3\big(b[s-T_s(\xi,\xi')]\big),
\end{aligned}\tag{2.4.40}$$

where W_1,W_2,W_3 are three independent Brownian motions, and $T_s(\xi,\xi')=\int_0^s du\,1\{\xi(u)=\xi'(u)\}$ is the intersection local time of ξ and ξ' up to time s. By combining (2.4.33) and (2.4.40), taking the expectation over W (i.e., over W_1,W_2,W_3) and using Fubini's theorem, we get

$$\begin{aligned}
E^W\left(\int_0^T ds\,\langle X^{*,f}(s),X^{*,\sigma_k g}(s)\rangle\right)&=\int_0^T E^{\xi,\xi'}_{p,q+k}\left(1\{\xi(s)=\xi'(s)\}e^{bT_s(\xi,\xi')}\right)\\
&\le e^{bT}E^{\xi,\xi'}_{p,q+k}\left(T_T(\xi,\xi')\right).
\end{aligned}\tag{2.4.41}$$

Clearly, the r.h.s. tends to zero as $\|k\|\to\infty$, because $T_T(\xi,\xi')\le T$ and $T_T(\xi,\xi')\to 0$ in probability w.r.t. ξ,ξ' as $\|k\|\to\infty$ for any fixed T.

Step 4: It remains to prove (2.4.36), which goes as follows. Let

$$(M(s))_{s\ge 0}\quad\text{with}\quad M(s)=\langle\underline{1},X^{*,f}(s)\rangle.\tag{2.4.42}$$

This is a (continuous-path square-integrable) non-negative martingale starting from a strictly positive and finite value (because $f=1_{\{p\}}$ has finite support). From the dual of (1.4.2) (recall Lemma 2.2.2), we have

$$dM(s)=\sum_{i\in I}dX_i^{*,f}(s)=\sum_{i,j\in I}a^*(i,j)[X_j^{*,f}(s)-X_i^{*,f}(s)]\,ds+\sum_{i\in I}\sqrt{b[X_i^{*,f}(s)]^2}\,dW_i(s).\tag{2.4.43}$$

The first term in the right-hand side is zero because $a^*(\cdot,\cdot)$ is doubly stochastic (being a random walk transition kernel (recall (1.4.9)) and $\langle\underline{1},X^{*,f}(s)\rangle<\infty$. Hence

$$M(s)\triangleq\widehat{W}(\tau(s))\tag{2.4.44}$$

with

$$\tau(s)=\int_0^s du\;b\sum_{i\in I}[X_i^{*,f}(u)]^2\tag{2.4.45}$$

and $\widehat{W}$ a Brownian motion adapted to the filtration of $X^{*,f}$. By the martingale convergence theorem, we have

$$\lim_{s\to\infty} M(s) = M(\infty) < \infty \quad W - a.s. \tag{2.4.46}$$

(use that $E^{X^{*,f}}(M(\infty)) \leq M(0) = \langle \underline{1}, f\rangle < \infty$). Combining this with (2.4.44), we conclude that

$$\lim_{s\to\infty} \tau(s) = \tau(\infty) < \infty \quad W - a.s. \tag{2.4.47}$$

This completes the proof of (2.4.36), hence of (2.4.29), and therefore also of (2.4.22). ∎

2.5 Properties of $G^{(m)}$ and $K^{(m)}$

Recall (1.6.5) and (1.6.8). We begin with the following statement:

Lemma 2.5.1 *If $a(\cdot,\cdot)$ is symmetric and transient, then $G^{(m)}(\cdot,\cdot)$ is strongly transient for all $m \geq 3$, i.e.,*

$$\sup_{x,y\in I^{(m)}} \sum_{z\in I^{(m)}} G^{(m)}(x,z)G^{(m)}(z,y) < \infty, \qquad m \geq 3. \tag{2.5.1}$$

Proof. Let

$$P_t^{(m)}(x,y) = P^{\eta^{(m)}}(\eta^{(m)}(t) = y \mid \eta^{(m)}(0) = x), \qquad x,y \in I^{(m)},\ t \geq 0, \tag{2.5.2}$$

be the transition probabilities of the differences random walk $\eta^{(m)} = (\eta^{(m)}(t))_{t\geq 0}$ defined in (1.6.2). We have

$$G^{(m)}(x,y) = m\int_0^\infty dt\, P_t^{(m)}(x,y), \qquad x,y \in I^{(m)}. \tag{2.5.3}$$

Compute

$$\begin{aligned}\frac{1}{m^2}\sum_{z\in I^{(m)}} G^{(m)}(x,z)G^{(m)}(z,y) &= \int_0^\infty ds_1 \int_0^\infty ds_2 \sum_{z\in I^{(m)}} P_{s_1}^{(m)}(x,z)P_{s_2}^{(m)}(z,y)\\ &= \int_0^\infty ds_1 \int_0^\infty ds_2\ P_{s_1+s_2}^{(m)}(x,y)\\ &= \int_0^\infty dt\, t\, P_t^{(m)}(x,y)\\ &\leq \int_0^\infty dt\, t\, P_t^{(m)}(0,0).\end{aligned} \tag{2.5.4}$$

Note that

$$P_t^{(m)}(0,0) = \sum_{i\in I} [P_t(0,i)]^m \tag{2.5.5}$$

with $P_t(i,j)$, $i,j \in I$, $t \geq 0$, the transition probabilities of a single random walk. Because the single random walk is symmetric and has exponential jump times (with mean 1), we have

$$\begin{aligned}&(i) \quad P_t(i,j) \leq P_t(0,0) \quad \forall i,j \in I,\ t \geq 0,\\ &(ii) \quad t \mapsto P_t(0,0) \text{ is non-increasing,}\end{aligned} \tag{2.5.6}$$

as is easily seen from the Fourier representation of $P_t(i,j)$, i.e.,

$$\begin{aligned} P_t(i,j) &= \sum_{n=0}^{\infty} e^{-t}\frac{t^n}{n!}\;(2\pi)^{-d}\int_{[-\pi,\pi)^d} dk\;\cos(i-j,k)\;[A(k)]^n \\ &= (2\pi)^{-d}\int_{[-\pi,\pi)^d} dk\;\cos(i-j,k)\;e^{-t[1-A(k)]} \end{aligned} \tag{2.5.7}$$

with $A(k)=\sum_{i\in I} a(0,i)\cos(i,k)\le 1$, $k\in[-\pi,\pi)^d$, and $(\cdot,\cdot)$ the inner product on $\mathbb{R}^d$. Via (2.5.6)(i), (2.5.5) gives

$$P_t^{(m)}(0,0)\le [P_t(0,0)]^{m-2}\sum_{i\in I} P_t(0,i)P_t(i,0)=[P_t(0,0)]^{m-2}\;P_{2t}(0,0), \tag{2.5.8}$$

which via (2.5.5)(ii) yields

$$\begin{aligned} \int_0^\infty dt\,t\,P_t^{(m)}(0,0) &\le \frac{1}{2}\int_0^\infty dt\;[P_t(0,0)]^{m-2}\int_0^{2t} ds\,P_s(0,0) \\ &\le \frac{1}{2}\left(\int_0^\infty dt\;[P_t(0,0)]^{m-2}\right)\left(\int_0^\infty ds\,P_s(0,0)\right). \end{aligned} \tag{2.5.9}$$

The right-hand side is finite by the transience of the single random walk. ∎

We next look at $K^{(m)}$ defined in (1.6.9).

Proposition 2.5.2 *Suppose that $a(\cdot,\cdot)$ is symmetric and transient. Then, for all $m\ge 2$, $K^{(m)}$ is a self-adjoint, positive and bounded operator on $\ell^2(S^{(m)})$.*

Proof. The symmetry of $K^{(m)}$ follows from the symmetry of $G^{(m)}$. Since $K^{(m)}$ is defined everywhere on $\ell^2(S^{(m)})$, it therefore is self-adjoint. The Fourier representation of $G^{(m)}$ reads

$$G^{(m)}(x,y)=\sum_{n=0}^{\infty}[a^{(m)}]^n(x,y)=(2\pi)^{-d}\int_{[-\pi,\pi)^d} dk\,e^{i(x-y,k)}\;[1-A^{(m)}(k)]^{-1}, \qquad x,y\in I^{(m)}, \tag{2.5.10}$$

with $A^{(m)}(k)=\sum_{x\in I^{(m)}} a^{(m)}(0,x)\cos(x,k)\le 1$, $k\in[-\pi,\pi)^d$. It follows that

$$\langle\mu,K^{(m)}\mu\rangle=(2\pi)^{-d}\int_{[-\pi,\pi)^d} dk\,[1-A^{(m)}(k)]^{-1}\left|\sum_{x\in I^{(m)}} e^{i(x,k)}\mu(x)\sqrt{\sharp^{(m)}(x)}\right|^2 \tag{2.5.11}$$

with $\langle\cdot,\cdot\rangle$ the inner product on $\ell^2(S^{(m)})$, which proves the positivity of $K^{(m)}$. To prove the boundedness of $K^{(m)}$, we consider the relation

$$\|\sqrt{K^{(m)}}\|^2=\sup_{\substack{\mu\in\ell^2(S^{(m)})\\ \langle\mu,\mu\rangle=1}}\langle\mu,K^{(m)}\mu\rangle \tag{2.5.12}$$

with $\|\cdot\|$ denoting the operator norm on $\ell^2(S^{(m)})$. Apply Cauchy-Schwarz twice, to obtain

$$
\begin{aligned}
&\langle \mu, K^{(m)} \mu \rangle \\
&\quad = \sum_{x,y \in S^{(m)}} \mu(x) \sqrt{\sharp^{(m)}(x)} G^{(m)}(x,y) \sqrt{\sharp^{(m)}(y)}\, \mu(y) \\
&\quad \le \sum_{x \in S^{(m)}} \left[\sum_{y \in S^{(m)}} \sharp^{(m)}(x) G^{(m)}(x,y) \mu^2(y) \right]^{\frac{1}{2}} \left[\sum_{y \in S^{(m)}} \mu^2(x) G^{(m)}(x,y) \sharp^{(m)}(y) \right]^{\frac{1}{2}} \\
&\quad \le \left[\sum_{x,y \in S^{(m)}} \sharp^{(m)}(x) G^{(m)}(x,y) \mu^2(y) \right]^{\frac{1}{2}} \left[\sum_{x,y \in S^{(m)}} \mu^2(x) G^{(m)}(x,y) \sharp^{(m)}(y) \right]^{\frac{1}{2}} \\
&\quad \le \left[\sum_{x \in S^{(m)}} \mu^2(x) \right] \sup_{x \in S^{(m)}} \sum_{y \in S^{(m)}} G^{(m)}(x,y) \sharp^{(m)}(y),
\end{aligned}
\tag{2.5.13}
$$

where in the last line we use the symmetry of $G^{(m)}$. The last sum is equal to the average total intersection local time (in pairs) of the m walks when their differences start in x. Clearly, the supremum is taken at $x = 0$, and equals $\sharp^{(m)}(0) G^{(2)}(0,0)$, because $G^{(2)}(0,0)$ is the average intersection local time for each pair. Hence

$$
\|\sqrt{K^{(m)}}\|^2 \le \sharp^{(m)}(0) G^{(2)}(0,0) < \infty. \tag{2.5.14}
$$

But, by the self-adjointness and positivity of $K^{(m)}$, we have (see Rudin [38], Chapter 12)

$$
\|\sqrt{K^{(m)}}\|^2 = \|[K^{(m)}]^n\|^{1/n} = \mathrm{spec}(K^{(m)}) \qquad \forall\, n \in \mathbb{N} \tag{2.5.15}
$$

with $\mathrm{spec}(\cdot)$ denoting the spectral radius in $\ell^2(S^{(m)})$. ∎

3 Variational representations

In Section 3.1 we identify $\bar{b}_m$ in terms of a variational problem. In Section 3.2 we prove that the m-th moment diverges at $b = \bar{b}_m$. In Section 3.3 we calculate the exponential growth rate of the m-th moment and prove that $\widetilde{b}_m = \bar{b}_m$. In Section 3.4 we study the m-dependence of b_m. In Section 3.5 we collect the results and prove Theorem 1.6.1.

