

Comparing risk measures

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Abstract

In this paper we compare two sets of risk measures with respect to the criteria of first and second order stochastic dominance. We observe that overall risk measures do not preserve consistent preference ordering between assets under the first order stochastic dominance rule, while the downside risk measures, with the exception of Expected Shortfall (ES), do preserve a consistent preference ordering under first order stochastic dominance. Further, risk measures except ES preserve consistent preference ordering between assets under the second order stochastic dominance rule, although for some of the downside risk measures such preference ordering is only partial.

KEY WORDS: stochastic dominance, risk measures, preference ordering

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1 Introduction

The notion of stochastic dominance is used for comparing choices under uncertainty. In classical utility theory, the notion links up to the concept of maximisation of continuous, non-decreasing and concave utility functions that describe rational and risk averse investors. In particular, the first order stochastic dominance of a risky asset X over a risky asset Y implies that a utility maximising investor having a continuous increasing utility function would always prefer X to Y or would remain indifferent between the two. Similarly, if X dominates over Y in the sense of second order stochastic dominance then all risk averse investors with a concave utility function would always prefer X to Y .¹

In this paper we compare several risk measures to examine whether assets preserve the same preference ordering under a specific risk measure as under the rules of first and second order stochastic dominance. Such comparison allows us to examine the extent to which choices made using the risk measures are consistent with the classical utility theory based choices.

Using several risk measures that are commonly found in the literature on financial risk, we are able to complement the existing literature with some interesting results. We observe that overall risk measures do not preserve the same preference ordering of assets as under the first order stochastic dominance (FOSD) rule, even under the simplifying assumption of normally distributed assets. Regarding downside risk measures, Fishburn (1977) has shown for lower partial moments and Koplanski and Kroll (2000) have shown for Value-at-Risk (VaR) that these measures retain a consistent preference ordering of assets under the FOSD rule. This implies that an investment choice based on lower partial moments and VaR is consistent with the choice of a utility maximising rational investor. In this paper we complement these results by further showing that choices based on Expected Shortfall (ES) is not consistent with the utility theory framework, since ES does not display any clear preference ordering under the first order stochastic dominance rule.

We also investigate risk measures for second order stochastic dominance and show that below the first crossing quantile of the assets (if such a crossing quantile exists), preference ordering with respect to VaR is consistent with the preference ordering under the second order stochastic dominance rule. This implies that in the tail regions of the assets, VaR-based selection of investment is consistent with the utility theory based selection of a risk averse investor under second order stochastic dominance criterion. This is interesting because it is the tail region which is most relevant from the perspective of risk management. In the case of normally distributed assets, VaR-based ordering is valid for a much larger area of the distribution, i.e., for all quantiles q such that $F(q) < 0.50$.

Like in the case of first order stochastic dominance, expected shortfall (ES) does not display any clear preference ordering for second order stochastic dominance rule as well. Thus, VaR is consistent with the framework of utility maximisation while ES is not.

The remainder of this paper is as follows: we present a brief outline of the concepts of first and second order stochastic dominance in Section 2. Section 3 describes the risk measures considered in this paper. In Section 4 we consider ordering of two assets with respect to

¹An extensive survey on stochastic dominance and utility theory is given in Levy (1992). A bibliographic list of the literature on stochastic dominance is given by Bawa (1982).

various risk measures when one asset is stochastically dominant over the other. Section 6 concludes the paper.

2 Stochastic dominance

Suppose that X and Y are two risky assets with distribution functions F_x and F_y respectively. Further, let μ_x and μ_y be the expected values and σ_x and σ_y the standard deviations of X and Y respectively.

2.1 First Order Stochastic Dominance (FOSD)

Asset X First Order Stochastically Dominates asset Y , denoted by X FOSD Y if $F_x(z) \leq F_y(z) \quad \forall z$, where $z \in (-\infty, \infty)$ with inequality for at least one z .

This implies that the probability of asset return X falling below a specified level z is smaller than that of asset return Y falling below the same level. Therefore all investors having continuous non-decreasing utility functions would either prefer X to Y or are indifferent between the two (Huang and Litzenberger, 1988; Ingersoll, 1987).

