Infinite volume limits of high-dimensional sandpile models

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Abstract: We study the Abelian sandpile model on $\mathbb{Z}^d$. In $d \geq 5$ we prove existence of the infinite volume addition operator, almost surely w.r.t the infinite volume limit μ of the uniform measures on recurrent configurations. We prove the existence of a Markov process with stationary measure μ, and study ergodic properties of this process. The main techniques we use are a connection between the statistics of waves and uniform two-component spanning trees and results on the uniform spanning tree measure on $\mathbb{Z}^d$.

Key-words: Abelian sandpile model, wave, addition operator, two-component spanning tree, loop-erased random walk, tail triviality.

1 Introduction

The Abelian sandpile model (ASM), introduced originally in [2] has been studied extensively in the physics literature, mainly because of its remarkable "self-organized" critical state. Many exact results were obtained by Dhar using the group structure of the addition operators acting on recurrent configurations introduced in [4], see e.g. [5] for a review. The relation between recurrent configurations and spanning trees, originally introduced by [17] has been used by Priezzhev to compute the stationary height probabilities of the two-dimensional model in the thermodynamic limit [20]. Later on, Ivashkevich, Ktitarev and Priezzhev introduced the concept of "waves" to study the avalanche statistics, and made a connection between two-component spanning trees and waves [8, 9]. In [21] this connection was used to argue that the critical dimension of the ASM is $d = 4$.

From the mathematical point of view, one is interested in the thermodynamic limit, both for the stationary measures and for the dynamics. Recently, in [1] the

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connection between recurrent configurations and spanning trees, combined with results of Pemantle [19] on existence and uniqueness of the uniform spanning forest measure on $\mathbb{Z}^d$ has led to the result that the uniform measures μ_V on recurrent configurations in finite volume have a unique thermodynamic (weak) limit μ. In [14] this was proved for an infinite tree, and a Markov process generated by Poissonian additions to recurrent configurations was constructed. A natural continuation of [1] is therefore to study the dynamics defined on μ-typical configurations. The first question here is to study the addition operator. We first prove that in $d \geq 3$ the addition operator a_x can be defined on μ-typical configurations. This turns out to be a rather simple consequence of the transience of the simple random walk. However, in order to construct a stationary process with the infinite volume addition operators, it is crucial that the measure μ is invariant under a_x. We prove that this is the case if avalanches are μ-a.s. finite. In order to obtain a.s. finiteness of avalanches, we prove that the statistics of waves has a bounded density with respect to the uniform two-component spanning tree. The final step is to show that one component of the uniform two-component spanning tree is a.s. finite in the infinite volume limit when $d > 4$. This is proved using Wilson's algorithm combined with a coupling of the two-component spanning tree with the (usual) uniform spanning tree.

Given existence of a_x, and stationarity of μ under its action, we can apply the formalism developed in [14] to construct a stationary process which is informally described as follows. Starting from a μ-typical configuration η, at each site $x \in \mathbb{Z}^d$ grains are added on the event times of a Poisson process N_t^x with mean $\varphi(x)$, where $\varphi(x)$ satisfies the condition

$$\sum_x \varphi(x) G(0,x) < \infty,$$

with G the Green function. The condition ensures that the number of topplings has finite expectation at any time $t > 0$. In this paper we further study the ergodic properties of this process. We show that tail triviality of the measure μ implies ergodicity of the process. We prove that μ has trivial tail in any dimension $d \geq 2$. For $2 \leq d \leq 4$ this is a rather straightforward consequence of the fact that the height-configuration is a (non-local) coding of the edge configuration of the uniform spanning tree, i.e., from the spanning tree in *infinite volume* one can reconstruct the infinite height configuration almost surely. This is not the case in $d > 4$ where we need a separate argument.

Our paper is organized as follows. We start with notations and definitions, recalling some basic facts about the ASM. In sections 3 and 4 we prove existence of the addition operator a_x and invariance of the measure μ. In section 5 we prove existence of inverse addition operators. In section 6 we make the precise link between avalanches and waves, in section 7 we prove that waves are finite if the uniform two-component spanning tree has a.s. a finite component. In section 8 we prove the required a.s. finiteness of the component of the origin in dimensions $d \geq 5$. In sections 9 and 10 we discuss tail triviality of the stationary measure, and correspondingly, ergodicity of the stationary process.

2 Notations, definitions

We consider the Abelian sandpile model, as introduced by Bak, Tang and Wiesenfeld and generalized by Dhar. In this model one starts from a toppling matrix Δ_{xy}, indexed by sites in $\mathbb{Z}^d$. In this paper Δ will always be the adjacency matrix (or minus the discrete lattice Laplacian):

$$\Delta_{xy} = \begin{cases} 2d & \text{if } x = y, \\ -1 & \text{if } |x-y| = 1, \\ 0 & \text{otherwise.} \end{cases}$$

A height configuration is a map $\eta : \mathbb{Z}^d \to \mathbb{N}$, and a stable height configuration is such that $\eta(x) \leq \Delta_{xx}$ for all $x \in \mathbb{Z}^d$. A site where $\eta(x) > \Delta_{xx}$ is called an unstable site.

All stable configurations are collected in the set Ω. We endow Ω with the product topology. For $V \subseteq \mathbb{Z}^d$, Ω_V denotes the stable configurations $\eta : V \to \mathbb{N}$. If $\eta \in \Omega$ and $W \subseteq \mathbb{Z}^d$, then η_W denotes the restriction of η to the subset W. We also use η_W for the restriction of $\eta \in \Omega_V$ to a subset $W \subseteq V$. The matrix Δ_V is the finite volume analogon of Δ, indexed now by the sites in V.

A toppling of a site x in volume V is defined on configurations $\eta : V \to \mathbb{N}$:

$$T_x(\eta)(y) = \eta(y) - (\Delta_V)_{xy} \tag{2.1}$$

A toppling is called legal if the site is unstable, otherwise it is called illegal. The stabilization of an unstable configuration is defined to be the stable result of a sequence of legal topplings, i.e.,

$$\mathcal{S}(\eta) = T_{x_n} \circ T_{x_{n-1}} \circ \ldots \circ T_{x_1}(\eta), \tag{2.2}$$

where all topplings are legal and such that $\mathcal{S}(\eta)$ is stable. That $\mathcal{S}(\eta)$ is well-defined follows from [4, 18], see also [6]. If η is stable, then, by definition $\mathcal{S}(\eta) = \eta$. The addition operator is define by

$$a_x \eta = \mathcal{S}(\eta + \delta_x) \tag{2.3}$$

As long as we are in finite volume, a_x is well-defined and $a_x a_y = a_y a_x$ (Abelianness).

The dynamics of the finite volume ASM is described as follows: at each discrete time step choose at random a site according to a probability measure $p(x) > 0, x \in V$, and apply a_x to the configuration. After time n the configuration is $\prod_{i=1}^n a_{X_1} \ldots a_{X_n} \eta$ where $X_1, \ldots, X_n$ is an i.i.d. sample of p. This gives a Markov chain with transition operator

$$Pf(\eta) = \sum_x p(x) f(a_x \eta) \tag{2.4}$$

Given a function $F(V)$ defined for all sufficiently large finite volumes in $\mathbb{Z}^d$, and taking values in a metric space with metric ρ, we say that $\lim_V F(V) = a$, if for all $\varepsilon > 0$ there exists W, such that $\rho(F(V), a) < \varepsilon$ whenever $V \supseteq W$. For a probability measure ν on Ω, $\mathbb{E}_\nu$ will denote expectation with respect to ν. The boundary of V is defined by $\partial V = \{y \in V : y \text{ has a neighbour in } V^c\}$, while its exterior boundary is $\partial_e V = \{y \in V^c : y \text{ has a neighbour in } V\}$.

2.1 Recurrent configurations

A stable configuration $\eta \in \Omega_V$ is called recurrent ($\in \mathcal{R}_V$) if it is recurrent in the Markov chain, or equivalently, if for any x there exists $n = n_x$ such that $a_x^n \eta = \eta$. The addition operators restricted to the recurrent configurations form an Abelian group and from that fact one easily concludes that the uniform measure μ_V on $\mathcal{R}_V$ is the unique invariant measure of the Markov chain. One can compute the number of recurrent configurations:

$$|\mathcal{R}_V| = \det(\Delta_V), \tag{2.5}$$

see [4]. Another important identity of [4] is the following. Denote by $N_V(x, y, \eta)$ the number of legal topplings at y needed to obtain $a_x\eta$ from $\eta+\delta_x$. Then the expectation satisfies

$$\mathbb{E}_{\mu_V}(N_V(x, y, \eta)) = G_V(x, y) = (\Delta_V)^{-1}_{xy}. \tag{2.6}$$

From this and the Markov inequality, one also obtains $G_V(x, y)$ as an estimate of the μ_V-probability that a site y has to be toppled if one adds at x. We also note that for our specific choice of Δ, G_V is $(2d)^{-1}$ times the Green function of simple random walk in V killed upon exiting V.

Recurrent configurations are characterized by the so-called burning algorithm [4]. A configuration η is recurrent if and only if it does not contain a so-called forbidden sub-configuration, that is, a subset $W \subseteq V$ such that for all $x \in W$:

$$\eta(x) \leq - \sum_{y \in W \setminus \{x\}} \Delta_{xy}. \tag{2.7}$$

From this explicit characterization, one easily infers a consistency property: if $\eta \in \mathcal{R}_V$ and $W \subseteq V$ then $\eta_W \in \mathcal{R}_W$. From that one naturally defines "recurrent configurations in infinite volume", as those configurations in Ω such that for all $V \subseteq \mathbb{Z}^d$, $\eta_V \in \mathcal{R}_V$. This set is denoted by $\mathcal{R}$.

2.2 Infinite volume: basic questions and results

In studying infinite volume limits of the ASM, the following questions are addressed. In this (non-exhaustive) list, any question can be asked only after a positive answer to all previous questions.

1. Do the measures μ_V weakly converge to a measure μ? Does μ concentrate on the set $\mathcal{R}$?

2. Is the addition operator a_x defined on μ-a.e. configuration $\eta \in \mathcal{R}$, and does it leave μ invariant? Does Abelianness still hold in infinite volume?

3. Can one define a natural Markov process on $\mathcal{R}$ with stationary distribution μ?

4. Has the stationary Markov process of question 3 good ergodic properties?

Question 1 is easily solved in $d = 1$, but unhappily, μ is trivial, concentrating on the single configuration that is identically 2. Hence no further questions on our list are relevant in that case. See [16] for a result on convergence to equilibrium in this case. For an infinite regular tree, the first three questions have been answered affirmatively and the fourth question remained open [14]. In dissipative models ($\Delta_{xx} > 2d$), all four questions are affirmatively answered when Δ_{xx} is sufficiently large [15].

For $\mathbb{Z}^d$, question 1 is positively answered in any dimension $d \geq 2$, using a correspondence between spanning trees and recurrent configurations and properties of the uniform spanning forest on $\mathbb{Z}^d$ [1]. The limiting measure μ is translation invariant. The proof of convergence in [1] in the case $d > 4$ is restricted to regular volumes, such as a sequence of cubes centered at the origin. In the appendix, we outline how to prove convergence along an arbitrary sequence of volumes using the result of [11].

In this paper we study questions 2,3 and 4 for $\mathbb{Z}^d$, $d \geq 5$, and all questions are affirmatively answered.

