A non-increasing Lindley-type equation

Maria Vlasiou[†]

April 27, 2005

Abstract: In this paper we study the Lindley-type equation $W = \max\{0, B - A - W\}$. Its main characteristic is that it is a non-increasing monotone function in its main argument W. Our main goal is to derive a closed-form expression of the steady-state distribution of W. In general this is not possible, so we shall state a sufficient condition that allows us to do so. We also examine stability issues, derive the tail behaviour of W, and briefly discuss how one can iteratively solve this equation by using a contraction mapping.

[†]EURANDOM, P.O. Box 513, 5600 MB Eindhoven, The Netherlands.

1 Introduction

Lindley's recursion [10] is $W_{n+1} = \max\{0, B_n - A_n + W_n\}$ and it describes the waiting time W_{n+1} of a customer in a single-server queue in terms of the waiting time of the previous customer, his or her service time B_n, and the interarrival time A_n between them. It is one of the fundamental and most well-studied equations in queuing theory. For a detailed study of Lindley's equation we refer to Asmussen [1], Cohen [6], and the references therein.

In this paper we study a Lindley-type equation that differs from the original Lindley equation only in the change of a plus sign into a minus sign. More precisely, we are interested in the recursion

$$W_{n+1} = \max\{0, B_{n+1} - A_n - W_n\} \tag{1.1}$$

and the main goal is to derive a closed-form expression for the steady-state limiting distribution of W. The implications of this minor difference are rather far reaching, since, in the particular case we are studying in this paper, Lindley's equation has a simple solution, while for our equation it is probably not possible to derive an explicit expression without making some additional assumptions. It is interesting to investigate the impact on the analysis of such a slight modification to the original equation.

There are various real-life applications that are described by Equation (1.1). This Lindley-type recursion arises naturally in a two-carousel bi-directional storage system, where a picker serves in turns two carousels; for details on this application see Park *et al.* [15]. In this situation, the recursion describes the waiting time W_n of the picker in a model involving two carousels alternately served by a single picker in terms of the rotation times B_n and the pick times A_n. Carousel models have been extensively studied during the last decades and there are several articles that are concerned with various relevant questions on these models. Indicative examples include the work by Litvak *et al.* [11, 12, 13], where the authors study the travel time needed to collect n items randomly located on a carousel under various strategies, and Jacobs *et al.* [8], who, by assuming a fixed number of orders, proposes a heuristic defining how many pieces of each item should be stored on the carousel in order to maximise the number of orders that can be retrieved without reloading. Another model that is described by (1.1) is a queuing model that involves a single server alternating between two service points. This model has already been introduced in [17]. In Section 2 we shall describe this model in detail, since it will be our main working example for the rest of the paper.

In the applied probability literature there has been a considerable amount of interest in generalisations of Lindley's recursion, namely the class of Markov chains described by the recursion $W_{n+1} = g(W_n, X_n)$. Our model is a special case of this general recursion and it is obtained by taking $g(w, x) = \max\{0, x - w\}$. Many structural properties of the recursion $W_{n+1} = g(W_n, X_n)$ have been derived. For example Asmussen and Sigman [2] develop a duality theory, relating the steady-state distribution to a ruin probability associated with a risk process. For more references in this domain, see for example Borovkov [3] and Kalashnikov [9]. An important assumption which is often made in these studies is that the function $g(w, x)$ is non-decreasing in its main argument w. For example, in [2] this assumption is crucial for their duality theory to hold. Clearly, in the example

$g(w, x) = \max\{0, x - w\}$ discussed here this assumption does not hold. For this reason, we believe that a detailed study of our recursion is of theoretical interest.

This paper is organised as follows. In Section 2 we shall give the setting of the problem and explain how this work complements the work that has been previously done on this Lindley-type equation. In Section 3 we prove that there exists a unique equilibrium distribution and that the system converges to it, irrespective of the initial state. We continue along the same lines in Section 4, where we look at the uniqueness issues of a solution to the limiting distribution of Equation (1.1) from an analytic point of view. Further on, in Section 5 we study some properties of the tail of the invariant distribution and in this way we conclude our study of the general case. In Section 6 we assume that the service times are exponentially distributed and derive the integral equation that our system satisfies. We use these results in Section 7, where we state a sufficient condition that the distribution of B_n should satisfy so that we can derive an explicit expression for the invariant distribution. We conclude in Section 8 with an overview of the results so far, remarks and further research possibilities.

2 The setting

It is perhaps more practical to view Equation (1.1) through an application that is described by it. This will allow us later on to use the terminology of this application in order to refer to the different elements of the equation and furthermore it will provide us with a background example that we can think of in order to develop our intuition.

To this end, we consider a system consisting of one server and two service points. At each service point there is an infinite queue of customers that needs to be served. The server alternates between the service points, serving one customer at a time. Before being served by the server, a customer must first undergo a preparation phase. Thus the server, after having finished serving a customer at one service point, may have to wait for the preparation phase of the customer at the other service point to be completed. Immediately after the server concludes his service at some working station another customer from the queue begins his preparation phase there. We are interested in the waiting time of the server. Let B_n denote the preparation time for the n-th customer and let A_n be the time the server spends on this customer. Then the waiting times W_n of the server satisfy (1.1). We assume that $\{A_n\}$ and $\{B_n\}$ are two mutually independent sequences of independent and identically distributed (i.i.d.) nonnegative random variables. For the sake of simplicity we shall write from now on $X_{n+1} = B_{n+1} - A_n$, $n \geqslant 1$, unless it is necessary to distinguish between the preparation and the service times. We shall also assume that $\mathbb{P}[X_n < 0] > 0$.

In the following we will only consider the system in equilibrium. Our aim is to calculate explicitly the distribution of the steady-state waiting time of the server. In Section 3 we prove that such a distribution indeed exists. For now, it suffices to say that obtaining a closed-form expression of the distribution is, in general, a non-trivial task. Let A, B and W denote the steady-state service, preparation and waiting time respectively. Then we have that

$$W \stackrel{\mathcal{D}}{=} \max\{0, B - A - W\}. \tag{2.1}$$

We call π_0 the mass of the steady-state waiting time distribution at zero. Additionally, for some random variable Y we denote its distribution and its density by F_Y and f_Y respectively.

