A Two-Priority Fluid Flow Model

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Abstract

We consider a fluid flow model processing two types of fluid at constant rate, where one type of fluid receives full service priority over the other. The two types of fluid are stored in separate infinite capacity buffers. The input is governed by an external Markovian environment process. We derive the Laplace-Stieltjes transform of the steady-state joint buffer content distribution by decomposing the problem into two subproblems: first, we study the buffer content processes over periods of time with positive leftover service capacity for the lower priority fluid, and then over periods of time with no leftover service capacity, after which we combine both results to get the joint distribution. We illustrate the results by a numerical example.

Key words: Fluid flow model, static priority service discipline, fluid process with jumps

1 Introduction

Fluid-flow models provide an important tool for the performance analysis of high-speed data networks, or large-scale production systems where a large number of relatively small jobs are processed. There has been a fervent research on fluid models related to telecommunication applications. Some of the pioneering works in the area are the papers by Anick, Mitra and Sondhi (1982), Kosten (1974), Mitra (1988). The survey paper of Kulkarni

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(1997) provides a good overview of the research on single-sever fluid models by this time. The study of fluid models with priorities is motivated by the performance analysis of asynchronous transfer mode (ATM) switches and internet protocol (IP) routers that support classes of traffic with different quality of service (QoS), e.g. Choi and Choi (1998), Elwalid and Mitra (1995), Kulkarni and Naraynan (1996), Zhang (1993). The employment of priority provides an efficient measure for congestion control in modern high-speed integrated networks like the Internet. For example, to ensure a good quality of service a provider may need to distinguish and prioritize between different traffic streams, such as video/voice, which is delay sensitive, and standard data transmission, which is loss sensitive. With the growing importance and diversity of the Internet traffic, the study of multi-class fluid models with priorities is a hot topic of immediate significance.

We study a single-node fluid-flow model with two types of fluid served according to a static priority service discipline, where type 1 fluid has higher priority than type 2 fluid. The service rate is constant and the input flows are regulated by an irreducible finite state Continuous Time Markov Chain (CTMC). We obtain exact analytic results for the steady-state joint distribution of the two buffer content processes. The same problem is the topic of the paper of Choi and Choi (1998) where the authors determine the Laplace-Stieltjes transform (LST) of the stationary joint buffer content distribution by a spectral decomposition technique. Here, we present a new probabilistic approach based on an embedded analysis of the system. We study separately the buffer content processes, first, over periods with positive leftover service capacity for type 2 fluid and then over periods with zero leftover service capacity. For the analysis during periods with positive leftover service capacity we show that the buffer content process is a *fluid process with jumps* to which the results of Tzenova, Adan and Kulkarni (2005) can be directly applied. The analysis during periods with no left-over capacity is facilitated by the fact that during these periods the lower priority buffer content can only increase. In addition to splitting the entire system analysis in two easier subproblems we believe that this approach is also more attractive from a practical point of view, since it involves only systems of first-order linear differential equations that are explicitly solved by standard well known techniques. Note that in case of ($K > 2$) priority classes the steady-state marginal distribution of the lowest class of fluid can be computed recursively by reducing the problem to a two-priority system with the higher priority class defined as the aggregate of the first $K-1$ priority classes of fluid. Thus the method of this paper can be used for more than two priority classes. Another important observation is that the tandem fluid model of H. van den Breg and Mandjes (1999), Elwalid and Mitra (1995), can be represented as a two-priority fluid model and hence can be analyzed using the techniques developed here.

For the same model Zhang (1993) develops an algorithm for the moments of the buffer content processes and, for the special case of a two-state governing Markov chain, obtains the distribution of the buffer contents in closed form. Assuming exponentially distributed busy periods of type 1 fluid, Elwalid and Mitra (1995) approximate the distribution of type 2 fluid by implementing results on the analysis of systems with non-prioritized service. Kulkarni and Narayanan (1996) consider the case of independent Markovian on-off sources and derive the LST of the steady-state marginal buffer-content distributions by noting that the down time of the server for type 2 fluid is the same as the busy period of type 1 fluid.

This paper is organized as follows. In the next section we describe the model in detail. In Section 3 we derive the LST of the total increase of type 2 fluid in the buffer during periods with no leftover service capacity. The latter represents the jumps of the type 2 buffer content process when embedded on time periods with positive leftover service capacity, which is analyzed in Section 4. Section 5 consists of some preliminary results that are necessary for the computation of the LST of the limiting joint fluid content distribution during periods with no leftover service capacity in Section 6. The results are illustrated by numerical examples in Section 7.

2 Model Description

We consider a single-node fluid flow model with a server that is always on and works at its full capacity μ. There are two different priority types of fluid entering two infinite capacity buffers that are emptied by the server. Their input rates are determined by the state of an external environment process $\{I(t),\ t \geq 0\}$ that is an irreducible CTMC with finite state space $S = \{1, \ldots, N\}$ and infinitesimal generator Q. While $I(t) = i,\ i \in S$, the input rate of type k fluid is $p_k(i) \geq 0,\ k = 1, 2$ and the overall net input rate to the two buffers is $r(i) = p_1(i) + p_2(i) - \mu$. We assume that type 1 fluid has a higher priority than type 2 fluid, i.e., the server allocates as much of its capacity μ as needed to serve type 1 fluid and then allocates the rest of the capacity to serve type 2 fluid. Let $r_1(i) = p_1(i) - \mu$ denote the net input rate of type 1 fluid while the environment is in state $i \in S$. Then type 2 fluid can be served only when there is no type 1 fluid in the buffer and the governing CTMC is in a state i with $r_1(i) < 0$. Therefore, we call these periods of time *on-periods* (from the point of view of type 2 fluid). To avoid trivialities we assume that every on-period is followed by an *off-period* which starts when the CTMC $\{I(t),\ t \geq 0\}$ jumps to a state $I(t) = i$ with $r_1(i) \geq 0$ in which there is no server capacity available to type 2.

To this end, we shall use the following partitioning of the state space S:

$$S'_- := \{i \in S : r_1(i) < 0\},\quad S'_+ := \{i \in S : r_1(i) \geq 0\},\quad N'_- := |S'_-|,\ N'_+ := |S'_+|,$$

$$S_0 := \{i \in S : r(i) = 0\},\ S_+ := \{i \in S : r(i) > 0\},\ S_- := \{i \in S : r(i) < 0\},$$

$$N_0 := |S_0|,\ N_+ := |S_+|,\ N_- := |S_-|.$$

The following matrices are used throughout the paper

$$R := \text{diag}[r(1), \ldots, r(N)],\quad R_1 := \text{diag}[r_1(1), \ldots, r_1(N)]. \tag{2.1}$$

Let π^I denote the limiting distribution of the CTMC $\{I(t),\ t \geq 0\}$. Then the system is stable if and only if the mean net input rate in steady-state is negative, $\pi^I Re < 0$, e.g., see Choi et al. (1998) [3]. We assume that this condition is satisfied.