3.1 Variational formula for $\bar{b}_m$

Proposition 3.1.1 *Suppose that $a(\cdot,\cdot)$ is symmetric and transient. Then $\bar{b}_m = m/\bar{\lambda}_m$ with*

$$
\bar{\lambda}_m = \sup_{\substack{\zeta \in \ell^1(S^{(m)}) \\ \zeta \neq 0}} \frac{\langle \zeta, [K^{(m)}]^2 \zeta \rangle}{\langle \zeta, K^{(m)} \zeta \rangle}. \tag{3.1.1}
$$

Proof. The proof comes in several steps. Throughout the proof we assume that $b\,\sharp^{(m)}(0) < m$.

Step 1: Recall the definition of $\bar{b}_m$ in (2.1.1) as well as the identity in (2.3.6). We begin by deriving a criterion for the property $E(\exp[bT^{(m)}(\infty)]) < \infty$ in terms of the discrete-time random walk

$$\eta^{(m),\odot} = (\eta^{(m),\odot}(i))_{i\in\mathbb{N}_0} \tag{3.1.2}$$

underlying the continuous-time random walk $\eta^{(m)} = (\eta^{(m)}(t))_{t\geq 0}$ defined in (1.6.2). To that end we perform the expectation over the jump times of $\eta^{(m)}$, which are independent of $\eta^{(m),\odot}$. Indeed, let

$$(\sigma_i)_{i\in\mathbb{N}_0} \tag{3.1.3}$$

be the successive discrete times at which $\eta^{(m),\odot}$ visits $S^{(m)}$ (put $\sigma_0 = 0$), and let M be the total number of visits to $S^{(m)}$ (which is random but finite a.s. by transience). Each visit to $S^{(m)}$ lasts a time τ that is exponentially distributed with mean $\frac{1}{m}$. Define

$$\Box^{(m),b}(x) = E\Big(\exp[b\,\sharp^{(m)}(x)\,\tau]\Big) = \frac{m}{m - b\,\sharp^{(m)}(x)}, \qquad x \in S^{(m)}. \tag{3.1.4}$$

Then we have

$$E\Big(\exp[bT^{(m)}(\infty)]\Big) = E\Big(\prod_{i=0}^{M}\Box^{(m),b}(\eta^{(m),\odot}(\sigma_i))\Big). \tag{3.1.5}$$

Step 2: In order to analyse the right-hand side of (3.1.5), we introduce the transition kernel of the Markov chain on $S^{(m)}$ obtained by observing $\eta^{(m),\odot}$ only when it visits $S^{(m)}$, which we denote by $P^{(m)}(\cdot,\cdot)$. Since $a(\cdot,\cdot)$ is symmetric, so is $P^{(m)}(\cdot,\cdot)$. By transience, this transition kernel is defective:

$$\sum_{y\in S^{(m)}} P^{(m)}(x,y)\begin{cases} \leq 1 & \text{if } \sharp^{(m)}(x) = 1,\\ = 1 & \text{otherwise.}\end{cases} \tag{3.1.6}$$

(The first line says that escape from $S^{(m)}$ is possible only when all walks are disjoint except one pair. This is because only one walk moves at a time.) In terms of $P^{(m)}(\cdot,\cdot)$ we can write

$$\begin{aligned} &E\Big(\prod_{i=0}^{M}\Box^{(m),b}(\eta^{(m),\odot}(\sigma_i))\Big)\\ &= \sum_{n=0}^{\infty}\sum_{x_0,\dots,x_n\in S^{(m)}}\delta_0(x_0)\Big(\prod_{i=1}^{n}P^{(m)}(x_{i-1},x_i)\Big)[1-P^{(m)}(x_n,S^{(m)})]\left(\prod_{i=0}^{n}\Box^{(m),b}(x_i)\right). \end{aligned} \tag{3.1.7}$$

Define the matrix

$$Q^{(m),b}(x,y) = \sqrt{\Box^{(m),b}(x)}\,P^{(m)}(x,y)\,\sqrt{\Box^{(m),b}(y)}, \qquad x,y \in S^{(m)}. \tag{3.1.8}$$

With this notation we can write, combining (3.1.5) and (3.1.7–3.1.8),

$$E\Big(\exp[bT^{(m)}(\infty)]\Big) = C_{m,b}\sum_{n=0}^{\infty}\Big\langle \delta_0, [Q^{(m),b}]^n R_m\Big\rangle \tag{3.1.9}$$

with $C_{m,b} = m/\sqrt{(m-b\,\sharp^{(m)}(0))(m-b)}$ and $R_m(\cdot) = [1-P^{(m)}(\,\cdot\,,S^{(m)})]1_{\{\sharp^{(m)}(\cdot)=1\}}$. The front factor, which arises from the endpoints in the second sum in (3.1.7), is harmless.

Step 3: Note the following:

Lemma 3.1.2 *$Q^{(m),b}$ is an irreducible, aperiodic, non-negative and symmetric matrix. As an operator acting on $\ell^2(S^{(m)})$ it is self-adjoint and bounded.*

Proof. Obvious. The aperiodicity follows from the fact that $P^{(m)}(x,x) > 0$ for some $x \in S^{(m)}$ with $\sharp^{(m)}(x) = 1$. ∎

Define

$$\bar{\chi}_m(b) = \lim_{n\to\infty} \frac{1}{n} \log[Q^{(m),b}]^n(x,y), \qquad x,y \in S^{(m)}. \tag{3.1.10}$$

Under the properties stated in Lemma 3.1.2, this limit exists, is in $\mathbb{R}$ and is the same for all $x,y \in S^{(m)}$ (Vere-Jones [45]). Moreover,

$$\sum_{n=0}^{\infty}[Q^{(m),b}]^n(0,0) < \infty \qquad \Longleftrightarrow \qquad \sum_{n=0}^{\infty}[Q^{(m),b}]^n(x,y) < \infty \quad \forall x,y \in S^{(m)}. \tag{3.1.11}$$

This leads to

$$\begin{aligned} \bar{\chi}_m(b) > 0 &\implies \sum_{n=0}^{\infty}[Q^{(m),b}]^n(0,0) = \infty, \\ \bar{\chi}_m(b) < 0 &\implies \sum_{n=0}^{\infty}[Q^{(m),b}]^n(0,0) < \infty. \end{aligned} \tag{3.1.12}$$

From (2.1.1), (2.3.6), (3.1.5), (3.1.9) and (3.1.12) we see that $\bar{b}_m$ is the solution of the equation $\bar{\chi}_m(b) = 0$. At the end of Section 3.2 the case $b = \bar{b}_m$ will be included in the top line of (3.1.12).

Step 4: Next, at $b = \bar{b}_m$ we have the following:

Lemma 3.1.3

$$\sup_{\substack{\nu\in\ell^1(S^{(m)})\\ \nu\neq 0}} \frac{\langle \nu, Q^{(m),\bar{b}_m}\nu\rangle}{\langle \nu,\nu\rangle} = 1. \tag{3.1.13}$$

Proof. Since $Q^{(m),\bar{b}_m}$ is a bounded operator on $\ell^2(S^{(m)})$, the function $\nu \mapsto \langle \nu, Q^{(m),\bar{b}_m}\nu\rangle$ is continuous on $\ell^2(S^{(m)})$. Since $\ell^1(S^{(m)}) \subset \ell^2(S^{(m)})$ is dense, it suffices to prove that

$$\sup_{\substack{\nu\in\ell^2(S^{(m)})\\ \langle\nu,\nu\rangle=1}} \langle \nu, Q^{(m),\bar{b}_m}\nu\rangle = 1. \tag{3.1.14}$$

[≤ 1]: First consider ν with finite support. Suppose that $\langle \nu, Q^{(m),\bar{b}_m}\nu\rangle \geq 1+\epsilon$ for some $\epsilon > 0$. Then, by the spectral theorem and Jensen's inequality,

$$\begin{aligned} \langle \nu, [Q^{(m),\bar{b}_m}]^{2n}\nu\rangle &= \int_{\mathbb{R}} \lambda^{2n}\, dE_{\nu,\nu}(\lambda) \geq \left(\int_{\mathbb{R}} \lambda\, dE_{\nu,\nu}(\lambda)\right)^{2n} \\ &= \langle \nu, Q^{(m),\bar{b}_m}\nu\rangle^{2n} \geq (1+\epsilon)^{2n} \qquad \forall\, n \in \mathbb{N} \end{aligned} \tag{3.1.15}$$

with $E_{\nu,\nu}$ the spectral measure associated with ν. Clearly this contradicts (3.1.10) with $\bar{\chi}_m(\bar{b}_m) = 0$, and so $\langle \nu, Q^{(m),\bar{b}_m}\nu\rangle \leq 1$ for ν with finite support. Since the ν's with finite support are dense in $\ell^2(S^{(m)})$, it follows that the supremum is ≤ 1.

$[\geq 1]$: Suppose that the supremum is $\leq 1-\epsilon$ for some $\epsilon > 0$. Then the spectrum of $Q^{(m),\bar{b}_m}$ is contained in $(-\infty, 1-\epsilon]$. Estimate

$$\begin{aligned} 0 \leq \langle \delta_0, [Q^{(m),\bar{b}_m}]^{2n+1}\delta_0\rangle &= \int_{-\infty}^{1-\epsilon} \lambda^{2n+1}\, dE_{\delta_0,\delta_0}(\lambda) \\ &\leq \int_0^{1-\epsilon} \lambda^{2n+1}\, dE_{\delta_0,\delta_0}(\lambda) \leq (1-\epsilon)^{2n+1} \qquad \forall n \in \mathbb{N}. \end{aligned} \tag{3.1.16}$$

But again this contradicts (3.1.10) with $\bar{\chi}_m(\bar{b}_m) = 0$. Hence the supremum is ≥ 1. ∎

Step 5: Putting $\mu = \sqrt{\Box^{(m),\bar{b}_m}}\,\nu$, we may rewrite (3.1.13) as (recall (3.1.4))

$$1 = \sup_{\substack{\mu \in \ell^1(S^{(m)}) \\ \mu \neq 0}} \frac{\langle \mu, P^{(m)}\mu\rangle}{\langle \mu, (\Box^{(m),\bar{b}_m})^{-1}\mu\rangle} = \sup_{\substack{\mu \in \ell^1(S^{(m)}) \\ \mu \neq 0}} \frac{\langle \mu,\mu\rangle - \langle \mu, (\mathbb{1} - P^{(m)})\mu\rangle}{\langle \mu,\mu\rangle - \frac{\bar{b}_m}{m}\langle \mu, \sharp^{(m)}\mu\rangle}. \tag{3.1.17}$$

Therefore

$$\bar{b}_m = \frac{m}{\bar{\lambda}_m} \qquad \text{with } \bar{\lambda}_m = \sup_{\substack{\mu \in \ell^1(S^{(m)}) \\ \mu \neq 0}} \frac{\langle \mu, \sharp^{(m)}\mu\rangle}{\langle \mu, (\mathbb{1} - P^{(m)})\mu\rangle}, \tag{3.1.18}$$

where the denominator is strictly positive because $P^{(m)}$ is irreducible. Let

$$\widehat{G}^{(m)} = \sum_{n=0}^{\infty} [P^{(m)}]^n = (\mathbb{1} - P^{(m)})^{-1}. \tag{3.1.19}$$

Because $P^{(m)}$ has spectral radius < 1 (due to the fact that $S^{(m)}$ is a uniformly transient set), we know that $\mathbb{1} - P^{(m)}$ is one-to-one on $\ell^1(S^{(m)})$ and that $\widehat{G}^{(m)}$ is a bounded operator on $\ell^1(S^{(m)})$. Therefore we can transform (3.1.18) via the change of variables $\mu = \widehat{G}^{(m)}\rho$:

$$\bar{\lambda}_m = \sup_{\substack{\rho \in \ell^1(S^{(m)}) \\ \rho \neq 0}} \frac{\langle (\widehat{G}^{(m)}\rho)^2, \sharp^{(m)}\rangle}{\langle \rho, \widehat{G}^{(m)}\rho\rangle}. \tag{3.1.20}$$

Finally, putting $\rho = \sqrt{\sharp^{(m)}}\,\zeta$ and using that $\widehat{G}^{(m)}(x,y) = G^{(m)}(x,y)$ for all $x,y \in S^{(m)}$ by the definition of $P^{(m)}$, we get the formula in Proposition 3.1.1. ∎

3.2 The m-th moment at $b = \bar{b}_m$

The case $b = \bar{b}_m$ can be included in the top line of (3.1.12) when 1 is the largest ℓ^1-eigenvalue of $Q^{(m),\bar{b}_m}$. Therefore we next consider the eigenvalue equation

$$\nu Q^{(m),\bar{b}_m} = \nu, \qquad \nu \in \ell^1(S^{(m)}),\ \nu > 0. \tag{3.2.1}$$

Lemma 3.2.1 *Suppose that $a(\cdot,\cdot)$ is symmetric and transient. If $\bar{b}_{m-1} > \bar{b}_m$, then* (3.2.1) *has a solution.*

Proof. The idea is to use the notion of a quasi-stationary distribution.