The main problem is to prove that avalanches are a.s. finite. This is done by a decomposition of avalanches into a sequence of waves (cf. [9, 10]), and studying the a.s. finiteness of the waves. The latter can be achieved by a two-component spanning tree representation of waves, as introduced in [9, 10]. We then study the uniform two-component spanning tree in infinite volume and prove that the component containing the origin is a.s. finite. This turns out to be sufficient to ensure finiteness of waves.

3 Existence of the addition operator

In this section we show convergence of the finite volume addition operators to an infinite volume addition operator when $d > 2$. This is actually very easy, but in order to make appropriate use of this infinite volume addition operator, we need to establish that μ is invariant under its action, and for the latter we need to show that avalanches are finite μ-a.s.

Given $\eta \in \Omega$, call $N_V(x, y, \eta)$ the number of topplings caused at y by addition at x in η, where we apply the finite (V)-volume rule, that is, grains falling out of V disappear. More precisely, let $a_{x,V}$ denote the addition operator acting on Ω_V, and for $\eta \in \Omega$ define

$$a_{x,V}\eta = (a_{x,V}\eta_V)\eta_{V^c}. \tag{3.1}$$

Then

$$\eta + \delta_x - \Delta_V N_V(x, \cdot, \eta) = a_{x,V}\eta, \tag{3.2}$$

where $\Delta_V(x, y) = \Delta(x, y)I[x \in V]I[y \in V]$ is the toppling matrix restricted to V. We start with the following simple lemma:

Lemma 3.3. *$N_V(x, y, \eta)$ is a non-decreasing function of V and depends only on η through η_V.*

Proof. Let $V \subseteq W$. Suppose we add a grain at x in configuration η. We perform topplings inside V until inside V the configuration is stable. The result of this procedure is a configuration $(a_{x,V}\eta_V)\xi_{V^c}$ Possibly $\xi_{V^c\cap W}$ is not stable, in that case we perform all the necessary topplings still needed to stabilize $(a_{x,V}\eta_V)\xi_{V^c\cap W}$ inside W. This can only cause (possibly) extra topplings at any site y inside V. $\square$

From Lemma 3.3 and by monotone convergence:

$$\mathbb{E}_\mu(\sup_V N_V(x,y,\eta)) = \lim_V \mathbb{E}_\mu(N_V(x,y,\eta)). \tag{3.4}$$

By weak convergence of μ_V to μ:

$$\begin{aligned}\lim_V \mathbb{E}_\mu(N_V(x,y,\eta)) &= \lim_V \lim_{W\supset V} \mathbb{E}_{\mu_W}(N_V(x,y,\eta)) \\ &\leq \lim_V \lim_{W\supset V} \mathbb{E}_{\mu_W}(N_W(x,y,\eta)) \\ &= \lim_W G_W(x,y) = G(x,y). \end{aligned} \tag{3.5}$$

In the last step we used that $d > 2$, otherwise $G_W(x,y)$ diverges as $W \uparrow \mathbb{Z}^d$. This proves that for all $x, y \in \mathbb{Z}^d$, μ-a.s. $N(x,y,\eta) = \sup_V N_V(x,y,\eta)$ is finite and hence

$$\mu\left(\forall x, y \in \mathbb{Z}^d : N(x,y,\eta) < \infty\right) = 1 \tag{3.6}$$

Therefore, on the event in (3.6), we can define

$$a_x\eta = \eta + \delta_x - \Delta N(x,\cdot,\eta). \tag{3.7}$$

It is easy to see that $a_x\eta$ is stable, using that $a_x\eta(y)$ is already determined by the number of topplings at y and its neighbours. We also get

$$a_x\eta = \lim_V a_{x,V}\eta, \quad \mu\text{-a.s.}, \tag{3.8}$$

where $a_{x,V}$ is defined in (3.1).

Note that with this definition, there can be infinite avalanches. However, if the volume increases, it cannot happen that the number of topplings at a fixed site diverges, and that is the only problem for defining a_x (a problem which could arise in $d = 2$). More precisely, an infinite avalanche that leaves eventually every finite region does not pose a problem for defining the addition operator. However, as we will see later on, infinite avalanches do cause problems in defining a stationary process. To define a_x we only need $d > 2$, however to exclude infinite avalanches our method will require $d > 4$.

It is obvious that a_x is well behaved with respect to translations, i.e.,

$$a_x = \theta_x \circ a_0 \circ \theta_{-x} \tag{3.9}$$

where θ_x is the shift on configurations: $\theta_x\eta(y) = \eta(y + x)$.

Integrating (3.7) over μ we easily obtain the following infinite volume analogue of Dhar's formula [4].

Proposition 3.10. *If μ is invariant under a_x, then*

$$\mathbb{E}_\mu(N(x,y,\eta)) = G(x,y) \tag{3.11}$$

At this point we cannot compose different a_x. Although a_x is well-defined a.s., it is not obvious that a_y can be applied on $a_x\eta$.

Proposition 3.12. *If μ is invariant under the action of a_0, then μ is also invariant under the action of a_x for all x, and there exists a μ-measure one set Ω', such that for any $\eta \in \Omega'$, and every $x_1, \ldots, x_n \in \mathbb{Z}^d$,*

$$a_{x_n} \circ a_{x_{n-1}} \circ \ldots a_{x_0}\eta$$

is well-defined.

Proof. If μ is invariant under a_0, then by translation invariance of μ, and by (3.9), it is invariant under all a_x. Define Ω_0 to be the set of those η where $a_x\eta$ is well-defined for all $x \in \mathbb{Z}^d$. For $n \geq 1$, define inductively the sets

$$\Omega_n = \Omega_{n-1} \cap \bigcap_{x\in\mathbb{Z}^d} a_x^{-1}(\Omega_{n-1}),$$

where a_x^{-1} here denotes inverse image (not to be confused with the inverse operator defined later). Since the a_x are measure preserving, it follows by induction that $\mu(\Omega_n) = 1$ for all n, and that compositions of length $n+1$ are well-defined on Ω_n. Therefore, $\Omega' = \cap_{n\geq 0}\Omega_n$ satisfies the properties stated. $\square$

The following proposition shows that if "avalanches are finite" (see later for the precise definition of avalanches) then Abelianness holds in infinite volume.

Proposition 3.13. *Assume that μ is invariant under a_0. Further assume that there exists a measure one set Ω' such that for any $\eta \in \Omega'$ and any $x \in \mathbb{Z}^d$, $a_x\eta$ is well-defined and there exists $V_x(\eta)$ such that for all $W \supseteq V_x(\eta)$*

$$a_x\eta = a_{x,W}\eta. \tag{3.14}$$

Then the set Ω' can be chosen such that $a_x\eta \in \Omega'$ for all $\eta \in \Omega'$ and all $x \in \mathbb{Z}^d$. Moreover,

$$a_x(a_y\eta) = a_y(a_x\eta) \tag{3.15}$$

Proof. It is straightforward that the set Ω' can be chosen such that $a_x\eta \in \Omega'$ for all $x \in \mathbb{Z}^d$. For $\eta \in \Omega'$ and for $W \supset V_y(\eta) \cup V_x(a_y\eta) \cup V_x(\eta) \cup V_y(a_x\eta)$,

$$a_x(a_y\eta) = a_{x,W}(a_{y,W}\eta) = a_{y,W}(a_{x,W}\eta) = a_y(a_x\eta). \tag{3.16}$$

$\square$

4 Invariance of μ under a_x

In order to define the addition operator, all we needed was the convergence of the finite volume Green function to infinite volume Green function. However, in the construction of a stationary process, it is essential that the candidate stationary measure (which in this case is the infinite volume limit of the uniform measures on recurrent configurations) is invariant under the action of a_x.

The following proposition shows that μ is indeed invariant, if there are no infinite avalanches μ-a.s. We define the avalanche cluster caused by addition at x to be the set

$$\mathcal{C}_x(\eta) = \{y \in \mathbb{Z}^d : N(x, y, \eta) > 0\} \tag{4.1}$$

We say that the avalanche at x is finite in η if $\mathcal{C}_x(\eta)$ is a finite set. We say that μ has the finite avalanche property, if for all $x \in \mathbb{Z}^d$, $\mu(|\mathcal{C}_x| < \infty) = 1$.

Proposition 4.2. *If μ has the finite avalanche property then for any local function f and for any $x \in \mathbb{Z}^d$,*

$$\int f(a_x\eta) d\mu = \int f(\eta) d\mu. \tag{4.3}$$

Proof. We have

$$\begin{aligned}
\int f(a_x\eta) d\mu &= \int f(a_{x,V}\eta) d\mu + \epsilon_1(V, f) \\
&= \int f(a_{x,V}\eta) d\mu_W + \epsilon_1(V, f) + \epsilon_2(V, W, f) \\
&= \int f(a_{x,W}\eta) d\mu_W + \epsilon_1(V, f) + \epsilon_2(V, W, f) + \epsilon_3(V, W, f)
\end{aligned}$$

Here ϵ_1 and ϵ_2 can be made arbitrarily small by (3.8) and by weak convergence. We also have

$$|\epsilon_3(V, W, f)| \le 2\|f\|_\infty \mu_W(a_{x,W} f \ne a_{x,V} f).$$

Next, by invariance of μ_W under the action of $a_{x,W}$,

$$\int f(a_{x,W}\eta) d\mu_W = \int f d\mu_W = \int f d\mu + \epsilon_4(W, f). \tag{4.4}$$

Here, by weak convergence, ϵ_4 can be made arbitrarily small. Therefore, combining the estimates, we conclude

$$\left| \int f(a_x\eta) d\mu - \int f(\eta) d\mu \right| \le C \limsup_V \limsup_{W \supset V} \mu_W(a_{x,W} f \ne a_{x,V} f). \tag{4.5}$$

Define the avalanche cluster in volume W by

$$\mathcal{C}_{x,W}(\eta) = \{y \in W : N_W(x, y, \eta) > 0\}, \quad \eta \in \mathcal{R}_W.$$

Let D_f denote the dependence set of the local function f. On the event $\mathcal{C}_{x,W}(\eta)\cap\partial V = \varnothing$ we have $a_{x,V}\eta = a_{x,W}\eta$. Hence, provided $D_f \subseteq V$, we have

$$\mu_W(a_{x,W}f \neq a_{x,V}f) \leq \mu_W(\mathcal{C}_{x,W} \cap \partial V \neq \varnothing).$$

The event on the right hand side is a cylinder event (only depends on heights in V). Therefore, the right hand side approaches $\mu(\mathcal{C}_x \cap \partial V \neq \varnothing)$, as $W \uparrow \mathbb{Z}^d$. By the assumptions of the proposition,

$$\lim_V \mu(\mathcal{C}_x \cap \partial V \neq \varnothing) = \mu(|\mathcal{C}_x| = \infty) = 0,$$

which completes the proof. ∎

5 Inverse addition operators

In this section we prove that a_x has an inverse defined μ-a.s., provided μ has the finite avalanche property. Recall that if there are no infinite avalanches, then for μ-a.e. η and every $x \in \mathbb{Z}^d$, there exists a finite set $V_x(\eta)$ such that $a_x\eta = a_{x,V_x(\eta)}\eta$. Define

$$a_{x,V}^{-1}\eta = (a_{x,V}^{-1}\eta_V)\eta_{V^c}$$

This is well-defined since $\eta_V \in \mathcal{R}_V$.