In [17] we have already examined the case where F_A is some general distribution and F_B is a phase-type distribution. Other work on this recursion includes the work on a two-carousel system by Park *et al.* [15], where the authors derive the steady-state distribution F_W, assuming that F_B is the uniform distribution on $[0, 1]$ and F_A is either exponential or deterministic. Keeping the carousel application in mind, in [18] we have extended this result. While keeping the assumptions made on F_B, we have allowed F_A to be some phase-type distribution. Here we would like to complement these results by letting now B follow some general distribution while the service time A is exponentially distributed with parameter μ.

One should pay special attention to Equation (2.1). If only the sign of W were different, then (2.1) would be Lindley's equation and we would be analysing the waiting time of a customer in an $M/G/1$ queue. Unfortunately, the standard methods that are used to derive the waiting time of a customer in the $M/G/1$ queue do not work here. In Section 6 we shall review some of these methods and explain in detail what are the intrinsic differences between the two equations that cause these problems.

We can now proceed with the first step in calculating the steady-state waiting time of the server, which is proving that such a distribution exists and is unique.

3 Stability

In order to prove that there is a unique equilibrium distribution, we need to address three issues: the existence of an invariant distribution, the uniqueness of it and the convergence to it, irrespective of the state of the system at zero.

3.1 Existence

To prove the existence of an equilibrium distribution, we first give the definition of *tightness.*

Definition:

A sequence μ_n of probability measures on R^+, $n \geqslant 1$, is said to be *tight* if for every $\epsilon > 0$ there is a number $M < \infty$ such that $\mu_n[0, M] \geqslant 1 - \epsilon$, for all n.

In other words, almost all the measure is included in a compact set.

Consider now the recursion $W_{n+1} = \max\{0, X_{n+1} - W_n\}$, where X_n is an i.i.d. sequence of almost surely finite random variables. Let $W_1 = w$ and $M > w$. Then, since $W_{n+1} \leqslant \max\{0, X_{n+1}\}$ for all $n \geqslant 1$, we have that

$$\mathbb{P}[W_{n+1} \leqslant M] \geqslant \mathbb{P}[\max\{0, X_{n+1}\} \leqslant M] = \mathbb{P}[\max\{0, X_1\} \leqslant M].$$

So we can choose M to be the maximum of w and the $1 - \epsilon$ quantile of $\max\{0, X_1\}$. Thus, the sequence $\mathbb{P}[W_n \leqslant x]$ is tight.

Moreover, since the function $g(w, x) = \max\{0, x - w\}$ is continuous in both x and w, the existence of an equilibrium distribution is a direct application of Theorem 4 of Foss and Konstantopoulos [7]. So there exists an almost surely finite random variable W, such that $W \overset{\mathcal{D}}{=} \max\{0, X_1 - W\}$.

3.2 Uniqueness

Suppose there are two solutions W^1, W^2, such that

$$\begin{aligned} W^1 &\overset{\mathcal{D}}{=} \max\{0, X - W^1\}, \\ W^2 &\overset{\mathcal{D}}{=} \max\{0, X - W^2\}. \end{aligned}$$

In order to show that W^1 and W^2 have the same distribution, we shall first construct two sequences of waiting times that converge to W^1 and W^2. Then by using coupling we shall show that these two sequences coincide after some finite time. This implies that they have the same limiting distribution.

To this end, define for $i = 1, 2$ the following quantities.

$$\tau = \inf\{n : W_n^1 = W_n^2 = 0\}, \qquad \hat{X}_n^i = \begin{cases} X_n^i, & n \leqslant \tau \\ X_n^1, & n > \tau \end{cases},$$

$$W_1^i = w^i, \qquad \text{and} \quad W_{n+1}^i = \max\{0, \hat{X}_{n+1}^i - W_n^i\},$$

where for every i and n, X_n^i is equal in distribution to X and w^i is a realisation of W^i. Then $\{W_n^i\}$, $i = 1, 2$, is a stationary sequence. We shall examine the system (W_n^1, W_n^2). We have

$$\mathbb{P}[\tau > m] \leqslant \mathbb{P}[X_k^1 \geqslant 0 \text{ or } X_k^2 \geqslant 0; \text{ for all } k \in 1, \ldots, m] = \mathbb{P}[X_1^1 \geqslant 0 \text{ or } X_1^2 \geqslant 0]^m = q_1^m$$

and $q_1 < 1$. Thus $\lim_{m\to\infty} \mathbb{P}[\tau > m] = 0$. Furthermore, note that for $n \geqslant \tau$ we have that $W_n^1 = W_n^2$. Therefore,

$$\begin{aligned} \mathbb{P}[W^1 \leqslant x] = \lim_{n\to\infty} \mathbb{P}[W_n^1 \leqslant x] &= \lim_{n\to\infty} \big(\mathbb{P}[W_n^1 \leqslant x; \tau \leqslant n] + \mathbb{P}[W_n^1 \leqslant x; \tau > n]\big) \\ &= \lim_{n\to\infty} \mathbb{P}[W_n^1 \leqslant x; \tau \leqslant n] \\ &= \lim_{n\to\infty} \mathbb{P}[W_n^2 \leqslant x; \tau \leqslant n] = \lim_{n\to\infty} \mathbb{P}[W_n^2 \leqslant x] = \mathbb{P}[W^2 \leqslant x], \end{aligned}$$

which means that there exists a unique invariant distribution for our equation.

3.3 Convergence

We need to show that a system that does not start in equilibrium will eventually converge to it. To achieve this, we will compare two systems that are identical, apart from the fact that one of them does not start in equilibrium while the other one does. To this end, assume that the random variables X_n are distributed as X and for $i = 1, 2$ let the process $\{W_n^i\}$ satisfy the recursion $W_{n+1}^i = \max\{0, X_{n+1} - W_n^i\}$. Furthermore, let $W_1^1 = y$ and $W_1^2 = w$, where w is a realisation of W while y is not. We also have that for every $n \geqslant 1$, $W_n^2 \overset{\mathcal{D}}{=} W$. We need to prove that

$$\lim_{n\to\infty} \mathbb{P}[W_n^1 \leqslant x] = \mathbb{P}[W \leqslant x].$$

As before, define $\tau = \inf\{n : W_n^1 = W_n^2 = 0\}$. Then

$$\mathbb{P}[\tau > m] \leqslant \mathbb{P}[X_k \geqslant 0;\ \text{for all } k \in 1, \ldots, m] = \mathbb{P}[X_1 \geqslant 0]^m = q_2^m, \tag{3.1}$$

where $q_2 < 1$. Therefore, we have that

$$\begin{aligned}\lim_{n\to\infty} \mathbb{P}[W_n^1 \leqslant x] &= \lim_{n\to\infty} \big(\mathbb{P}[W_n^1 \leqslant x; \tau \leqslant n] + \mathbb{P}[W_n^1 \leqslant x; \tau > n]\big) \\ &= \lim_{n\to\infty} \mathbb{P}[W_n^2 \leqslant x; \tau \leqslant n] \\ &= \lim_{n\to\infty} \mathbb{P}[W_n^2 \leqslant x] = \mathbb{P}[W \leqslant x].\end{aligned}$$

In the argument above, the second equality is valid because

$$\lim_{n\to\infty} \mathbb{P}[W_n^1 \leqslant x; \tau > n] \leqslant \lim_{n\to\infty} \mathbb{P}[\tau > n] = 0.$$

From this we can also conclude that $q_2 = \mathbb{P}[X_1 \geqslant 0]$ is a bound of the rate of convergence to the equilibrium distribution.