Let $X_k(t)$ denote the amount of type $k,\ k = 1, 2$, fluid in the buffer at time t and consider the Markov process $\{(X_1(t), X_2(t), I(t)),\ t \geq 0\}$. Clearly, the service of type 1 fluid is not affected by the presence of type 2 fluid in the system and the results from the classical single buffer fluid model, as given in Kulkarni (1997) and Kulkarni et al. (2002) [8, 10], can be directly applied to the analysis of $X_1(t)$. In this paper we obtain the LST of the joint steady-state distribution of the two buffer content processes

$$\tilde{F}_i(s_1, s_2) := \lim_{t\to\infty} E(e^{-s_1 X_1(t)} e^{-s_2 X_2(t)}; I(t) = i)$$

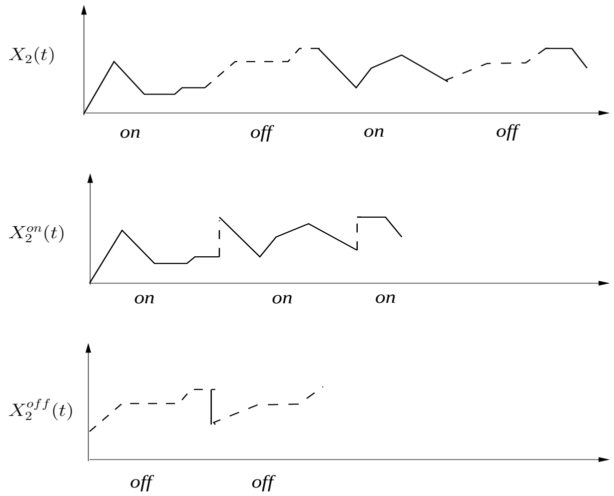

Figure 2.1: Type 2 buffer content process $X_2(t)$ embedded over on-periods and off-periods.

in two parts. We consider the embedded processes over on-periods and over off-periods denoted by $X_k^{on}(t)$, $I^{on}(t)$ and by $X_k^{off}(t)$, $I^{off}(t)$, respectively (see Fig. 2.1). Clearly, $X_1^{on}(t) \equiv 0$ and $X_2^{on}(t)$ is a fluid process with jumps as in Tzenova et al. (2005) [13], where a jump occurs at the end of an on-period and represents the total accumulation of type 2 fluid during an off-period (that is skipped when embedded to on-periods only). These observations facilitate the study of

$$\tilde{F}_i^{on}(s_2) := E(e^{-s_2 X_2^{on}}; I^{on} = i) = \lim_{t\to\infty} E(e^{-s_2 X_2^{on}(t)}; I^{on}(t) = i). \tag{2.2}$$

Next, we study the embedded processes over off-periods, during which type 2 can only increase,

$$\tilde{F}_i^{off}(s_1, s_2) := E(e^{-s_1 X_1^{off}} e^{-s_2 X_2^{off}}; I^{off} = i) = \lim_{t\to\infty} E(e^{-s_1 X_1^{off}(t)} e^{-s_2 X_2^{off}(t)}; I^{off}(t) = i). \tag{2.3}$$

We put these results together to compute

$$\tilde{F}_i(s_1, s_2) = p^{on} \tilde{F}_i^{on}(s_2) + p^{off} \tilde{F}_i^{off}(s_1, s_2) \tag{2.4}$$

where p^{on} (p^{off}) denotes the long-run fraction of time the system is in an on-period (off-period). Note that p^{on} can be computed directly from a standard fluid model with only the first type of fluid as

$$p^{on} = \lim_{t\to\infty} P(X_1(t) = 0, I(t) \in S'_-). \tag{2.5}$$

In the following section we study the accumulation of type 2 fluid in the buffer during an off-period that corresponds to the jumps in the type 2 buffer content process when restricted over on-periods only.

3 Accumulation of Type 2 Fluid during an Off-Period

During an off-period type 2 fluid is not served and it only accumulates in the buffer. Let $A_2(t)$ denote the amount of type 2 fluid that accumulates in the buffer by time t during an off-period. In this section we compute the LST of $A_2(T)$ for an off-period of length T. Define

$$\psi_{ji}(x,y) := P(A_2(T) \le y, I(T) = i | I(0) = j, X_1(0) = x),\ \ x,y \ge 0,\ \ j,i \in S, \tag{3.6}$$

and

$$\tilde{\psi}_{ji}(x,s) := E(e^{-sA_2(T)}; I(T) = i | I(0) = j,\ \ X_1(0) = x),\ \ Re(s) \ge 0;\ \ j,i \in S. \tag{3.7}$$

Clearly, $\psi_{ji}(x,y) = \tilde{\psi}_{ji}(x,s) = 0$ if $i \in S'_+$. For the purposes of the following result we define the column vectors

$$\tilde{\psi}_i(x,s) := [\tilde{\psi}_{1i}(x,s), \ldots, \tilde{\psi}_{Ni}(x,s)]^t,\ \ i \in S'_-.$$

Theorem 3.1 *For fixed $i \in S'_-$ the LST $\tilde{\psi}_i(x,s)$ satisfies the following system of differential equations*

$$R_1 \frac{\partial \tilde{\psi}_i}{\partial x}(x,s) + (Q - sP_2)\tilde{\psi}_i(x,s) = 0 \tag{3.8}$$

with boundary conditions

$$\tilde{\psi}_{ji}(0,s) = \delta_{ji},\ \ for\ j \in S'_-, \tag{3.9}$$

where δ_{ji} is the Kronecker symbol and $P_2 := diag[p_2(1), \ldots, p_2(N)]$.

Proof: Let $x > 0,\ j \in S$ and $i \in S'_-$. After conditioning on a small time interval of length $h > 0$ we have

$$\tilde{\psi}_{ji}(x,s) = \sum_{k\neq j} q_{jk} h E(e^{-s(A_2(T)+p_2(k)h)}; I(T)=i | I(0)=k,\ X_1(0)=x+r_1(k)h)$$

$$+(1+q_{jj}h+o(h))E(e^{-s(A_2(T)+p_2(j)h)}; I(T)=i | I(0)=j,\ X_1(0)=x+r_1(j)h)+o(h).$$

Using the notation of (3.7) and rearranging the last equation we get

$$\frac{e^{sp_2(j)h}\tilde{\psi}_{ji}(x,s)-\tilde{\psi}_{ji}(x+r_1(j)h,s)}{h} = \sum_{k\in S} q_{jk}\tilde{\psi}_{ki}(x+r_1(k)h,s)+o(1).$$

Next, we substitute $e^{sp_2(j)h} = 1+sp_2(j)h+o(h)$ and let $h\to 0$ to get

$$-r_1(j)\frac{\partial\tilde{\psi}_{ji}}{\partial x}(x,s)+sp_2(j)\tilde{\psi}_{ji}(x,s) = \sum_{k\in S} q_{jk}\tilde{\psi}_{ki}(x,s).$$

In vector notation this equation is equivalent to (3.8). The boundary conditions (3.9) follow from the definition of $A_2(T)$. Given $X_1(0)=0$ and $I(0)\in S'_-$ it is clear that the length of the off-period is 0 and therefore $A_2(T)=0$. $\diamondsuit$

It is well known that the solution to (3.8) and (3.9) can be computed by standard spectral decomposition in terms of the generalized eigenvalues λ and eigenvectors ϕ of the system (3.8) that solve $(\lambda R_1 + Q - sP_2)\phi = 0$, see Kulkarni (1997) [8]. Therefore in the sequel we will skip the explicit solutions to similar systems of first-order differential equations.