Lemma 5.1. *Suppose that μ has the finite avalanche property.*

1. *For μ almost every η there exists $V_0 = V_0(\eta)$ such that $a_{x,V}^{-1}\eta = a_{x,V_0}^{-1}(\eta)$ for all $V \supseteq V_0$.*
2. *If we define $a_x^{-1}\eta = a_{x,V_0(\eta)}^{-1}(\eta)$, then μ-a.s. $a_x^{-1}(a_x\eta) = a_x(a_x^{-1}\eta) = \eta$.*
3. *As operators in $L_2(\mu)$, $a_x^* = a_x^{-1}$, i.e., the a_x are unitary operators.*

Proof. We will prove that

$$\lim_{V_0} \mu\left(\exists V \supseteq V_0 : a_{x,V}^{-1}(\eta) \neq a_{x,V_0}^{-1}(\eta)\right) = 0, \tag{5.2}$$

what is sufficient for the first statement, by monotonicity in V_0 of the event in (5.2). Write

$$\begin{aligned}
&\mu\left(\exists V \supseteq V_0 : a_{x,V}^{-1}(\eta) \neq a_{x,V_0}^{-1}(\eta)\right)\\
&\quad = \mu\left(\exists V \supseteq V_0 : a_{x,V}^{-1}(a_x\eta) \neq a_{x,V_0}^{-1}(a_x\eta)\right)\\
&\quad = \mu\left(\exists V \supseteq V_0 : a_{x,V}^{-1}(a_x\eta) \neq a_{x,V_0}^{-1}(a_x\eta) \text{ and } \forall V \supseteq V_0 : a_{x,V}(\eta) = a_{x,V_0}(\eta)\right) \qquad (5.3)\\
&\qquad\quad + \epsilon_{V_0}\\
&\quad = \epsilon_{V_0}.
\end{aligned}$$

Here we used the invariance of μ under a_x in the first step. The last step follows because if $a_x\eta = a_{x,V}\eta = a_{x,V_0}\eta$, then

$$a_{x,V}^{-1}(a_x\eta) = a_{x,V}^{-1}(a_{x,V}\eta) = \eta = a_{x,V_0}^{-1}(a_{x,V_0}\eta) = a_{x,V_0}^{-1}(a_x\eta). \tag{5.4}$$

Next, for ϵ_{V_0} we have

$$\epsilon_{V_0} \leq \mu\left(\exists V \supseteq V_0 : a_{x,V}\eta \neq a_{x,V_0}\eta\right) \tag{5.5}$$

which converges to zero as $V_0 \uparrow \mathbb{Z}^d$, by the finite avalanche property. This proves the first statement of the lemma. To prove the second statement first remark that by the definitions of a_x and a_x^{-1}, for μ almost every η there exists a (sufficiently large) V, such that

$$a_x(a_x^{-1}\eta) = a_{x,V}(a_{x,V}^{-1}\eta) = \eta = a_{x,V}^{-1}(a_{x,V}\eta) = a_x^{-1}(a_x\eta). \tag{5.6}$$

The last statement of the Lemma is an obvious consequence of the first two. □

The above lemma proves that as operators on $L_2(\mu)$, the a_x generate a unitary group, which we denote by G.

6 Waves and avalanches

The goal of Sections 6, 7 and 8 is to prove the following theorem.

Theorem 6.1. *Suppose $d > 4$. Then $\mu(|C_x| < \infty) = 1$ for all $x \in \mathbb{Z}^d$.*

Remark 6.2. *The assumption $d > 4$ can be replaced by the condition that $d \geq 3$ and the conclusion of Proposition 7.11 (ii) holds.*

In order to prove that avalanches are almost surely finite, we decompose avalanches into waves. We prove that almost surely, there is a finite number of waves, and that all waves are almost surely finite. Without loss of generality, we assume that $x = 0$ (the origin), and we drop indices referring to x from our notation.

We first recall the definition of a wave, cf. [9, 10]. Consider a finite volume W, and add a grain at site 0 in a stable configuration. If the site becomes unstable, then topple it once and topple all other unstable sites except 0. It is easy to see that in this procedure a site can topple at most once. The toppled sites form what is called the first wave. Next, if 0 has to be toppled again, we start another wave, and so on until 0 is stable.

We define $\alpha_W(\eta)$ to be the number of waves caused by addition at 0 in the volume W. By definition, α_W is the number of topplings at 0 in W, caused by addition at 0, that is $\alpha_W(\eta) = N_W(0,0,\eta)$. For fixed W, let $\mathcal{C}_W(\eta)$ denote the avalanche cluster in volume W. We decompose $\mathcal{C}_W$ as

$$\mathcal{C}_W(\eta) = \bigcup_{i=1}^{\alpha_W(\eta)} \Xi_W^i(\eta), \tag{6.3}$$

where $\Xi^i_W(\eta)$ is the i-th wave in W caused by addition at 0.

We can define waves in infinite volume as we defined the toppling numbers and avalanches in Section 3, by monotonicity in the volume. More precisely, the defintion is as follows. By the arguments of section 3, Ξ^1_W is a non-decreasing function of W, and therefore we can define $\Xi^1 = \cup_W \Xi^1_W = \lim_W \Xi^1_W$. If 0 is unstable after the first wave (now considered in infinite volume), we condsider the second wave $\widetilde{\Xi}^2_W$ with the finite volume rule. Again we have a nondecreasing family, and we set $\Xi^2 = \cup_W \widetilde{\Xi}^2_W = \lim_W \widetilde{\Xi}^2_W$. Note that if $|\Xi^1| < \infty$, we have $\widetilde{\Xi}^2_W = \Xi^2_W$ for all large W, and consequently, $\Xi^2 = \lim_W \Xi^2_W$. We similarly define $\widetilde{\Xi}^i_W$ as the result of the i-th wave with the finite volume rule after the first $i-1$ waves have been carried out in infinite volume. We let $\Xi^i = \cup_W \widetilde{\Xi}^i_W = \lim_W \widetilde{\Xi}^i_W$. Again, under the assumption $|\Xi^j| < \infty$, $1 \le j < i$, we also have $\Xi^i = \lim_W \Xi^i_W$. For convenience, we define Ξ^i_W or Ξ^i as the empty set, whenever such waves do not exist.

The easy part in proving finiteness of avalanches is to show that the number of waves is finite. Since $\alpha_W(\eta)$ is non-decreasing in W, it has a pointwise limit $\alpha(\eta)$, and as before,

$$\mathbb{E}_\mu(\alpha) \le G(0,0) < \infty. \tag{6.4}$$

This implies $\alpha < \infty$ μ-a.s.

In order to prove that $\mathcal{C}(\eta)$ is finite μ-a.s., we show, by induction on i, that all sets $\Xi^i(\eta)$ are finite μ-a.s. We base the proof on the following proposition, proved in Sections 7 and 8.

Proposition 6.5. *Let $d > 4$. For $i \ge 1$ we have*

$$\lim_V \limsup_{W \supset V} \mu_W(\Xi^i_W \not\subseteq V) = 0. \tag{6.6}$$

Noting that $\{\Xi^1 \subseteq V\}$ is a local event, Proposition 6.5 with $i = 1$ implies that $\mu(|\Xi^1| < \infty) = 1$. Assume now that $\mu(|\Xi^j| < \infty) = 1$, $1 \le j < i$. Then

$$\mu(\Xi^i \not\subseteq V) \le \mu(\Xi^i \not\subseteq V,\, \Xi^j \subseteq V',\, 1 \le j < i) + \mu(\Xi^j \not\subseteq V' \text{ for some } 1 \le j < i). \tag{6.7}$$

By the induction hypothesis, the second term on the right hand side can be made arbitrarily small by choosing V' large. For fixed V', the event in the first term is a local event (only depends on sites in $V' \cup \partial_e V'$, if $V' \supseteq V$). Therefore, the first term in (6.7) equals

$$\lim_W \mu_W(\Xi^i_W \not\subseteq V,\, \Xi^j_W \subseteq V',\, 1 \le j < i) \le \limsup_W \mu_W(\Xi^i_W \not\subseteq V). \tag{6.8}$$

Here the right hand side goes to 0 as $V \nearrow \mathbb{Z}^d$, by Proposition 6.5, proving that $\mu(|\Xi^i| < \infty) = 1$. Finiteness of all waves proved, we can pass to the limit in (6.3) and obtain the decomposition

$$\mathcal{C}(\eta) = \bigcup_{i=1}^{\alpha(\eta)} \Xi^i(\eta). \tag{6.9}$$

It follows that $\mu(|\mathcal{C}| < \infty) = 1$, which completes the proof of Theorem 6.1 assuming Proposition 6.5.

7 Finiteness of waves

In this section we prove Proposition 6.5 showing that waves are finite, assuming Proposition 7.11 below. The proof of Proposition 7.11 is completed in Section 8. To prove that waves are finite, we use their representation as two-component spanning trees [8, 9], which we now describe. Consider a configuration $\eta_W \in \mathcal{R}_W$ with $\eta_0 = 2d$, and suppose we add a particle at 0. Consider the first wave, which is entirely determined by the recurrent configuration $\eta_{W\setminus\{0\}}$. The result of the first wave on $W \setminus \{0\}$ is given by

$$S^1_W(\eta) = \Big(\prod_{j\sim 0} a_{j,W\setminus\{0\}}\Big)\eta_{W\setminus\{0\}}. \tag{7.1}$$

Next we associate to any $\eta_{W\setminus\{0\}} \in \mathcal{R}_{W\setminus\{0\}}$ a tree $\mathcal{T}_W(\eta_{W\setminus\{0\}})$, that will represent a wave starting at 0. For the definition, we use Majumdar and Dhar's tree construction [17].

Denote by $\widehat{W}$ the graph obtained from $\mathbb{Z}^d$ by identifying all sites in $\mathbb{Z}^d \setminus (W \setminus \{0\})$ to a single site δ_W (removing loops). By [17], there is a one-to-one map between recurrent configurations $\eta_{W\setminus\{0\}}$ and spanning trees of $\widehat{W}$. The correspondence is given by following the spread of an avalanche started at δ_W. Initially, each neighbour of δ_W receives a number of grains equal to the number of edges connecting it to δ_W, which results in every site toppling exactly once. The spanning tree records the sequence in which topplings have occurred. There is some flexibility in how to carry out the topplings (and hence in the correspondence with spanning trees), and here we make a specific choice in accordance with [9]. Namely, we first transfer grains from δ_W only to the neighbours of 0, and carry out all possible topplings. We call this the *first phase*. The set of sites that topple in the first phase is precisely a wave started at 0. Now transfer grains from δ_W to the boundary sites of W, which will cause topplings at all sites that were not in the wave; this is the second phase.

The two phases can alternatively be described via the burning algorithm of Dhar [4], which in the above context looks as follows. For convenience, let $\widetilde{W}$ denote the graph obtained by identifying all sites in $\mathbb{Z}^d \setminus W$ to a single site δ_W. That is, $\widehat{W}$ can be obtained from $\widetilde{W}$ by identifying 0 and δ_W. We start with all sites of $\widetilde{W}$ declared unburnt. At step 0 we burn 0 (the origin). At step t, we

$$\text{burn all sites } y \text{ for which } \eta_y > \text{number of unburnt neighbours of } y. \tag{7.2}$$

The process stops at some step $T = T(\eta_{W\setminus\{0\}})$. The sites that burn up to time T is precisely the sites toppling in the first phase. We continue by burning δ_W in step $T+1$, and then repeating (7.2) as long as there are unburnt sites.