One should note at this point that the stochastic process $\{W_n\}$ is a (possibly delayed) regenerative process with the time points where $W_n = 0$ being the regeneration points. Since $\mathbb{P}[X_n < 0] > 0$ the process is moreover aperiodic and from (3.1) it follows that it has a finite mean cycle length. Therefore, from the standard theory on regenerative processes it follows that the limiting distribution exists; see for example Corollary 1.5 in Asmussen [1, p. 171]. However, the coupling method that we have presented in this section has the advantage that it allows us to obtain an estimate of the rate of convergence to the invariant distribution. The essential feature remains though that, because we have an aperiodic regenerative process with a finite mean cycle length, we can couple the system with the stationary version of the process.

The above results on uniqueness and convergence are fairly general since we did not have to impose any conditions on the distributions of X or W. Nonetheless, it only proves that there exists a unique invariant distribution F_W; i.e. there is only one element of the class of all distribution functions that satisfies Equation (2.1). If we demand though that F_W is continuous, then we can, in fact, prove more. We can expand the class of distributions to the class of measurable bounded functions and prove that the solution of (2.1) is still unique. The proof of this is the subject of the following section.

4 The recursion revisited

The aim of this section is to examine the set of functions that satisfy (2.1). Note, first, that for $x \geqslant 0$ Equation (2.1) yields that $F_W(x) = \mathbb{P}[W \leqslant x] = \mathbb{P}[X - W \leqslant x]$. Assuming that F_W is continuous, then the last term is equal to $1 - \mathbb{P}[X - W \geqslant x]$, which gives us that

$$F_W(x) = 1 - \int_x^\infty \mathbb{P}[W \leqslant y - x] dF_X(y) = 1 - \int_x^\infty F_W(y - x) dF_X(y).$$

This means that the invariant distribution of W, provided that it is continuous, satisfies the functional equation

$$F(x) = 1 - \int_x^\infty F(y - x) dF_X(y). \tag{4.1}$$

Therefore, there exists at least one function that is a solution to (4.1). The question remains though whether there exist other functions, not necessarily distributions, that satisfy (4.1). The following theorem clarifies this matter.

Theorem 1. *There is a unique measurable bounded function $F : [0, \infty) \to R$ that satisfies the functional equation*

$$F(x) = 1 - \int_x^\infty F(y - x) dF_X(y).$$

Proof. Let us consider the space $\mathcal{L}^\infty([0, \infty))$, i.e. the space of measurable and bounded functions on the real line with the norm

$$\|F\| = \sup_{t \geqslant 0} |F(t)|.$$

In this space we define the mapping

$$(\mathcal{T}F)(x) = 1 - \int_x^\infty F(y - x) dF_X(y).$$

Then for two arbitrary functions F_1 and F_2 in this space we have

$$\begin{aligned}
\|(\mathcal{T}F_1) - (\mathcal{T}F_2)\| &= \sup_{x \geqslant 0} |(\mathcal{T}F_1)(x) - (\mathcal{T}F_2)(x)| \\
&= \sup_{x \geqslant 0} \left| \int_x^\infty [F_2(y - x) - F_1(y - x)] \, dF_X(y) \right| \\
&\leqslant \sup_{x \geqslant 0} \int_x^\infty \sup_{t \geqslant 0} |F_2(t) - F_1(t)| dF_X(y) \\
&= \|F_1 - F_2\| \; \sup_{x \geqslant 0} (1 - F_X(x)) \\
&\leqslant \|F_1 - F_2\| \; (1 - F_X(0)).
\end{aligned}$$

Thus

$$\|(\mathcal{T}F_1) - (\mathcal{T}F_2)\| \leqslant \|F_1 - F_2\| \; (1 - F_X(0)) = \|F_1 - F_2\| \; \mathbb{P}(B > A).$$

Since $\mathbb{P}(B > A) < 1$ we have a contraction mapping. Furthermore, we know that $\mathcal{L}^\infty([0, \infty))$ is a Banach space, therefore by the Fixed Point Theorem we have that (4.1) has a unique solution. □

The set of continuous and bounded functions on $[0, \infty)$ with the norm $\|F\| = \sup_t |F(t)|$ is also a Banach space, since it is a closed subspace of $\mathcal{L}^\infty([0, \infty))$. The above theorem, combined with the results of Section 3.2, assures us that should we find a continuous solution to (4.1) that belongs to $\mathcal{L}^\infty([0, \infty))$, then this solution is necessarily a distribution.

One should also note the usefulness of the above result in calculating numerically the invariant distribution. Since we have a contraction mapping, we can evaluate the distribution of W by successive iterations. One can start from some (trivial) distribution and substitute it into the right-hand side of (4.1). This will produce the second term of the iteration, and so on. The convergence to the invariant distribution is geometrically fast and the rate is bounded by the probability $\mathbb{P}(B > A)$. Note that we retrieve with this method the same bound for the rate of convergence as the one that we had in Section 3.3, namely the probability q_2.

Now we have the theoretical background that is required in order to proceed with determining the distribution of W. In the following section we shall first discuss though the tail behaviour of this distribution under various assumptions on the random variable B and later on we will proceed with the calculation of a closed-form expression for F_W.

5 Tail behaviour

We are interested in the tail asymptotics of W. In other words, we would like to know when we can estimate the probability that W exceeds some large value x by using only information from the given distributions of A and B.