4 Type 2 Buffer Content Process during an On-Period

Consider the low priority fluid content process during on-periods, i.e., while $X_1(t)=0$ and $I(t)\in S'_-$. The embedded process $X_2^{on}(t)$ is a *fluid process with jumps* as treated in Tzenova et al. (2005) [13]. A jump occurs at the end of an on-period and the size of the jump is equal to $A_2(T)$, the total amount of type 2 fluid accumulated during the corresponding off-period of length T. Let $J_{ji}(z)$ denote the distribution of the jump size $A_2(T)$ during an off-period of length T that starts in state $j\in S'_+$ and ends in state

$i \in S'_-$,

$$J_{ji}(z) := P(A_2(T) \leq z, I(T) = i | I(0) = j), \;\; z \geq 0, \;\; j \in S'_+, i \in S'_-.$$

Clearly, $J_{ji}(z) = \psi_{ji}(0, z)$ where $\psi_{ji}(0, z)$ is as in Eq. (3.6). In this section we determine $\tilde{F}^{on}_i(s_2)$ as defined in Eq. (2.2) above. The next theorem gives the differential equations satisfied by the row vector $F^{on}(x) := [F^{on}_i(x), \;\; i \in S'_-]$ where

$$F^{on}_i(x) := \lim_{t\to\infty} P(X^{on}_2(t) \leq x, \;\; I^{on}(t) = i), \;\; x \geq 0, i \in S'_-. \tag{4.10}$$

We use matrix convolution defined as follows. Suppose that $A(x) = [A_{ij}(x)]$ and $B(x) = [B_{ij}(x)]$ are two matrices of functions where the number of columns of $A(x)$ is equal to the number of rows of $B(x)$. Then their *convolution* $A * B(x)$ is the matrix $C(x) = [C_{ij}(x)]$ with elements

$$C_{ij}(x) = \sum_k \int_0^x A_{ik}(x - t) dB_{kj}(t).$$

To simplify the notation we also use the sub-matrices

$$R_{--} := [R_{ij}, \;\; i, j \in S'_-], Q_{--} := [Q_{ij}, \;\; i, j \in S'_-], Q_{-+} := [Q_{ij}, \;\; i \in S'_-, \;\; j \in S'_+],$$

$$J_{+-}(z) := [J_{ij}(z), \;\; i \in S'_+, \;\; j \in S'_-].$$

For the proofs of the subsequent results we refer to Theorems 2.2, 2.3, and 2.4, respectively, in Tzenova et al (2005) [13].

Theorem 4.1 *The vector $F^{on}(x)$ satisfies the system of differential equations*

$$\frac{dF^{on}}{dx}(x) R_{--} = F^{on}(x) Q_{--} + F^{on} * (Q_{-+} J_{+-})(x), \tag{4.11}$$

with boundary conditions

$$F^{on}_j(0) = 0, \;\; \textit{if } j \in S'_- \cap S_+,$$

$$F^{on}(0)[Q_{--}]_{\cdot,j} + F^{on}(0)[Q_{-+} J_{+-}(0)]_{\cdot,j} = 0, \;\; \textit{if } j \in S'_- \cap S_0.$$

After we take the LST of both sides of Eq. (4.11) we have

$$s(\tilde{F}^{on}(s) - F^{on}(0)) R_{--} = \tilde{F}^{on}(s) Q_{--} + \tilde{F}^{on}(s) Q_{-+} \tilde{J}_{+-}(s),$$

which is equivalent to

$$\tilde{F}^{on}(s)[sR_{--} - Q_{--} - Q_{-+}\tilde{J}_{+-}(s)] = sF^{on}(0)R_{--}.$$

Here the elements of the matrix $\tilde{J}_{+-}(s)$ are given by $\tilde{J}_{ji}(s) = \tilde{\psi}_{ji}(0, s),\ j \in S'_+,\ i \in S'_-$, from Eq. (3.7). To find the remaining $F^{on}_j(0),\ j \in S'_- \cap S_-$ first note that $S'_- \cap S_- = S_-$ and therefore we need $|S_-| = N_-$ equations. The following result is required to derive these equations.

Theorem 4.2 *The generalized eigenvalue problem*

$$(sR_{--} - Q_{--} - Q_{-+}\tilde{J}_{+-}(s))\phi = 0$$

has exactly N_- solutions $(s_1, \phi_1), \ldots, (s_{N_-}, \phi_{N_-})$, with $s_1 = 0, Re(s_i) > 0,\ i = 2, \ldots, N_-$ and $\phi_i \neq 0$.

Define $\Gamma_{--} := [\Gamma_{ij},\ i, j \in S'_-]$ with elements

$$\Gamma_{ij} := \sum_{k \in S'_+} q_{ik} m_{kj},$$

where m_{kj} is the mean size of a jump that starts in k and ends in j. Clearly m_{kj} can be computed by Theorem 3.1 as $-\frac{\partial}{\partial s}\tilde{\psi}_{ji}(x, s)|_{x=0,s=0}$.

Theorem 4.3 *The LST $\tilde{F}^{on}(s)$ is given by the solution to*

$$\tilde{F}^{on}(s)(sR_{--} - Q_{--} - Q_{-+}\tilde{J}_{+-}(s)) = sF^{on}(0)R_{--},$$

where the unknowns $F^{on}_i(0),\ i \in S'_-$ are determined from the following set of equations

$$F^{on}_i(0) = 0,\ i \in S'_- \cap S_+,$$

$$F^{on}(0)[Q_{--}]_{\cdot,i} + F^{on}(0)[Q_{-+}J_{+-}(0)]_{\cdot,i} = 0,\ i \in S'_- \cap S_0,$$

$$F^{on}(0)R_{--}\phi_i = 0,\ \textit{for } i = 2, \ldots, N_-,$$

$$F^{on}(0)R_{--}e = \pi(R_{--} + \Gamma_{--})e.$$

Before we proceed with the analysis of the two buffers content processes during off-periods we collect some necessary preliminary results in the next section.

5 Auxiliary Results

Consider ν_{ij} - the probability that the environment process is in state $j \in S'_+$ at the beginning of the next off-period given that the current on-period starts at state $i \in S'_-$, see Fig. 5.2 where T_n denotes the beginning of the n-th on-period and T'_n the beginning of the n-th off-period. The next result follows by first-step analysis.