Following Majumdar and Dhar's construction [17], we assign to each $y \in W \setminus \{0\}$ burnt at time t a unique neighbour y' (called the parent of y) burnt at time $t-1$. This defines a spanning subgraph of $\widetilde{W}$ with two tree components, having roots 0 and δ_W. Identifying 0 and δ_W yields a spanning tree of $\widehat{W}$, also representing $\eta_{W\setminus\{0\}}$. We denote by $\mathcal{T}_W(\eta_{W\setminus\{0\}})$ the component with root 0 (the origin). With slight abuse of language, we refer to the two-component graph as a *two-component spanning tree.*

We can generalize the above construction to further waves as follows. For $k \geq 2$, define

$$S_W^k(\eta) = \Big(\prod_{j\sim 0} a_{j,W\setminus\{0\}}\Big)^k \eta_{W\setminus\{0\}}. \tag{7.3}$$

If there exists a k-th wave, then its result on $W \setminus \{0\}$ is given by (7.3). Applying the above constructions to $S_W^{k-1}(\eta)$, we obtain that the k-th wave (if there is one) is represented by $\mathcal{T}_W(S_W^{k-1}(\eta))$.

We will prove that $\mathcal{T}_W$ has a weak limit $\mathcal{T}$, which is almost surely finite. But first let us show that this is actually sufficient for finiteness of all waves.

Consider the first wave, and let $W \supset V$. By construction, $\Xi_W^1(\eta)$ is precisely the vertex set of $\mathcal{T}_W(\eta_{W\setminus\{0\}})$, hence

$$\mu_W(\Xi_W(1,\eta) \not\subseteq V) = \mu_W(\mathcal{T}_W(\eta_{W\setminus\{0\}}) \not\subseteq V). \tag{7.4}$$

Here the right hand side is determined by the distribution of $\eta_{W\setminus\{0\}}$ under μ_W. This is different from the law of $\eta_{W\setminus\{0\}}$ under $\mu_{W\setminus\{0\}}$, which is simply the uniform measure on $\mathcal{R}_{W\setminus\{0\}}$. It is latter that we can get information about using the correspondence to spanning trees. Indeed, under $\mu_{W\setminus\{0\}}$, the spanning tree corresponding to $\eta_{W\setminus\{0\}}$ is uniformy distributed on the set of spanning trees of $\widehat{W}$. In order to translate our results back to μ_W, we show that the former distribution has a bounded density with respect to the latter. This will be a consequence of the following lemma.

Lemma 7.5. *There is a constant $C(d) > 0$ such that for all $d \geq 3$*

$$\sup_{V\subseteq\mathbb{Z}^d} \frac{|\mathcal{R}_{V\setminus\{0\}}|}{|\mathcal{R}_V|} \leq C(d) \tag{7.6}$$

Proof. By Dhar's formula (2.5),

$$|\mathcal{R}_{V\setminus\{0\}}| = \det(\Delta_{V\setminus\{0\}}) = \det(\Delta'_V)$$

where Δ'_V denotes the matrix indexed by sites $y \in V$ and defined by $(\Delta'_V)_{yz} = (\Delta_{V\setminus\{0\}})_{yz}$ for $y, z \in V \setminus \{0\}$, and $(\Delta'_V)_{0z} = (\Delta'_V)_{z0} = \delta_{0z}$. Clearly,

$$\Delta_V + P = \Delta'_V$$

where P is a matrix which has only non-zero entries P_{yz} for $y, z \in N = \{u : |u| \leq 1\}$. Moreover, $\max_{y,z\in V} P(0,y) \leq 2d-1$. Hence

$$\frac{|\mathcal{R}_{V\setminus\{0\}}|}{|\mathcal{R}_V|} = \frac{\det(\Delta_V + P)}{\det(\Delta_V)} = \det(I + G_V P),$$

where $G_V = (\Delta_V)^{-1}$. Here $(G_V P)_{yz} = 0$ unless $z \in N$. Therefore

$$\det(I + G_V P) = \det(I + G_V P)_{u\in N, v\in N} \tag{7.7}$$

By transience of the simple random walk in $d \geq 3$, we have $\sup_V \sup_{y,z} G_V(y,z) \leq G(0,0) < \infty$, and therefore the determinant of the finite matrix $(I + G_V P)_{u\in N,v\in N}$ in (7.7) is bounded by a constant depending on d. $\square$

We note that an alternative proof of Lemma 7.5 can be given based on the following idea. Consider the graph $\bar{W}$ obtained by adding an extra edge e between 0 and δ_W in $\widetilde{W}$. Then the ratio in (7.6) can be expressed in terms of the probability that a uniformly chosen spanning tree of $\bar{W}$ contains e. By standard spanning tree results [3, Theorem 4.1], the latter is the same as the probability that a random walk in $\bar{W}$ started at 0 first hits δ through e.

We continue with the bounded density argument. For any configuration $\sigma_{W\setminus\{0\}} \in \mathcal{R}_{W\setminus\{0\}}$ we have

$$\mu_W(\eta_{W\setminus\{0\}} = \sigma_{W\setminus\{0\}}) = \frac{1}{|\mathcal{R}_W|} \left|\{k \in \{1, \dots, 2d\} : (k)_0(\sigma)_{W\setminus\{0\}} \in \mathcal{R}_W\right|. \tag{7.8}$$

Therefore,

$$\frac{\mu_W(\eta_{W\setminus\{0\}} = \sigma_{W\setminus\{0\}})}{\mu_{W\setminus\{0\}}(\eta_{W\setminus\{0\}} = \sigma_{W\setminus\{0\}})} \leq \frac{|\mathcal{R}_{W\setminus\{0\}}|}{|\mathcal{R}_W|} 2d \leq C, \tag{7.9}$$

where, by (7.6), $C > 0$ does not depend on σ or on W. From this estimate, it follows that

$$\frac{\mu_W(\mathcal{T}_W(\eta_{W\setminus\{0\}}) \not\subseteq V)}{\mu_{W\setminus\{0\}}(\mathcal{T}_W(\eta_{W\setminus\{0\}}) \not\subseteq V)} \leq C. \tag{7.10}$$

For a more convenient notation, we let $\nu_W^{(0)}$ denote the probability measure assigning equal mass to each spanning tree of $\widehat{W}$, or alternatively, to each two-component spanning trees of $\widetilde{W}$. We can view $\nu_W^{(0)}$ as a measure on $\{0,1\}^{\mathbb{E}^d}$ in a natural way, where $\mathbb{E}^d$ is the set of edges of $\mathbb{Z}^d$. By the Majumdar-Dhar correspondence [17], $\nu_W^{(0)}$ corresponds with the measure $\mu_{W\setminus\{0\}}$, and the law of $\mathcal{T}_W$ under $\mu_{W\setminus\{0\}}$ is that of the component of 0 under $\nu_W^{(0)}$. We keep the notation $\mathcal{T}_W$ when referring to $\nu_W^{(0)}$. In Section 8 we prove the following Proposition.

Proposition 7.11. *(i) For any $d \geq 1$, the limit $\lim_W \nu_W^{(0)} = \nu^{(0)}$ exists.*

(ii) Assume $d > 4$. The component $\mathcal{T}$ of 0 under $\nu^{(0)}$ satisfies $\nu^{(0)}(|\mathcal{T}| < \infty) = 1$.

By Proposition 7.11 (i), we have

$$\lim_{W \supset V} \mu_{W\setminus\{0\}}(\mathcal{T}_W(\eta_{W\setminus\{0\}}) \not\subseteq V) = \lim_{W \supset V} \nu_W^{(0)}(\mathcal{T}_W \not\subseteq V) = \nu^{(0)}(\mathcal{T} \not\subseteq V). \tag{7.12}$$

By Proposition 7.11 (ii), the right hand side of (7.12) goes to zero as $V \nearrow \mathbb{Z}^d$, and together with (7.10) and (7.4), we obtain the $i = 1$ case of (6.6).

Finiteness of the other waves follows similarly. For $k \geq 2$ we have

$$\begin{aligned} \mu_W(\Xi_W^k(\eta) \not\subseteq V) &\leq \mu_W(\mathcal{T}_W(S_W^{k-1}\eta) \not\subseteq V) \\ &\leq C\mu_{W\setminus\{0\}}(\mathcal{T}_W(S_W^{k-1}\eta) \not\subseteq V) \\ &= C\mu_{W\setminus\{0\}}(\mathcal{T}_W(\eta) \not\subseteq V), \end{aligned} \tag{7.13}$$

where the last step follows by invariance of $\mu_{W\setminus\{0\}}$ under $\prod_{j\sim 0} a_j$. We have already seen that the right hand side of (7.13) goes to zero, which completes the proof of Proposition 6.5 assuming Proposition 7.11.

8 Finiteness of two-component spanning trees

In this Section we complete the arguments for finiteness of avalanches by proving Proposition 7.11, which amounts to showing that the weak limit of $\mathcal{T}_W$ is almost surely finite.

Let ν_W denote the probability measure assigning equal weight to each spanning tree of $\widetilde{W}$. ν_W is known as the uniform spanning tree measure in W with wired boundary conditions [3].

8.1 Wilson's algorithm

We use the algorithm below, due to Wilson [23], to analyze random samples from $\nu_W^{(0)}$ and ν_W.

Let G be a finite connected graph. By simple random walk on G we mean the random walk which at each step jumps to a random neighbour, chosen uniformly. For a path $\omega = [\omega_1, \dots, \omega_m]$ on G, define $\mathrm{LE}(\omega)$ as the path obtained by erasing loops chronologically from ω. We call $\mathrm{LE}(\omega)$ the *loop-erasure* of ω. Pick a vertex $r \in G$, called the *root*. Enumerate the vertices of G as $x_1, \dots, x_k$. Let $(S_n^{(i)})_{n \geq 1}$, $1 \leq i \leq k$ be independent simple random walks started at $x_1, \dots, x_k$, respectively. Let

$$T^{(1)} = \min\{n \geq 0 : S_n^{(1)} = r\},$$

and set

$$\gamma^{(1)} = \mathrm{LE}(S^{(1)}[0, T^{(1)}]).$$

Now recursively define $T^{(i)}$, $\gamma^{(i)}$, $i = 2, \dots, k$ as follows. Let

$$T^{(i)} = \min\{n \geq 0 : S_n^{(i)} \in \cup_{1 \leq j < i} \gamma^{(j)}\},$$

and

$$\gamma^{(i)} = \mathrm{LE}(S^{(i)}[0, T^{(i)}]).$$

(If $x_i \in \cup_{1 \leq j < i} \gamma^{(j)}$, then $\gamma^{(i)}$ is the single point x_i.) Let $T = \cup_{1 \leq i \leq k} \gamma^{(i)}$. Then T is a spanning tree of G and is uniformly distributed [23].

Applying the algorithm with $G = \widetilde{W}$ and root δ_W gives a sample from ν_W. Similarly, applying the method with $G = \widehat{W}$ and root δ_W we get a sample from $\nu_W^{(0)}$. In the latter case, we can imagine the construction happening in $\widetilde{W}$, where 0 is also considered part of the boundary. In other words, the two-component spanning tree is built from loop-erased random walks in $\widetilde{W}$ who attach either to a piece at 0, or to a piece at the boundary.

One can extend the algorithm to the case where G is an infinite graph, on which simple random walk is transient [3]. In this case, one chooses the root to be at infinity, and note that loop-erasure makes sense for paths that visit each site finitely many times.

8.2 The weak limit

Denote the two-component spanning tree in W by F_W. We denote by T_W the wired uniform spanning tree in $\widetilde{W}$. We regard δ_W as the root of T_W. If x is closer to the root than y (in graph distance) then we say that y is a descendant of x, and x is an ancestor of y. For a set $B \subseteq W$ we write $\mathrm{desc}(B; T_W)$ for the set of descendants of vertices in B. We sometimes think of the edges of T_W being directed towards the root. We write $V_N = [-N, N]^d \cap \mathbb{Z}^d$.