Step 1: Consider the matrix

$$Q^{(m),\bar{b}_m,\otimes}(x,y) = \frac{1}{N_{m,\bar{b}_m}} Q^{(m),\bar{b}_m}(x,y), \qquad N_{m,\bar{b}_m} = \sup_{x\in S^{(m)}} \sum_{y\in S^{(m)}} Q^{(m),\bar{b}_m}(x,y). \tag{3.2.2}$$

This is an irreducible defective probability kernel. By introducing a cemetery state ∂, we can extend $Q^{(m),\bar{b}_m,\otimes}$ to a non-defective probability kernel on $S^{(m)} \cup \{\partial\}$. Let $(Z_n)_{n\in\mathbb{N}_0}$ denote the corresponding Markov chain starting in 0, and let

$$\nu_n = \mathcal{L}(Z_n \mid Z_n \neq \partial), \qquad n \in \mathbb{N}_0. \tag{3.2.3}$$

If we manage to show that ($\mathcal{P}$ denotes the set of probability measures)

$$\lim_{n\to\infty} \nu_n = \nu_\infty \quad \text{ in } \mathcal{P}(S^{(m)}), \tag{3.2.4}$$

then, because

$$\nu_{n+1} = \frac{\nu_n Q^{(m),\bar{b}_m,\otimes}}{\langle \nu_n Q^{(m),\bar{b}_m,\otimes}, 1\rangle}, \tag{3.2.5}$$

we get that

$$\nu_\infty Q^{(m),\bar{b}_m,\otimes} = \lambda_\infty \nu_\infty, \qquad \lambda_\infty > 0,\ \nu_\infty > 0, \tag{3.2.6}$$

with $\lambda_\infty = \langle \nu_\infty Q^{(m),\bar{b}_m,\otimes}, 1\rangle = 1/N_{m,\bar{b}_m}$ the probability of no defection to ∂ (per step) in the quasi-stationary distribution ν_∞. Hence ν_∞ solves (3.2.1).

Step 2: To prove (3.2.4), we use a criterion in Ferrari, Kesten and Martinez [22], Theorem 1, according to which it is enough to prove that there exist $\delta > 0$ and $D < \infty$ (depending on m) such that

$$P_0(\tau_0 > n \mid \tau_\partial > n) \leq De^{-\delta n} \qquad \forall n \in \mathbb{N}_0 \tag{3.2.7}$$

with P_0 the law of $(Z_n)_{n\in\mathbb{N}_0}$ given $Z_0 = 0$, and τ_0, τ_∂ the first hitting times of $0, \partial$ (time zero excluded).

For $K \subset \{1,\dots,m\}$ with $0 < |K| < m$, let

$$V_K = \{x \in S^{(m)} \colon \quad \text{in site } x \text{ there exist } i \in K, j \in K^c \text{ such that walks } i \text{ and } j \text{ coincide}\} \tag{3.2.8}$$

with $K^c = \{1,\dots,m\} \setminus K$. We will prove that there exist $\delta > 0$ and $D_K < \infty$ such that

$$P_0(\tau_{V_K} > n \mid \tau_\partial > n) \leq D_K e^{-\delta n} \qquad \forall n \in \mathbb{N}_0. \tag{3.2.9}$$

Since

$$\bigcap_{0<|K|<m} V_K = \{0\}, \tag{3.2.10}$$

we have $\{\tau_0 > n\} \subset \cup_{0<|K|<m} \{\tau_{V_K} > n\}$, and so (3.2.9) implies (3.2.7) with $D = \sum_K D_K$.

To prove (3.2.9), write

$$\begin{aligned} &P_0(\tau_{V_K} > n \mid \tau_\partial > n) \\ &= \frac{\sum_{y_1,\dots,y_n\in S^{(m)}\setminus V_K} Q^{(m),\bar{b}_m,\otimes}(0,y_1) Q^{(m),\bar{b}_m,\otimes}(y_1,y_2) \times \cdots \times Q^{(m),\bar{b}_m,\otimes}(y_{n-1},y_n)}{\sum_{y_1,\dots,y_n\in S^{(m)}} Q^{(m),\bar{b}_m,\otimes}(0,y_1) Q^{(m),\bar{b}_m,\otimes}(y_1,y_2) \times \cdots \times Q^{(m),\bar{b}_m,\otimes}(y_{n-1},y_n)}. \end{aligned} \tag{3.2.11}$$

We may drop the $\otimes$ because the normalisation factor in (3.2.2) cancels out. After that the denominator in (3.2.11) equals

$$\langle \delta_0, [Q^{(m),\bar{b}_m}]^n 1 \rangle \geq \langle \delta_0, [Q^{(m),\bar{b}_m}]^n \delta_0 \rangle = [\bar{\chi}_m(\bar{b}_m) + o(1)]^n = \exp[o(n)] \tag{3.2.12}$$

because $\bar{\chi}_m(\bar{b}_m) = 0$. It therefore suffices to prove that the numerator in (3.2.11) satisfies the exponential bound.

Now, on $S^{(m)} \setminus V_K$ we have

$$T^{(m)}(\sigma_n) = T^{(m),K}(\sigma_n) + T^{(m),K^c}(\sigma_n) \qquad \forall n \in \mathbb{N} \tag{3.2.13}$$

with σ_n the time of the n-th visit to $S^{(m)}$ (recall (3.1.3)) and $T^{(m),K}(\sigma_n)$ the total intersection local time (in pairs) up to time σ_n of the walks indexed by K, and similarly for K^c. Therefore, retracing the calculations in Steps 1 and 2 of the proof of Proposition 3.1.1, we find that

$$\text{numerator (3.2.11)} \leq E_0\left(\exp\left[\bar{b}_m\left\{T^{(m),K}(\sigma_n) + T^{(m),K^c}(\sigma_n)\right\}\right] 1_{\{\sigma_n<\infty\}}\right). \tag{3.2.14}$$

The inequality comes from using (3.2.13) and afterwards dropping the restriction to $S^{(m)} \setminus V_K$. Next, apply Hölder's inequality to get, for $\epsilon > 0$,

$$\begin{aligned} &\text{numerator (3.2.11)} \\ &\quad \leq E_0\left(\exp\left[\bar{b}_m\left\{T^{(m),K}(\infty) + T^{(m),K^c}(\infty)\right\}\right] 1_{\{\sigma_n<\infty\}}\right) \\ &\quad \leq E_0\left(\exp\left[(1+\epsilon)\bar{b}_m\left\{T^{(m),K}(\infty) + T^{(m),K^c}(\infty)\right\}\right]\right)^{1/(1+\epsilon)} P_0(\sigma_n < \infty)^{\epsilon/(1+\epsilon)}. \end{aligned} \tag{3.2.15}$$

The expectation in the right-hand side factors because $T^{(m),K}(\infty)$ and $T^{(m),K^c}(\infty)$ are independent, and each factor is finite when ϵ is picked so small that $(1+\epsilon)\bar{b}_m < \widehat{b}_{m-1}$, because $|K|, |K^c| \leq m-1$. On the other hand, the probability in the right-hand side tends to zero exponentially fast with n because $S^{(m)}$ is a uniformly transient set. ∎

By (2.1.1) and Proposition 2.4.1, we have $E^{\nu_\Theta}([X_0]^m) < \infty$ for $b < b_m$ and $E^{\nu_\Theta}([X_0]^m) = \infty$ for $b > b_m$. With the help of Lemma 3.2.1 we can now include $b = b_m$.

Lemma 3.2.2 *$E^{\nu_\Theta}([X_0]^m) = \infty$ at $b = b_m$.*

Proof. By Proposition 2.4.1(b), we have $b_m = \bar{b}_m$. Suppose first that $b_{m-1} > b_m$. Then, by Lemma 3.2.1,

$$\nu Q^{(m),b_m} = \nu, \quad \nu \in \ell^1(S^{(m)}),\ \nu > 0, \tag{3.2.16}$$

and hence

$$\sum_{n=0}^{\infty} \sum_{x \in S^{(m)}} \nu(x)[Q^{(m),b_m}]^n(x,y) = \sum_{n=0}^{\infty} \nu(y) = \infty \qquad \forall y \in S^{(m)}. \tag{3.2.17}$$

By the irreducibility of $Q^{(m),b_m}$, this implies that

$$\sum_{n=0}^{\infty} [Q^{(m),b_m}]^n(0,0) = \infty, \tag{3.2.18}$$

which shows, via (3.1.5) and (3.1.9), that

$$E^{\xi^{(m)}}\Big(\exp[b_m T^{(m)}(\infty)]\Big) = \infty. \tag{3.2.19}$$

Now use Proposition 2.4.1(c) to obtain from (3.2.19) that $E^{\nu_\Theta}([X_0]^m) = \infty$ at $b = b_m$.

Suppose next that $b_{m-1} = b_m$, but $b_{m'-1} > b_m$ for some $m' < m$. We may assume that m' is the largest such index. Then $b_{m'-1} > b_{m'}$, and so the above argument tells us that $E^{\nu_\Theta}([X_0]^{m'}) = \infty$ at $b = b_{m'}$. But $b_{m'} = b_m$ and $m' < m$, and so we again get $E^{\nu_\Theta}([X_0]^m) = \infty$ at $b = b_m$.

Finally, if $b_{m'} = b_m$ for all $m' < m$, then $b_2 = b_m$. However, at $b = b_2$ we have $E^{\nu_\Theta}([X_0]^2) = \infty$, as is easily seen from Lemma 2.3.2, Proposition 2.4.1(c) and (3.1.9), because $S^{(2)} = \{0\}$. Therefore once again $E^{\nu_\Theta}([X_0]^m) = \infty$ at $b = b_m$. ∎

3.3 Growth rate of the m-th moment and $\widetilde{b}_m = \bar{b}_m$

In this section we show that

$$\lim_{t\to\infty} \frac{1}{t} \log E^{\xi^{(m)}}\Big(\exp[bT^{(m)}(t)]\Big) = \chi_m(b) \tag{3.3.1}$$

exists and can be expressed in terms of a variational problem. We will analyse this variational problem and show that

$$\widetilde{b}_m = \sup\{b > 0\colon\ \chi_m(b) = 0\} \tag{3.3.2}$$

(recall (2.1.1) and (2.3.6)) coincides with $\bar{b}_m$. Together with Proposition 2.4.1(b) this will show that all three critical values in (2.1.1) coincide.