Suppose that for some finite constant $\kappa \geqslant 0$

$$\frac{\mathbb{P}[B > x + y]}{\mathbb{P}[B > x]} \overset{x\to\infty}{\longrightarrow} e^{-\kappa y}.$$

Then

$$\frac{\mathbb{P}[e^B > e^x \cdot e^y]}{\mathbb{P}[e^B > e^x]} \overset{x\to\infty}{\longrightarrow} (e^y)^{-\kappa},$$

which means that e^B is regularly varying with index $-\kappa$. For the random variable B this means that if $\kappa = 0$, then B is long-tailed, and thus, in particular, heavy-tailed. If $\kappa > 0$, then B is light-tailed, but not lighter than an exponential tail.

For the tail of the waiting time we have that $\mathbb{P}[W > x] = \mathbb{P}[B - (W + A) > x]$ which implies that

$$\mathbb{P}[e^W > e^x] = \mathbb{P}[e^B e^{-(W+A)} > e^x]. \tag{5.1}$$

It is known that if $X > 0$ is a regularly varying random variable with index $-\kappa$, $\kappa \geqslant 0$, and $Y > 0$ is independent of X with $\mathbb{E}[Y^{\kappa+\epsilon}]$ finite for some $\epsilon > 0$, then XY is regularly varying with index $-\kappa$; see Breiman [4, Proposition 3] and in particular Cline and Samorodnitsky [5, Corollary 3.6]. Specifically,

$$\mathbb{P}[X \cdot Y > x] \sim \mathbb{E}[Y^\kappa]\mathbb{P}[X > x].$$

So (5.1) now becomes

$$\mathbb{P}[e^W > e^x] \sim \mathbb{P}[e^B > e^x]\mathbb{E}[e^{-\kappa(W+A)}]$$

or

$$\mathbb{P}[W > x] \sim \mathbb{P}[B > x]\mathbb{E}[e^{-\kappa W}]\mathbb{E}[e^{-\kappa A}].$$

In other words, the tail of W behaves asymptotically as the tail of B, multiplied by a constant. One can write the above result in terms of the tail of X. It suffices to note that

$$\mathbb{P}[X > x] = \mathbb{P}[B - A > x] = \mathbb{P}[e^B e^{-A} > e^x],$$

and since e^B is regularly varying with index $-\kappa$ we have that the above expression is asymptotically equal to $\mathbb{P}[B > x]\mathbb{E}[e^{-\kappa A}]$. The above findings are summarised in the following proposition.

Proposition 1. *Let* e^B *be regularly varying with index* $-\kappa$. *Then for the tail of* W *we have that*

$$\mathbb{P}[W > x] \sim \mathbb{P}[X > x]\,\mathbb{E}[e^{-\kappa W}].$$

Another case that is particularly interesting is when e^B is rapidly varying with index $-\infty$. This means that

$$\lim_{x\to\infty} \frac{\mathbb{P}[e^B > e^x \cdot e^y]}{\mathbb{P}[e^B > e^x]} = \lim_{x\to\infty} \frac{\mathbb{P}[B > x+y]}{\mathbb{P}[B > x]} = \begin{cases} 0, & \text{if } y > 0; \\ 1, & \text{if } y = 0; \\ \infty, & \text{if } y < 0. \end{cases}$$

This is equivalent to letting the index κ that was given previously go to infinity. For the random variable B this means that B is extremely light tailed. That would be the case if, for example, the tail of B is given by $\mathbb{P}[B > x] = e^{-x^2}$. As before, we are interested in deriving the asymptotic behaviour of the tail of W in terms of the tail of X. We shall first prove the following lemma.

Lemma 1. e^B *is rapidly varying* $\Rightarrow$ e^X *is rapidly varying.*

Proof. It suffices to show that for $y > 0$,

$$\lim_{x\to\infty} \frac{\mathbb{P}[X > x+y]}{\mathbb{P}[X > x]} = 0.$$

We have that

$$\frac{\mathbb{P}[X > x+y]}{\mathbb{P}[X > x]} = \frac{\mathbb{P}[B - A > x+y]}{\mathbb{P}[B - A > x]} = \frac{\int_0^\infty \mathbb{P}[B > x+y+z]dF_A(z)}{\int_0^\infty \mathbb{P}[B > x+z]dF_A(z)}. \tag{5.2}$$

Since e^B is rapidly varying and $y > 0$ then we have that

$$\lim_{x\to\infty} \frac{\mathbb{P}[B > x+y+z]}{\mathbb{P}[B > x+z]} = 0,$$

or in other words, for every $\delta > 0$ there is a finite constant η_δ, such that if $x + z \geqslant \eta_\delta$, then $\mathbb{P}[B > x+y+z] \leqslant \delta\mathbb{P}[B > x+z]$. By taking the limit of (5.2) for x going to infinity, we have that

$$\lim_{x\to\infty} \frac{\mathbb{P}[X > x+y]}{\mathbb{P}[X > x]} \leqslant \lim_{x\to\infty} \frac{\delta \int_0^\infty \mathbb{P}[B > x+z]dF_A(z)}{\int_0^\infty \mathbb{P}[B > x+z]dF_A(z)} = \delta,$$

which proves the assertion, since the left-hand side of the above expression is independent of δ, and δ can be chosen to be arbitrarily small. □

To derive the tail asymptotics we shall first decompose the tail of W as follows.

$$\begin{aligned} \mathbb{P}[W > x] &= \mathbb{P}[X - W > x] = \mathbb{P}[X - W > x\,;\, W = 0] + \mathbb{P}[X - W > x\,;\, W > 0] \\ &= \mathbb{P}[X > x]\mathbb{P}[W = 0] + \mathbb{P}[X - W > x\,;\, 0 < W < \epsilon] + \mathbb{P}[X - W > x\,;\, W \geqslant \epsilon], \end{aligned} \tag{5.3}$$

for some $\epsilon > 0$. Since the last two terms of the right-hand side of (5.3) are positive, we can immediately conclude that

$$\liminf_{x\to\infty} \frac{\mathbb{P}[W > x]}{\mathbb{P}[X > x]\mathbb{P}[W = 0]} \geqslant 1.$$

For the upper limit we first observe that

$$\mathbb{P}[X - W > x\,;\, 0 < W < \epsilon] \leqslant \mathbb{P}[X > x]\,\mathbb{P}[0 < W < \epsilon]$$

and that

$$\mathbb{P}[X - W > x\,;\, W \geqslant \epsilon] \leqslant \mathbb{P}[X > x + \epsilon]\,\mathbb{P}[W \geqslant \epsilon].$$

Furthermore, since e^X is rapidly varying, we have that for $\epsilon > 0$ the limit of $\mathbb{P}[X > x + \epsilon]$ over $\mathbb{P}[X > x]$ tends to zero as x tends to infinity, or in other words

$$\mathbb{P}[X > x + \epsilon] = o(\mathbb{P}[X > x]).$$

Combining the above arguments we obtain from (5.3) that

$$\limsup_{x\to\infty} \frac{\mathbb{P}[W > x]}{\mathbb{P}[X > x]\mathbb{P}[W = 0]} \leqslant 1 + \frac{\mathbb{P}[0 < W < \epsilon]}{\mathbb{P}[W = 0]} = 1,$$

since the left-hand side does not depend on ϵ and the inequalities in $\mathbb{P}[0 < W < \epsilon]$ are strict. The above results are summarised in the following proposition.