It is well-known that T_W has a weak limit T as $W \uparrow \mathbb{Z}^d$, called the (wired) uniform spanning forest (USF) on $\mathbb{Z}^d$ [19, 3]. We denote its law by ν. When $d \geq 3$, the USF can be constructed directly by Wilson's method in $\mathbb{Z}^d$, rooted at infinity [3, Theorem 5.1].

Similarly, F_W has a weak limit F. To see this, let W_n be an increasing sequence of finite volumes exhausting $\mathbb{Z}^d$. If B is a finite set of edges, [3, Corollary 4.3] implies that $\nu^{(0)}_{W_n}(B \subseteq F_{W_n})$ is increasing in n. This is sufficient to imply the existence of a limit $\nu^{(0)}$ independent of the sequence W_n, and the limit is uniquely determined by the conditions

$$\nu^{(0)}(B \subseteq F) = \lim_{n \to \infty} \nu^{(0)}_{W_n}(B \subseteq F_{W_n}),$$

as B varies over finite edge-sets (see the discussion in [3, Section 5]). This proves part (i) of Proposition 7.11. When $d \geq 3$, the configuration under $\nu^{(0)}$ can again be constructed by Wilson's method directly, by [3, Theorem 5.1]. Since 0 is part of the boundary, the simple random walks in this construction are either killed when they hit the component growing at 0, or they escape to infinity.

8.3 Finiteness of $\mathcal{T}$

For part (ii) of Proposition 7.11, we prove

$$\lim_{N \to \infty} \nu^{(0)}(\mathcal{T}_0 \subseteq V_N) = 1. \tag{8.1}$$

The proof of (8.1) is based on a coupling of $\nu^{(0)}$ and ν that arises from applying Wilson's algorithm with the same random walks in the two cases and with a suitable common enumeration of sites.

Let $1 \leq M < N$. We define the event

$$G(M, N) = \{\mathrm{desc}(V_M; T) \subseteq V_N\}.$$

In other words, $G(M, N)$ is the event that there exists a connected set $V_M \subseteq D \subseteq V_N$, such that there is no directed edge of T from $\mathbb{Z}^d \setminus D$ to D. By [3, Theorem 10.1], each component of the USF has one end, meaning that there are no two disjoint infinite paths within any component. This implies that for any $M \geq 1$,

$$\lim_{N \to \infty} \nu(G(M, N)) = 1. \tag{8.2}$$

We enumerate the sites in the following way. Let $x_1, \dots, x_l$ be an enumeration of the sites of ∂V_N. We let $x_{l+1}, x_{l+2}, \dots$ list the rest of the sites arbitrarily. As before, $(S^{(i)}_n)_{n\geq 1}$ denotes the random walk started at x_i, common to both constructions. Let $T^{(i)}$ and $\widehat{T}^{(i)}$ be the hitting times in the construction of T and F, respectively. Let $\gamma^{(i)}$ and $\widehat{\gamma}^{(i)}$ denote the corresponding families of loop-erased paths in the two constructions. The two families are determined by the same random walks, and we denote by Pr the probability law that governs both of them.

Consider the construction of T, and condition on $G(M,N)$. In terms of the paths $\gamma^{(i)}$, the conditioning can be written as

$$G(M,N) = \left\{ \left(\cup_{i=1}^{l} \gamma^{(i)} \right) \cap V_M = \varnothing \right\}.$$

The right hand side is in fact an implicit condition on the random walks $S^{(i)}$. We claim that if we further condition on the paths $\gamma^{(i)}$, $1 \leq i \leq l$, then we have

$$\Pr\big(\widehat{\gamma}^{(i)} \neq \gamma^{(i)} \text{ for some } 1 \leq i \leq l \,\big|\, \gamma^{(i)}, 1 \leq i \leq l \big) \leq \frac{C}{M^{d-4}}, \tag{8.3}$$

for some constant C, uniformly in the $\gamma^{(i)}$. Equivalently, we show

$$\Pr\big(S^{(i)}_n = 0 \text{ for some } 0 \leq n \leq T^{(i)},\ 1 \leq i \leq l \,\big|\, \gamma^{(i)}, 1 \leq i \leq l \big) \leq \frac{C}{M^{d-4}}. \tag{8.4}$$

To see that (8.3) and (8.4) are indeed equivalent, note that if $S^{(i)}[0, T^{(i)}]$ does not hit 0 for $1 \leq i \leq l$, then we have $\widehat{T}^{(i)} = T^{(i)}$ and $\widehat{\gamma}^{(i)} = \gamma^{(i)}$ for $1 \leq i \leq l$. On the other hand, if j is the smallest index such that $S^{(j)}[0, T^{(j)}]$ hits 0, then $\widehat{T}^{(j)} < T^{(j)}$ and $\widehat{\gamma}^{(j)} \neq \gamma^{(j)}$.

In order to show (8.4), we first fix $1 \leq j \leq l$ and prove a bound on

$$\Pr\big(S^{(j)}_n = 0 \text{ for some } 0 \leq n \leq T^{(j)} \,\big|\, \gamma^{(i)}, 1 \leq i \leq l \big). \tag{8.5}$$

By the definition of the $\gamma^{(i)}$, the expression in (8.5) in fact equals

$$\Pr\big(S^{(j)}_n = 0 \text{ for some } 0 \leq n \leq T^{(j)} \,\big|\, \gamma^{(i)}, 1 \leq i \leq j \big). \tag{8.6}$$

We analyze (8.6) using a description of the conditional distribution of a random walk given its loop-erasure (see [13]). This requires a few definitions. For $D \subseteq \mathbb{Z}^d$ and $y, z \in D \cup \partial D$, let $\mathcal{P}(D, y, z)$ denote the collection of all paths $\eta = [\eta_0, \dots, \eta_s]$ such that $\eta_0 = y$, $\eta_s = z$ and $\eta[0, s) \subseteq D$. For $y, z \in D \cup \partial D$ let $G_D(y, z)$ be the Green function for simple random walk started at y and killed at its first exit time T_D from D. We have

$$G_D(y,z) = \mathbb{E}^y \sum_{0 \leq n \leq T_D} I[S_n = z] = \sum_{\eta \in \mathcal{P}(D,y,z)} (2d)^{-|\eta|}.$$

In the last expression, $|\eta|$ denotes the number of steps in the path η. We also define the escape probability

$$\mathrm{Es}_D(y, B) = \mathbb{P}^y(S(n) \notin B, 1 \leq n < T_D).$$

Let $A \subseteq \mathbb{Z}^d$, $x \in \mathbb{Z}^d$ and let $\gamma = [\gamma_0, \dots, \gamma_m]$ be a self-avoiding path with $\gamma_0 = x$ and $\gamma[0, m) \subseteq A$. Let $S[0, \infty)$ denote simple random walk started at x. Let LE_m denote the operation of creating the first m steps of the loop-erased path, defined when there are at least m steps in the loop-erasure. It is simple to deduce that for any m-step self-avoiding path γ,

$$\Pr(\mathrm{LE}_m(S[0, T_A]) = \gamma) = (2d)^{-|\gamma|} \left[\prod_{p=0}^{|\gamma|} G_{A \setminus \gamma[0,p-1]}(\gamma_p, \gamma_p) \right] \mathrm{Es}_A(\gamma_m, \gamma[0, m]). \quad (8.7)$$

To see this, observe that one can decompose the random walk path starting at x and ending in A^c into its loop erasure γ, a family of loops $\eta_p \in \mathcal{P}(A \setminus \gamma[0, p-1], \gamma_p, \gamma_p)$, and the portion from the endpoint of γ to A^c (if any). Summing over the possible loops attached at every vertex γ_p, $0 \leq p \leq |\gamma|$ gives (8.7).

A small modification of (8.7) gives an expression for the probability that the loop at γ_p visits 0, when the loop-erasure is γ. Define

$$\widetilde{G}_D(y, z) = \sum_{\substack{\eta \in \mathcal{P}(D,y,z) \\ \eta \text{ visits } 0}} (2d)^{-|\eta|}.$$

Then

$$\begin{aligned} &\Pr(\mathrm{LE}_m(S[0, T_A]) = \gamma \text{ and the loop at } \gamma_p \text{ visits } 0) \\ &\quad = (2d)^{-|\gamma|} \widetilde{G}_{A \setminus \gamma[0,p-1]}(\gamma_p, \gamma_p) \left[\prod_{q \neq p} G_{A \setminus \gamma[0,q-1]}(\gamma_q, \gamma_q) \right] \mathrm{Es}_A(\gamma_m, \gamma[0, m]). \end{aligned} \quad (8.8)$$

Let T_γ denote the last time that $S(n)$ visits γ_{m-1}. Then equations (8.7) and (8.8) imply

$$\Pr(S[0, T_\gamma] \text{ visits } 0 \,|\, \mathrm{LE}_m(S[0, T_A]) = \gamma) \leq \sum_{p=0}^{m-1} \frac{\widetilde{G}_{A \setminus \gamma[0,p-1]}(\gamma_p, \gamma_p)}{G_{A \setminus \gamma[0,p-1]}(\gamma_p, \gamma_p)}. \quad (8.9)$$

We analyze the right hand side of (8.9) further. First note that $G_D(y, y) \geq 1$, due to the contribution of the 0-step walk. We also have

$$\begin{aligned} \widetilde{G}_D(y, y) &= \sum_{\eta_1 \in \mathcal{P}(D,y,0)} (2d)^{-|\eta_1|} \sum_{\substack{\eta_2 \in \mathcal{P}(D,0,y) \\ \eta_2 \text{ does not return to } 0}} (2d)^{-|\eta_2|} \\ &\leq \sum_{\eta_1 \in \mathcal{P}(D,y,0)} (2d)^{-|\eta_1|} \sum_{\eta_2 \in \mathcal{P}(D,0,y)} (2d)^{-|\eta_2|} \\ &= G(y, 0; D)\, G(0, y; D) \leq G(y)^2, \end{aligned}$$

where $G(y)$ is the Green function in $\mathbb{Z}^d$. This yields

$$\Pr(S[0, T_\gamma] \text{ visits } 0 \,|\, \mathrm{LE}_m(S[0, T_A]) = \gamma) \leq \sum_{p=0}^{m-1} G(\gamma_p)^2. \quad (8.10)$$

From this we obtain

$$\Pr(S[0,T_A] \text{ visits } 0 \,|\, \mathrm{LE}(S[0,T_A]) = \gamma) \le \sum_{0 \le p < |\gamma|} G(\gamma_p)^2 \tag{8.11}$$

for any finite or infinite self-avoiding path γ. When γ is finite, this follows from the case $\gamma_m \in A$, when $T_A = T_\gamma + 1$, and when γ is infinite, we let $m \to \infty$. Applying (8.11) with $A = \mathbb{Z}^d \setminus \cup_{1 \le i < j} \gamma^{(i)}$, the expression in (8.6) is less than

$$\sum_{y \in \bar{\gamma}^{(i)}} G(y)^2, \tag{8.12}$$

where $\bar{\gamma}$ denotes the vertices of the path γ, excluding the last one, if $\gamma^{(i)}$ is finite. Summing this over $1 \le i \le l$, and using that the $\bar{\gamma}^{(i)}$ are disjoint and $\gamma^{(i)} \cap V_M = \varnothing$, we obtain that the left hand side of (8.4) is less than

$$\sum_{i=1}^{l} \sum_{y \in \bar{\gamma}^{(i)}} G(y)^2 \le \sum_{y \in \mathbb{Z}^d \setminus V_M} G(y)^2. \tag{8.13}$$

Using $d > 4$ and the well-known fact $G(y) \le C|y|^{d-2}$ [12, Theorem 1.5.4], we obtain the claim in (8.4).