First order stochastic dominance of X on Y gives rise to the following equivalent statements (Huang and Litzenberger, 1988; Ingersoll, 1987).

1. X FOSD Y
2. $F_x(z) \leq F_y(z) \quad \forall z$
3. $X \stackrel{d}{=} Y + \epsilon$, $\epsilon \geq 0$, where $\stackrel{d}{=}$ denotes distributional equivalence.

In above, 3 implies that $\mu_x \geq \mu_y$, but the converse is not true.

2.2 Second order stochastic dominance (SOSD)

Asset X Second Order Stochastically Dominates asset Y , denoted by X SOSD Y , if

$$\mu_x = \mu_y \tag{1}$$

$$\int_0^z F_x(x) dx \leq \int_0^z F_y(x) dx \quad \forall z \tag{2}$$

It can be proved that if the above conditions are satisfied then all risk averse investors having utility functions whose first derivatives are continuous will prefer X to Y and vice versa (Huang and Litzenberger, 1988; Ingersoll, 1987).

The following equivalent statements arise from the definition of second order stochastic dominance of X on Y (Huang and Litzenberger, 1988; Ingersoll, 1987).

1. X SOSD Y

2. $\mu_x = \mu_y$ and $\int_0^z F_x(x)dx \leq \int_0^z F_y(x)dx \quad \forall z$
3. $X \stackrel{d}{=} Y + \nu$, where $E[\nu|X] = 0$

A direct implication of 3 above is $\sigma_x^2 \leq \sigma_y^2$. Thus if X SOSD Y then $\mu_x = \mu_y$, and $\sigma_x^2 \leq \sigma_y^2$; however the converse is not true.

Having briefly defined the concepts of first and second order stochastic dominance we now discuss the various risk measures in the following section.

3 Risk measures

We consider two sets of risk measures, viz., overall risk measures and downside risk measures. We take the definitions of these from Dhaene et al. (2003).

- An *overall risk measure* is a measure of the “distance” between the risky situation and the corresponding risk-free situation when both favourable and unfavourable discrepancies are taken into account.
- A *downside risk measure* is a measure of “distance” between the risky situation and the corresponding risk-free situation when only unfavourable discrepancies contribute to the “risk”.

The various risk measures considered under each category are as follows.

3.1 Overall risk measures

1. Variance σ^2 , given by

$$\sigma^2 = \int_{-\infty}^{\infty} (x - \mu)^2 f(x) dx$$

where $f(x)$ is the pdf and μ the mean of X . Variance, or standard deviation is the most classical measure of risk (Markowitz, 1952). In the seminal work of Markowitz (1952) standard deviation was used as a measure of risk of the portfolio. The use of standard deviation was a major academic breakthrough as it allowed the “diversification” theory of modern portfolio theory to be presented in a mathematical framework.

2. Market risk β , given by

$$\beta = \rho_{X,R} \frac{\sigma_R}{\sigma_X}$$

where $\rho_{X,R}$ is the correlation coefficient between X and R , the market portfolio. Beta measures the market risk, or systematic risk, of a security or portfolio in comparison to the market as a whole.

3. Interquartile range (IQR), given by

$$IQR = F^{-1}(.75) - F^{-1}(.25)$$

where $F(\cdot)$ is the cdf of X . This measure is sometimes used as a measure of overall risk when the second moment is not bounded. For example, for symmetric α -stable distributions with $1 < \alpha < 2$, standard deviation does not exist, but it can be approximated by IQR (Fama and Roll, 1968).

3.2 Downside risk measures

Risk measures considered under this category are lower partial moments of second, first and zeroth orders, Value-at-Risk and expected shortfall. Lower partial moment of order n is computed at some fixed quantile q , and defined as the n^{th} moment below q . Developed by Bawa (1975) and studied rigorously by Fishburn (1977), lower partial moments measure risk by a probability weighted mean of the deviations below a specified target level q . The higher the n , the higher is the risk aversion. Risk neutrality is reflected by $n = 1$ and risk seeking behaviour by $n = 0$.