Proposition 2. *Let e^B be rapidly varying with index $-\infty$. Then for the tail of W we have that*

$$\mathbb{P}[W > x] \sim \mathbb{P}[X > x]\,\mathbb{P}[W = 0].$$

In the case when e^B was regularly varying, it was possible to express the tail of W also in terms of the tail of B – instead of the tail of X– simply by applying Breiman's result. In this situation though, this does not seem to be so straightforward. However, in some special situations it is indeed possible to derive the tail of X in terms of the tail of B, and consequently use this form for the tail asymptotics of the waiting time. In the following we shall give one particular example where it is possible to do so.

Assume that A is exponentially distributed with parameter μ and the tail of B is given by $\mathbb{P}[B > x] = e^{-x^p}$, where $p > 1$. In this example we shall limit ourselves to $p = 2$. However, the extension to the set of natural numbers is almost straightforward. For the tail of X we have that

$$\begin{aligned}\mathbb{P}[X > x] = \mathbb{P}[B - A > x] = &\int_0^\infty \mu e^{-\mu y} e^{-(x+y)^2} dy \\ &= e^{-x^2} \int_0^\infty \mu e^{-\mu y - y^2} e^{-2xy} dy = e^{-x^2} \frac{1}{x} \int_0^\infty \mu e^{-\mu \frac{u}{x} - \frac{u^2}{x^2}} e^{-2u} du.\end{aligned}$$

Note that the prefactor e^{-x^2} is equal to the tail of B and that the integral at the right-hand side behaves asymptotically like $\frac{\mu}{2}$, as x goes to infinity. In other words, we have that

$$\mathbb{P}[X > x] \sim \mathbb{P}[B > x] \frac{\mu}{2x}.$$

For p being any natural number greater than 1, the procedure is exactly the same. For $p = n$ the change of variables that will be the most adequate is $x^{n-1}y = u$ and the asymptotic behaviour of X in this situation is given by

$$\mathbb{P}[X > x] \sim \mathbb{P}[B > x]\frac{\mu}{nx^{n-1}}.$$

6 Derivation of the integral equation

In this section we derive the equation that we shall work with later on and we compare this equation with the analogous equation for the $M/G/1$ single-server queue. Furthermore, we examine various methods that are traditionally used for the single-server queue, but do not seem to be very helpful in our case.

In Section 4 we had already derived the integral equation (4.1) that our model satisfies. However, this form of the integral equation is not the best option to work with, since the distribution of X, that appears in the integral, will only complicate the calculations. Therefore it is now useful to distinguish between the random variables A and B. To begin with, consider Equation (2.1). Then for the distribution of W we have that

$$\begin{aligned} F_W(x) &= \mathbb{P}[W \leqslant x] = \mathbb{P}[B - W - A \leqslant x] \\ &= \int_0^\infty \int_0^\infty \mathbb{P}[B \leqslant x + z + y] dF_A(z) dF_W(y) \\ &= \pi_0 \int_0^\infty \mathbb{P}[B \leqslant x + z] dF_A(z) + \int_{0+}^\infty \int_0^\infty \mathbb{P}[B \leqslant x + y + z] dF_A(z) dF_W(y). \quad (6.1) \end{aligned}$$

It seems natural that the first two cases one might be interested in are the cases that are analogous to the M/G/1 and the G/M/1 single-server queue. In our model that means that we allow either the service time A or the preparation time B to be exponentially distributed. Here we are concerned with the first case, while in [17] we have already covered the second case. The methods that were described there were surprisingly simple. One approach is by exploiting the memoryless property of the exponential distribution in order to directly derive the Laplace transform of F_W. Unfortunately, if the situation is reversed and we assume that A, instead of B, is exponentially distributed, then this simple calculation fails at its very first step. The other approach that was mentioned in [17] is by defining a Markov chain, based on the number of exponential phases that the customer has to complete during his preparation time when the server returns to that service point. Also this approach is heavily relying on the fact that B has some structure that can be exploited, and the memoryless property comes in handy again. Nonetheless, if B follows some general distribution, this method is not applicable either. We would have to incorporate in the Markov chain the –unknown– remaining preparation time, which includes too little information to make any calculations possible. Therefore, the case that we are considering in this paper needs special attention.

Assume now that A is exponentially distributed with parameter μ, i.e. $f_A(x) = \mu e^{-\mu x}$. Since A has a density, then one can easily show that W has a density too as follows. From Equation (2.1) we readily have that

$$\mathbb{P}[W \leqslant x] = \int_{-\infty}^\infty \mathbb{P}[A \geqslant y - x]\, dF_{B-W}(y).$$

Since A has a density then the integral

$$\int_{-\infty}^{\infty} \frac{d}{dx}\mathbb{P}[A \geqslant y - x]\, dF_{B-W}(y)$$

exists and is the density of F_W. Then (6.1) becomes

$$\begin{aligned}
F_W(x) &= \pi_0 \int_0^\infty F_B(x+z)\mu e^{-\mu z} dz + \int_0^\infty f_W(y) \int_0^\infty F_B(x+y+z)\mu e^{-\mu z} dz dy \\
&= \mu\pi_0 e^{\mu x} \int_0^\infty F_B(x+z) e^{-\mu(x+z)} dz \\
&\quad + \int_0^\infty \mu e^{\mu(x+y)} f_W(y) \int_0^\infty F_B(x+z+y) e^{-\mu(x+z+y)} dz dy \\
&= \mu\pi_0 e^{\mu x} \int_x^\infty F_B(u) e^{-\mu u} du + \int_0^\infty \mu e^{\mu(x+y)} f_W(y) \int_{x+y}^\infty F_B(u) e^{-\mu u} du dy.
\end{aligned}$$