Now we can complete the proof of Proposition 7.11 (ii). Observe that on the event

$$G(M,N) \cap \{\widehat{\gamma}^{(i)} = \gamma^{(i)}, 1 \le i \le l\}$$

we have $\mathcal{T}_0 \subseteq V_N$. Therefore, by (8.3),

$$\Pr\big(\mathcal{T}_0 \subseteq V_N \,\big|\, G(M,N)\big) \ge 1 - \frac{C}{M^{d-4}}. \tag{8.14}$$

Choosing M large and then N large, (8.2) and (8.14) imply (8.1) and completes the proof.

9 Tail triviality of μ

In Section 10 we are going to need the $d > 4$ part of the following theorem.

Theorem 9.1. *The measure μ is tail trivial for any $d \ge 2$.*

Proof. [Case $2 \le d \le 4$*]* The proof is based on the fact that the uniform spanning forest measure ν is tail trivial [3, Theorem 8.3]. Let $\mathcal{X} \subseteq \{0,1\}^{\mathbb{E}^d}$ denote the set of spanning trees of $\mathbb{Z}^d$ with one end. Recall the uniform spanning forest measure ν from Section 8. It was shown by Pemantle [19] that when $2 \le d \le 4$, the measure ν is concentrated on $\mathcal{X}$.

It is shown in [1] that there is a mapping $\psi : \mathcal{X} \to \Omega$ such that μ is the image of ν under ψ. Moreover, ψ has the following property. Let $T_x = T_x(\omega)$ denote the tree consisting of all ancestors of x and its $2d$ neighbours in ω. In other words, T_x is the union of the paths leading from x and its neighbours to infinity. It follows from the results in [1] that $\eta_x = (\psi(\omega))_x$ is a function of T_x alone.

Assume that $f(\eta)$ is a bounded tail measurable function. Then for any n, f is a function of $\{\eta_x : \|x\|_\infty \geq n\}$ only. This means that $f(\eta) = f(\psi(\omega)) = g(\omega)$ is a function of the family $\{T_x(\omega) : \|x\|_\infty \geq n\}$. Let $\mathcal{F}_k = \sigma(\omega_e : e \cap [-k,k]^d = \varnothing)$. For $1 \leq k < n$ consider the event

$$E_{n,k} = \bigcap_{x:\|x\|_\infty \geq n} \{T_x \cap [-k,k]^d = \varnothing\}.$$

Observe that $E_{n,k} \in \mathcal{F}_k$, and $gI[E_{n,k}]$ is $\mathcal{F}_k$-measurable. Using that ω has a single end ν-a.s., it is not hard to check that for any $k \geq 1$

$$\lim_{n\to\infty} \nu(E_{n,k}) = 1.$$

Letting $n \to \infty$, this implies that there is an $\mathcal{F}_k$-measurable function $\hat{g}_k$, such that $g = \hat{g}_k$ ν-a.s. Since this holds for any $k \geq 1$, tail triviality of ν implies that g is constant ν-a.s. Letting c denote the constant, this imples

$$\mu(f(\eta) = c) = \nu(f(\psi(\omega)) = c) = 1,$$

which completes the proof in the case $2 \leq d \leq 4$.

[Case $d > 4$] The above simple proof does not work when $d > 4$, due to the fact that there is no coding of the sandpile configuration in terms of the USF in infinite volume. Nevertheless, it turns out that a coding is possible by adding extra randomness to the USF, namely, a random ordering of its components. Due to the presence of this random ordering, however, we have not been able to deduce tail triviality of μ directly from tail triviality of ν.

We start by recalling results from [1]. Let $\mathcal{X}$ denote the set of spanning forests of $\mathbb{Z}^d$ with infinitely many components, where each component is infinite and has a single end. The USF measure ν is concentrated on $\mathcal{X}$ [3]. Given $x \in \mathbb{Z}^d$ and $\omega \in \mathcal{X}$, let $T_x^{(1)}(\omega), \dots, T_x^{(k)}(\omega)$ denote the trees consisting of all ancestors of x and its $2d$ neighbours in ω. Here $k = k_x(\omega) \geq 1$. Each $T_x^{(i)}$ is a union of infinite paths starting at x or a neighbour of x, and has a unique vertex $v_x^{(i)}$ that is the first point common to all paths.. Let $F_x^{(i)}(\omega)$ denote the tree consisting of all descendants of $v_x^{(i)}$ in ω. Let $\mathcal{F}$ denote the collection of finite rooted trees in $\mathbb{Z}^d$. Let Σ_l denote the set of permutations of the symbols $\{1, \dots, l\}$.

It follows from the proofs of Lemma 3 and Theorem 1 in [1] that the sandpile height at x is a function of $\{F_x^{(i)}(\omega), v_x^{(i)}(\omega)\}_{i=1}^k$ and a random $\sigma_x \in \Sigma_k$, in the following sense. There are functions $\psi_l : \mathcal{F}^l \times \Sigma_l$, $l = 1, 2, \dots$ such that if σ_x is a uniform random element of Σ_k, given ω, then

$$\eta_x = \psi_{k_x}((F_x^{(1)}, v_x^{(1)}), \dots, (F_x^{(k)}, v_x^{(k)}), \sigma_x) \tag{9.2}$$

has the distribution of the height variable at x under μ. Here it is convenient to think of σ_x as a random ordering of those components of ω that contain at least one neighbour of x. Then one can also view η_x as a function of $\{T_x^{(i)}\}_{i=1}^k$ and σ_x.

Next we turn to a description of the joint distribution of $\{\eta_x\}_{x\in A_0}$ for $A_0 \subseteq \mathbb{Z}^d$ finite. Let A denote the set of those site that are either in A_0 or have a neighbour in A_0. Let $\mathcal{C}^{(1)}, \dots, \mathcal{C}^{(K)}$ denote the components of the USF intersecting A, with $K = K_A(\omega)$. Each $\mathcal{C}^{(i)}$ contains a unique vertex $v_A^{(i)}$ where the paths from $A \cap \mathcal{C}^{(i)}$ to infinity first meet. Let $F_A^{(i)}$ denote the tree consisting of all descendants of $v_A^{(i)}$. Each rooted tree $(F_x^{(j)}, v_x^{(j)})$, $x \in A_0, 1 \le j \le k_x$ is a subtree of some $F_A^{(i)}$, $1 \le i \le K$ and the former are determined by the latter. Let $\sigma_A \in \Sigma_K$ be uniformly distributed, given ω. For each $x \in A_0$, σ_A induces a permutation in Σ_{k_x}, by restriction. It follows from the results in [1] that the height configuration in A_0 is a function of $\{(F_A^{(i)}, v_A^{(i)})\}_{i=1}^K$ and σ_K. Moreover, the joint distribution of $\{\sigma_x\}_{x\in A_0}$ is the one induced by σ_A.

From the above we obtain the following description of μ in terms of the USF and a random ordering of its components. Let $\omega \in \mathcal{X}$ be distributed according to ν. Given ω, we define a random partial ordering $\prec_\omega$ on $\mathbb{Z}^d$ in the following way. Let $\mathcal{C}^{(1)}, \mathcal{C}^{(2)}, \dots$ be an enumeration of the components of ω, and let $U_1, U_2, \dots$ be i.i.d. random variables, given ω, having the uniform distribution on $[0, 1]$. Define the random partial order $\prec_\omega$ depending on ω and $\{U_i\}_{i\ge 1}$ by letting $x \prec_\omega y$ if and only if $x \in \mathcal{C}^{(i)}$, $y \in \mathcal{C}^{(j)}$ and $U_i < U_j$. Even though $\prec_\omega$ is defined for sites, it is simply an ordering of the components of ω. The distribution of $\prec$ is in fact uniquely characterized by the property that it induces the uniform permutation on any finite set of components, and one could define it by this property, without reference to the U's. This in turn shows that the distribution is independent of the ordering $\mathcal{C}^{(1)}, \mathcal{C}^{(2)}, \dots$ initially chosen.

Let $\mathcal{Q} = \{0, 1\}^{\mathbb{Z}^d \times \mathbb{Z}^d}$ denote the space of binary relations (where for $q \in \mathcal{Q}$ we interpret $q(x, y) = 1$ as $x \prec y$, and $q(x, y) = 0$ otherwise). We denote the joint law of $(\omega, \prec)$ on $\mathcal{X} \times \mathcal{Q}$ by $\widetilde{\nu}$. From the couple $(\omega, \prec)$, we can recover the random permutations σ_x as follows. If $v_x^{(1)}, \dots, v_x^{(k)}$ are as defined earlier, then

$$(\sigma_x(1), \dots, \sigma_x(k)) = (j_1, \dots, j_k)$$

if and only if

$$v_x^{(j_1)} \prec_\omega \cdots \prec_\omega v_x^{(j_k)}.$$

The sandpile height configuration is a function of the couple $(\omega, \prec_\omega)$. Using the above σ_x in (9.2) gives $\{\eta_x\}_{x\in\mathbb{Z}^d}$ with distribution μ. In other words, there is a $\widetilde{\nu}$-a.s. defined function $\psi : \mathcal{X} \times \mathcal{Q} \to \Omega$ such that μ is the image of $\widetilde{\nu}$ under ψ.

Before we start the argument proper, we need to recall some further terminology from [1]. Given finite rooted trees $(\bar{F}, \bar{v}) = (F_i, v_i)_{i=1}^K$ and a finite set of sites A, define the events

$$D(\bar{v}) = \{v_1, \dots, v_K \text{ are in distinct components of } \omega\},$$
$$B(\bar{F}, \bar{v}) = D(\bar{v}) \cap \{F_A^{(i)} = F_i, v_A^{(i)} = v_i \text{ for } 1 \le i \le K\},$$

For $\Lambda \subseteq \mathbb{Z}^d$ finite we also define

$$D_\Lambda(\bar{v}) = \{v_1, \dots, v_K \text{ are in distinct components of } \omega_\Lambda\},$$
$$B_\Lambda(\bar{F}, \bar{v}) = D_\Lambda(\bar{v}) \cap \{F_A^{(i)} = F_i, v_A^{(i)} = v_i \text{ for } 1 \le i \le K\},$$

where ω_Λ is the wired UST in Λ.

Recall that tail triviality is equivalent to the following [7, page 120]. For any cylinder event E' and $\varepsilon > 0$ there exists n such that (with $V_n = [-n, n]^d \cap \mathbb{Z}^d$) for any event $R' \in \mathcal{F}_{V_n^c}$ we have

$$|\mu(E' \cap R') - \mu(E')\mu(R')| \le \varepsilon. \tag{9.3}$$

Fix E' and ε, and for the moment also fix n and R'. Let $E = \psi^{-1}(E')$ and $R = \psi^{-1}(R')$. Let A_0 denote the set of sites on which E' depends, and let A be the set of sites that are either in A_0 or have a neighbour in A_0.