4. Second Lower Partial Moment or Semi-variance (SLPM)

$$SLPM = \int_{-\infty}^q (q - x)^2 f(x) dx = 2 \int_{-\infty}^q (q - x) F(x) dx$$

The equivalence between the two expressions in the definition of SLPM is shown in Appendix A.

5. First Lower Partial Moment (FLPM)

$$FLPM = \int_{-\infty}^q (q - x) f(x) dx = \int_{-\infty}^q F(x) dx$$

The equivalence between the two expressions in the definition of FLPM is shown in Appendix A.

6. Zeroth Lower Partial Moment (ZLPM)

$$ZLPM = \int_{-\infty}^q f(x) dx = F(q)$$

7. Value-at-Risk (VaR): If $F(q)$ is fixed at p , then the inverse of ZLPM gives Value-at-Risk (VaR) as

$$VaR_p = -F^{-1}(p) = -q$$

VaR is defined as the maximum potential loss to an investment with a pre-specified confidence level $(1 - p)$.

8. Expected Shortfall (ES): When the return distribution is continuous, ES at confidence

level $(1 - p)$ is defined as

$$\begin{aligned} \text{ES}_p &= E(x|x \leq VaR_p) \\ &= \int_{-\infty}^q x \frac{f(x)}{F(q)} dx \\ &= q - \frac{1}{F(q)} \text{FLPM} \end{aligned}$$

The equivalence between the two expressions in the definition of ES is shown in Appendix A.

4 Comparing risk measures when X dominates Y in the sense of first order stochastic dominance

Asset X is said to dominate asset Y under a specific risk measure ρ if $\rho_x \leq \rho_y$ where ρ_x and ρ_y are the values of ρ for asset returns X and Y respectively. If X dominates over Y under a specific risk measure ρ , then X is said to be preferred to Y .

In this section we compare if the preference ordering under the specific risk measure is same as that under the first order stochastic dominance rule which lead to the preference of X over Y if X FOSD Y .

For the overall risk measures considered in this paper, it is easy to observe that for unknown F_x and F_y , the first order stochastic dominance of X on Y does not lead to an unambiguous ordering between assets with respect to any of the overall risk measures. However, as shown in Proposition 1 below, the special assumption of normality allows us to establish ambiguous relationships between assets with respect to the overall risk measures.

Proposition 1 *If X FOSD Y , $X \sim N(\mu_x, \sigma_x^2)$ and $Y \sim N(\mu_y, \sigma_y^2)$, then following relationships hold.*

$$\sigma_x = \sigma_y \tag{3}$$

$$\beta_x \geq \beta_y \text{ if } \rho_{x,R} \geq \rho_{y,R} \tag{4}$$

$$\beta_x \leq \beta_y \text{ if } \rho_{x,R} \leq \rho_{y,R} \tag{4}$$

$$IQR_x = IQR_y \tag{5}$$

Proof:

See Appendix B.1

Proposition (1) implies that for the overall risk measures, assets do not preserve the preference order consistent with the preference ordering under the first order stochastic dominance rule, even under the simplifying assumption of normally distributed asset returns. With respect to σ and IQR , investors would be indifferent between the two assets since the risk measures are equal. The preference ordering with respect to market risk β is consistent with preference ordering with respect to first order stochastic dominance only if the correlation coefficient

between X and the market portfolio is less than that between Y and the market portfolio, otherwise it is inconsistent.

As far as the downside risk measures are concerned, some results already exist in the literature. From Fishburn (1977) and Kaplanski and Kroll (2000), we know that regardless of the distribution of X and Y , assets can be unambiguously ordered with respect to SLPM, FLPM, ZLPM and VaR. Further the ordering is consistent with the FOSD rule. We complement these results by establishing a new result that ES does not lead to any unambiguous ordering of assets. We enlist all these results in Proposition 2.

Proposition 2 *If X FOSD Y , then regardless of the distribution of X and Y following relationships hold.*

$$SLPM_x \leq SLPM_y \quad (6)$$

$$FLPM_x \leq FLPM_y \quad (7)$$

$$ZLPM_x \leq ZLPM_y \quad (8)$$

$$VaR_x \leq VaR_y \quad (9)$$

$$ES_x \begin{matrix} \geq \\ \leq \end{matrix} ES_y \quad (10)$$

Proof:

Relationships (6), (7) and (8) follow from Fishburn (1977). Relationship (9) follows from Kaplanski and Kroll (2000). For derivation of (10) see Appendix B.2.