For the remainder of the paper we shall also need to assume that F_B is a continuous function. Therefore, we can differentiate with respect to x using Leibniz's rule to obtain

$$\begin{aligned}
f_W(x) &= \mu^2\pi_0 e^{\mu x} \int_x^\infty F_B(u) e^{-\mu u} du - \mu\pi_0 F_B(x) \\
&\quad + \mu^2 \int_0^\infty e^{\mu(x+y)} f_W(y) \int_{x+y}^\infty F_B(u) e^{-\mu u} du dy - \mu \int_0^\infty F_B(x+y) f_W(y) dy
\end{aligned}$$

or

$$f_W(x) = \mu F_W(x) - \mu\pi_0 F_B(x) - \mu \int_0^\infty F_B(x+y) f_W(y) dy. \tag{6.2}$$

What makes this equation troublesome to solve is the plus sign that appears in the integral at the right-hand side. If we would be dealing with the classic $M/G/1$ single-server queue, then the equation for the $M/G/1$ queue that is analogous to (6.2) would be identical apart from this sign. This difference nonetheless is of great importance when we try to derive the waiting-time distribution. It is not possible to derive a linear differential equation of f_W by differentiating (6.2), since we will not be able to avoid having some integral at the right-hand side. Taking Laplace transforms is also not useful since the integral at (6.2) is not a convolution (as it would be, if only the sign in the argument of F_B were different). Therefore, we are not able to directly obtain an expression for the Laplace transform of W. Moreover, it does not seem possible to factorise the transformed equation into terms that are analytic either in the right half complex plane or in the left half plane (this would allow us to use the Wiener-Hopf technique in order to derive the Laplace transform of W). However, Equation (6.2) can be reduced to a *generalised Wiener-Hopf* equation. It is known that the following equation

$$\int_0^\infty \left(k(x-y) + F_B(x+y)\right) f_W(y) dy = -\pi_0 F_B(x) \qquad (x > 0) \tag{6.3}$$

is equivalent to a generalised Wiener-Hopf equation (see Noble [14, p. 233]). Equation (6.2) reduces to Equation (6.3), if we let the kernel $k(x)$ be the function

$$k(x) = \frac{\delta(x)}{\mu} - \mathbb{1}_{\{x>0\}} - \frac{F_W(0)}{1 - F_W(0)},$$

where $\delta(x)$ is the Dirac δ-function and $\mathbb{1}_{\{x>0\}}$ is the indicator function of the set $\{x > 0\}$. This is indeed the case, since we have that

$$\int_0^\infty \left(\frac{\delta(x-y)}{\mu} - \mathbb{1}_{\{x>y\}} - \frac{F_W(0)}{1-F_W(0)} \right) f_W(y)dy = \frac{f_W(x)}{\mu} - [F_W(x) - F_W(0)] - F_W(0),$$

which is exactly in the form of Equation (6.2). We were unable though to solve this generalised Wiener-Hopf equation.

It is interesting to note at this point that Equation (6.2) is a Fredholm integral equation with infinite domain. For a textbook treatment of Fredholm integral equations, the interested reader is referred to Tricomi [16]. It is well-known that such equations can be solved by the method of successive iterations. We have already observed this in Section 4, where it was shown that Equation (6.2) satisfies a contraction mapping. Therefore, successive iterations provide us with a series of functions that converge (geometrically fast) to the unique solution of (6.2).

From the previous discussion it is apparent that none of the well-known techniques that work for Lindley's equation will help us find a closed-form expression for F_W. Therefore we will limit ourselves to studying under which conditions we can derive an explicit formula for F_W. As discussed in [14], the generalised Wiener-Hopf equation can be solved in special cases. In the following section we shall study a class of distribution functions F_B for which such a solution is possible.

7 The distribution of W

Before we begin with the analysis, we first define the class $\mathcal{M}$ as the collection of distribution functions F on $[0, \infty)$ that have the following property. For every $x, y \geqslant 0$, we can decompose the tail of the distribution as follows

$$\overline{F}(x+y) = 1 - F(x+y) = \sum_{i=1}^{n} g_i(x)h_i(y),$$

where for every i, g_i and h_i are arbitrary measurable functions (that can even be constants). Of course, by demanding that F is a distribution we have implicitly made some assumptions on the functions g_i and h_i, but these assumptions are, for the time being, of no real importance.

The class $\mathcal{M}$ is particularly rich. To begin with, one can show that all phase-type distributions are included in $\mathcal{M}$. Then all the individual functions g_i and h_i have a nice interpretation. Let F be a phase-type distribution. Such a distribution F is defined in terms of a Markov jump proces $J(x)$, $x \geqslant 0$, with finite state space $E \cup \Delta$, such that Δ is the set of absorbing states and E the set of transient states. Then F is the distribution of the time until absorption. It is usually assumed that the process starts in E; see Asmussen [1, Chapter 3]. For our purpose, suppose that we have an $n+1$-state Markov chain, where state 0 is absorbing and states $\{1, \ldots, n\}$ are not. Then

$$\overline{F}(x) = \mathbb{P}[J(x) \text{ is not absorbed}].$$

So we have that

$$\begin{aligned}\overline{F}(x+y) &= \mathbb{P}[J(x+y) \in \{1,\dots,n\}] \\ &= \sum_{i=1}^{n} \mathbb{P}[J(x+y) \in \{1,\dots,n\} \mid J(x)=i]\mathbb{P}[J(x)=i] \\ &= \sum_{i=1}^{n} \mathbb{P}[J(y) \in \{1,\dots,n\} \mid J(0)=i]\mathbb{P}[J(x)=i] \\ &= \sum_{i=1}^{n} h_i(y) g_i(x),\end{aligned}$$

with

$$\begin{aligned}h_i(y) &= \mathbb{P}[J(y) \in \{1,\dots,n\} \mid J(0)=i] \\ g_i(x) &= \mathbb{P}[J(x)=i].\end{aligned}$$