In order to pose the problem in a form suitable for a large deviation analysis, we recall the definition of the differences random walk $(\eta^{(m)}(t))_{t\geq 0}$ in (1.6.2) and the intersection function $\sharp^{(m)}$ in (1.6.6). Using (2.3.7), we have the identity

$$T^{(m)}(t) = \int_0^t \sharp^{(m)}\Big(\eta^{(m)}(s)\Big)\, ds. \tag{3.3.3}$$

Next, we define the empirical measure

$$L_t^{(m)} = \frac{1}{t}\int_0^t \delta_{\eta^{(m)}(s)} ds. \tag{3.3.4}$$

Lemma 3.3.1 *Suppose that $a(\cdot,\cdot)$ is symmetric. Then $(L_t^{(m)})_{t\geq 0}$ satisfies the weak large deviation principle on $\mathcal{P}(I^{(m)})$ with rate function*

$$J^{(m)}(\nu) = \Big\langle \nu^{\frac12}, m(\mathbb{1} - a^{(m)})\nu^{\frac12}\Big\rangle, \qquad \nu \in \mathcal{P}(I^{(m)}), \tag{3.3.5}$$

where $a^{(m)}(\cdot,\cdot)$ is the transition kernel defined in (1.6.3). $J^{(m)}$ is bounded on $\ell^2(I^{(m)})$ and when restricted to $\ell^1(I^{(m)})$ is continuous in the ℓ^1-topology.

Proof. See Deuschel and Stroock [20], Section 4.2. The rate function is given by (3.3.5) because $m(a^{(m)} - \mathbb{1})$ is the generator of the Markov process $(\eta^{(m)}(t))_{t\geq 0}$ and $a^{(m)}(\cdot,\cdot)$ is symmetric (recall (1.6.1)). The latter is crucial for having the explicit formula in (3.3.5). The boundedness of $J^{(m)}$ is obvious. The continuity of $J^{(m)}$ follows from the fact that $m(\mathbb{1}-a^{(m)})$ is a bounded operator from $\ell^2(I^{(m)})$ into $\ell^1(I^{(m)})$, in combination with the fact that if $\nu_n \to \nu$ as $n \to \infty$ in ℓ^1-norm, then $\nu_n^{\frac{1}{2}} \to \nu^{\frac{1}{2}}$ in ℓ^2-norm. To see the latter, write

$$\|\nu_n^{\frac{1}{2}} - \nu^{\frac{1}{2}}\|^2 = \sum_x \Big(\sqrt{\nu_n(x)} - \sqrt{\nu(x)} \Big)^2 = 2 - 2\sum_x \sqrt{\nu_n(x)\nu(x)}. \tag{3.3.6}$$

If $\nu_n \to \nu$ in ℓ^1-norm, then Fatou's lemma gives $\liminf_{n\to\infty} \sum_x \sqrt{\nu_n(x)\nu(x)} \geq \sum_x \nu(x) = 1$. Hence $\lim_{n\to\infty} \|\nu_n^{\frac{1}{2}} - \nu^{\frac{1}{2}}\|_2^2 = 0$. ∎

Lemma 3.3.1 leads us to the following identification:

Lemma 3.3.2

$$\chi_m(b) = \sup_{\nu\in\mathcal{P}(I^{(m)})} \Big\{ b\,\langle \nu, \sharp^{(m)}\rangle - J^{(m)}(\nu) \Big\}. \tag{3.3.7}$$

Proof. For ease of notation we drop the superscript (m). We cannot apply Varadhan's lemma directly to (3.3.1–3.3.4), since we only have a *weak* large deviation principle. This problem can be handled via a standard compactification argument, as follows.

Let $(\eta_N^+(t))_{t\geq 0}, (\eta_N^-(t))_{t\geq 0}$ be the differences random walks obtained by wrapping $(\eta(t))_{t\geq 0}$ around the torus $\Lambda_N = ([-N,N)^d \cap I)^{\frac{1}{2}m(m-1)}$ (recall that $I = \mathbb{Z}^d$), respectively, by killing it when it hits $\partial\Lambda_N$, the boundary of Λ_N. Let $T_N^+(t), T_N^-(t)$ be the quantities corresponding to $T(t)$ in (3.3.3) for these two processes. Then, by the large deviation principle in Lemma 3.3.1 restricted to Λ_N, we have for every N:

$$\lim_{t\to\infty} \frac{1}{t} \log E\Big(\exp[bT_N^+(t)] \Big) = S_N^+, \qquad \lim_{t\to\infty} \frac{1}{t} \log E\Big(\exp[bT_N^-(t)] \Big) = S_N^-, \tag{3.3.8}$$

with

$$S_N^+ = \sup_{\nu_N\in\mathcal{P}(\Lambda_N)} \Big\{ b\,\langle \nu_N, \sharp\rangle - J_N(\nu_N) \Big\}, \qquad S_N^- = \sup_{\substack{\nu_N\in\mathcal{P}(\Lambda_N)\\ \nu_N(\partial\Lambda_N)=0}} \Big\{ b\,\langle \nu_N, \sharp\rangle - J_N(\nu_N) \Big\}. \tag{3.3.9}$$

Here, J_N is the analogue of J in (3.3.5) restricted to Λ_N, i.e.,

$$J_N(\nu_N) = \Big\langle \nu_N^{\frac{1}{2}}, m(\mathbb{1}-a_N)\nu_N^{\frac{1}{2}} \Big\rangle, \qquad \nu_N \in \mathcal{P}(\Lambda_N), \tag{3.3.10}$$

with $a_N(\cdot,\cdot)$ the periodised transition kernel

$$a_N(x,y) = \sum_{z\in(2N)I^{(m)}} a(x, y+z), \qquad x,y \in \Lambda_N. \tag{3.3.11}$$

Now, obviously

$$T_N^+(t) \geq T(t) \geq T_N^-(t), \tag{3.3.12}$$

because the interesection local time increases by wrapping and decreases by killing. Consequently,

$$S_N^+ \geq S \geq S_N^- \tag{3.3.13}$$

with S the right-hand side of (3.3.7). Hence, to prove (3.3.1) and (3.3.7) it suffices to show that

$$\liminf_{N\to\infty} S_N^- \geq S, \qquad \limsup_{N\to\infty} S_N^+ \leq S. \tag{3.3.14}$$

Lower bound: It suffices to show that for every $\nu \in \mathcal{P}(I^{(m)})$ there exists a sequence (ν_N), with $\nu_N \in \mathcal{P}(\Lambda_N)$ and $\nu_N(\partial\Lambda_N) = 0$ for each N, such that

$$w - \lim_{N\to\infty} \nu_N = \nu, \quad \lim_{N\to\infty} J_N(\nu_N) = J(\nu). \tag{3.3.15}$$

Indeed, with the help of Fatou's lemma this gives

$$\liminf_{N\to\infty} S_N^- \geq \liminf_{N\to\infty} \Big\{ b\,\langle \nu_N, \sharp\rangle - J_N(\nu_N) \Big\} \geq b\,\langle \nu, \sharp\rangle - J(\nu), \tag{3.3.16}$$

and so we get the first half of (3.3.14) after taking the supremum over ν afterwards.

For the given ν, the sequence (ν_N) is chosen as follows. Put $\nu_N(x) = \nu(x)$ for all $x \in \Lambda_N \setminus (\partial\Lambda_N \cup \{0\})$ and $\nu_N(0) = \nu(0) + \sum_{x\notin\Lambda_N\cup\partial\Lambda_N} \nu(x)$. Then the first half of (3.3.15) is obvious. For the second half, since J is continuous in the ℓ_1-topology it suffices to show that $\lim_{N\to\infty}[J_N(\nu_N) - J(\nu_N)] = 0$. To that end, we estimate

$$\begin{aligned}
0 &\leq J(\nu_N) - J_N(\nu_N) \\
&= m\,\langle \nu_N^{\frac{1}{2}}, (a_N - a)\nu_N^{\frac{1}{2}}\rangle \\
&\leq m \sum_{x,y\in\Lambda_N} \nu_N(x)(a_N - a)(x,y) \\
&\leq m \sup_{x\in\Lambda_N} \sum_{y\in\Lambda_N} (a_N - a)(x,y) \\
&\leq m \sup_{x\in\Lambda_N} \sum_{\substack{y\in I^{(m)} \\ \|y-x\|_\infty \geq N}} a(x,y) = \delta_N,
\end{aligned} \tag{3.3.17}$$

where in the third line we use the Cauchy-Schwarz inequality and the symmetry of a_N and a, and in the fifth line we use (3.3.11) and the shift-invariance of $a(\cdot,\cdot)$ (note that on the sublattice with spacing $2N$ containing $y \in \Lambda_N$, the site closest to x is captured by the supremum over $x \in \Lambda_N$ and the sum over $y \in I^{(m)}$ with $\|y - x\|_\infty < N$). Obviously, $\lim_{N\to\infty} \delta_N = 0$, which completes the proof of the first half of (3.3.14).

Upper bound: Estimate, with the help of (3.3.17),

$$\begin{aligned}
S_N^+ &= \sup_{\nu_N\in\mathcal{P}(\Lambda_N)} \Big\{ b\,\langle \nu_N, \sharp\rangle - J_N(\nu_N) \Big\} \\
&\leq \sup_{\nu_N\in\mathcal{P}(\Lambda_N)} \Big\{ b\,\langle \nu_N, \sharp\rangle - J(\nu_N) \Big\} + \delta_N \\
&\leq S + \delta_N.
\end{aligned} \tag{3.3.18}$$

Let $N \to \infty$ to obtain the second half of (3.3.14). ∎

It follows from (3.3.7) that $b \mapsto \chi_m(b)$ is nondecreasing and convex on $[0,\infty)$. Since it is finite, it is also continuous on $[0,\infty)$. Obviously, $\chi_m(0) = 0$. The critical value $\widetilde{b}_m$ is given by (3.3.2). It further follows from (3.3.7) that

$$b\,\sharp^{(m)}(0) - J^{(m)}(\delta_0) \leq \chi_m(b) \leq b\,\sharp^{(m)}(0), \tag{3.3.19}$$

where $J^{(m)}(\delta_0) = m(1 - a^{(m)}(0,0)) = m$ (use (3.3.5) and (1.6.3), and recall that $a(0,0) = 0$ as assumed below (1.4.9)).

Proposition 3.3.3 *Suppose that $a(\cdot,\cdot)$ is symmetric and transient. Then $\widetilde{b}_m = \bar{b}_m$.*

Proof. Changing variables in (3.3.7) by putting $\nu = \mu^2$, we get

$$\chi_m(b) = \sup_{\substack{\mu \in \ell^2(I^{(m)}) \\ \langle \mu,\mu \rangle = 1}} F^{(m)}(\mu) \tag{3.3.20}$$

with

$$F^{(m)}(\mu) = b\,\langle \mu^2, \sharp^{(m)} \rangle - m\langle \mu, (\mathbb{1} - a^{(m)})\mu \rangle. \tag{3.3.21}$$

Define

$$\widetilde{\lambda}_m = \sup_{\substack{\mu \in \ell^2(I^{(m)}) \\ \mu \neq 0}} \frac{\langle \mu^2, \sharp^{(m)} \rangle}{\langle \mu, (\mathbb{1} - a^{(m)})\mu \rangle}, \tag{3.3.22}$$

where the denominator is strictly positive because $a^{(m)}(\cdot,\cdot)$ is irreducible. It follows from (3.3.20–3.3.21) that

$$\begin{aligned} b\,\widetilde{\lambda}_m > m &\implies \chi_m(b) > 0, \\ b\,\widetilde{\lambda}_m \leq m &\implies \chi_m(b) = 0. \end{aligned} \tag{3.3.23}$$

Hence $\widetilde{b}_m = m/\widetilde{\lambda}_m$. Change variables in (3.3.22) by putting $\mu = G^{(m)}\rho$. Then, because $G^{(m)} = (\mathbb{1} - a^{(m)})^{-1}$, we get

$$\widetilde{\lambda}_m \geq \sup_{\substack{\rho \in \ell^1(S^{(m)}) \\ \rho \neq 0}} \frac{\langle (G^{(m)}\rho)^2, \sharp^{(m)} \rangle}{\langle \rho, G^{(m)}\rho \rangle}. \tag{3.3.24}$$

Here the inequality arises because we restrict the support of ρ to $S^{(m)}$: $G^{(m)}\rho$ does not run through all of $\ell^2(S^{(m)})$. Note here that, because $G^{(m)}$ is strongly transient by Lemma 2.5.1, $\rho \in \ell^1(S^{(m)})$ implies that $\mu = G^{(m)}\rho \in \ell^2(I^{(m)})$.