We first have a closer look at the event E. We define

$$S(\bar{F}, \bar{v}, \sigma) = B(\bar{F}, \bar{v}) \cap \{v_{\sigma(1)} \prec \cdots \prec v_{\sigma(K)}\},$$
$$\mathcal{G}_E = \{(\bar{F}, \bar{v}, \sigma) : S(\bar{F}, \bar{v}, \sigma) \subseteq E\},$$
$$\mathcal{G}_E(r) = \{(\bar{F}, \bar{v}, \sigma) \in \mathcal{G}_E : F_i \subseteq V_r \text{ for } 1 \le i \le K\}.$$

Here E is a disjoint union of the events $S(\bar{F}, \bar{v}, \sigma)$ over $(\bar{F}, \bar{v}, \sigma) \in \mathcal{G}_E$. By the definition of $\prec$ we have

$$\mu(E') = \widetilde{\nu}(E) = \sum_{(\bar{F}, \bar{v}, \sigma) \in \mathcal{G}_E} \frac{1}{K!} \nu(B(\bar{F}, \bar{v})).$$

We also define an analogue of S in a finite volume Λ. Assume that the relation $\prec_\partial$ is prescribed on the exterior boundary of Λ. For any realization of the wired UST ω_Λ there is a unique extension of $\prec_\partial$ into Λ, denoted $\prec_\Lambda$, where $x \prec_\Lambda y$ if and only if they are connected (in ω_Λ) to boundary vertices $w(x)$ and $w(y)$ satisfying $w(x) \prec_\partial w(y)$. Using this extension, we define

$$S_\Lambda(\bar{F}, \bar{v}, \sigma) = B_\Lambda(\bar{F}, \bar{v}) \cap \{v_{\sigma(1)} \prec_\Lambda \cdots \prec_\Lambda v_{\sigma(K)}\}.$$

We let $\widetilde{\nu}_{\Lambda, \prec_\partial}$ denote the law of $(\omega_\Lambda, \prec_\Lambda)$ with boundary condition $\prec_\partial$.

Introduce

$$G = G(r) = \{F_A^{(i)} \subseteq V_r \text{ for } 1 \le i \le K\},$$

where we asume that $A_0 \subseteq V_r \subseteq V_n$. Now $E \cap G$ is a disjoint union of the events $S(\bar{F}, \bar{v}, \sigma)$ over $(\bar{F}, \bar{v}, \sigma) \in \mathcal{G}_E(r)$. Since A_0 is fixed, we can choose r large enough so that $\nu(G(r)^c) \le \varepsilon$.

Turning to R, we define

$$\mathcal{H} = \mathcal{H}_n = \bigcup_{x \in V_n^c} \bigcup_{i=1}^{k_x} \text{vertex set of } T_x^{(i)}$$
$$\mathcal{D} = \mathcal{D}_n = \mathbb{Z}^d \setminus \mathcal{H}_n.$$

The occurrence of R is determined by the collection of edges joining vertices in $\mathcal{H}$ together with the restriction of $\prec$ to $\mathcal{H}$. We also introduce for $r < m < n$ and $V_m \subseteq \Lambda \subseteq V_n$ the events

$$F_\Lambda = \{\mathcal{D}_n = \Lambda\} \quad \text{and} \quad F = F(n,m) = \cup_\Lambda F_\Lambda = \{\mathcal{H}_n \cap V_m = \varnothing\}.$$

In other words, F is the event that the portion of ω determining the sandpile configuration in V_n^c does not intersect V_m. The value of m will be chosen later. It is easy to see that for fixed m one can choose n large enough, so that $\nu(F^c) \le \varepsilon$. This is because $F(n,m)$ is monotone increasing in n, and $\cap_{n=m+1}^\infty F(n,m)^c = \varnothing$, since each component of the USF has a single end.

In addition to G and F we need a third auxillary event. Let

$$J = \{\forall x, y \in V_r : \text{if } x \leftrightarrow y \text{ then they are connected inside } V_m\},$$

where $x \leftrightarrow y$ means that x and y are in the same component of the USF. Using again that each component of ω has one end, for large enough m we have $\nu(J^c) \le \varepsilon\varepsilon_1$, where we have set $\varepsilon_1 = \varepsilon_1(r) = \varepsilon/|\mathcal{G}_E(r)|$. Define the event

$$J_0 = F \cap \big\{\nu\big(J^c \,\big|\, \omega_{\mathcal{H}_n}\big) \le \varepsilon_1\big\},$$

where $\omega_{\mathcal{H}_n}$ denotes the configuration on the set of edges touching $\mathcal{H}_n$. By Markov's inequality,

$$\nu(J_0^c) \le \nu(F^c) + \nu\Big(F \cap \big\{\nu\big(J^c \,\big|\, \omega_{\mathcal{H}_n}\big) \ge \varepsilon_1\big\}\Big) \le \varepsilon + \frac{\nu(J^c)}{\varepsilon_1} \le 2\varepsilon.$$

Choosing r large enough, m large enough and n large enough, we have

$$|\mu(E' \cap R') - \widetilde{\nu}(E \cap G \cap R \cap J_0))| \le 4\varepsilon. \tag{9.4}$$

Recall that we regard the edges of ω being directed towards infinity. By the definition of $\mathcal{H}$, there are no directed edges from $\mathcal{H}$ to $\mathcal{D}$. Therefore, given the restriction of ω to $\mathcal{H}$, the conditional law of ω in $\mathcal{D}$ is that of the wired uniform spanning tree in $\mathcal{D}$ (denoted ν_D). One can see this by using Wilson's method rooted at inifnity to first generate the configuration on $\mathcal{H}$, and then the configuration in $\mathcal{D}$.

Note that the event F_Λ only depends on the portion of ω outside Λ. We want to rewrite the second term on the left hand side of (9.4) by conditioning on F_Λ, the portion of ω outside Λ, and the restriction of $\prec$ to $\mathbb{Z}^d \setminus \Lambda$. By the previous paragraph, the conditional distribution of $(\omega, \prec)$ inside Λ is given by $\widetilde{\nu}_{\Lambda,\prec_\partial}$, where $\prec_\partial$ is determined by the conditioning.

The above implies

$$\widetilde{\nu}(E \cap G \cap R \cap J_0) = \sum_{V_m \subseteq \Lambda \subseteq V_n} \int_{R \cap J_0 \cap F_\Lambda} \widetilde{\nu}_{\Lambda,\prec_\partial}(E \cap G)\, d\widetilde{\nu}. \tag{9.5}$$

Next we further analyze

$$\widetilde{\nu}_{\Lambda,\prec_\partial}(E \cap G) = \sum_{(\bar{F},\bar{v},\sigma)\in\mathcal{G}_E(r)} \widetilde{\nu}_{\Lambda,\prec_\partial}(S_\Lambda(\bar{F},\bar{v},\sigma)). \tag{9.6}$$

Our aim is to show that the summand in (9.6) is close to $\nu(B(\bar{F},\bar{v}))/K!$, uniformly in Λ and the boundary condition, if m is large enough. For this we turn to a description of the event $B_\Lambda(\bar{F},\bar{v})$ in terms of Wilson's algorithm. This part of the proof is similar to the proof of Lemma 3 in [1], however it does not seem possible to use that result directly.

Fix $(F_i, v_i)_{i=1}^K$ and $\sigma \in \Sigma_K$. Enumerate the vertices in $\cup_{i=1}^K F_i$, starting with $v_1, \dots, v_K$ and followed by the rest of the vertices $y_1, y_2, \dots$ in an arbitrary order. We apply Wilson's method with root at the wired vertex of Λ, and the above enumeration. Let $S^{(i)}$, $i = 1, \dots, K$ be independent simple random walks started at v_i. Let $\gamma_\Lambda^{(i)}$ denote the loop-erasure of $S^{(i)}$ up to its first exit time from Λ. We define an event C_Λ whose occurrence is equivalent to the occurrence of $B_\Lambda(\bar{F},\bar{v})$, by Wilson's method. Since the event $D_\Lambda(\bar{v})$ has to occur, we require that for $i = 1, \dots, K$, $S^{(i)}$ upto its first exit time be disjoint from $\cup_{1\le j<i}\gamma_\Lambda^{(i)}$. In addition, the fact that $B_\Lambda(\bar{F},\bar{v})$ has to occur, gives conditions on the paths starting at $y_1, y_2, \dots$, namely, these paths have to realize the events $(F_A^{(i)}, v_A^{(i)}) = (F_i, v_i)$, given the paths $\{\gamma_\Lambda^{(i)}\}_{i=1}^K$. These implicit conditions define C_Λ. We write Pr for probabilities associated with random walk events, and we couple the constructions in different volumes by using the same infinite random walks $S^{(i)}$. Analogously we define the $\Lambda = \mathbb{Z}^d$ version, C, which corresponds to $B(\bar{F},\bar{v})$.

Let $W_\Lambda^{(i)}$ denote the vertex where $S^{(i)}$ exits Λ. Then we have

$$\widetilde{\nu}_{\Lambda,\prec_\partial}(S_\Lambda(\bar{F},\bar{v},\sigma)) = \Pr(C_\Lambda,\, W_\Lambda^{(\sigma(1))} \prec \cdots \prec W_\Lambda^{(\sigma(K))}). \tag{9.7}$$

For $r < l < m$ we consider the event C_{V_l}, and write C_l for short. It is not hard to see that $\lim_\Lambda I[C_\Lambda] = I[C]$, Pr-a.s., which implies that for l large enough, $\Pr(C_l \triangle C) \le \varepsilon_1$.. The difference between the right hand side of (9.7) and

$$\Pr(C_l,\, W_\Lambda^{(\sigma(1))} \prec \cdots \prec W_\Lambda^{(\sigma(K))}) \tag{9.8}$$

is at most $2\varepsilon_1$. Recall that $\Lambda \supseteq V_m$, and $m > l$. By conditioning on the first exit points from V_l, (9.8) can be written as

$$\Pr(C_l) \Pr\big(W_\Lambda^{(\sigma(1))} \prec \cdots \prec W_\Lambda^{(\sigma(K))} \,\big|\, W_l^{(1)}, \dots, W_l^{(K)}\big). \tag{9.9}$$

The first factor here differs from $\Pr(C) = \nu(B(\bar{F},\bar{v}))$ by at most ε_1. If m is large with respect to l, the value of the second factor is essentially independent of σ. This is because by a standard coupling argument, the distributions of $W_\Lambda^{(i)}$ and $W_\Lambda^{(j)}$ given $W_l^{(i)}$ and $W_l^{(j)}$ (respectively), can be made arbitrarily close in total variation distance. This implies that the difference between (9.9) and

$$\Pr(C) \Pr\big(W_\Lambda^{(1)} \prec \cdots \prec W_\Lambda^{(K)} \,\big|\, W_l^{(1)}, \dots, W_l^{(K)}\big)$$

is at most ε_1, if m is large enough, uniformly in Λ.

Observe that if $W_\Lambda^{(i)} \leftrightarrow W_\Lambda^{(j)}$ for some $1 \le i < j \le K$, then the event J^c occurs. Since the integration in (9.5) is over a subset of J_0, we have

$$\Pr\bigl(C_\Lambda,\, W_\Lambda^{(i)} \leftrightarrow W_\Lambda^{(j)} \text{ for some } 1 \le i < j \le K\bigr) \le \widetilde{\nu}_{\Lambda,\prec_\partial}(J^c) \le \varepsilon_1. \tag{9.10}$$

It follows that for some universal constant C, if m is large enough

$$\bigl|\Pr\bigl(C_\Lambda,\, W_\Lambda^{(\sigma(1))} \prec \cdots \prec W_\Lambda^{(\sigma(K))}\bigr) - \Pr(C)/K!\bigr| \le C\varepsilon_1.$$

Now an application of (9.7) and (9.6) implies

$$|\widetilde{\nu}_{\Lambda,\prec_\partial}(E \cap G) - \widetilde{\nu}(E \cap G)| \le C\varepsilon,$$

uniformly in Λ. Therefore, the difference between the right hand side of (9.5) and

$$\widetilde{\nu}(E \cap G) \sum_{V_m \subseteq \Lambda \subseteq V_n} \widetilde{\nu}(R \cap J_0 \cap F_\Lambda) = \widetilde{\nu}(E \cap G)\widetilde{\nu}(R \cap J_0)$$

is at most $C\varepsilon$.