However, $\mathcal{M}$ includes more distribution functions apart from the phase-types. A well-known distribution that is not phase-type but has a rational Laplace transform (cf. Asmussen [1, p. 87]) is the distribution with a density proportional to $(1+\sin x)e^{-x}$. So, let the density be $f(x) = c(1+\sin x)e^{-x}$, where

$$c^{-1} = \int_0^\infty (1+\sin x)e^{-x}dx = \frac{3}{2}.$$

Then the distribution is given by

$$F(x) = 1 - \frac{e^{-x}(2+\sin x+\cos x)}{3}$$

and one can easily check now that $\overline{F}(x+y)$ can be decomposed into a finite sum of products of functions of x and of functions of y. In fact, all functions with rational Laplace transforms are included in this class. To see this, let the function $f(x)$ have the Laplace transform

$$\hat{f}(s) = \frac{P(s)}{Q(s)},$$

where $P(s)$ and $Q(s)$ are polynomials in s with deg$[P]<$ deg$[Q]$. Let now the roots of $Q(s)$ be $q_1,\dots,q_n$ with multiplicities $m_1,\dots,m_n$ respectively. Then $\hat{f}(s)$ can be decomposed as follows:

$$\hat{f}(s) = \frac{c_1^1}{(s-q_1)} + \frac{c_2^1}{(s-q_1)^2} + \cdots + \frac{c_{m_1}^1}{(s-q_1)^{m_1}} + \frac{c_1^2}{(s-q_2)} + \cdots + \frac{c_{m_n}^n}{(s-q_n)^{m_n}},$$

where the constants c_j^i are given by

$$c_j^i = \frac{1}{(m_i-j)!}\,\frac{d^{m_i-j}}{ds^{m_i-j}}\left[(s-q_i)^{m_i}\frac{P(s)}{Q(s)}\right]\Bigg|_{s=q_i}.$$

Then $f(x)$ is simply the function

$$f(x) = \sum_{i=1}^{n} \sum_{j=1}^{m_i} \frac{c_j^i \ x^{j-1}}{(j-1)!} \, e^{q_i x},$$

which clearly belongs in $\mathcal{M}$.

Denote by β and γ_i, $i = 1, \ldots, n$, the Laplace transforms of the functions F_B and g_i respectively. Then the following theorem holds.

Theorem 2. *Assume that $F_B \in \mathcal{M}$, is continuous, and that for every $i = 1, \ldots, n$ the functions $h_i(y)$ are bounded and*

$$\int_0^\infty |g_i(x)| dx < \infty.$$

Then the distribution of W is given by

$$F_W(x) = 1 - e^{\mu x} \int_x^\infty e^{-\mu s} \left(\mu \pi_0 \overline{F}_B(s) + \mu \sum_{i=1}^{n} c_i g_i(s) \right) ds,$$

where the constants π_0 and c_i, $i = 1, \ldots, n$, are a solution to the linear system of equations

$$2\pi_0 - \mu \pi_0 \ \beta(\mu) + \mu \sum_{i=1}^{n} c_i \gamma_i(\mu) = 1$$

and for $i = 1, \ldots, n$,

$$\begin{aligned} c_i = \mu \pi_0 & \int_0^\infty h_i(x) \left(\overline{F}_B(x) - \mu \int_x^\infty e^{-\mu(s-x)} \overline{F}_B(s) ds \right) dx \\ & + \mu \sum_{j=1}^{n} c_j \int_0^\infty h_i(x) \left(g_j(x) - \mu \int_x^\infty e^{-\mu(s-x)} g_j(s) ds \right) dx. \end{aligned} \tag{7.1}$$

Proof. Since $F_B \in \mathcal{M}$, (6.2) becomes

$$\begin{aligned} f_W(x) &= \mu F_W(x) + \mu \pi_0 \overline{F}_B(x) - \mu \pi_0 + \mu \int_0^\infty \overline{F}_B(x+y) f_W(y) dy - \mu \int_0^\infty f_W(y) dy \\ &= \mu F_W(x) + \mu \pi_0 \overline{F}_B(x) - \mu \pi_0 + \mu \sum_{i=1}^{n} g_i(x) \int_0^\infty h_i(y) f_W(y) dy - \mu(1 - \pi_0), \end{aligned}$$

or

$$f_W(x) = \mu F_W(x) + \mu \pi_0 \overline{F}_B(x) + \mu \sum_{i=1}^{n} c_i g_i(x) - \mu, \tag{7.2}$$

where $c_i = \int_0^\infty h_i(y) f_W(y) dy$.

Equation (7.2) is a linear differential equation of first order that satisfies the initial condition $F_W(0) = \pi_0$. Its solution is given by

$$F_W(x) = e^{\mu x} \int_0^x e^{-\mu s} \left(\mu \pi_0 \overline{F}_B(s) + \mu \sum_{i=1}^{n} c_i g_i(s) - \mu \right) ds + \pi_0 e^{\mu x}. \tag{7.3}$$

We can rewrite the previous equation as follows.

$$
\begin{aligned}
F_W(x) &= e^{\mu x}\int_0^x e^{-\mu s}\left(\mu\pi_0\overline{F}_B(s)+\mu\sum_{i=1}^n c_i g_i(s)\right)ds+(\pi_0-1)e^{\mu x}+1\\
&= e^{\mu x}\left(2\pi_0-\mu\pi_0\;\beta(\mu)+\mu\sum_{i=1}^n c_i\gamma_i(\mu)-1\right)\\
&\quad - e^{\mu x}\int_x^\infty e^{-\mu s}\left(\mu\pi_0\overline{F}_B(s)+\mu\sum_{i=1}^n c_i g_i(s)\right)ds+1. \qquad (7.4)
\end{aligned}
$$

There are $n+1$ unknown terms in the above equation, the probability π_0 and the constants c_i for $i=1,\ldots,n$. These constants are a solution to the linear system of $n+1$ equations

$$\lim_{x\to\infty} F_W(x)=1, \qquad (7.5)$$

or equivalently,

$$2\pi_0-\mu\pi_0\;\beta(\mu)+\mu\sum_{i=1}^n c_i\gamma_i(\mu)=1, \qquad (7.6)$$

and, for $i=1,\ldots,n$,

$$c_i=\int_0^\infty h_i(y)f_W(y)dy \quad \text{for } i=1,\ldots,n. \qquad (7.7)$$

For the fact that Equation (7.6) is both necessary and sufficient for (7.5) to hold, one only needs to note that

$$\lim_{x\to\infty}\sum_{i=1}^n c_i g_i(x)=0.$$

Moreover, Equation (7.7) can be rewritten as follows. We substitute f_W by using (7.2). For the distribution F_W that appears in the latter equation we use Equation (7.4), after simplifying this one by using (7.6). With this straightforward calculation we derive the constants c_i in the form that they appear in (7.1).

Let us now call the system formed by Equations (7.6) and (7.7) Σ. We can show that Σ has at least one solution by constructing one as follows. From Section 3.1 we know that there exists at least one invariant distribution for W. This distribution, by definition, satisfies the condition that its limit at infinity equals one and it also satisfies Equation (7.3). Then it is clear that it also satisfies Σ, therefore Σ has at least one solution.