Putting $\rho = \sqrt{\sharp^{(m)}}\zeta$ with $\zeta \in \ell^1(S^{(m)})$, we obtain

$$\widetilde{\lambda}_m \geq \sup_{\substack{\zeta \in \ell^1(S^{(m)}) \\ \zeta \neq 0}} \frac{\langle \zeta, [K^{(m)}]^2\zeta \rangle}{\langle \zeta, K^{(m)}\zeta \rangle}. \tag{3.3.25}$$

Comparing with (3.1.1), we have thus found that $\widetilde{\lambda}_m \geq \widehat{\lambda}_m$. Consequently, $\widetilde{b}_m \leq \bar{b}_m$. However, it is obvious from (2.1.1) that $\widetilde{b}_m \geq \bar{b}_m$. Hence we get $\widetilde{b}_m = \bar{b}_m$ and $\widetilde{\lambda}_m = \bar{\lambda}_m$. ∎

3.4 Analysis of $m \mapsto b_m$

The remaining steps in this section are the following three propositions.

Proposition 3.4.1 *Suppose that $a(\cdot,\cdot)$ is symmetric and transient. Then $b_m = \bar{b}_m = \widetilde{b}_m = \frac{m}{\lambda_m}$ with λ_m the spectral radius of $K^{(m)}$ in $\ell^2(S^{(m)})$.*

Proof. It follows from Propositions 2.4.1(b), 3.1.1 and 3.3.3 that $b_m = \bar{b}_m = \widetilde{b}_m = \frac{m}{\lambda_m}$ with λ_m given by the right-hand side of (3.1.1). We saw in the proof of Lemma 3.1.3 that the supremum may be taken over $\nu \in \ell^2(S^{(m)})$. Putting $\mu = \sqrt{K^{(m)}}\nu$ in (3.1.1), we see that $\lambda_m = \|\sqrt{K^{(m)}}\|^2$. Now recall (2.5.15). ∎

Proposition 3.4.2 *Suppose that $a(\cdot,\cdot)$ is symmetric and transient. Then the spectral radius of $K^{(m)}$ is an eigenvalue if and only if $b_{m-1} > b_m$.*

Proof. The fact that $b_{m-1} > b_m$ implies that the spectral radius of $K^{(m)}$ is an eigenvalue follows from Lemma 3.2.1 via the change of variables in Step 5 in the proof of Proposition 3.1.1. Indeed, Lemma 3.2.1 shows that the supremum in (3.1.1) is then actually attained.

To get the reverse, we use Lemma 3.2.1 and Proposition 3.4.1. The idea is to imbed the variational problem with label $m-1$ into the one with label m. To that end, consider m random walks but ignore the intersection local time the m-th random walk has with the others. By repeating the large deviation analysis in Section 3.3, we get a formula for $\widetilde{\chi}_{m-1}(b)$ of the same form as in (3.3.7) but with the first term $\langle \nu, \sharp^{(m)} \rangle$ replaced by

$$\langle \nu, \sharp^{(m),(m-1)} \rangle, \tag{3.4.1}$$

where (compare with (1.6.6))

$$\sharp^{(m),(m-1)}(z) = \sum_{1 \le p < q \le m-1} 1_{\{z_p - z_q = 0\}}, \qquad z = (z_p - z_q)_{1 \le p < q \le m}, \quad z_p, z_q \in I. \tag{3.4.2}$$

In this way we can represent $\chi_m(b)$ and $\chi_{m-1}(b)$ via a variational problem on the same space:

$$\begin{aligned} \chi_m(b) &= \sup_{\nu \in \mathcal{P}(I^{(m)})} \Big\{ b \langle \nu, \sharp^{(m)} \rangle - J^{(m)}(\nu) \Big\}, \\ \chi_{m-1}(b) &= \sup_{\nu \in \mathcal{P}(I^{(m)})} \Big\{ b \langle \nu, \sharp^{(m),(m-1)} \rangle - J^{(m)}(\nu) \Big\}. \end{aligned} \tag{3.4.3}$$

Now consider these two variational problems at $b = b_m$. Suppose that $K^{(m)}$ has a maximal eigenvalue. Then this eigenvalue is unique, and so is its corresponding eigenvector. Consequently, the first variational problem has a unique maximiser, say $\bar{\nu} \in \mathcal{P}(I^{(m)})$, which is strictly positive. For $\epsilon \in (0, \bar{\nu}(0))$, put

$$\mathcal{U}_\epsilon = \{\nu \in \mathcal{P}(I^{(m)}) \colon |\nu(0) - \bar{\nu}(0)| < \epsilon\}, \tag{3.4.4}$$

and let

$$\begin{aligned} A_\epsilon &= \sup_{\nu \in \mathcal{P}(I^{(m)}) \setminus \mathcal{U}_\epsilon} \Big\{ b_m \langle \nu, \sharp^{(m),(m-1)} \rangle - J^{(m)}(\nu) \Big\}, \\ B_\epsilon &= \sup_{\nu \in \mathcal{U}_\epsilon} \Big\{ b_m \langle \nu, \sharp^{(m),(m-1)} \rangle - J^{(m)}(\nu) \Big\}. \end{aligned} \tag{3.4.5}$$

Since $\bar{\nu}$ is the unique maximiser, we have

$$0 = \chi_m(b_m) > \sup_{\nu \in \mathcal{P}(I^{(m)}) \setminus \mathcal{U}_\epsilon} \Big\{ b_m \langle \nu, \sharp^{(m)} \rangle - J^{(m)}(\nu) \Big\} \ge A_\epsilon. \tag{3.4.6}$$

Since $\sharp^{(m)}(0) > \sharp^{(m),(m-1)}(0)$, we have

$$0 = \chi_m(b_m) \ge \sup_{\nu \in \mathcal{U}_\epsilon} \Big\{ b_m \langle \nu, \sharp^{(m)} \rangle - J^{(m)}(\nu) \Big\} > B_\epsilon. \tag{3.4.7}$$

Combining (3.4.6–3.4.7) with the observation that $\chi_{m-1}(b_m) = A_\epsilon \vee B_\epsilon$, we find that $\chi_{m-1}(b_m) < 0$ and hence that $b_{m-1} > b_m$. ∎

Proposition 3.4.3 *Suppose that $a(\cdot,\cdot)$ is transient. Then*
(a) $b_2 = 2/G^{(2)}(0,0)$.
Suppose that $a(\cdot,\cdot)$ is symmetric and transient. Then
(b) $b_2 \geq b_3 \geq b_4 \geq \cdots > 0$.
(c) $b_2 \leq (m-1)b_m \leq 2/G^{(m)}(0,0) < 2$.
(d) $\lim_{m\to\infty}(m-1)b_m$ *exists.*

Proof. (a) The formula for b_2 is obvious because $S^{(2)} = \{0\}$ and $\sharp^{(2)}(0) = 1$, giving $\lambda_2 = G^{(2)}(0,0)$.

(b) The fact that $m \mapsto b_m$ is nonincreasing is trivial from (2.1.1) and Lemma 2.3.2. The fact that $b_m > 0$ for all m follows from (c).

(c) We prove that

$$\sharp^{(m)}(0)G^{(m)}(0,0) \leq \lambda_m \leq \sharp^{(m)}(0)G^{(2)}(0,0), \tag{3.4.8}$$

which implies the claim. The lower bound is obtained by picking $\mu = \delta_0$ in (2.5.12) and using (2.5.15). The upper bound follows from (2.5.14–2.5.15).

(d) The proof is an adaptation of the argument in Carmona and Molchanov [6], Chapter III, Section 1.3. The key is the following inequality:

Lemma 3.4.4 *If*

$$m = \sum_{i=1}^{r} n_i m_i \qquad \text{with } n_i, m_i \in \mathbb{N} \text{ and } m_i \geq 2 \text{ for } i = 1,\ldots,r, \tag{3.4.9}$$

then

$$\chi_m\left(\frac{b}{m-1}\right) \leq \sum_{i=1}^{r} n_i\,\chi_{m_i}\left(\frac{b}{m_i-1}\right) \qquad \forall b > 0. \tag{3.4.10}$$

Proof. Let $\mathcal{P}$ denote the collection of all partitions of $\{1,\ldots,m\}$ into n_i groups of m_i integers for $i = 1,\ldots,r$. For $P \in \mathcal{P}$, write

$$P = (P_{ij})_{i=1,\ldots,r,j=1,\ldots,n_i} \tag{3.4.11}$$

to label these groups, so that

$$\{1,\ldots,m\} = \cup_{i=1}^{r} \cup_{j=1}^{n_i} P_{ij} \tag{3.4.12}$$

with

$$|P_{ij}| = m_i, \quad i = 1,\ldots,r,\ j = 1,\ldots,n_i. \tag{3.4.13}$$

The cardinality of $\mathcal{P}$ is

$$|\mathcal{P}| = N = \frac{m!}{\prod_{i=1}^{r} n_i!\, m_i^{n_i}}. \tag{3.4.14}$$

Moreover,

$$\sum_{P\in\mathcal{P}}\sum_{j=1}^{n_i}\sum_{\substack{k,l\in P_{ij}\\1\le k<l\le m}} 1 = N_i, \quad i=1,\dots,r, \tag{3.4.15}$$

with

$$N_i = n_i\,\frac{m_i(m_i-1)}{m(m-1)}\,N. \tag{3.4.16}$$

Now, let

$$\delta_i = \frac{m-1}{m_i-1}\,\frac{1}{N}. \tag{3.4.17}$$

Then, by (3.4.9),

$$\sum_{i=1}^{r}\delta_i N_i = 1. \tag{3.4.18}$$

Using (3.4.15) and (3.4.18), we may write

$$\begin{aligned} T^{(m)}(t) &= \sum_{1\le k<l\le m}\int_0^t 1_{\{\xi_k(s)=\xi_l(s)\}}\,ds \\ &= \sum_{P\in\mathcal{P}}\sum_{i=1}^{r}\delta_i\sum_{j=1}^{n_i}\sum_{\substack{k,l\in P_{ij}\\1\le k<l\le m}}\int_0^t 1_{\{\xi_k(s)=\xi_l(s)\}}\,ds. \end{aligned} \tag{3.4.19}$$

With the help of (3.4.19), we may estimate

$$\begin{aligned} &E^{\xi^{(m)}}\left(\exp\left[bT^{(m)}(t)\right]\right) \\ &= E^{\xi^{(m)}}\left(\prod_{P\in\mathcal{P}}\prod_{i=1}^{r}\exp\left[b\delta_i\sum_{j=1}^{n_i}\sum_{\substack{k,l\in P_{ij}\\1\le k<l\le m}}\int_0^t 1_{\{\xi_k(s)=\xi_l(s)\}}\,ds\right]\right) \\ &\le \prod_{P\in\mathcal{P}} E^{\xi^{(m)}}\left(\prod_{i=1}^{r}\exp\left[Nb\delta_i\sum_{j=1}^{n_i}\sum_{\substack{k,l\in P_{ij}\\1\le k<l\le m}}\int_0^t 1_{\{\xi_k(s)=\xi_l(s)\}}\,ds\right]\right)^{1/N} \\ &= \prod_{P\in\mathcal{P}}\prod_{i=1}^{r} E^{\xi^{(m)}}\left(\exp\left[Nb\delta_i\sum_{j=1}^{n_i}\sum_{\substack{k,l\in P_{ij}\\1\le k<l\le m}}\int_0^t 1_{\{\xi_k(s)=\xi_l(s)\}}\,ds\right]\right)^{1/N} \\ &= \prod_{i=1}^{r} E^{\xi^{(m)}}\left(\exp\left[Nb\delta_i\sum_{j=1}^{n_i}\sum_{\substack{k,l\in P^*_{ij}\\1\le k<l\le m}}\int_0^t 1_{\{\xi_k(s)=\xi_l(s)\}}\,ds\right]\right), \end{aligned} \tag{3.4.20}$$

where P^* is any representative partition. Here, the third line uses Hölder's inequality, the fourth line uses that the factors labelled by i are independent, and the fifth line uses that the expectations for given P do not depend on the choice of P. Taking logarithms, dividing by t and letting $t\to\infty$ on both sides of (3.4.20), we obtain

$$\chi_m(b) \le \sum_{i=1}^{r} n_i\,\chi_{m_i}\left(Nb\delta_i\right) \tag{3.4.21}$$