Using the choice of r and the choice of n again, we get

$$|\mu(E' \cap R') - \mu(E')\mu(R')| \le C\varepsilon,$$

proving the claim in the case $d > 4$. $\square$

10 Ergodicity of the stationary process

Arrived at this point, we can apply the results in [14], and we obtain the following.

Theorem 10.1. *Let $\varphi : \mathbb{Z}^d \to (0, \infty)$ be an addition rate such that*

$$\sum_x \varphi(x) G(0, x) < \infty. \tag{10.2}$$

Then the following hold.

1. *The closure of the operator on $L_2(\mu)$ defined on local functions by*

$$L_\varphi f = \sum_x \varphi(x)(a_x - I) f \tag{10.3}$$

is the generator of a stationary Markov process $\{\eta_t : t \ge 0\}$.

2. *If φ satisfies (10.2), then let $N_t^\varphi(x)$ denote Poisson processes with rate $\varphi(x)$ that are independent (for different x). The limit*

$$\eta_t = \lim_{V \uparrow \mathbb{Z}^d} \left[\prod_{x \in V} a_x^{N_t^\varphi(x)} \right] \eta \tag{10.4}$$

exists a.s. with respect to the product of the Poisson process measures on N_t^φ with the stationary measure μ on the $\eta \in \Omega$. Moreover, η_t is a cadlag version of the process with generator L_φ.

Let $\{\eta_t : t \geq 0\}$ be the stationary process with generator $L_\varphi = \sum_x \varphi(x)(a_x - I)$. We recall that a process is called ergodic if every (time-)shift invariant measurable set has measure zero or one. For a Markov process, this is equivalent to the following: if $S_t f = f$ for all $t > 0$, then f is constant μ-a.s. This in turn is equivalent to the statement that $Lf = 0$ implies f is constant μ-a.s. The tail σ-field on Ω is defined as usual:

$$\mathcal{F}_\infty = \bigcap_{n \in \mathbb{N}} \sigma\{\eta(x) : |x| \geq n\} \tag{10.5}$$

A function f is tail measurable if its value does not change by changing the configuration in a finite number of sites, i.e., if

$$f(\eta) = f(\xi_V \eta_{V^c})$$

for every ξ and $V \subseteq \mathbb{Z}^d$ finite.

Theorem 10.6. *The stationary process of Theorem 10.1 is mixing.*

Proof. Recall that G denotes the group generated by the unitary operators a_x on $L_2(\mu)$. Consider the following statements.

1. The process $\{\eta_t : t \geq 0\}$ is ergodic.
2. The process $\{\eta_t : t \geq 0\}$ is mixing.
3. Any G-invariant function is μ-a.s. constant.
4. μ is tail trivial.

Then we have the following implications: 1, 2 and 3 are equivalent and 4 implies 3. This will complete the proof, because 4 holds by Theorem 9.1.

It is easy to see that on $L_2(\mu)$,

$$L^* = \sum_x \varphi(x)(a_x^{-1} - I). \tag{10.7}$$

Hence L and L^* commute, i.e., L is a normal operator. The equivalence of 1 and 2 then follows immediately, see [22]. To see the equivalence of 1 and 3: suppose $Lf = 0$, then, using invariance of μ under a_x

$$\langle Lf | f \rangle = -\frac{1}{2} \sum_x \varphi(x) \int (a_x f - f)^2 d\mu = 0. \tag{10.8}$$

Similarly, $Lf = 0$ implies $L^* f = 0$, hence

$$\langle L^* f | f \rangle = -\frac{1}{2} \sum_x \varphi(x) \int (a_x^{-1} f - f)^2 d\mu = 0, \tag{10.9}$$

which shows the invariance of f under a_x and a_x^{-1}, and thus under the action of G. Finally, to prove the implication $4 \Rightarrow 3$, we will show that a function invariant under the action of G is tail measurable. Suppose $f : \mathcal{R} \to \mathbb{R}$, and $f(a_x\eta) = f(\eta) = f(a_x^{-1}\eta)$ for all x. If η and ζ are elements of $\mathcal{R}$ and differ in a finite number of coordinates, then

$$\zeta = \prod_x a_x^{\zeta(x)-\eta(x)}\eta \tag{10.10}$$

and hence $f(\eta) = f(\zeta)$. This proves that f is tail measurable. □

A Appendix

In this section we indicate how to extend the argument of [1] in the case $d > 4$ and prove $\lim_\Lambda \mu_\Lambda = \mu$. This boils down to showing that (18) and (19) in [1] (referred to as (18)[1], etc. below) hold with the limit taken through arbitrary volumes. Most of the argument in [1] has been carried out for general volumes, and we only mention the differences. We use the notation introduced in Section 9.

We start with the extension of (18)[1]. Let $x, y \in \mathbb{Z}^d$ be fixed, and let $S^{(1)}$ and $S^{(2)}$ be independent simple random walks starting at x and y, respectively. Let $T_\Lambda^{(1)}$ and $T_\Lambda^{(2)}$ be the first exit times from Λ for these random walks. The required extension of (18)[1] follows from an extension of (27)[1], which in turn follows from the statement

$$\lim_{\delta\to 0}\limsup_\Lambda \Pr\Big(1-\delta \le \frac{T_\Lambda^{(1)}}{T_\Lambda^{(2)}} \le 1+\delta\Big) = 0. \tag{A.1}$$

Statement (A.1) is proved in [11].

For the extension of (19)[1], we recall from Section 9 the events $B_\Lambda(\bar{F}, \bar{v})$ and $B(\bar{F}, \bar{v})$ defined for a collection $(F_i, v_i)_{i=1}^K$. Let $S^{(i)}$, $i = 1, \ldots, K$ be independent random walks started at v_i, respectively. Let $T_\Lambda^{(i)}$ be the exit time of $S^{(i)}$ from Λ. Also recall the random walk events C_Λ and C, and that C_m and $T_m^{(i)}$ are short for C_Λ and $T_\Lambda^{(i)}$ when $\Lambda = [-m, m]^d \cap \mathbb{Z}^d$. By the arguments in [1], the required extension of (19)[1] follows, once we show an extension of (32)[1], namely that for any permutation $\sigma \in \Sigma_K$

$$\lim_{m\to\infty}\lim_\Lambda \Pr\left(C_m, T_\Lambda^{\sigma(1)} < \cdots < T_\Lambda^{\sigma(K)}\right) = \Pr(C)\frac{1}{K!}. \tag{A.2}$$

Observe that C_m and the collection $\widetilde{T}_{\Lambda,m}^{(i)} = T_\Lambda^{(i)} - T_m^{(i)}$, $i = 1, \ldots, K$ are conditionally independent, given $\{S^{(i)}(T_m^{(i)})\}_{i=1}^K$. Therefore, using (A.1), the left hand side of (A.2) equals

$$\lim_{m\to\infty}\lim_\Lambda \Pr(C_m)\Pr\left(\widetilde{T}_{\Lambda,m}^{\sigma(1)} < \cdots < \widetilde{T}_{\Lambda,m}^{\sigma(K)}\right). \tag{A.3}$$

By a standard coupling argument, the second probability approaches $1/K!$ for any fixed m, and hence the limit in (A.3) equals $P(C)/K!$. This completes the proof of the required extension of (19)[1].

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References

[1] Athreya, S.R. and Járai, A.A.: Infinite volume limit for the stationary distribution of Abelian sandpile models. Commun. Math. Phys. **249**, 197-213 (2004).

[2] Bak, P., Tang, K. and Wiesefeld, K.: Self-organized criticality. Phys. Rev. A **38**, 364-374 (1988).

[3] Benjamini, I., Lyons, R., Peres, Y. and Schramm, O.: Uniform spanning forests. Ann. Prob. **29** 1–65 (2001).

[4] Dhar D.: Self organized critical state of sandpile automaton models. Phys. Rev. Letters **64**, 1613-1616 (1990) .

[5] Dhar D.: The Abelian sandpile and related models. Physica A **263**, 4-25 (1999).

[6] Gabrielov, A.: Asymmetric Abelian avalanches and sandpiles. Preprint (1994).

[7] Georgii, H.-O.: Gibbs measures and phase transitions. de Gruyter, Berlin (1988).

[8] Ivashkevich, E.V., Ktitarev, D.V. and Priezzhev, V.B.: Critical exponents for boundary avalanches in two-dimensional Abelian sandpile. J. Phys. A **27** L585–L590 (1994).

[9] Ivashkevich, E.V., Ktitarev, D.V. and Priezzhev, V.B.: Waves of topplings in an Abelian sandpile. Physica A **209** 347–360 (1994).

[10] Ivaskevich, E.V. and Priezzhev, V.B.: Introduction to the sandpile model. Preprint (2003) http://arxiv.org/abs/cond-mat/9801182.

[11] Járai, A.A. and Kesten, H.: A bound for the distribution of the hitting time of arbitrary sets by random walk. Preprint (2004).

[12] Lawler, G.F.: *Intersections of random walks.* Birkhäuser, softcover edition (1996).

[13] Lawler, G.F.: Loop-erased random walk. In *Perplexing problems in Probability*, Progress in Probability, Vol. 44, Birkhäuser, Boston, (1999).

[14] Maes, C., Redig, F. and Saada, E.: The Abelian sandpile model on an infinite tree. Ann. Probab. **30**, 2081–2107, (2002).

[15] Maes, C., Redig, F. and Saada, E.: The infinite volume limit of dissipative Abelian sandpiles. Comm. Math. Phys. **244**, 395–417 (2004).

[16] Maes, C., Redig F., Saada, E. and Van Moffaert, A.: On the thermodynamic limit for a one-dimensional sandpile process. Markov Process. Related Fields **6** 1–22 (2000).

[17] Majumdar, S.N. and Dhar, D.: Equivalence between the Abelian sandpile model and the $q \to 0$ limit of the Potts model. Physica A **185** 129–145 (1992).

[18] Meester, R., Redig, F. and Znamenski, D.: The Abelian sandpile: a mathematical introduction. Markov Process. Related Fields **6**, 1-22, (2000).

[19] Pemantle, R.: Choosing a spanning tree for the integer lattice uniformly. Ann. Prob.**19** 1559–1574, (1991).

[20] Priezzhev, V.B.: Structure of two-dimensional sandpile. I. Height probabilities. J. Stat. Phys.**74**, 955–979 (1994).

[21] Priezzhev, V.B.: The upper critical dimension of the Abelian sandpile model. J. Stat. Phys.**98**, 667-684 (2000).

[22] Rosenblatt, M.: Transition probability operators. Proc. Fifth Berkely Symposium Math. Stat. Prob., **2**, 473-483, (1967).

[23] Wilson, D.B.: Generating random spanning trees more quickly than the cover time. In *Proceedings of the Twenty-Eighth ACM Symposium on the Theory of Computing* 296–303. ACM, New York, (1996).