Theorem 5.1 *The absorption probabilities $\nu_{kj},\ k \in S'_-,\ j \in S'_+$ are given by the solution to*

$$\sum_{k \in S'_-} q_{ik}\nu_{kj} + q_{ij} = 0,\ i \in S'_-,\ j \in S'_+.$$

Consider $\alpha_{jk}(x)$ - the probability that the environment process is in state $k \in S'_-$ at the beginning of the next on-period given that its current state is $j \in S$ and there is an amount x of type one fluid in the buffer. We use the following notation

$$\alpha_k(x) := [\alpha_{ik}(x),\ i \in S]^t, \quad \frac{d\alpha_k}{dx}(x) := \left[\frac{d\alpha_{ik}}{dx}(x),\ i \in S\right]^t,\ k \in S'_-.$$

Theorem 5.2 *For a fixed $k \in S'_-$ the column vector $\alpha_k(x)$ satisfies the differential equations*

$$R_1 \frac{d\alpha_k}{dx}(x) + Q\alpha_k(x) = 0,\ x \geq 0,\ k \in S'_-, \tag{5.12}$$

with boundary conditions

$$\alpha_{ik}(0) = \delta_{ik},\ i \in S'_-. \tag{5.13}$$

Proof: The proof is based on conditioning on a small interval of length $h > 0$ and using the fact that the probabilities of two or more transitions in the CTMC $\{I(t),\ t \geq 0\}$ over an interval of length h are $o(h)$, so

$$\alpha_{ik}(x) = \sum_{j \in S, j \neq i} q_{ij}h\alpha_{jk}(x + r_1(j)h) + (1 + q_{ii}h)\alpha_{ik}(x + r_1(i)h) + o(h).$$

After rearranging the last equation and letting $h \to 0$ we get

$$-r_1(i)\frac{d}{dx}\alpha_{ik}(x) = \sum_{j \in S} q_{ij}\alpha_{jk}(x),$$

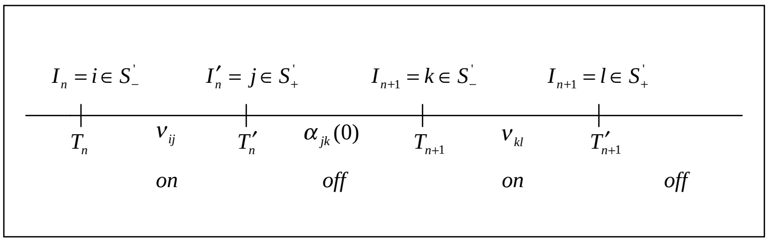

Figure 5.2: On- and off-periods for the two-priority fluid model; embedded processes

which can be written in matrix form to obtain (5.12). The boundary conditions (5.13) follow directly from the definition of $\alpha_{ik}(0)$. $\diamondsuit$

Now, consider the discrete time processes $\{I'_n,\ n \geq 0\}$ and $\{I_n,\ n \geq 0\}$ defined by the state of the governing CTMC at the beginning T'_n of the n-th off-period and at the beginning T_n of the n-th on-period, respectively. The next result follows trivially by noting that there is no type 1 fluid in the buffer at the beginning of on- and off-periods. Therefore knowing the state of the environment $I'_n := I(T'_n) = k$ and $I_n := I(T_n) = i$ is enough to determine I'_{n+1} and I_{n+1}.

Theorem 5.3 $\{I'_n,\ n \geq 0\}$ *and* $\{I_n,\ n \geq 0\}$, *are irreducible time-homogeneous Markov chains with state spaces* S'_+ *and* S'_-, *respectively. Their respective transition probability matrices* $P' = [p'_{kl},\ k, l \in S'_+]$ *and* $P = [p_{ij},\ i, j \in S'_-]$ *are given by*

$$p'_{kl} = \sum_{i \in S'_-} \alpha_{ki}(0)\nu_{il},\ k, l \in S'_+,$$

$$p_{ij} = \sum_{k \in S'_+} \nu_{ik}\alpha_{kj}(0),\ i, j \in S'_-.$$

The irreducibility of $\{I'_n,\ n \geq 0\}$ and $\{I_n,\ n \geq 0\}$ implies that the existance of their limiting probablity vectors denoted by $\pi' := [\pi'_k,\ k \in S'_+]$ and $\pi := [\pi_i,\ i \in S'_-]$, respectively.

The time between the beginning of two consecutive on-periods is called *a cycle*. Let τ_i denote the mean length of a cycle that starts in state i, u_i - the mean length of an on-period that starts in state i, and d_k - the mean length of an off-period that starts in state k. Then u_i is the mean first passage time into the set S'_+ of the CTMC $\{I(t),\ t \geq 0\}$, and is given by the unique solution to $\sum_{j \in S'_-} q_{ij} u_j = -1,\ i \in S'_-$, e.g., see Kulkarni (1995) [9]. Also, d_k can be found directly as the mean first passage time (to an empty buffer and a state of the environment in S'_-) in a standard fluid model with only the first type of fluid, see Theorem 4.2 in Kulkarni and Tzenova (2002) [10]. Thus, clearly

$$\tau_i = u_i + \sum_{k \in S'_+} \nu_{ik} d_k,\ i \in S'_-.$$

Before we discuss the two buffer content processes during an off-period in the next section we obtain

μ_{kj} - mean time between two consecutive type $k(\in S'_+)$ off-periods that end in $j \in S'_-$,

μ_j - mean time between two consecutive off-periods that end in $j \in S'_-$.

Lemma 5.4 *For all* $j \in S'_-,\ k \in S'_+$

$$\mu_{kj} = \frac{\sum_{l \in S'_+} \pi'_l d_l}{\pi'_k \alpha_{kj}(0)}, \tag{5.14}$$

$$\mu_j = \frac{\sum_{l \in S'_+} \pi'_l d_l}{\pi_j}. \tag{5.15}$$

Proof: Since time is restricted to only time spent in off-periods, the Elementary Renewal Theorem gives the mean number of off-periods per time unit in steady state as 1/E(Length of an off-period). Now, the long-run fraction of off-periods that start in state $k \in S'_+$ and end in state $j \in S'_-$ in steady state is given by $\pi'_k \alpha_{kj}(0)$. Also, the mean length of an off-period is given by $\sum_{l \in S'_+} \pi'_l d_l$. Hence

$$\frac{1}{\mu_{kj}} = \frac{\pi'_k \alpha_{kj}(0)}{\sum_{l \in S'_+} \pi'_l d_l},$$

which is equivalent to Eq. (5.14). Eq. (5.15) can be obtained similarly. $\diamondsuit$

Note that

$$\sum_{k \in S'_+} \pi'_k \alpha_{kj}(0) = \pi_j, \quad \sum_{j \in S'_-} \pi_j \nu_{jk} = \pi'_k, \text{ and } \frac{1}{\mu_j} = \sum_{k \in S'_-} \frac{1}{\mu_{kj}}.$$

6 The Two Buffer Content Processes during an Off-Period

In this section we compute $\tilde{F}_i^{off}(s_1, s_2)$ as defined in Eq. (2.3) above, where $X_k^{off}(t)$ and $I^{off}(t)$ are the buffer content processes and the governing environment embedded over off-periods only (see Fig. 2.1). In addition, consider the process $Y(t) := I(T_n'),\ T_n' \leq t < T_{n+1}'$ where T_n' denotes the beginning of the n-th off-period (see Fig. 5.2). Clearly, $Y(t)$ stays the same within a given off-period and the following on-period and denotes the state in which the most recent off-period has started. For a fixed $k \in S_+'$, we refer to the off-periods that start in k as *type k off-periods.* First, we consider an off-period of type k and determine

$$\tilde{G}_{kj}(s_1, s_2) := \lim_{t\to\infty} E(e^{-s_1 X_1^{off}(t)} e^{-s_2 A_2^{off}(t)}; I^{off}(t) = j, Y^{off}(t) = k),\ k \in S_+',\ j \in S,$$

where $A_2^{off}(t)$ denotes the total amount of type 2 fluid accumulated in the buffer by time t during the current off-period (of type k). To simplify notation, in the rest of this section X_1, X_2, Y, and A_2 will represent X_1^{off}, X_2^{off}, Y^{off}, and A_2^{off}, unless indicated otherwise. Also let $\tilde{B}_{kj}(s_2)$ be the LST of type 2 fluid that is added to the buffer within an arbitrary type k off-period of length T that finishes in state j, i.e.,

$$\tilde{B}_{kj}(s_2) := E(e^{-s_2 A_2(T)}; I(T) = j, Y(0) = k),\ k \in S_+',\ j \in S.$$