(recall (3.3.1) and (3.4.13)). Inserting (3.4.17) and replacing b by $b/(m-1)$, we arrive at (3.4.10). ∎

We now show why Lemma 3.4.4 implies the claim in Proposition 3.4.3(d). An immediate consequence of Lemma 3.4.4 and the definition of b_m,

$$b_m = \sup\{b > 0\colon\ \chi_m(b) = 0\} \tag{3.4.22}$$

(recall (3.3.2) and use that $b_m = \bar{b}_m$), is that, subject to (3.4.9),

$$(m-1)b_m \geq \min_{i=1,\dots,r} (m_i - 1)b_{m_i} \tag{3.4.23}$$

Let $c = \limsup_{m\to\infty}(m-1)b_m$. Pick $\epsilon > 0$ arbitrary, and pick $m_1 = m_1(\epsilon)$ such that

$$(m_1 - 1)b_{m_1} \geq c - \epsilon. \tag{3.4.24}$$

For $m > m_1$, let $n_1 = \lceil m/m_1 \rceil$. Since $m \leq n_1 m_1 < m + m_1$, we may estimate

$$\begin{aligned}(m-1)b_m \geq (m-1)b_{n_1 m_1} &= \frac{m-1}{n_1 m_1 - 1}\,(n_1 m_1 - 1)b_{n_1 m_1}\\ &\geq \frac{m-1}{n_1 m_1 - 1}\,(m_1 - 1)b_{m_1} > \frac{m-1}{m + m_1 - 1}(c - \epsilon).\end{aligned} \tag{3.4.25}$$

Here, the first inequality uses that $m \mapsto b_m$ is non-increasing, while the second inequality uses (3.4.23) for $r = 1$. Let $m \to \infty$ to obtain that $\liminf_{m\to\infty}(m-1)b_m \geq c - \epsilon$. Let $\epsilon \downarrow 0$ to get $\lim_{m\to\infty}(m-1)b_m = c$. Note that $c = \sup_{m\geq 2}(m-1)b_m$. ∎

3.5 Proof of Theorem 1.6.1

Proof. (a) Combine (2.1.1), Proposition 2.4.1(c) and Lemma 3.2.2.

(b) Combine (2.3.6), (3.3.1), (3.3.2), Propositions 2.4.1(b) and 3.3.3, and the fact that $b \mapsto \chi_m(b)$ is continuous.

(c–d) Combine Propositions 3.4.2–3.4.3.

(e) Use Lemma 3.3.2 and (3.3.19). ∎

4 Survival versus extinction

In Section 4.1 we prove Theorem 1.5.3. In Section 4.2 we prove Theorem 1.6.3.

4.1 Proof of Theorem 1.5.3

In Section 4.1.1 we introduce the Palm distribution of our process $X = (X(t))_{t\geq 0}$. In Section 4.1.2 we use the Palm distribution to identify b_*. In Section 4.1.3 we prove Theorem 1.5.3.

4.1.1 Palm distribution and its stochastic representation

By size-biasing our process

$$X = (X(t))_{t\geq 0} \tag{4.1.1}$$

at the origin at time T, we obtain a process

$$\widehat{X}^T = (\widehat{X}^T(t))_{0\leq t\leq T} \tag{4.1.2}$$

satisfying

$$dP\big((\widehat{X}^T(t))_{0\leq t\leq T} \in \cdot\,\big) = \frac{1}{\Theta}\, X_0(T)\, dP\big((X(t))_{0\leq t\leq T} \in \cdot\,\big). \tag{4.1.3}$$

In what follows we construct a stochastic representation of $\widehat{X}^T$ in terms of a random walk and an SSDE, leading to a stochastic representation of the Palm distribution of X, i.e., the law of $\widehat{X}^T(T)$.

Step 1: Fix $T>0$. Let $\zeta = (\zeta(t))_{t\geq 0}$ be the random walk on I with transition kernel $a(\cdot,\cdot)$ and jump rate 1, starting at the origin and independent of W. Given ζ, let

$$\widehat{X}^{\zeta,T}(t) = \{\widehat{X}_i^{\zeta,T}(t)\}_{i\in I} \tag{4.1.4}$$

be the solution of the SSDE

$$\begin{aligned} d\widehat{X}_i^{\zeta,T}(t) &= \sum_{j\in I} a(i,j)[\widehat{X}_j^{\zeta,T}(t) - \widehat{X}_i^{\zeta,T}(t)]\,dt + \sqrt{b\widehat{X}_i^{\zeta,T}(t)}\,dW_i(t) \\ &\qquad + b\widehat{X}_i^{\zeta,T}(t)\,1\{\zeta(T-t)=i\}\,dt, \quad i\in I,\ t\geq 0. \end{aligned} \tag{4.1.5}$$

Here, the difference with the SSDE in (1.4.2) is the presence of the last term, which produces a ζ-dependent random potential. Like (1.4.2), if $\widehat{X}_i^{\zeta,T}(0)\in\mathcal{E}_1$ (the space of configurations defined in (1.4.4)), then (4.1.5) has a unique strong solution on $\mathcal{E}_1$ with continuous paths (compare with Theorem 1.4.1(a)). Let

$$P(\cdot) = \text{ law of } \zeta \text{ on } I^{[0,\infty)}, \quad Q^{\zeta,T}(\cdot) = \text{ law of } \widehat{X}^{\zeta,T}(T) \text{ on } [0,\infty)^I \text{ for given } \zeta. \tag{4.1.6}$$

Then

$$Q^T(\cdot) = \int Q^{\zeta,T}(\cdot)\,P(d\zeta) = \text{ law of } \widehat{X}^T(T) \text{ on } [0,\infty)^I \tag{4.1.7}$$

is the Palm distribution of X.

Similarly as in Lemma 2.2.1, we have a representation formula:

Lemma 4.1.1 *Given a realisation of the random walk ζ, the process $(\widehat{X}^{\zeta,T}(T))_{T\geq 0}$ starting from $\widehat{X}^{\zeta,T}(0)=\underline{\Theta}$ can be represented as the following functional of the Brownian motions:*

$$\widehat{X}_i^{\zeta,T}(T) = \Theta e^{-\frac{1}{2}bT} E_i^{\xi}\left(\exp\left[\sqrt{b}\int_0^T \sum_{j\in I} 1_{\{\xi(T-s)=j\}}dW_j(s) + b\int_0^T 1_{\{\xi(T-s)=\zeta(T-s)\}}ds\right]\right), \tag{4.1.8}$$

where $\xi=(\xi(t))_{t\geq 0}$ is the random walk on I with transition kernel $a(\cdot,\cdot)$ and jump rate 1, and the expectation is over ξ conditioned on $\xi(0)=i$ (ξ and W are independent).

Proof. The proof uses Itô-calculus and is similar to that of Lemma 2.2.1. ∎

Step 2: The link between $X(T)$ and $\widehat{X}^T(T)$ is given by the following identity, which expresses the size-biasing at the origin at time T as required in (4.1.3):

Proposition 4.1.2 *For all measurable finite $f\colon [0,\infty)^I \to \mathbb{R}$ that depend on finitely many coordinates,*

$$E^{\zeta,W}(f(\widehat{X}^{\zeta,T}(T))) = \frac{1}{\Theta} E^W(X_0(T) f(X(T))). \tag{4.1.9}$$

Proof. For polynomial f, the identity easily follows from (4.1.8) via a calculation of the type presented in Section 2.3. It turns out, however, that for fixed time the m-th moment grows too fast with m to be distribution determining. Therefore we need an argument to show that the identity holds in the generality stated. This goes via an approximation argument, as follows.

Let $\eta = (\{\eta_i(n)\}_{i\in I})_{n\in\mathbb{N}_0}$ be the discrete-time Markov chain on $[0,\infty)^I$ defined by the recursion formula

$$\eta_i(n+1) = \sum_{j\in I} a(i,j) V_j(n)\, \eta_j(n), \quad i \in I,\, n \in \mathbb{N}_0, \tag{4.1.10}$$

where

$$V = (\{V_i(n)\}_{i\in I})_{n\in\mathbb{N}_0} \tag{4.1.11}$$

are independent copies of an i.i.d. random field on $[0,\infty)^I$ whose components are equal in distribution to the random variable

$$V_0 = e^{\sqrt{b}Z - \frac{1}{2}b} \quad \text{with } Z \text{ standard normal.} \tag{4.1.12}$$

As initial configuration we take $\eta(0) \in \mathcal{E}_1$. Note that iteration of the recursion formula in (4.1.10) gives us a representation similar to (2.2.1), with V replacing W and the average running over the discrete-time random walk with transition kernel $a(\cdot,\cdot)$. This will be useful later on for the approximation argument.

To construct the stochastic representation of the Palm distribution of η, let $\zeta = (\zeta(n))_{n\in\mathbb{N}_0}$ be the discrete-time random walk on I with transition kernel $a(\cdot,\cdot)$ starting at 0, and let $\widehat{V}_0$ be the size-biased version of the random variable V_0:

$$E(f(\widehat{V}_0)) = E(V_0 f(V_0)) \qquad \forall f\colon [0,\infty) \to \mathbb{R} \text{ measurable with bounded support.} \tag{4.1.13}$$

Fix $N \in \mathbb{N}_0$. Given ζ, let

$$\{\widehat{\eta}_i^{\zeta,N}(n)\}_{i\in I} \tag{4.1.14}$$

be given by the recursion formula

$$\begin{aligned} &\widehat{\eta}_i^{\zeta,N}(n+1) \\ &= \sum_{j\in I} a(i,j) \left[V_j(n) 1_{\{\zeta(N-n)\neq j\}} + \widehat{V}_j(n) 1_{\{\zeta(N-n)=j\}}\right] \widehat{\eta}_i^{\zeta,N}(n), \quad i \in I,\, n \in \mathbb{N}_0, \end{aligned} \tag{4.1.15}$$

where

$$\widehat{V} = (\{\widehat{V}_i(n)\}_{i\in I})_{n\in\mathbb{N}_0} \tag{4.1.16}$$

are independent copies of an i.i.d. random field on $[0,\infty)^I$ whose components are equal in distribution to the random variable $\widehat{V}_0$. Note that iteration of the recursion formula in (4.1.15) gives us a representation similar to (4.1.8). Let

$$Q^{\zeta,N}(\cdot) = \text{ law of } \widehat{\eta}^{\zeta,N}(N) \text{ on } [0,\infty)^I. \tag{4.1.17}$$

Then

$$Q^N(\cdot) = \int Q^{\zeta,N}(\cdot)\, P(d\zeta) = \text{ law of } \widehat{\eta}^N(N) \text{ on } [0,\infty)^I \tag{4.1.18}$$

is the law of the size-biasing of η at then origin at time N. Indeed:

Lemma 4.1.3 *For all measurable finite $f\colon [0,\infty)^I \to \mathbb{R}$ that depend on finitely many coordinates,*

$$E^{\zeta,V}(f(\widehat{\eta}^{\zeta,N}(N))) = \frac{1}{\Theta} E^V(\eta_0(N) f(\eta(N))). \tag{4.1.19}$$

Proof. This claim, which is the discrete-time analogue of the claim in Proposition 4.1.2, can be proved with the help of Birkner [4], Lemma 1. There the recursion formula (4.1.15) for $\widehat{\eta}^{\zeta,N}(N)$ is derived in terms of $\widehat{V}_0$, the size-biased version of V_0 (recall (4.1.12–4.1.13)). ∎