Proposition (2) indicates that the preference ordering of assets with respect to the downside risk measures is consistent with that under FOSD, except for expected shortfall. Thus, the choices made using the lower partial moments of second, first and zeroth order and VaR are consistent with the choice made under the utility theory framework. However expected shortfall does not lead to an unambiguous choice and therefore expected shortfall is not consistent with the classical utility based approach of asset selection.

5 Comparing risk measures when X dominates Y in the sense of second order stochastic dominance

If asset X dominates asset Y in the sense of second order stochastic dominance, then for unknown F_x and F_y , the risk measure IQR does not provide any unambiguous ordering of assets. As far as the other measures are concerned, ordering of assets by these measures is possible under the SOSD rule; however such an ordering may be only partial in case of some measures. We present these observations in Proposition 3 below.

Proposition 3 *Suppose that X SOSD Y and \exists a first crossing quantile at \bar{q} . Then regardless*

of the distribution of X and Y , the following relationships hold.²

$$\sigma_x \leq \sigma_y \tag{11}$$

$$\beta_x \geq \beta_y \text{ if } \rho_{x,R} \geq \rho_{y,R}$$

$$\beta_x \leq \beta_y \text{ if } \rho_{x,R} \leq \rho_{y,R} \tag{12}$$

$$\text{SLPM}_x \leq \text{SLPM}_y \tag{13}$$

$$\text{FLPM}_x \leq \text{FLPM}_y \tag{14}$$

$$\text{ZLPM}_x \leq \text{ZLPM}_y, \text{ below } \bar{q} \tag{15}$$

$$\text{VaR}_x \leq \text{VaR}_y, \text{ below } \bar{q} \tag{16}$$

$$\text{ES}_x \geq \text{ES}_y \text{ below } \bar{q} \tag{17}$$

Proof:

Relationship (11) is an implication of the definition of SOSD. Relationship (12) follows from (11). Fishburn (1977) has established relationships (13) and (14). Inequality (15) and (16) follow from the fact that below the first crossing quantile $F_x(q) \leq F_y(q)$, and hence X FOSD Y below the first crossing quantile. The proof of (17) is more involved and detailed in Appendix C.

Thus, under the second order stochastic dominance of X on Y , we can order assets under overall risk measures σ and β and downside risk measures SLPM and FLPM in an unambiguous manner without any explicit distributional assumption. In this case, σ , SLPM and FLPM clearly preserve the preference ordering that is given by second order stochastic dominance rule. In case of the market risk β , the preference ordering is consistent only if $\rho_{x,R} \leq \rho_{y,R}$, otherwise it is inconsistent with the SOSD rule.

As far as the other measures are concerned, under unknown asset returns distributions, the downside risk measures ZLPM, VaR and ES retains only partial ordering of assets, that is, below the first crossing quantile of the two distributions. Below the first crossing quantile, the preference ordering under ZLPM and VaR is consistent with the SOSD rule, but ES displays an ordering that is inconsistent with the SOSD rule.

If asset returns are normally distributed, then we can show that IQR and ZLPM preserve the same preference ordering as under the SOSD rule for the entire range of the distributions. Under normality assumption, VaR preserves the consistent ordering for the region defined by the quantiles q such that $\Pr\{X \leq q\} < 0.50$. For ES, normality assumption does not lead to any clear ordering. This is stated in Proposition 4 below.

Proposition 4 *If X SOSD Y and X and Y are both normally distributed, then following relationships hold.*

²We thank Simon Polbennikov for pointing out that there might be infinitely many crossing quantiles in case of two infinitely inter winning CDFs whereby one SOSD the other infinitely many times.

$$IQR_x \leq IQR_y \tag{18}$$

$$ZLPM_x \leq ZLPM_y \tag{19}$$

$$VaR_x \leq VaR_y \text{ for } p < 0.5 \tag{20}$$

$$ES_x \geq ES_y \text{ for } p = 0.5 \text{ For } p < 0.5, ES_x \begin{matrix} \geq \\ \leq \end{matrix} ES_y \tag{21}$$

where p denotes probability level.