Clearly, $\tilde{B}_{kj}(s_2) = 0$ if $j \in S_+'$ and from Eq. (3.7) we have

$$\tilde{B}_{kj}(s_2) = \pi_k' \tilde{\psi}_{kj}(0, s_2),\ j \in S_-',$$

where $\tilde{\psi}_{kj}(0, s_2)$ are found from Theorem 3.1. For a fixed $k \in S_+'$ we intorduce the row vectors $\tilde{B}_k(s_2) := [\tilde{B}_{kj}(s_2), j \in S]$, $\tilde{G}_k(s_1, s_2) := [\tilde{G}_{kj}(s_1, s_2),\ j \in S]$. To further facilitate the notation in the sequel we write $\alpha_{kj} := \alpha_{kj}(0)$. In the next theorem we find $\tilde{G}_k(s_1, s_2)$ as a solution to a linear system of equations.

Theorem 6.1 *$\tilde{G}_k(s_1, s_2)$ is given by the solution to the following linear system of equations*

$$\tilde{G}_k(s_1, s_2)(s_1 R_1 + s_2 P_2 - Q) = le_k - m\tilde{B}_k(s_2), \tag{6.16}$$

where $l := \sum_{j\in S'} \frac{\nu_{jk}}{\mu_j}$, e_k *is the standard unity vector with 1 at position* k *and* $m = 1/\sum_{i\in S'_+} \pi'_i d_i$.

Proof: Let $k \in S'_+$ be fixed. Consider

$$G_{kj}(x_1,a_2) := \lim_{t\to\infty} P(X_1^{off}(t) < x_1, A_2^{off}(t) < a_2; I^{off}(t) = j, Y^{off}(t) = k),\ j \in S.$$

Following standard techniques we can derive the following differential equations

$$r_1(j)\frac{\partial}{\partial x_1}G_{kj}(x_1,a_2) + r_2(j)\frac{\partial}{\partial a_2}G_{k,j}(x_1,a_2) =$$

$$\sum_{i\in S} q_{ij}G_{ki}(x_1,a_2) - \delta_{jk}\sum_{l:r_1(l)\geq 0}\ \sum_{i:r_1(i)<0} \nu_{ij} r_1(i)\frac{\partial}{\partial x_1}G_{li}(0,\infty),\ j \in S'_+. \tag{6.17}$$

and

$$r_1(j)\frac{\partial}{\partial x_1}G_{kj}(x_1,a_2) + r_2(j)\frac{\partial}{\partial a_2}G_{kj}(x_1,a_2) - r_1(j)\frac{\partial}{\partial x_1}G_{kj}(0,a_2) = \sum_{i\in S} q_{ij}G_{ki}(x_1,a_2),\ j \in S'_-. \tag{6.18}$$

Before we take the Laplace Transform (LT) of Eq. (6.18) we need to evaluate the integral

$$\int_{x_1=0}^{\infty}\int_{a_2=0}^{\infty}\frac{\partial}{\partial x_1}G_{kj}(0,a_2)e^{-s_1x_1}e^{-s_2a_2}dx_1da_2 = \frac{1}{s_1}\int_0^{\infty}\frac{\partial}{\partial x_1}G_{kj}(0,a_2)e^{-s_2a_2}da_2 =$$
$$\frac{1}{s_1}\frac{\partial}{\partial x_1}G_{kj}(0,\infty)\frac{1}{\pi'_k\alpha_{kj}}\int_0^{\infty}\frac{\pi'_k\alpha_{kj}\frac{\partial}{\partial x_1}G_{kj}(0,a_2)}{\frac{\partial}{\partial x_1}G_{kj}(0,\infty)}e^{-s_2a_2}da_2. \tag{6.19}$$

Next we note that

$$\pi'_k\alpha_{kj}\frac{\frac{\partial}{\partial x_1}G_{kj}(0,a_2)}{\frac{\partial}{\partial x_1}G_{kj}(0,\infty)}$$

is equal to the c.d.f. of the total accumulation of type 2 fluid during an off-period, and the off-period is of type k and ends in state j. Hence, the integral in (6.19) equals $\tilde{B}_{kj}(s_2)/s_2$, and (6.19) can be written as

$$\frac{1}{s_1s_2}\frac{\partial}{\partial x_1}G_{kj}(0,\infty)\frac{\tilde{B}_{kj}(s_2)}{\pi'_k\alpha_{kj}}.$$

Taking LT of both sides of Eqs. (6.17) and (6.18) leads to

$$r_1(j)s_1G^*_{kj}(s_1,s_2) + r_2(j)s_2G^*_{kj}(s_1,s_2) =$$

$$\sum_{i\in S} q_{ij} G^*_{ki}(s_1,s_2) - \frac{\delta_{jk}}{s_1 s_2} \sum_{l:r_1(l)\geq 0} \sum_{i:r_1(i)<0} \nu_{ij} r_1(i) \frac{\partial}{\partial x_1} G_{li}(0,\infty),\ j \in S'_+,$$

$$r_1(j)s_1 G^*_{kj}(s_1,s_2) + r_2(j)s_2 G^*_{kj}(s_1,s_2) - \frac{r_1(j)}{s_1 s_2} \frac{\partial}{\partial x_1} G_{kj}(0,\infty) \frac{\tilde{B}_{kj}(s_2)}{\pi'_k \alpha_{kj}} =$$

$$\sum_{i\in S} q_{ij} G^*_{ki}(s_1,s_2),\ j \in S'_-.$$

Next, note that

$$-r_1(j) \frac{\partial}{\partial x_1} G_{kj}(0,\infty),\ j \in S'_-$$

represents the mean number of type k off-periods that end in state j, $j \in S'_-$, per time unit in steady state and therefore, from the classic theory of Renewal processes, we have

$$-r_1(j) \frac{\partial}{\partial x_1} G_{kj}(0,\infty) = \frac{1}{\mu_{kj}},$$

where μ_{kj} is given by (5.14). Now, the above equations become

$$r_1(j)s_1 G^*_{kj}(s_1,s_2) + r_2(j)s_2 G^*_{kj}(s_1,s_2) =$$

$$\sum_{i\in S} q_{ij} G^*_{ki}(s_1,s_2) + \frac{\delta_{jk}}{s_1 s_2} \sum_{l:r_1(l)\geq 0} \sum_{i:r_1(i)<0} \nu_{ij} \frac{1}{\mu_{li}},\ j \in S'_+, \tag{6.20}$$

$$r_1(j)s_1 G^*_{kj}(s_1,s_2) + r_2(j)s_2 G^*_{kj}(s_1,s_2) + \frac{m\tilde{B}_{kj}(s_2)}{s_1 s_2} =$$

$$\sum_{i\in S} q_{ij} G^*_{ki}(s_1,s_2),\ j \in S'_-, \tag{6.21}$$

where in the last equation we use Lemma 5.4 to get

$$-r_1(j) \frac{\partial}{\partial x_1} G_{kj}(0,\infty) \frac{1}{\pi'_k \alpha_{kj}} = \frac{1}{\mu_{kj} \pi'_k \alpha_{kj}} = \frac{1}{\sum_{l\in S'_+} \pi'_l d_l} (= m).$$

Finally, (6.20) and (6.21) can be written in matrix form to obtain (6.16).