In order to prove Proposition 4.1.2 via Lemma (4.1.1), what we need to check first is that

$$\widehat{V}_0 = e^{\sqrt{b}Z + \frac{1}{2}b} \qquad \text{with } Z \text{ standard normal}, \tag{4.1.20}$$

since this implies that the discrete version of Lemma (4.1.1) is satisfied by $\widehat{\eta}^{\zeta,N}$. Comparing (4.1.12) with (4.1.20), we see that the size-biasing changes $-\frac{1}{2}b$ to $\frac{1}{2}b$ in the exponent. To see why this is so, we note that by size-biasing the diffusion

$$dY(t) = \sqrt{bY^2(t)}\, dW(t), \qquad Y(0) = 1, \tag{4.1.21}$$

with respect to the time horizon T, we get the diffusion

$$d\widehat{Y}(t) = b\widehat{Y}(t)\, dt + \sqrt{b\widehat{Y}^2(t)}\, dW(t), \qquad \widehat{Y}(0) = 1, \tag{4.1.22}$$

observed at time T. Indeed, $h(x) = x$ is harmonic for the diffusion in (4.1.21). The h-transform of a diffusion with generator G has generator

$$G^h(f) = \frac{1}{h} G(hf). \tag{4.1.23}$$

Since

$$G = \frac{1}{2} bx^2 \frac{\partial^2}{\partial x^2} \tag{4.1.24}$$

is the generator of the diffusion in (4.1.21), an explicit calculation with $h(x) = x$ gives

$$G^h = bx \frac{\partial}{\partial x} + \frac{1}{2} bx^2 \frac{\partial^2}{\partial x^2}, \tag{4.1.25}$$

which is the generator of the diffusion in (4.1.22). Now observe that $V_0 \triangleq Y(1)$ and $\widehat{V}_0 \triangleq \widehat{Y}(1)$, to get the claim in (4.1.20).

To complete the proof of Proposition 4.1.2, all we need to do is to show that a speeded up version of the pair $(\eta, \widehat{\eta}^N)$ converges to the pair $(X, \widehat{X}^T)$, so that (4.1.19) implies (4.1.9). This goes as follows.

For $L \in \mathbb{N}$, let $a^{(L)}(\cdot,\cdot)$ and $V_0^{(L)}$ be given by

$$\begin{aligned} a^{(L)}(i,j) &= \frac{1}{L} a(i,j) + \left(1 - \frac{1}{L}\right) \delta(i,j), \\ V_0^{(L)} &= \exp\left[\sqrt{\frac{b}{L}}\, Z - \frac{1}{2}\frac{b}{L}\right]. \end{aligned} \tag{4.1.26}$$

Follow the same construction as in (4.1.10–4.1.18) to obtain a pair $(\eta^{(L)}, \widehat{\eta}^{N,(L)})$ satisfying the analogue of (4.1.19):

$$E^{\zeta,V}(f(\widehat{\eta}^{\zeta,N,(L)}(N))) = \frac{1}{\Theta} E^V\left(\eta_0^{(L)}(N) f(\eta^{(L)}(N))\right). \tag{4.1.27}$$

The key is now the observation that

$$\left(\eta^{(L)}(\lceil Lt \rceil)\right)_{0 \le t \le T} \Longrightarrow (X(t))_{0 \le t \le T} \qquad \text{as } L \to \infty \tag{4.1.28}$$

and

$$\left(\widehat{\eta}^{\lceil Lt \rceil,(L)}(\lceil Lt \rceil)\right)_{0 \le t \le T} \Longrightarrow (\widehat{X}(t))_{0 \le t \le T} \qquad \text{as } L \to \infty. \tag{4.1.29}$$

Clearly, (4.1.27–4.1.29) imply (4.1.9).

To see why (4.1.28) is true, we iterate (4.1.10):

$$\eta_i(m) = \sum_{j_1,\ldots,j_m} a(i,j_1) \times \cdots \times a(j_{m-1},j_m)\, V_{j_1} \times \cdots \times V_{j_m}. \tag{4.1.30}$$

Replace $a(\cdot,\cdot), V_i$ by $a^{(L)}(\cdot,\cdot), V_i^{(L)}$, and note that that

$$\begin{aligned} &V_{j_1}^{(L)} \times \cdots \times V_{j_m}^{(L)} \triangleq \exp\left[\sqrt{\frac{b}{L}}\,(Z_1 + \cdots + Z_m) - \frac{1}{2}\frac{b}{L}\, m\right] \text{ for all } j_1,\ldots,j_m \in I \\ &\text{with } Z_k \sim N(0,1),\ k = 1,\ldots,m, \text{ i.i.d.} \end{aligned} \tag{4.1.31}$$

For $m = \lceil Lt \rceil$ we get weak convergence, as $L \to \infty$, to

$$\exp\left[\sqrt{bt}\, Z - \frac{1}{2} bt\right] \qquad \text{with } Z \sim N(0,1) \tag{4.1.32}$$

(note that the mean is fixed at 1). Further note that

$$\left(a^{(L)}(\cdot,\cdot)\right)^{\lceil Lt \rceil} \to e^{a(\cdot,\cdot)t} \qquad \text{as } L \to \infty \tag{4.1.33}$$

pointwise on $I \times I$. Applying (4.1.31–4.1.33) to the L-version of (4.1.30), we get the convergence in (4.1.28) for fixed t with the help of the discrete analogue of (4.1.8). Since both the limiting process and the L-approximant processes are Markov, it follows that the convergence holds for the processes as well.

To see why (4.1.29) is true, note that

$$\sup_L E\left(\left[\eta_0^{(L)}(\lceil Lt\rceil)\right]^2\right) < \infty. \tag{4.1.34}$$

Indeed, the square of (4.1.31) converges to the square of (4.1.32), and hence the expectation of the square of the exponential in the right-hand side of the L-version of (4.1.30) is bounded, uniformly in L, by a constant times $\exp[\frac{1}{2}bt]$. By (4.1.34), $\eta_0^{(L)}(\lceil Lt\rceil)$ is uniformly integrable, and hence (4.1.28) yields (4.1.29). ∎

Step 3: The reason why the Palm distribution is convenient is the following important fact, which is immediate from Proposition 4.1.2:

Lemma 4.1.4 *$X(T) \Longrightarrow \delta_{\underline{0}}$ as $T \to \infty$ if and only if $\widehat{X}^T(T) \Longrightarrow \delta_{\underline{\infty}}$ as $T \to \infty$.*

Proof. For a proof we refer to Kallenberg [35]. The intuition is the following. First note that, by translation invariance and irreducibility, the statement is equivalent to $X_0(T) \Longrightarrow \delta_0$ if and only if $\widehat{X}_0^T(T) \Longrightarrow \delta_\infty$. Next, if X locally dies out, then (because the mean is preserved) it clusters, i.e., it develops high peaks at rare sites. After size-biasing, as in Proposition 4.1.2, these peaks cause explosion, which is why $\widehat{X}$ diverges. Conversely, if X locally survives, then it does not cluster. The size-biasing therefore does not cause explosion, which is why $\widehat{X}$ does not diverge. ∎

From Lemma 4.1.4 we obtain the following criterion for survival vs. extinction of X.

Lemma 4.1.5 *Let $P^T(\cdot)$ denote the law of $X(T)$ on $[0,\infty)^I$ (recall (4.1.6–4.1.7)). Then*

$$(P^T)_{T\geq 0} \text{ survives} \iff (Q^T)_{T\geq 0} \text{ is tight} \iff (Q^{\zeta,T})_{T\geq 0} \text{ is tight } \zeta - a.s. \tag{4.1.35}$$

In Section 4.1.2 we use Lemma 4.1.5 to identify b_*.

4.1.2 Identification of b_*

Abbreviate

$$M^\xi(T) = \Theta e^{-\frac{1}{2}bT} \exp\left[\sqrt{b}\int_0^T \sum_{j\in I} 1_{\{\xi(T-s)=j\}} dW_j(s)\right]. \tag{4.1.36}$$

This is a martingale in W for every fixed ξ, with $M^\xi(0) = \Theta$. In terms of this quantity, the representation formulas for X (in (2.2.1)) and $\widehat{X}$ (in (4.1.8)) read

$$\begin{aligned} X_i(T) &= E_i^\xi(M^\xi(T)), \\ \widehat{X}_i^{\zeta,T}(T) &= E_i^\xi\left(M^\xi(T)e^{bT^{\xi,\zeta}(T)}\right), \end{aligned} \tag{4.1.37}$$

where $T^{\xi,\zeta}(T)$ is the intersection local time of ξ and ζ up to time T.

Proposition 4.1.6 *X survives for $0 < b < b_*$ and locally dies out for $b > b_*$, where*

$$b_* = \sup\{b > 0\colon\ (Q^{\zeta,T})_{T\geq 0} \text{ is tight } \zeta - a.s.\}. \tag{4.1.38}$$

Proof. This follows from (4.1.6) and (4.1.35). Note that tightness is tail trivial. ∎

Equation (4.1.38) is to be compared with the formulas for b_m and b_{**} given in (1.6.18) and (1.6.20). However, (4.1.38) is more difficult to analyse. For one, since b appears with both signs in (4.1.36), it is not a priori obvious that b_* defines a unique transition: for this we need to show that if $(Q^{\zeta,T})_{T\geq 0}$ is tight ζ-a.s. for some $b>0$, then the same is true for all smaller values of b. Fortunately, the latter property can be shown to hold with the help of a coupling technique put forward in Cox, Fleischmann and Greven [9]. There it is shown that, for processes with fixed mean (Θ in our case), "more noise causes the process to be more spread out and hence to be more prone to extinction". More precisely, it is proved that two systems of the type in (1.4.8), with diffusion functions g_1, g_2 satisfying $g_1 \geq g_2$, have the property

$$E^{X,g_1}\left(e^{-\lambda X_0(t)}\right) \geq E^{X,g_2}\left(e^{-\lambda X_0(t)}\right) \quad \forall \lambda > 0, \tag{4.1.39}$$

where $E^{X,g}$ is expection over X driven by g. Therefore, if the right-hand side converges to 1 as $t\to\infty$, then so does the left-hand side and, conversely, if the left-hand side remains bounded away from 1 as $t\to\infty$, then so does the right-hand side. Applying this to $g_1(x)=b_1x^2$ and $g_2(x)=b_2x^2$, we get the required monotonicity.

Our next observation is the following:

Lemma 4.1.7 $b_* \geq b_{**}$.

Proof. Take the expectation over W in the second line of (4.1.37), use Fubini and the fact that (4.1.36) is a martingale in W, to obtain

$$E^W(\widehat{X}_i^{\zeta,T}(T)) = E_i^{\xi}\left(e^{bT^{\xi,\zeta}(T)}\right). \tag{4.1.40}$$

Let $b<b_{**}$. Then, according to (1.6.20), the right-hand side is ζ-a.s. bounded in T. Consequently, $(\widehat{X}_i^{\zeta,T}(T))_{T\geq 0}$ is a tight family of random variables ζ-a.s. Thus, by (4.1.6), $(Q^{\zeta,T})_{T\geq 0}$ is tight ζ-a.s. Therefore, by Lemma 4.1.5, $b\leq b_*$. ∎

In Section 4.2 we will prove that $b_{**} > b_2$. Together with Lemma 4.1.7 this will imply that $b_* > b_2$, which is the claim in Theorem 1.6.3.

4.1.3 Proof of Theorem 1.5.3

Proof. We look at the corresponding parts of Theorem 1.5.2.
(a–b) Combine Proposition 2.4.1(a) with Proposition 4.1.6.
(c–d) These follow from Lemma 2.2.2 along the lines of Cox, Klenke and Perkins [11]. The key property is the self-duality expressed in Lemma 2.2.2. Indeed, self-duality translates convergence of the process into convergence of the process starting from a configuration with finite mass. For the latter, martingale convergence arguments can be applied to get (d). After that, (c) follows from (b) and (d).
(e) This follows from (d) and the representation formula in Lemma 2.2.1, which show that ν_Θ tends to $\delta_{\underline{\infty}}$ as $\Theta\to\infty$ because of the presence of the prefactor Θ in (2.2.1). ∎

4.2 Proof of Theorem 1.6.3

Proof. We derive a variational expression for b_{**} and prove that $b_{**} > b_2$.