Proof:

See Appendix D.

Thus, imposition of the simplifying assumption of normal distribution allows us to use IQR and ZLPM to provide complete preference ordering between assets in a consistent manner as the SOSD rule. Further, imposition of normality assumption implies that preference ordering with respect to VaR is consistent for a large area, for $p < 0.5$, and that ES does not preserve preference ordering consistent with SOSD rule.

6 Conclusion

In this paper we compare a set of commonly used risk measures with respect to the criteria of stochastic dominance of first and second orders. We show that the overall risk measures do not always display a consistent preference ordering under the FOSD condition, even after imposing the simplifying assumption of normally distributed asset returns. However, regardless of the asset return distributions the preference ordering displayed by all the downside risk measures except for ES is consistent as under the first order stochastic dominance rule. Thus, all the downside risk measures except ES provide efficient selection of risky options while ES cannot order the options.

We observe that regardless of the asset return distributions, all risk measures except for IQR and ES display consistent preference ordering of assets under the SOSD rule. In this case, standard deviation, β (depending on the correlation coefficient of the assets with the market portfolio), SLPM and FLPM show clear preference ordering for the entire distribution while the ordering under ZLPM and VaR are only partial, i.e., below the first crossing quantile. Under the simplifying assumption of normally distributed assets, this preference ordering under ZLPM is valid for the entire distribution and the one under VaR is valid for a much bigger area of the distribution, defined by all quantiles q such that $F(q) < 0.5$. Further, under the assumption of normality, IQR preserves a consistent preference ordering for the entire distribution. The downside risk measure ES does not display any clear preference ordering, even under the simplifying assumption of normal distribution.

The most important highlight of these results is that ES may not be a suitable measure to choose between two risky options while other downside risk measures, including VaR provide suitable investment choice, consistent with the classical utility theory framework.

References

- Bawa VS, 1975. Optimal Rules for Ordering Uncertain Prospects. *Journal of Financial Economics* **2**(1):95–121.
- Bawa VS, 1982. Stochastic Dominance: A Research Bibliography. *Management Science* **28**:698–712.
- Dhaene J, Goovaerts MJ, Kaas R, 2003. Economic Capital Allocation Derived from Risk Measures. *North American Actuarial Journal* **7**(2):44–56.
- Fama EF, Roll R, 1968 September. Some Properties of Symmetric Stable Distributions. *Journal of the American Statistical Association* **63**:817–836.
- Fishburn PC, 1977. Mean-Risk Analysis with Risk Associated with Below Target Returns. *American Economic Review* **67**(2):116–126.
- Huang CF, Litzenberger RH, 1988. *Foundations for Financial Economics*. Elsevier Science Publishing Co.
- Ingersoll JE, 1987. *Theory of Financial Decision Making*. Rowman & Littlefield Publishers.
- Kaplanski G, Kroll Y, 2000 Spring. VaR Risk Measures versus Traditional Risk Measures: an Analysis and Survey. *Journal of Risk* **4**(3).
- Levy H, 1992. Stochastic Dominance and Expected Utility: Survey and Analysis. *Management Science* **38**(4):555–593.
- Markowitz H, 1952. Portfolio Selection. *The Journal of Finance* **7**:77–91.

Appendix A: Equivalence between different expressions in the definitions of various downside risk measures

A.1 SLPM

$$\begin{aligned}
 \text{SLPM}(q) &= \int_{-\infty}^q (q-x)^2 f(x) dx \\
 &= q^2 F(q) - 2q \left\{ qF(q) - \int_{-\infty}^q F(x) dx \right\} + \left\{ |x^2 f(x)|_{-\infty}^q - 2 \int_{-\infty}^q xF(x) dx \right\} \\
 &= 2q \int_{-\infty}^q F(x) dx - 2 \int_{-\infty}^q xF(x) dx \\
 &= 2 \int_{-\infty}^q (q-x)F(x) dx
 \end{aligned}$$