$\diamondsuit$

We proceed with the study of X_2 and I at the *beginning* of an arbitrary off-period. Consider a given state $k \in S'_+$ and denote by $H_k(x)$ the steady-state probability that $X_2 < x$ and $I = k$ at the beginning of an arbitrary off-period.

Theorem 6.2 *For a fixed state $k \in S'_+$ and buffer content of type 2, $x \geq 0$ at the beginning of an off-period*

$$H_k(x) = \frac{\sum_{i \in S'_-} F_i^{on}(x) q_{ik}}{\sum_{i \in S'_-} F_i^{on}(\infty) q_{ik}} \pi'_k, \tag{6.22}$$

where $F_i^{on}(x)$ is as in Eq. (4.10).

Proof: Consider an arbitrary off-period of type k. Denote by M the total number of initiations per time unit of type k off-periods that start with $X_2 < x$. Such initiations can occur only from states in an on-period with $X_2 < x$, i.e., from states $(X_1 = 0, X_2 < x, I = i)$, for some $i \in S'_-$. The long-run fraction of time the system is in such states is $p^{on} \sum_{i \in S'_-} F_i^{on}(x)$ and the rate to state $(X_1 = 0, X_2 < x, I = k)$ is q_{ik}. Hence

$$M = p^{on} \sum_{i \in S'_-} F_i^{on}(x) q_{ik}.$$

Now, let N denote the number of times per time unit an off-period starts in state k. Clearly,

$$N = p^{on} \sum_{i \in S'_-} F_i^{on}(\infty) q_{ik}.$$

Thus, the conditional probability that an off-period starts with $X_2 < x$ given that it starts in state k is equal to M/N. We obtain the joint steady-state probability $H_k(x)$ as given in Eq. (6.22) by recalling that π'_k denotes the long-run fraction of number of off-periods that start in k. ♢

After we take LST of Eq. (6.22) we obtain the following Corollary.

Corollary 6.3 *The LST of the amount of type 2 fluid in the buffer at the beginning of an arbitrary off-period that starts in state k in steady-state is given by*

$$\tilde{H}_k(s) = \frac{\sum_{i \in S'_-} \tilde{F}_i^{on}(s) q_{ik}}{\sum_{i \in S'_-} F_i^{on}(\infty) q_{ik}} \pi'_k.$$

Now, we are ready to formulate the main result of this section which gives the LST of the limiting joint distribution of the two buffer content processes during off-periods. The result follows immediately by noting that for an arbitrary off-period of type k the amount of type 2 fluid at the beginning of the off-period is independent of the amount of type 2 fluid accumulated in the buffer.

Theorem 6.4 *For all $i \in S$*

$$\tilde{F}_i^{off}(s_1, s_2) = E(e^{-s_1 X_1^{off}} e^{-s_2 X_2^{off}}; I^{off} = i) = \sum_{k \in S'_+} \tilde{H}_k(s_2) \frac{\tilde{G}_{ki}(s_1, s_2)}{\pi'_k}.$$

Now the LST of the limiting distribution of $(X_1(t), X_2(t), I(t))$ can be computed by putting together the two components of Eq. (2.4), namely $\tilde{F}_i^{on}(s_2)$ and $\tilde{F}_i^{off}(s_2)$ as found above.

7 Examples

For illustration purposes we consider a two-priority fluid model where the fluid of type k is generated by a Markov on-off source with on-times Exp(α_k) and off-times Exp(β_k). Such on/off sources are frequently used to model traffic streams in telecommunication systems. The sources behave independently. While the k-th source is on (off) fluid of type k enters the buffer at rate $p_k > 0$ (zero). We assume that $p_1 \geq \mu$ to avoid trivialities and to insure off-periods, i.e. $S'_- \neq \emptyset$. The system is stable if

$$\sum_{k=1}^{2} \frac{p_k \beta_k}{\alpha_k + \beta_k} < \mu.$$

We present numerical results for identical sources with $\alpha_k = \alpha = 4, \beta_k = \beta = 1,\ k = 1, 2$; the input rate of the first source is $p_1 = 8$ and we compare the two cases of $p_2 = 1$ and $p_2 = 15$. The idea is to observe the impact on the system of the lower priority input rate by keeping the rest of parameters constant. The possible range for the server capacity, $\sum_{k=1}^{2} \frac{p_k \beta_k}{\alpha_k + \beta_k} < \mu \leq p_1$, in each case is $1.8 < \mu \leq 8$ and $4.6 < \mu \leq 8$, respectively. The numerical results are obtained by numerical inversion of the LSTs using the EULER algorithm of Abate and Whitt (1992).

The governing CTMC with state space $S = \{(0,0), (0,1), (1,0), (1,1)\}$ has limiting distribution

$$\pi = \frac{1}{(\alpha + \beta)^2} [\alpha^2, \alpha\beta, \alpha\beta, \beta^2] = [0.64, 0.16, 0.16, 0.04].$$

Let X_2 denote the steady-state type 2 buffer content. Figures 7.3 and 7.4 show the joint and the marginal steady-state probabilities, $P(X_2 < t, I = i),\ i \in S$ and $P(X_2 < t)$, for the two cases of p_2 when the service capacity is $\mu = 7.5$. As expected these probabilities decrease as type 2 input rate p_2 increases and $P(X_2 < t, I = i) \to \pi_i$ as $t \to \infty$. Figures 7.5 and 7.6 show the behavior of $P(X_2 < t)$ for each case as the server capacity μ changes. As μ increases the marginal distributions also increase for each t. This implies that in steady state X_2 decreases stochastically as μ increases. An interesting characteristic of the system is the steady-state probability of no output of type 2 (the long-run fraction of time type 2 fluid does not pass through the server), $P(X_2 = 0, I = (0,0)) + P(X_2 = 0, I = (1,0))$. Figure 7.7 gives a comparison of these as μ varies from 4.75 to 7.55 in the three cases $p_2 = 1,\ p_2 = 6,\ p_2 = 15$. It supports the intuition that smaller values of p_2 lead to higher values of $P(X_2 = 0, I = (0,0)) + P(X_2 = 0, I = (1,0))$.