Step 1: According to (1.6.19–1.6.20), we have

$$b_{**} = \sup\Big\{b > 0:\ E^{\xi'}\Big(\exp\Big[b\int_0^\infty 1_{\{\xi(t)=\xi'(t)\}}dt\Big]\Big) < \infty \ \ \xi - a.s.\Big\}. \tag{4.2.1}$$

Expand the expectation in powers of b to write

$$\begin{aligned} &E^{\xi'}\Big(\exp\Big[b\textstyle\int_0^\infty\ 1_{\{\xi(t)=\xi'(t)\}}dt\Big]\Big) \\ &= \sum_{N=0}^{\infty} b^N \textstyle\int_0^\infty ds_1 \dots \int_0^\infty ds_N\ 1_{\{s_1\le\dots\le s_N\}} \\ &\qquad \times P^{\xi'}(\xi(s_1)=\xi'(s_1),\dots,\xi(s_N)=\xi'(s_N)) \\ &= \sum_{N=0}^{\infty} b^N \textstyle\int_0^\infty ds_1 \dots \int_0^\infty ds_N\ 1_{\{s_1\le\dots\le s_N\}} \\ &\qquad \times p_{s_1}(0,\xi(s_1))p_{s_2-s_1}(\xi(s_1),\xi(s_2)) \times\dots\times p_{s_N-s_{N-1}}(\xi(s_{N-1}),\xi(s_N)) \end{aligned} \tag{4.2.2}$$

with $p_t(i,j) = P^{\xi'}(\xi'(t)=j \mid \xi'(0)=i)$, $i,j\in I$, the transition kernel of the random walk. We will show that $\frac{1}{N}$ times the logarithm of the N-fold integral in (4.2.2) converges ξ-a.s. to a constant C as $N\to\infty$. By (4.2.1), this will imply that $b_{**} = \exp[-C]$.

Step 2: Let $(\tau_n)_{n\in\mathbb{N}}$ be i.i.d. positive random variables with probability law

$$P(\tau_1\in dt) = \frac{p_{2t}(0,0)}{\frac{1}{2}G(0,0)}\ dt. \tag{4.2.3}$$

Then the N-fold integral in the last line of (4.2.2) can be rewritten as

$$\Big[\frac{1}{2}G(0,0)\Big]^N E^{\tau_1,\dots,\tau_N}\Big(\prod_{n=1}^N \frac{p_{\tau_n}(\Delta_n)}{p_{2\tau_n}(0)}\Big), \tag{4.2.4}$$

where $p_t(j-i) = p_t(i,j)$ and

$$\Delta_n = \xi\Big(\sum_{m=1}^n \tau_m\Big) - \xi\Big(\sum_{m=1}^{n-1}\tau_m\Big), \qquad n\in\mathbb{N}, \tag{4.2.5}$$

with $\tau_0 = 0$. The expectation in (4.2.4) seems well suited for a large deviation analysis, but the problem is that the Δ_n are correlated because ξ is fixed. In order to circumvent this problem, we introduce i.i.d. copies of ξ (independent of $\{\tau_n\}_{n\in\mathbb{N}}$):

$$\{\widetilde{\xi}_n\}_{n\in\mathbb{N}} \qquad \text{with} \qquad \widetilde{\xi}_n = (\widetilde{\xi}_n(t))_{t\ge0}, \tag{4.2.6}$$

and define

$$\widetilde{\Delta}_n = \widetilde{\xi}_n(\tau_n), \qquad n\in\mathbb{N}. \tag{4.2.7}$$

Since ξ has i.i.d. increments over disjoint time intervals, the probability law of $(\widetilde{\Delta}_n)_{n\in\mathbb{N}}$ as a function of $(\widetilde{\xi}_n)_{n\in\mathbb{N}}$ is the same as the probability law of $(\Delta_n)_{n\in\mathbb{N}}$ as a function of ξ. Therefore, if we consider

$$E^{\tau_1,\dots,\tau_N}\Big(\prod_{n=1}^N \frac{p_{\tau_n}(\widetilde{\Delta}_n)}{p_{2\tau_n}(0)}\Big) \tag{4.2.8}$$

and manage to prove that this expectation has a $(\widetilde{\xi}_n)_{n\in\mathbb{N}}$-a.s. constant growth rate in N, then this growth rate is also the ξ-a.s. growth rate of the expectation in (4.2.4). Indeed, the two expectations in (4.2.4) and (4.2.8), viewed as sequences labelled by N, have the same probability law and so the a.s. convergence carries over.

Step 3: Let us introduce the empirical process

$$R_N^{\{\widetilde{\xi}_n\}_{n\in\mathbb{N}}} = \frac{1}{N}\sum_{n=1}^{N} \delta_{(\tau_m,\widetilde{\xi}_m)_{n\le m<N+n}}, \qquad N\in\mathbb{N}, \tag{4.2.9}$$

with periodic boundary conditions on $1,\dots,N$. This is a random element of $\mathcal{P}([(0,\infty)\times I^{[0,\infty)}]^{\mathbb{N}})$, the set of shift-invariant probability measures on $[(0,\infty)\times I^{[0,\infty)}]^{\mathbb{N}}$. In terms of this quantity the expectation in (4.2.8) can be written as

$$E^{\tau_1,\dots,\tau_N}\Big(\exp\Big[N\int\int(\pi_1 R_N^{\{\widetilde{\xi}_n\}_{n\in\mathbb{N}}})(d\tau,d\widetilde{\xi})\log\frac{p_\tau(\widetilde{\xi}(\tau))}{p_{2\tau}(0)}\Big]\Big), \tag{4.2.10}$$

where $\pi_1 Q$ is the projection of Q onto $(0,\infty)\times I^{[0,\infty)}$.

According to Deuschel and Stroock [20] Theorem 5.4.27, $(R_N^{\{\widetilde{\xi}_n\}_{n\in\mathbb{N}}})_{N\in\mathbb{N}}$ satisfies the large deviation principle on $\mathcal{P}([(0,\infty)\times I^{[0,\infty)}]^{\mathbb{N}})$ with rate function $h(\cdot\mid(\mu\times\nu)^{\mathbb{N}})$, i.e., the specific relative entropy with respect to $(\mu\times\nu)^{\mathbb{N}}$, where

$$\begin{aligned}&\mu \text{ is the law of } \tau_1 \text{ given in (4.2.3)},\\ &\nu \text{ is the law of } \widetilde{\xi}_1.\end{aligned} \tag{4.2.11}$$

Therefore, applying Varadhan's lemma in combination with the ergodic theorem for $\widetilde{\xi}$, we obtain

$$\lim_{N\to\infty}\frac{1}{N}\log(4.2.10) = S \tag{4.2.12}$$

with

$$S = \sup_{\substack{Q\in\mathcal{P}([(0,\infty)\times I^{[0,\infty)}]^{\mathbb{N}})\\ \pi^{\downarrow}Q=\nu^{\mathbb{N}}}}\Big[\int\int(\pi_1 Q)(d\tau,d\widetilde{\xi})\log\frac{p_\tau(\widetilde{\xi}(\tau))}{p_{2\tau}(0)} - h(Q\mid(\mu\times\nu)^{\mathbb{N}})\Big], \tag{4.2.13}$$

where $\pi^{\downarrow}Q$ is the projection of Q onto $[I^{[0,\infty)}]^{\mathbb{N}}$. [1] Note that Varadhan's lemma applies because $(t,x)\mapsto p_t(x)/p_{2t}(0)$ is continuous and bounded from above (see Deuschel and Stroock [20] Theorem 2.1.10).

Since the integral in (4.2.13) depends on $\pi_1 Q$ only, the variational problem reduces to $Q=q^{\mathbb{N}}$:

$$S = \sup_{\substack{q\in\mathcal{P}((0,\infty)\times I^{[0,\infty)})\\ \pi^{\downarrow}q=\nu}}\Big[\int\int q(d\tau,d\widetilde{\xi})\log\frac{p_\tau(\widetilde{\xi}(\tau))}{p_{2\tau}(0)} - h(q\mid\mu\times\nu)\Big], \tag{4.2.14}$$

[1] For details on how the restriction $\pi^{\downarrow}Q=\nu^{\mathbb{N}}$ arises from the ergodic theorem, see e.g. Comets [8], Greven and den Hollander [28], Seppäläinen [39]. These papers derive large deviation principles for random processes in quenched random media, showing that an average may be taken over the random medium, but subject to the restriction that all cylinders have the correct frequency as dictated by the ergodic theorem.

where $\pi^{\downarrow}q$ is the projection of q onto $I^{[0,\infty)}$. Collecting (4.2.1–4.2.2), (4.2.4), (4.2.8), (4.2.10–4.2.12) and (4.2.14), we arrive at

$$b_{**} = \exp[-\log\{\tfrac{1}{2}G(0,0)\} - S] = b_2 e^{-S} \tag{4.2.15}$$

(recall (1.5.6)). This completes the derivation of a variational expression for b_{**}.

Step 4: To prove that $b_{**} > b_2$, we must show that $S < 0$. For any q with $\pi^{\downarrow}q = \nu$, we have

$$h(q \mid \mu \times \nu) = \int \nu(d\widetilde{\xi})\ h(q_{\widetilde{\xi}} \mid \mu) \tag{4.2.16}$$

with $q_{\widetilde{\xi}}$ the probability law on $(0,\infty)$ obtained from q by conditioning on $\widetilde{\xi}$. Writing $\mu(d\tau) = \mu(\tau)d\tau$ and $q_{\widetilde{\xi}}(d\tau) = q_{\widetilde{\xi}}(\tau)d\tau$, we therefore have

$$\begin{aligned}
&\int\int q(d\tau, d\widetilde{\xi}) \log \tfrac{p_\tau(\widetilde{\xi}(\tau))}{p_{2\tau}(0)} - h(q \mid \mu \times \nu)\\
&\quad = \int \nu(d\widetilde{\xi}) \int d\tau\ q_{\widetilde{\xi}}(\tau) \log \left[\tfrac{p_\tau(\widetilde{\xi}(\tau))}{p_{2\tau}(0)}\, \tfrac{\mu(\tau)}{q_{\widetilde{\xi}}(\tau)}\right]\\
&\quad = \int \nu(d\widetilde{\xi}) \int d\tau\ q_{\widetilde{\xi}}(\tau) \log \left[\tfrac{p_\tau(\widetilde{\xi}(\tau))}{\frac{1}{2}G(0,0)}\, \tfrac{1}{q_{\widetilde{\xi}}(\tau)}\right]\\
&\quad \leq \log \int \nu(d\widetilde{\xi}) \int d\tau\ \tfrac{p_\tau(\widetilde{\xi}(\tau))}{\frac{1}{2}G(0,0)}\\
&\quad = \log \int_0^\infty dt \sum_{x\in I} \tfrac{(p_t(x))^2}{\frac{1}{2}G(0,0)}\\
&\quad = \log \int_0^\infty dt\ \tfrac{p_{2t}(0)}{\frac{1}{2}G(0,0)}\\
&\quad = \log 1 = 0,
\end{aligned} \tag{4.2.17}$$

where we use Jensen's inequality and (4.2.11). Hence, $S \leq 0$. However, the inequality in (4.2.17) is strict because it is not possible that

$$q_{\widetilde{\xi}}(\tau) = C p_\tau(\widetilde{\xi}(\tau)) \qquad \nu(d\widetilde{\xi}) q_{\widetilde{\xi}}(\tau) d\tau - a.s. \tag{4.2.18}$$

Indeed, $\int q_{\widetilde{\xi}}(\tau)d\tau = 1$ ν-a.s. while $\int p_\tau(\widetilde{\xi}(\tau))d\tau$ is not constant ν-a.s. Consequently, $S < 0$, which completes the proof of Theorem 1.6.3. ∎

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