A.2 FLPM

$$\begin{aligned}
 \text{FLPM}(q) &= \int_{-\infty}^q (q-x)f(x) dx \\
 &= q \int_{-\infty}^q f(x) dx - \left\{ |xF(x)|_{-\infty}^q - \int_{-\infty}^q F(x) dx \right\} \\
 &= \int_{-\infty}^q F(x) dx
 \end{aligned}$$

A.3 ES

$$\begin{aligned}
 \text{ES}(q) &= \int_{-\infty}^q x \frac{f(x)}{F(q)} dx \\
 &= \left| \frac{xF(x)}{F(q)} \right|_{-\infty}^q - \int_{-\infty}^q \frac{F(x)}{F(q)} dx \\
 &= q - \frac{1}{F(q)} \int_{-\infty}^q F(x) dx \\
 &= q - \frac{\text{FLPM}(q)}{F(q)}
 \end{aligned}$$

Appendix B: Derivation of the relationships in Proposition 1 and Proposition 2

B.1 Overall risk measures σ , β and IQR

$X \sim N(\mu_x, \sigma_x^2)$, $Y \sim N(\mu_y, \sigma_y^2)$. Due to the first order stochastic dominance of X on Y,

$$\begin{aligned}
 F_x(z) &\leq F_y(z) \quad \forall z, \quad \text{with inequality for at least one } z \\
 \int_{-\infty}^z \frac{1}{\sigma_x \sqrt{2\pi}} \exp^{-\frac{1}{2}\left(\frac{x-\mu_x}{\sigma_x}\right)^2} dx &\leq \int_{-\infty}^z \frac{1}{\sigma_y \sqrt{2\pi}} \exp^{-\frac{1}{2}\left(\frac{y-\mu_y}{\sigma_y}\right)^2} dy \\
 \Phi\left(\frac{z-\mu_x}{\sigma_x}\right) &\leq \Phi\left(\frac{z-\mu_y}{\sigma_y}\right) \quad \Phi(x) \text{ being the cdf of } N(O, 1). \\
 \left(\frac{z-\mu_x}{\sigma_x}\right) &\leq \left(\frac{z-\mu_y}{\sigma_y}\right) \quad \text{by the property of standard normal curve} \\
 (\sigma_y - \sigma_x)z + (\sigma_y\mu_x - \sigma_x\mu_y) &\geq 0, \quad \forall z \in (-\infty, \infty)
 \end{aligned}$$

Since this positivity constraint has to be satisfied for all $z \in (-\infty, \infty)$, therefore the only possibility is

$$\begin{aligned}
 \sigma_x - \sigma_y &= 0 \\
 \sigma_x &= \sigma_y
 \end{aligned}$$

From this it follows

$$\begin{aligned}
 \beta_x &\geq \beta_y \quad \text{if } \rho_{x,R} \geq \rho_{y,R} \\
 \beta_x &\leq \beta_y \quad \text{if } \rho_{x,R} \leq \rho_{y,R} \\
 IQR_x &= IQR_y
 \end{aligned}$$

B.2 Proof of relationship (10) in Proposition 2

$$\begin{aligned}
 ES_x(q) &= q - \frac{FLPM_x(q)}{F_x(q)} \\
 ES_y(q) &= q - \frac{FLPM_y(q)}{F_y(q)} \\
 ES_x(q) - ES_y(q) &= \frac{FLPM_y(q)}{F_y(q)} - \frac{FLPM_x(q)}{F_x(q)} \\
 &\stackrel{\geq}{\leq} 0 \text{ as } FLPM_x(q) \leq FLPM_y(q) \text{ and } F_x(q) \leq F_y(q)
 \end{aligned}$$

Appendix C. Proof of (17) in Proposition 3

Suppose that X SOSD Y and \exists a first crossing quantile \bar{q} . In that case it holds that

$$F_x(q) \leq F_y(q) \quad \forall q < \bar{q}$$

At \bar{q} ,

$$F_x(\bar{q}) = F_y(\bar{q})$$

Therefore

$$ES_x(\bar{q}) = \bar{q} - \frac{1}{F_x(\bar{q})} \text{FLPM}_x(\bar{q})$$

And

$$ES_y(\bar{q}) = \bar{q} - \frac{1}{F_y(\bar{q})} \text{FLPM}_y(\bar{q})$$

Taking the difference

$$\begin{aligned} ES_x(\bar{q}) - ES_y(\bar{q}) &= -\frac{1}{F_x(\bar{q})} [\text{FLPM}_x(\bar{q}) - \text{FLPM}_y(\bar{q})] \\ &\geq 0 \text{ since } \text{FLPM}_x(\bar{q}) \leq \text{FLPM}_y(\bar{q}) \text{ due to } X \text{ SOSD } Y. \end{aligned}$$