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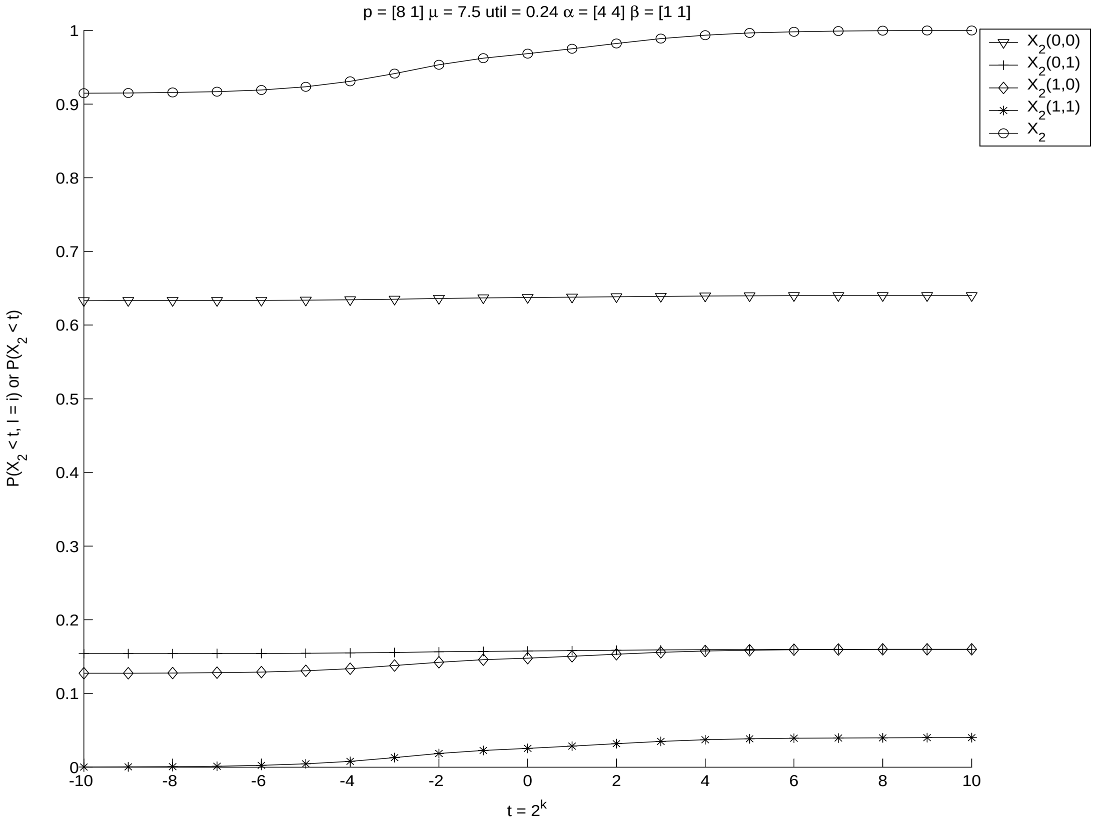

Figure 7.3: Steady-state distributions $P(X_2 < t, I = i \in S)$ and $P(X_2 < t)$ where $t = 2^k$; $p_1 = 8, p_2 = 1$, utilization:$= \sum_{i=1}^{2} \frac{p_i \beta_i}{\mu(\alpha_i + \beta_i)} = 0.24$

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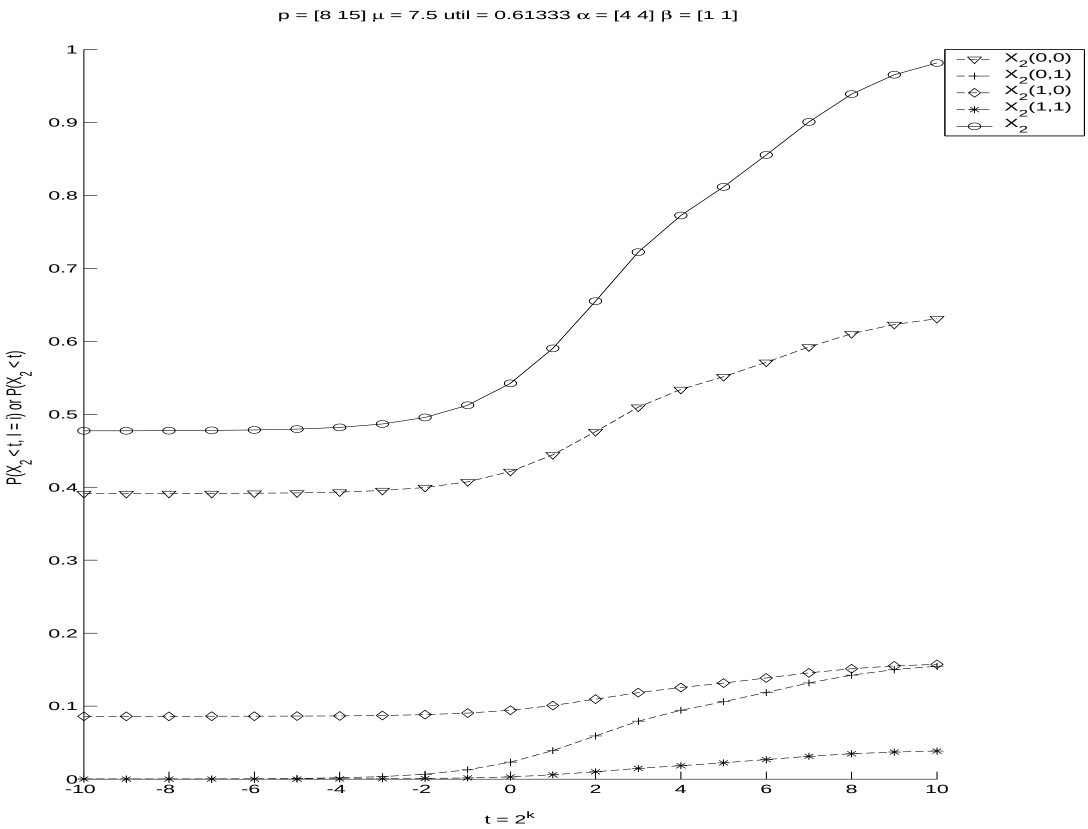

Figure 7.4: Steady-state distributions $P(X_2 < t, I = i \in S)$ and $P(X_2 < t)$ where $t = 2^k$; $p_1 = 8, p_2 = 15$, utilization:$= \sum_{i=1}^{2} \frac{p_i \beta_i}{\mu(\alpha_i + \beta_i)} = 0.61333$