In Section 4 we have already explained that if one finds a continuous and bounded solution to (2.1), then this solution is necessarily a distribution. To complete the proof, it remains to show that these conditions apply to (7.3). First of all, (7.3) is clearly a continuous function and since $\lim_{x\to\infty} F_W(x)=1$ and $0\leqslant F_W(0)=\pi_0<\infty$, it is also bounded. Therefore (7.3) is a distribution.

$\square$

Remark 1. The conditions that appear in Theorem 2 guarantee that all the integrals that appear in the intermediate calculations and in Σ are well defined. In particular, one should note that demanding that

$$\int_0^\infty |g_i(x)|dx < \infty$$

implies that the random variable B has a finite mean, $\gamma_i(\mu)$ and $\beta(\mu)$ exist and are finite numbers, and that

$$\int_0^\infty h_i(x)\overline{F}_B(x)dx \qquad \text{and} \qquad \int_0^\infty h_i(x)g_j(x)dx$$

exist and are finite (cf. Equation (7.1)).

Remark 2. We have explained in the proof why Σ has at least one solution, but we have not excluded the possibility that Σ has multiple solutions. However, this is not necessary. The fact that (7.3) is necessarily the *unique* invariant distribution guarantees that the (possible) multiple solutions of Σ will make the term $\sum_{i=1}^n c_i g_i(s)$ unique. It does not seem possible to derive answers –in an algebraic way– from Equation (7.1) about the uniqueness of the solution of Σ. More information about the functions g_i would probably allow us to decide upon this matter.

8 Final comments

In this paper we have considered the Lindley-type equation $W = \max\{0, B-A-W\}$. The main characteristic of this equation, that deviates from the standard literature, is that it is non-increasing in its main argument. This fact produces some surprising results when analysing the equation.

Here we have examined various issues that are related to this equation, without limiting ourselves to the specific distributional conditions on the random variables that appear. For the general case we have shown that there is a unique invariant distribution and that the system converges to it. Furthermore, we gave an upper bound on the rate of convergence. We have further shown that if we demand that this invariant distribution is continuous, then $W = \max\{0, B-A-W\}$ satisfies an integral equation that is a contraction mapping. This is particularly useful when one tries to approximate numerically the invariant distribution. We have also studied the tail behaviour of W both for heavy-tailed and (extremely) light-tailed distributions of the preparation time B. We have shown that in both cases, the tail behaviour of W is asymptotically proportional to the tail behaviour of the random variable $B-A$ and in some cases it is possible to express the asymptotics in terms of the random variable B alone.

Although sometimes the methods that apply in Lindley's equation seem to work here as well (as it was the case in [17]), this is not true in general. It seems that a closed-form expression for the distribution F_W of W cannot be easily derived. We review some of the standard methods that provide us with answers in Lindley's equation and yet fail to do so with this Lindley-type equation only to derive in the end a sufficient condition that allows us to compute F_W explicitly.

One can apparently ask for this equation all the questions that have been asked for Lindley's equation so far. Yet it is not clear that it will always be possible to derive (similar) answers. In future work we shall try to remove the condition that the sequences $\{A_n\}$ and $\{B_n\}$ are mutually independent and examine the difficulties that arise there. Furthermore, for the method that we followed in this paper, it was essential that F_B has an unbounded support. Otherwise the decomposition of $F_B(x+y)$ in products of functions would not be possible for every x and every y in the support. It would be interesting to also investigate which method can provide us with answers in the case where F_B is defined over a bounded interval.

Acknowledgements

The author would like to thank I.J.B.F. Adan, O.J. Boxma, T. Matsoukas, J. Wessels and A.P. Zwart for their helpful and constructive comments and suggestions.

References

[1] Asmussen, S. (2003). *Applied Probability and Queues.* Springer-Verlag, New York.

[2] Asmussen, S. and Sigman, K. (1996). Monotone stochastic recursions and their duals. *Probability in the Engineering and Informational Sciences*, **10,** 1–20.

[3] Borovkov, A.A. (1998). *Ergodicity and Stability of Stochastic Processes.* John Wiley & Sons, Chichester.

[4] Breiman, L. (1965). On some limit theorems similar to the arc-sin law. *Theory of Probability and its Applications* **10,** 323–331.

[5] Cline, D.B.H. and Samorodnitsky, G. (1994). Subexponentiality of the product of independent random variables. *Stochastic Processes and their Applications* **49,** 75–98.

[6] Cohen, J.W. (1982). *The Single Server Queue.* Elsevier Science Publishers, Amsterdam.

[7] Foss, S. and Konstantopoulos, T. (2004). An overview of some stochastic stability methods. *Journal of the Operations Research Society of Japan* **47,** 275–303.

[8] Jacobs, D.P., Peck, J.C. and Davis, J.S. (2000). A simple heuristic for maximizing service of carousel storage. *Computers and Operations Research* **27,** 1351–1356.

[9] Kalashnikov, V. (2002). Stability bounds for queueing models in terms of weighted metrics. In: Suhov, Y. (editor). *Analytic Methods in Applied Probability.* Amer. Math. Soc. Transl. Ser. 2, **207,** 77–90. Amer. Math. Soc., Providence, RI.

[10] Lindley, D.V. (1952). The theory of queues with a single server. *Proceedings Cambridge Philosophical Society* **48,** 277–289.

[11] Litvak, N. (2001). Collecting n items randomly located on a circle. Ph.D. thesis, Eindhoven University of Technology. Available at `http://w3.tue.nl/nl/diensten/bib/`.

[12] Litvak, N. and Adan, I. (2001). The travel time in carousel systems under the nearest item heuristic. *Journal of Applied Probability* **38,** 45–54.

[13] Litvak, N. and van Zwet, W.R. (2004). On the minimal travel time needed to collect n items on a circle. *Annals of Applied Probability* **14,** 881–902.

[14] Noble, B. (1958). *Methods Based on the Wiener-Hopf Technique for the Solution of Partial Differential Equations.* Pergamon Press, London.

[15] Park, B.C., Park, J.Y. and Foley, R.D. (2003). Carousel system performance. *Journal of Applied Probability* **40,** 602–612.

[16] Tricomi, F.G. (1970). *Integral Equations.* Fifth printing. Interscience Publishers, Inc., New York.

[17] Vlasiou, M. and Adan, I.J.B.F. (2005). An alternating service problem. *Probability in the Engineering and Informational Sciences*, to appear.

[18] Vlasiou, M., Adan, I.J.B.F. and Wessels, J. (2004). A Lindley-type equation arising from a carousel problem. *Journal of Applied Probability* **41,** 1171–1181.