Fixing probability p such that $p < F_x(\bar{q})$, suppose that

$$\begin{aligned} p &= F_x(q_1) = F_y(q_2) \\ q_1 &\geq q_2 \\ \text{FLPM}_x(q_1) &\leq \text{FLPM}_y(q_1) \\ &\leq \text{FLPM}_y(q_2), \text{ since } X \text{ SOSD } Y \\ ES_x(p) - ES_y(p) &= (q_1 - q_2) - \frac{1}{p} [\text{FLPM}_x(q_1) - \text{FLPM}_y(q_2)] \\ &\geq 0 \end{aligned}$$

Appendix D: Proof of Proposition 4

Suppose that X SOSD Y and X and Y are normally distributed with expected values μ_x and μ_y and standard deviations σ_x and σ_y respectively.

D.1 IQR

$$\begin{aligned} IQR_x &= F_x^{-1}(.75) - F_x^{-1}(.25) \\ IQR_y &= F_y^{-1}(.75) - F_y^{-1}(.25) \end{aligned}$$

Due to normality of X and Y ,

$$\begin{aligned}
F_x^{-1}(.75) &= \mu_x + \sigma_x \Phi^{-1}(.75) \\
F_x^{-1}(.25) &= \mu_x + \sigma_x \Phi^{-1}(.25) \\
F_y^{-1}(.75) &= \mu_y + \sigma_y \Phi^{-1}(.75) \\
F_x^{-1}(.25) &= \mu_y + \sigma_y \Phi^{-1}(.25) \\
IQR_x - IQR_y &= (\sigma_x - \sigma_y) (\Phi^{-1}(.75) - \Phi^{-1}(.25)) \\
&\leq 0, \text{ since } \sigma_x \leq \sigma_y \text{ due to } X \text{ SOSD } Y \\
IQR_x &\leq IQR_y
\end{aligned}$$

D.2 ZLPM, VaR and ES

Two normal cdfs with equal means but unequal standard deviations can have only one crossing point. This crossing point can be precisely determined, and given by $x = \mu$, where $\mu = \mu_x = \mu_y$ the equal mean of X and Y , such that

$$F_x(\mu) = F_y(\mu) = 0.50$$

This implies that upto the point μ , the cdf of the normal distribution with lower standard deviation (in this case F_x) lies below the cdf of the normal distribution with higher standard deviation (in this case F_y). Above the point μ F_x is above F_y . Thus below the probability level $p = 0.50$, we have the following:

$$\begin{aligned}
F_x(q) &\leq F_y(q) \\
ZLPM_x(q) &\leq ZLPM_y(q)
\end{aligned}$$

Further,

$$\begin{aligned}
F_x^{-1}(p) &\geq F_y^{-1}(p) \text{ for } p \leq 0.50 \\
-F_x^{-1}(p) &\leq -F_y^{-1}(p) \\
VaR_x(p) &\leq VaR_y(p)
\end{aligned}$$

At the crossing quantile μ ,

$$\begin{aligned}
ES_x(\mu) &= \mu - \frac{FLPM_x(\mu)}{F_x(\mu)} \\
ES_y(\mu) &= \mu - \frac{FLPM_y(\mu)}{F_y(\mu)} \\
ES_x(\mu) - ES_y(\mu) &= \frac{1}{0.5} (-FLPM_x(\mu) + FLPM_y(\mu)) \\
&\geq 0 \text{ since } FLPM_x(\mu) \leq FLPM_y(\mu)
\end{aligned}$$

Thus,

$$ES_x \geq ES_y$$

Below and above the crossing quantile, ES does not observe any unambiguous ordering.