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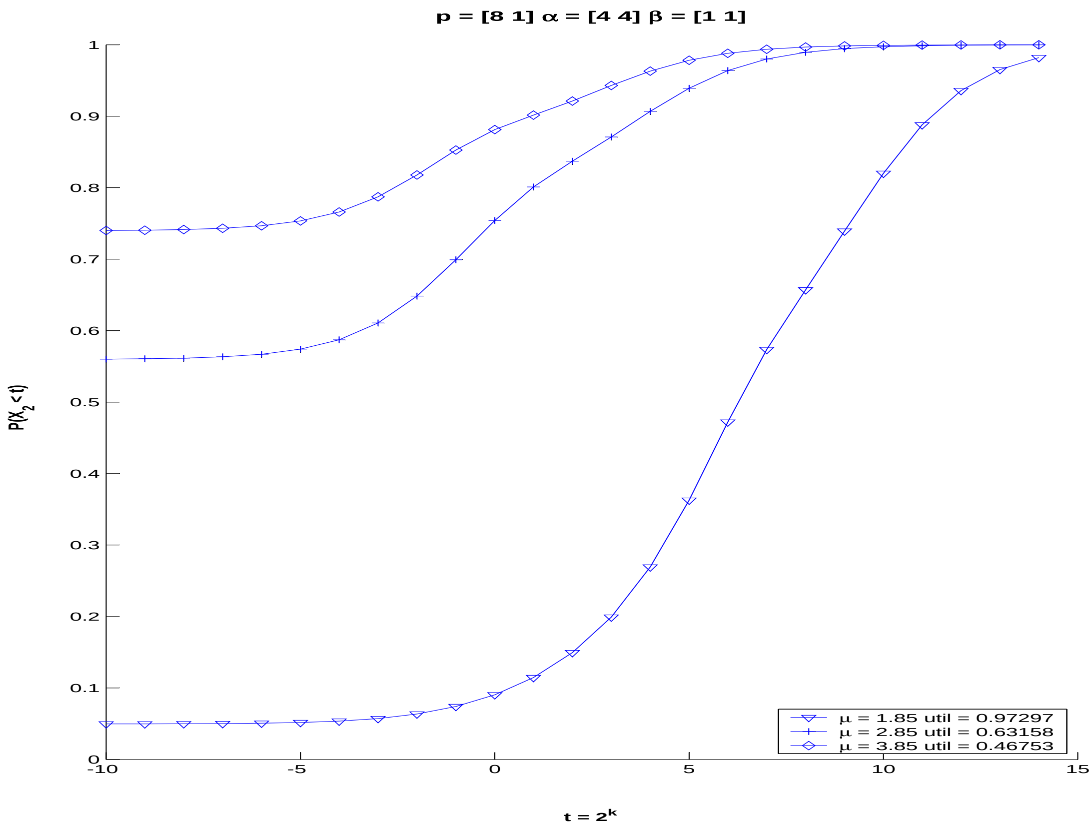

Figure 7.5: The marginal steady-state distribution $P(X_2 < t)$, $t = 2^k$, for different values of μ in the case $p_1 = 8, p_2 = 1$; the traffic intensity $\sum_{i=1}^{2} \frac{p_i \beta_i}{\mu(\alpha_i + \beta_i)}$ is given in the legend by util.

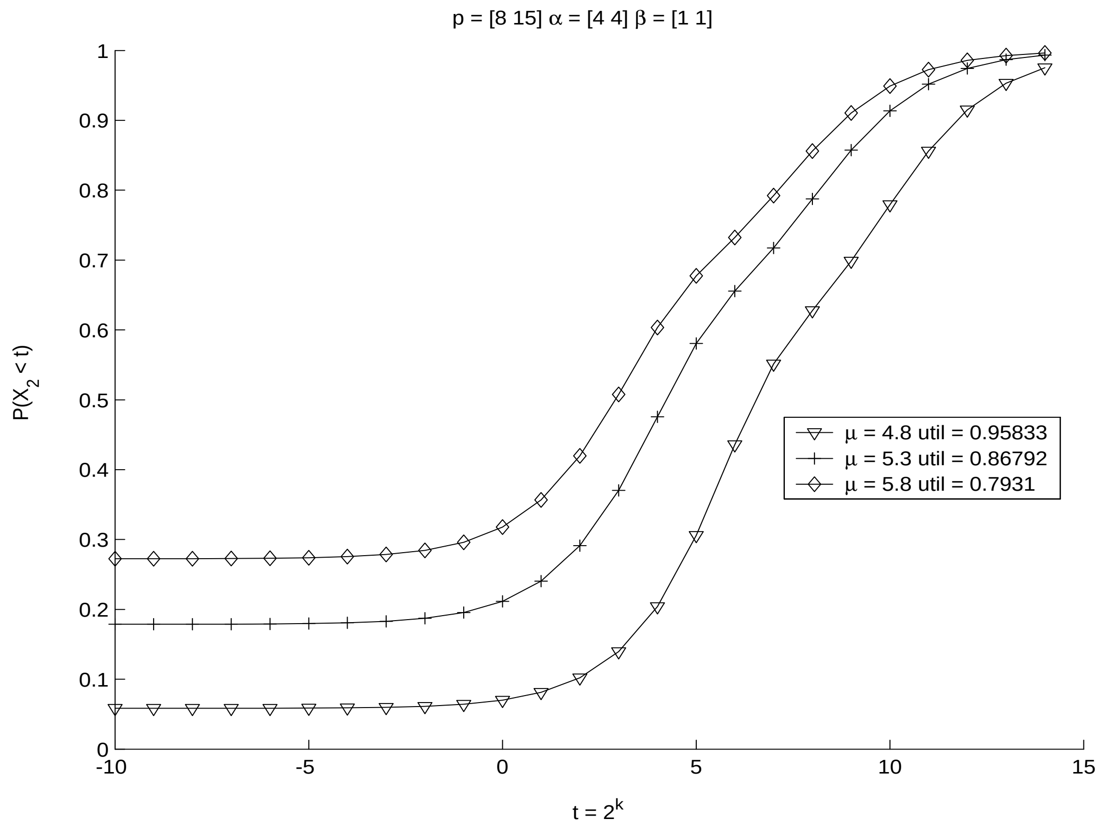

Figure 7.6: The marginal steady-state distribution $P(X_2 < t)$, $t = 2^k$, for different values of μ in the case $p_1 = 8, p_2 = 15$; the traffic intensity $\sum_{i=1}^{2} \frac{p_i \beta_i}{\mu(\alpha_i + \beta_i)}$ is given in the legend by util.

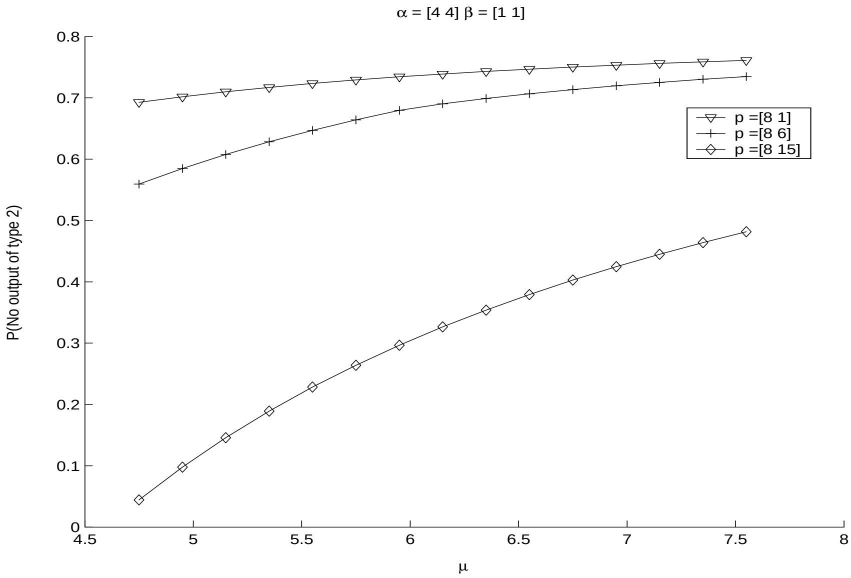

Figure 7.7: Comparison of the long-run probabilities of no output of type 2, $P(X_2 = 0, I = (0,0)) + P(X_2 = 0, I = (1,0))$, for the three cases as μ is varied between 4.75 and 7.55.